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GAUSSIAN EQUIVALENT REPRESENTATION OF FUNCTIONAL INTEGRALS IN QUANTUM PHYSICS

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A new method is proposed for systematic approximate calculations of a large class of non-Gaussian functional integrals beyond the region of perturbative expansion. This method provides a good accuracy of the lowest approximation, obtained in a simple way, which represents a generalization of the variational estimation if the functionals are real. In contrast to the variational approach, this method is applicable to complex functionals and theories with ultraviolet divergencies. Higher-order corrections to the lowest approximation are evaluated by a regular scheme. This method is applied to different problems of theoretical physics: the polaron problem in solid states, the phase-transition phenomenon in quantum field models and the investigation of wave transmission in randomly distributed media.

Предложен новый, непертурбативный метод систематического вычисления широкого класса функциональных интегралов, применяемых в квантовой физике. Метод обеспечивает хорошую точность в низшем приближении, получаемом несложным путем. В случае вещественных функционалов оно представляет собой обобщение вариационного принципа. Предлагаемый метод выгодно отличается от вариационных подходов применимостью для комплексных функционалов и в теориях с расходимостями. Поправки высших порядков к низшему приближению вычисляются по регулярной схеме. Метод применен к ряду задач из различных областей теоретической физики: теории полярона в физике твердого тела, изучению фазового перехода в скалярной модели φ^4 в теории поля и к исследованию распространения волн в стохастических средах.

INTRODUCTION

In modern theoretical physics the formulation of quantum theory relying on the original classical system is mainly distinguished in two mutually complementary ways. One of them is the method of canonical quantization (CQ),

where the field dynamic variables are considered as operators satisfying certain commutation relations and defined on a Hilbert space of states. Many papers have been devoted to CQ and for history and details we refer readers to [1,2].

The second formalism of quantization is Feynman's method of path integrals (PI) [3,4]. The basic idea of the Feynman formulation of PI is that the quantum motion of a particle is considered as the sum of all quantum transitions along all possible classical trajectories with amplitudes proportional to

$$A\,[\mathbf{x}] \propto \exp\left(i \frac{S\,[\mathbf{x}]}{\hbar} \right),$$

where $S\,[\mathbf{x}]$ is the classic action taken on a given trajectory $\mathbf{x}$. The total transition amplitude is supposed to be proportional to the integral

$$A \propto \int_{\Gamma} \delta \mathbf{r} \exp\left\{ \frac{i}{\hbar} S\,(t, \mathbf{x}) \right\}, \tag{1}$$

where the classical action

$$S\,(t, \mathbf{r}) = \int_0^t d\tau \left\{ \frac{m^2}{2} |\dot{\mathbf{r}}(\tau)|^2 - V\,[\mathbf{r}(\tau)] \right\} \tag{2}$$

is taken along the given path $\mathbf{r}(\tau)$. Integration in (1) is performed over the space Γ of all possible «path-trajectories» $\mathbf{r}(\tau)$ with $0 < \tau < t$ for which $\mathbf{r}(0) = \mathbf{x}_0$ and $\mathbf{r}(t) = \mathbf{x}$. This representation attracts much attention because it is close to the classical theory, having both the physical clarity and the fine compact mathematical formulation. These advantages stimulated applications of PI to various problems in quantum physics [4].

From a technical point of view, the PI formalism of quantization represents an essential attempt to go out beyond the perturbation expansion and becomes effective for describing systems with infinite numbers of degrees of freedom.

In mathematics Wiener [5] was the first to introduce in 1920, the conception of PI to describe Brownian motion. Dirac first suggested a representation of a particle propagator in terms very close to PI techniques [6]. The systematic development of quantum mechanics (QM) within the PI approach belongs to Feynman.

In quantum physics, Feynman [3] formulated non-relativistic QM on the language of PI (in other words, the functional or continual integrals) and showed that this approach is completely equivalent to the solution of the Schrödinger equation. One of the main reasons for the popularity of «path integrals» is the understanding that classical mechanics becomes an approximation of QM in Feynman's formulation if one applies the method of «stationary phases» to the latter. At the classic limit $\hbar \to 0$ the leading contribution to PI is given by

the stationary points of the phase function $S(t, \mathbf{r})$ in (1), which is the solution of Newton's classical equation of motion.

In 1949 Feynman used the functional integral (FI) for construction of covariant QED (the Feynman diagrams) [4]. After that, acknowledging the dignity of this new approach, Kac (1951) suggested a FI of Wiener's type for representation of the evolution operator in Euclidean space [7].

Path integration has come a long way since the 1950s. Probably the most famous early application of FI in statistical physics was to the **polaron** (a nonrelativistic electron «dressed» by the surrounding quanta of lattice vibrations in ionic crystals). In polaron theory, FI not only helps to formulate the answer qualitatively, but also remains the best way to calculate the answer more exactly than other methods. It is a tractable field theory; the benefits obtained from using FI are entirely analogous to those gotten in quantum field theory (QFT). But contrary to the polaron problem, all steps for QFT are more difficult because of the divergences, the vector character of the fields and also gauge problems.

A number of investigators [8,9] independently came to the formulation of QFT in terms of FI considering variational estimations for Green functions.

A relatively simple way to represent the Green function of a quantized field within FI was suggested in [10], where the equivalence of FI over bosonic fields to the averaging over vacuum states of these fields is proved.

A new understanding of FI occured in [11,12], where the evolution operator of the model $P(\varphi)_2$ in Euclidean metrics was represented in FI form as follows

$$\exp\{-\beta H\} = \int d\sigma_0 \exp\{-\int dx : P(\varphi) :\},$$

$$d\sigma_0 = C\,\delta\varphi \exp\left\{\frac{1}{2}\int dx\,[(\nabla\varphi)^2 + m^2\varphi^2]\right\}, \qquad (3)$$

where $d\sigma_0$ is a Gaussian measure of integration, generated by the action $S_0(\varphi) = \frac{1}{2}\int dx\,[(\nabla\varphi)^2 + m^2\varphi^2]$ of a free bosonic field and $\int dx : P(\varphi) :$ introduced for a certain renormalization of the classic interaction $\int dx\, P(\varphi)$. This definition of FI in (3) which allows the removal of interaction divergences coming from low-order «tadpole-type» diagrams is the essentially new and important feature of construction by Glimm and Jaffe.

Next important step in the application of FI in QFT was made in the quantization of Yang-Mills fields. A consequent scheme of quantization for a massless Yang-Mills field was constructed in 1967 by Faddeev, Popov [13] and De Witt [14] within the PI approach. FI turned out to be the shortest and most convenient method for constructing the Feynman rules for perturbation expansion in gauge field theories. This method played an important role in the

investigations of Slavnov [15], Taylor [16], Lee and Zinn-Justin [17], t'Hooft and Weltman [18]. In these papers a generalized Ward-Takakhashi identity was obtained, various methods of invariant regularization were developed and a procedure renormalizing the perturbation series was built. Within FI there has also been an attempt made to construct a quantum theory of gravitation [19].

In the 1970s, techniques based on the original ideas of Peak and Inomata [20], and Duru and Kleinert [21] for solving certain non-Gaussian FIs occuring in QM have attracted much attention. Standard examples of QM considered in this approach are defined by using Bessel- and Legendre-type diffusion processes, other than the Wiener process often used on these subjects. These results do not require the machinery of stochastic analysis and can be treated in a quick, transparent way. A development of this method is assumed in [22], where certain non-Gaussian integrals with potentials like $\sim 1/r^2$ or of the Morse-type have been derived rigorously by using techniques of changing dimension and time in FIs.

Excellent monographs and review papers have been devoted to FI in quantum theory [13], [23—29].

Although many points concerning the correct mathematical definition and practical calculation of FIs still remain open, it becomes clear that the description of a quantum system within the FI method is as convenient as using linear operators acting on vectors of Hilbert space within the method of CQ.

We summarize the above, stressing in particular that:

— the FI is a *convenient conception for the qualitative consideration* of quantum theories owing to the simplicity of using the WKB approximation, the evident relativistic covariance of the fomulation and the ease with which some specific constraints can be taken into account (e.g., introduction of «ghosts» in Yang-Mills theory);

— the FI can serve as a *practical tool for the quantitative estimation* of characteristics of quantum systems because of the possibilities of reducing some dynamical variables by exact integration (e.g., in the polaron problem), changing space/time (for the inverse-square potential in QM) and the convenience of computer calculations for imaginary-time sum over paths [32], etc.

In the present paper we consider mostly the second aspect of the application of FIs in quantum physics.

1. APPROXIMATE METHODS FOR CALCULATING FUNCTIONAL INTEGRALS

A great number of problems of modern physics can be formulated in terms of the FI approach. These problems have a common feature: their solution can be obtained in the form of a functional integral, which is defined on the

Gaussian measure. The most general form of a typical functional integral can be written as follows:

$$Z(g) = C_0 \int \delta\varphi \exp\left\{ -\frac{1}{2} (\varphi D_0^{-1} \varphi) + gW[\varphi] \right\}, \tag{4}$$

where

$$(\varphi D_0^{-1} \varphi) = \int_\Gamma dx \int_\Gamma dy \, (\varphi(x) \, D_0^{-1}(x, y) \, \varphi(y))$$

and $\Gamma \in R^d$ ($d = 1, 2, \ldots$). The Gaussian functional measure

$$d\sigma_0 = C_0 \, \delta\varphi \exp\left\{ -\frac{1}{2} (\varphi D_0^{-1} \varphi) \right\} \tag{5}$$

is defined by a Green function $D_0(x, y)$ corresponding to a differential operator $D_0^{-1}(x, y)$ with appropriate boundary conditions. The normalization constant C_0 is chosen in such a way that

$$\int d\sigma_0 = 1 \quad \text{or,} \quad Z(0) = 1.$$

In the standard nonrelativistic quantum mechanics the interaction functional W is usually defined by a potential $gU(\varphi)$:

$$W[\varphi] = g \int_\Gamma dx U(\varphi(x)), \tag{6}$$

where the coupling constant g is real. In other and more interesting cases (for example, polaron, bound states in QFT, stochastic processes, etc.) the interaction functional $W[\varphi]$ usually represents more complicated dependence on $\varphi(x)$.

Up to now exact calculations of functional integrals of this type are known [30] only for a quite limited class of interaction functionals: for quadratic forms of interaction leading to the pure Gaussian integral and very limited numbers of potentials (Coulomb potential and some others), for which the path integral can be reduced to the Gaussian integral after a definite change of variables. For others various approximate methods should be applied.

The contemporary progress in the computer hardwares and effective softwares for the numerical simulation technique enables one to obtain numerical calculations of (4) with sufficient accuracy although the practical implementation of this approach is very laborious. Besides, direct numerical (lattice) simulation is bound up with the difficulties of the continuous limit in lattice discretization and limited computer resources.

The development of analytical methods is very important because only analytical methods permit us to investigate qualitative features of quantum physical systems and indicate effective ways for improvement of numerical algorithms. Much efforts have been devoted to construct analytic methods for calculating the characteristics of quantum system within the FI formalism.

Among numerous approximate analytical FI methods we can list the following popular approaches: the standard perturbation expansion over g, the quasi-classic WKB approximation, the $1/N$ expansion, the instanton approximation and the variational methods (e.g., the Gaussian effective potential and space time transformation).

However, these methods have some limitations. For example, the WKB method cannot be used to study high-order quantum effects, the $1/N$ expansion gives a low convergence of the approximation series at real space-dimension numbers $N = 3$.

The standard perturbative method usually provides the perturbation series

$$Z(g) = \sum_{n=0}^{\infty} g^n Z_n$$

having the practical sense for a weak interaction $g << 1$ when only a few of the lowest terms Z_n is enough for getting $Z(g)$ with an acceptable accuracy. In addition, the calculation of Z_n for large n really is not more a simple task.

If the FI (4) is real, then the problems of such kind are studied by means of variational methods, which are popular due to their clear physical meaning and relatively simple calculations. However, the variational technique does not provide a regular prescription for choosing the trial functionals and it also does not allow one to control the accuracy of the estimation. Moreover, there is a class of problems (most of the QFT models with ultraviolet (UV) divergencies, complex and nonhermitean functionals, and so on), where the variational methods cannot be applied at all because the Jensen inequality no longer holds for these nonreal actions.

Our goal is to develop a universal method to calculate this path integral for any and especially large g. Sometimes it is possible to hear an opinion that in the strong coupling regime $g \to \infty$ the integral of the type (4) loses its Gaussian character and another non-Gaussian measure should be introduced. For example, it can be like this

$$d\sigma = C\, \delta\varphi \exp\{-\int_{\Gamma} dx\, \varphi^4(x)\}.$$

We want to claim that it is not true in the case of integrals of the type (4), where

- the highest derivative of the differential operator $D_0^{-1}(x, y)$ is 2ν, i.e. $D_0^{-1}(x, y) \sim \partial^{2\nu}$ for $\nu \geq 1$;
- the interaction functional $W[\varphi]$ depends only on $\varphi(x)$ and does not contain derivatives like $\partial\varphi(x)$.

Really, let us bring semiqualitative arguments. Let $f_n(x)$ is an orthonormal system of eigenfunctions of the operator D_0^{-1}:

$$D_0^{-1} f_n(x) = \frac{1}{D_n} f_n(x),$$

$$(f_n, f_m) = \int_\Gamma dx\, f_n(x)\, f_m(x) = \delta_{nm}. \tag{7}$$

The eigennumbers D_n satisfy the following asymptotics

$$D_n = O\left(\frac{1}{n^{2\nu}}\right) \quad \text{for} \quad n \to \infty.$$

Let us introduce the representation

$$\varphi(x) = \sum_n f_n(x) \sqrt{D_n}\, u_n, \tag{8}$$

where $\{u_n, (n = 0, 1, \ldots)\}$ are a new denumerate set of variables. We have

$$(\varphi D_0^{-1} \varphi) = \sum_n |u_n|^2,$$

$$Z(g) = C_0 \int \prod_n du_n \exp\left\{-\frac{1}{2}\sum_n |u_n|^2 + gW[\varphi]\right\}.$$

We expand $\varphi(x)$ as follows

$$\varphi(x) = \phi_N(x) + \phi_{>N}(x),$$

$$\phi_N(x) = \sum_{n<N} f_n(x) \sqrt{D_n}\, u_n, \tag{9}$$

$$\phi_{>N}(x) = \sum_{n>N} f_n(x) \sqrt{D_n}\, u_n \sim \sum_{n>N} \frac{f_n(x)}{n^\nu} u_n = O\left(\frac{1}{N^\nu}\right),$$

where N is a large number. Then we have

$$d\sigma = C_0 \prod_n du_n \exp\left\{-\frac{1}{2}\sum_n |u_n|^2\right\} = d\sigma_N\, d\sigma_{>N},$$

$$d\sigma_N = C_N \prod_{n<N} du_n \exp\left\{-\frac{1}{2}\sum_{n<N} |u_n|^2\right\},$$

$$d\sigma_{>N} = C_{>N} \prod_{n>N} du_n \exp\left\{-\frac{1}{2}\sum_{n>N} |u_n|^2\right\},$$

$$\int d\sigma_N = \int d\sigma_{>N} = 1.$$

The interaction functional can be represented as follows

$$W[\varphi] = g\int_\Gamma dx U(\phi_N(x)) + O\left(\frac{1}{N^\nu}\right). \tag{10}$$

Thus the functional integral under consideration can be approximated

$$Z(g) \sim Z_N(g) = \int d\sigma_N \exp\{gW[\phi_N]\} \tag{11}$$

and

$$Z(g) = \lim_{N\to\infty} Z_N(g).$$

One can see that the existence of this limit does not depend on the value of the coupling constant g and for any large g there exists a number $N(g)$ so that

$$Z(g) = Z_N(g) + O\left(\frac{1}{g}\right), \tag{12}$$

i.e., the functional measure can be considered as a Gaussian measure.

As a result we can conclude that the path integrals of the type (4) for any g can be considered as functional integrals over a Gaussian measure. Thus we can expect that there exists a representation of the initial functional integral (4)

$$Z(g) = C_g \int \delta\varphi \exp\left\{-\frac{1}{2}(\varphi D_g^{-1}\varphi) + W_g[\varphi]\right\} \tag{13}$$

with another C_g, D_g^{-1} and $W_g[\varphi]$, for which the main contributions from the interaction functional $gW[\varphi]$ should be accumulated in the operator D_g^{-1} and the perturbation corrections over the new interaction $W_g[\varphi]$ should be small. Our problem is to find this representation.

For this aim we shall use the idea that *the normal ordering of the Hamiltonian means essentially that the main quantum contributions to the ground state or vacuum of the system are taken into account.*

In the language of the FI it means that the conception of normal ordering with respect to a given Gaussian measure should be formulated and next problem is to represent the functional integral (4) in the form (13), where

- the Gaussian measure is defined by the operator D_g^{-1},
- the interaction functional $W_g[\varphi]$ is written in the normal form with respect to the Gaussian measure with D_g^{-1} and it does not contain quadratic terms over φ, i.e. $W_g[\varphi] = O(\varphi^4)$ for $\varphi \to 0$.

This representation we shall call the Gaussian equivalent representation of functional integrals. In section 2 all definitions will be formulated.

This method will be applied to the following problems:

- investigation of the behaviour of the polaron in ionic crystals in quantum statistics,
- phase transitions and phase restructure in quantum field models,
- propagation of waves in a stochastic medium with stochastically distributed centres in radiophysics.

2. GAUSSIAN EQUIVALENT REPRESENTATION OF FUNCTIONAL INTEGRALS

The main content of this Section is the development of the method of **Gaussian equivalent representation** (GER) of FIs and its application to the investigation of the ground state (vacuum) of various QFT and QM models in order to study nonperturbative phenomena such as the strong coupling regime, phase structure and phase transitions.

The GER method is a type of generalization of the variational technique, but in contrast to the latter, it is efficient for QFT models with UV divergencies and to theories with nonhermitean and nonlocal actions (stochastic and dissipative processes), where variational methods cannot be used.

This method is characterized by a high accuracy of the lowest approximation, which can be obtained by simple and rapid calculations. It gives a regular prescription for calculation of higher order corrections to the lowest approximation and can be considered as the next step in the development of approximate calculation methods.

2.1. General Formalism. Considering many theoretical problems in statistical physics [33], quantum field theory and mathematical physics one

deals with a class of functional integrals defined on a Gaussian measure. We shall consider functional integrals of the general type (4) as follows

$$Z_\Gamma(g) = C_0 \int \delta\varphi \exp\left\{ -\frac{1}{2}(\varphi D_0^{-1}\varphi) + gW_0[\varphi] \right\} =$$

$$= \int d\sigma_0 \exp\{gW_0[\varphi]\}. \tag{14}$$

Here we have introduced the following notation for the Gaussian measure

$$d\sigma_0 = C_0\, \delta\varphi \exp\left\{ -\frac{1}{2}(\varphi D_0^{-1}\varphi) \right\} =$$

$$= \frac{1}{\sqrt{\det D_0}} \prod_x d\varphi(x) \exp\left\{ -\frac{1}{2} \int_\Gamma \int dx\, dy\, \varphi(x)\, D_0^{-1}(x, y)\, \varphi(y) \right\}. \tag{15}$$

The Gaussian measure is normalized in such a way that $\int d\sigma_0 \cdot 1 = 1$. The integration in (14) is performed over functions $\varphi(x)$ defined on a region $\Gamma \subseteq \mathbf{R}^d$ $(d = 1, 2, \ldots)$. Usually the region Γ is chosen as a multidimensional box: $\Gamma = \{x : a_j \le x_j \le b_j,\ (j = 1, \ldots, d)\}$.

A differential operator $D_0^{-1}(x, y)$ is defined on functions $\varphi(x)$ with appropriate boundary conditions. For example, the operator

$$D_0^{-1}(x, y) = \left(-\frac{\partial^2}{\partial x^2} + m_0^2 \right) \delta(x - y) \tag{16}$$

acts on functions satisfying some periodic boundary conditions. The corresponding Green function $D_0(x, y)$ satisfies the equation

$$\int_\Gamma dy\, D_0^{-1}(x, y)\, D_0(y, z) = \delta(x - z)$$

and ensures definite boundary conditions.

The parameter g is a coupling constant. The interaction functional $W_0[\varphi]$ can be written in a general form

$$W_0[\varphi] = \int d\mu_a\, e^{i(a\varphi)}, \tag{17}$$

where we have introduced the notation

$$(a\varphi) = \int_\Gamma dy\, a(y)\, \varphi(y),$$

and $d\mu_a$ is a functional measure. For example, for a potential having a Fourier transfrom one can write

$$W_0[\varphi] = \int_\Gamma dx U[\varphi(x)] = \int_\Gamma dx \int \frac{dk}{2\pi} \tilde{U}(k) \exp\{i \int_\Gamma dy\, k\varphi(y)\, \delta(x-y)\}.$$

FI in representation (14) is well defined as a perturbation expansion over the coupling constant g. Thus, physically acceptable results can be obtained only in the weak coupling regime $g \ll 1$. In this case the Gaussian measure $d\sigma_0$ (15) gives the main contribution in FI and corrections can be calculated by using a perturbation expansion.

The task is to give a representation of this integral in the strong coupling regime [34]. Our idea is that the FI beyond the perturbation regime remains of the Gaussian type but with another Green function in the measure. In other words, we want to obtain a representation in which all main contributions of strong interaction are concentrated in the measure.

Let us perform the following transformations of the integral (14):

$$\varphi(x) \to \varphi(x) + b(x),$$
$$D_0^{-1}(x, y) \to D^{-1}(x, y), \tag{18}$$

where $b(x)$ is an arbitrary function and $D(x, y)$ is an appropriate Green function of the differential operator D^{-1}:

$$\int_\Gamma dy\, D^{-1}(x, y)\, D(y, z) = \delta(x - z)$$

providing the same boundary conditions.

Transformations (18) represent in a certain sense a functional analogue of standard canonical transformations made in the Hamiltonian formalism. The functional integral (14) takes the form

$$Z_\Gamma(g) = \sqrt{\det \frac{D}{D_0}} \exp\left\{-\frac{1}{2}(bD_0^{-1}b)\right\} \cdot \int d\sigma \exp\{gW_1[\varphi, b, D]\}, \tag{19}$$

where

$$d\sigma = C\, \delta\varphi \exp\left\{-\frac{1}{2}(\varphi D^{-1}\varphi)\right\},$$
$$dW_1[\varphi, b, D] = gW[\varphi + b] - (bD_0^{-1}\varphi) - \frac{1}{2}(\varphi [D_0^{-1} - D^{-1}]\, \varphi), \tag{20}$$

with the normalization condition $\int d\sigma \cdot 1 = 1$.

The tadpole Feynman diagrams give the main quantum contributions into background energy of the system under consideration or, in other words, into the formation of the background state or vacuum. The mathematical problem is to take them into account correctly. In the quantum theory the main divergences given by tadpole vacuum diagrams are efficiently eliminated out of consideration if the normal- ordered product of operators is introduced into the interaction Hamiltonian. Following this, the interaction functional in (19) should be written in the normal-ordered form. Thus we should introduce in W_1 the concept of the normal product according to the given Gaussian measure $d\sigma$. It can be done in the following way

$$: e^{i(a\varphi)} : = e^{i(a\varphi)} e^{\frac{1}{2}(aDa)}. \tag{21}$$

This definition leads to the following relations

$$\int d\sigma : e^{i(a\varphi)} : = 1, \qquad \int d\sigma : \varphi(x_1) \ldots \varphi(x_n) : = 0.$$

After these transformations the functional in the integrand can be rewritten

$$gW_1 = g \int d\mu_a \, e^{i(ab) - \frac{1}{2}(aDa)} : e_2^{i(a\varphi)} : +$$

$$+ \left[g \int d\mu_a \, e^{i(ab) - \frac{1}{2}(aDa)} - \frac{1}{2} ([D_0^{-1} - D^{-1}] D) \right] +$$

$$+ \left[ig \int d\mu_a \, e^{i(ab) - \frac{1}{2}(aDa)} (a\varphi) - (bD_0^{-1}\varphi) \right] -$$

$$- \frac{1}{2} : \left[g \int d\mu_a \, e^{i(ab) - \frac{1}{2}(aDa)} (a\varphi)^2 + (\varphi [D_0^{-1} - D^{-1}] \varphi) \right] : \tag{22}$$

where $e_2^z = e^z - 1 - z - \frac{z^2}{2}$.

Now we introduce the concept of the «correct form» of the action in the FI. We demand that the linear and quadratic terms on the integration variables $\varphi(x)$ should be absent in the interaction functional W_1 in (22). This requirement is argued in the same way. The system under consideration should be near its equilibrium point so that any linear terms on the variable $\varphi(x)$ must be absent. The quadratic configurations $\sim \varphi^2$ determine the Gaussian oscillator character of the equilibrium point and all of them are concentrated in the Gaussian measure $d\sigma$ only. Therefore, they should not appear in the interaction functional and

$$W_I \sim \varphi^3 \quad \text{for} \quad \varphi \to 0.$$

Thus the «correct form» requirement is satisfied if the following equations are held

$$g \int d\mu_a \, ia(x) \, e^{i(ab) - \frac{1}{2}(aDa)} - \int_\Gamma dy \, D_0^{-1}(x, y) \, b(y) = 0,$$

$$g \int d\mu_a \, a(x) \, a(y) \, e^{i(ab) - \frac{1}{2}(aDa)} + D_0^{-1}(x, y) - D^{-1}(x, y) = 0. \tag{23}$$

These equations provide the removal of the linear and quadratic terms from the interaction functional. Let us introduce the following functional and its correlation functions:

$$\hat{W}[b] = \int d\mu_a \exp\left\{ i(ab) - \frac{1}{2}(aDa) \right\},$$

$$\omega_n(x_1, \ldots, x_n) = \frac{\delta^n}{\delta b(x_1) \cdot \ldots \cdot \delta b(x_n)} \hat{W}[b]. \tag{24}$$

Equations (24) can be written in the form

$$b(x) = g \int_\Gamma dy \, D_0(x, y) \, \omega_1(y),$$

$$D(x_1, x_2) = D_0(x_1, x_2) + g \iint_\Gamma dy_1 \, dy_2 D_0(x_1, y_1) \, \omega_2(y_1, y_2) \, D(y_2, x_2). \tag{25}$$

These equations determine the new Green function $D(x_1, x_2)$ and the function $b(x)$ in (22). Finally the new representation for FI in (14) can be rewritten in the form:

$$Z_\Gamma(g) = \exp\{E_0\} \cdot \int d\sigma \exp\{g W_I[\varphi]\}, \tag{26}$$

where

$$E_0 = \frac{1}{2} \ln \det\left(\frac{D}{D_0}\right) - \frac{1}{2}(bD_0^{-1}b) - \frac{1}{2}([D_0^{-1} - D^{-1}]D) + g\hat{W}[b],$$

$$gW_I[\varphi] = g \int d\mu_a \, e^{i(ab) - \frac{1}{2}(aDc)} : e_2^{i(a\varphi)} : . \tag{27}$$

The representation of the interaction functional in the normal product form means that

$$\int d\sigma \, W_I[\varphi] = 0.$$

The function E_0 defines the «energy» of the zero approximation. Next corrections to the leading term in (26) can be calculated by using a perturbation expansion over the new interaction functional W_I.

It should be stressed that representations (14) and (26) are equivalent. Therefore the mathematical object $Z_\Gamma(g)$ has at least two different representations (14) and (26). In principle other representations may exist if equation (26) has a more distinct solution. In this case we give preference to the representation in which the perturbation corrections connected with gW or gW_I are minimal for given parameters.

All our transformations are valid for real and complex functions and functionals in the FI.

In the case of real FIs representation (26) leads to the following conclusion. Using Jensen's inequality one can get

$$Z_\Gamma(g) \geq \exp\{E_0\}, \tag{28}$$

so that E_0 defines the lowest estimation for our FI.

On the other hand, one can easily check that (26) defines the minimum of the function E_0. Thus, inequality (28) is the variational estimate of the initial FI. Moreover, representation (26) makes it possible to calculate the perturbation corrections to E_0 by developing the functional integral (26) over W_I.

2.2. The GER Method for Calculating the Partition Function. In this Section we develop the main techniques of the GER method especially for calculating the partition function in QM and QS. In other words, we deal with integrals where the field variable is the coordinate of a particle $\mathbf{r}(t)$ which is parameterized by the one-dimensional parameter t. For simplicity one can choose the symmetrical interval $-T < t < T$. The parameter T is connected with time in QM or the inverse temperature $2T = \beta$ in QS.

The partition function plays an important role in QS. For a wide class of quantum mechanical and quantum statistical problems describing the interaction of a quantum particle with a field or the propagation of waves and quantum particles through a media with random or stochastic admixtures the partition function can be represented in the form of a FI of the following general type

$$Z_T(g) =$$

$$= C_0 \int\limits_{\mathbf{r}(-T)=\mathbf{r}(T)} \delta\mathbf{r} \exp\left\{-\frac{1}{2}\int\limits_{-T}^{T} dt\, \dot{\mathbf{r}}^2(t) + \frac{g}{2}\iint\limits_{-T}^{T} dt\, ds\, V(\mathbf{r}(t) - \mathbf{r}(s);\, t-s)\right\}. \tag{29}$$

The standard normalization is $Z_T(0) = 1$. The integration in (29) is performed over all «paths» in a d-dimensional space satisfying periodic boundary conditions.

The kinetic term in the Gaussian measure can be written in the form

$$\int_{-T}^{T} dt\, \dot{\mathbf{r}}^2(t) = \iint_{-T}^{T} dt\, ds\, \mathbf{r}(t)\, D_0^{-1}(t, s)\, \mathbf{r}(s),$$

$$D_0^{-1}(t, s) = -\frac{\partial^2}{\partial t^2}\, \delta(t - s). \tag{30}$$

The Green function $D_0(t, s)$ corresponding to the differential operator $D_0^{-1}(t, s)$ and satisfying the periodic boundary conditions is

$$D_0(t, s) = -\frac{1}{2}\, |t - s| - \frac{ts}{2T} \xrightarrow{T\to\infty} -\frac{1}{2}\, |t - s|. \tag{31}$$

The Fourier transform of this Green function is

$$\tilde{D}_0(p^2) = \int_{-\infty}^{\infty} dt\, e^{ipt} D_0(t) = \frac{1}{2}\left[\frac{1}{(p + i0)^2} + \frac{1}{(p - i0)^2} \right] \to \frac{1}{p^2}\,. \tag{32}$$

The parameter g is a coupling constant. In QM and QS, the potentials describing the influence of a field interaction or media on a quantum particle usually have a general form like $V(\mathbf{r} - \mathbf{r}'; t - t')$. So, we will consider this class of potentials further. The potential $V(\mathbf{r}(t) - \mathbf{r}(s); t - s)$ in (29) is assumed to have the Fourier representation

$$V(\mathbf{r}(t) - \mathbf{r}(s); t - s) = \int \frac{d\mathbf{k}}{(2\pi)^d}\, \tilde{V}(\mathbf{k}; t - s)\, e^{i\mathbf{k}\mathbf{R}(t, s)} = \int d\mathcal{K}(\mathbf{k}; t - s)\, e^{i\mathbf{k}\mathbf{R}(t, s)},$$

$$d\mathcal{K}(\mathbf{k}; t - s) = \frac{d\mathbf{k}}{(2\pi)^d}\, \tilde{V}(\mathbf{k}; t - s),$$

$$\mathbf{R}(t, s) = \mathbf{r}(t) - \mathbf{r}(s). \tag{33}$$

Thus the initial FI in (29) can be rewritten as

$$Z_T(g) = \int d\sigma_0 \exp\{gW_0[\mathbf{r}]\}, \tag{34}$$

where

$$d\sigma_0 = C_0\, \delta\mathbf{r} \exp\left\{ -\frac{1}{2} \iint_{-T}^{T} dt\, ds\, \mathbf{r}(t)\, D_0^{-1}(t, s)\, \mathbf{r}(s) \right\}, \tag{35}$$

$$gW_0[\mathbf{r}] = \frac{g}{2} \iint_{-T}^{T} dt\, ds \int d\mathcal{K}(\mathbf{k}; t - s)\, e^{i\mathbf{k}\mathbf{R}(t, s)} \tag{36}$$

and the normalization condition is $\int d\sigma_0 \cdot 1 = 1$.

Now we are ready to apply the GER method to this FI. Note that for the potentials $V(\mathbf{r}; t)$ of type (33) having their maximum at $\mathbf{r} = 0$ we do not need to introduce the function $\mathbf{b}(t)$, i.e., $\mathbf{b}(t) = 0$. According to the GER method a new Gaussian measure should be introduced into the integral (34) as follows

$$d\sigma = C\, \delta\mathbf{r} \exp\left\{ -\frac{1}{2} \iint_{-T}^{T} dt\, ds\, \mathbf{r}(t)\, D^{-1}(t-s)\, \mathbf{r}(s) \right\}. \tag{37}$$

The normalization constant C is $\int d\sigma \cdot 1 = 1$.

Second, we introduce the «normal-ordered» form of the potential (33) in the following way

$$e^{i\mathbf{k}\mathbf{R}(t,\,s)} = :e^{i\mathbf{k}\mathbf{R}(t,\,s)}: \exp\left[-\mathbf{k}^2 F(t-s)\right], \tag{38}$$

where

$$F(t-s) = D(0) - D(t-s),$$

$$\int d\sigma\, R_i(t, s)\, R_j(t, s) = 2\delta_{ij} F(t-s).$$

In particular, the next relations are valid:

$$\int d\sigma :e^{i\mathbf{k}\mathbf{R}(t,\,s)}: = 1,$$

$$r_i(t)\, r_j(s) = :r_i(t)\, r_j(s): + \delta_{ij} D(t-s), \qquad i, j = 1 \ldots d.$$

The functional $\hat{W}[b]$ in (24) becomes

$$\hat{W}[b] = \frac{1}{2} \iint_{-T}^{T} dt\, ds \int d\mathcal{K}(\mathbf{k}; t-s) \exp\left[-\mathbf{k}^2 F(t-s)\right] e^{i\mathbf{k}(\mathbf{b}(t) - \mathbf{b}(s))}. \tag{39}$$

Its second correlation function is

$$\frac{\delta^2}{\delta b_i(t)\, \delta b_j(s)} \hat{W}[\mathbf{b}]\Big|_{b=0} = -\delta_{ij} \left[\delta(t-s) \int_{-\infty}^{\infty} d\tau\, \Phi(\tau) - \Phi(t-s)\right], \tag{40}$$

where

$$\Phi(\tau) = \frac{1}{d} \int d\mathcal{K}(\mathbf{k}; \tau)\, \mathbf{k}^2 \exp\left[-\mathbf{k}^2 F(\tau)\right].$$

Equation (26) defining «the correct form» of the interaction functional becomes

$$\tilde{\Sigma}(p^2) = \frac{1}{d} \int_{-\infty}^{\infty} d\tau \left[1 - \cos(p\tau)\right] \int d\mathcal{K}(\mathbf{k}; \tau)\, \mathbf{k}^2 \exp\left[-\mathbf{k}^2 F(\tau)\right], \tag{41}$$

$$F(\tau) = \int_0^{\infty} \frac{dp}{\pi} \frac{1 - \cos(p\tau)}{p^2 + g\, \tilde{\Sigma}(p^2)}. \tag{42}$$

Equations (41) and (42) define the Green function $D(\tau)$. For the asymptotic cases of weak ($g \to 0$) and strong ($g \to \infty$) interaction regimes, these equations may admit analytic solutions because one needs only that their behaviour be within the accuracy of the first several leading order terms such as $\sim g$, g^2 or, $\sim 1/g$, $1/g^2$. In general, these are not solvable analytically as they are nonlinear integral equations over functionals, but their solutions may be obtained by developing some numerical techniques. For example, the fixed-point method of consequent iterations can be used. Starting from guess function $\tilde{\Sigma}_0(p^2)$ we can calculate the iterations:

$$\tilde{\Sigma}_{n+1}(p^2) = \frac{1}{d} \int_{-\infty}^{\infty} d\tau \, [1 - \cos(p\tau)] \int d\mathcal{K}(k; \tau) \, \mathbf{k}^2 \exp[-\mathbf{k}^2 F_n(\tau)],$$

$$F_{n+1}(\tau) = \int_0^{\infty} \frac{dp}{\pi} \cdot \frac{1 - \cos(p\tau)}{p^2 + g\,\tilde{\Sigma}_n(p^2)}. \tag{43}$$

This procedure can be developed for numerical solutions of (41) and (42). In this case, however, the initial guess functions $F_0(\tau)$ and $\tilde{\Sigma}_0(p^2)$ should be chosen reasonably, i.e., the iteration process (43) has to converge to solutions

$$\tilde{\Sigma}(p^2) = \tilde{\Sigma}_{\infty}(p^2) = \lim_{n\to\infty} \tilde{\Sigma}_n(p^2), \tag{44}$$

$$F(t) = F_{\infty}(t) = \lim_{n\to\infty} F_n(t). \tag{45}$$

For a reasonable choice of guess functions, it is useful to investigate asymptotics of solutions for equations (41) and (42). An example of analytic and numerical solution of (41) and (42) is given in Section 3.3 within the polaron problem.

Substitution of (37)—(42) into (34) and the requirement that the new interaction functional to be written in the «correct form» (see Section 2.1) lead to the new representation of the initial FI

$$Z_T(g) = \exp(-2TE_0(g)) \cdot J_T(g),$$

$$J_T(g) = \int d\sigma \exp\{gW_I[\mathbf{r}]\}, \tag{46}$$

where the interaction functional looks as

$$gW_I[\mathbf{r}] = \frac{g}{2} \iint_{-T}^{T} dt\, ds \int d\mathcal{K}(\mathbf{k}; t - s) \exp[-\mathbf{k}^2 F(t - s)] : e_2^{i\mathbf{k}\mathbf{R}(t, s)} :. \tag{47}$$

The function $E_0(g)$ being «the leading-order energy» or the energy in the zero approximation is

$$E_0(g) = d \int_0^\infty \frac{dp}{2\pi} \left[\ln \frac{\tilde{D}_0(p^2)}{\tilde{D}(p^2)} + p^2 \tilde{D}(p^2) - 1 \right] +$$

$$+ \frac{g}{2} \int_{-\infty}^{\infty} d\tau \int d\mathcal{K}(\mathbf{k}; \tau) \exp [- \mathbf{k}^2 F(\tau)]. \qquad (48)$$

Thus the Gaussian equivalent representation of the initial FI in (34) is defined by (46)—(48). For a given potential $V(\mathbf{r})$ we have a pure mathematical problem to solve (41), (42) and find the Green function $D(t, s)$. Then we can compute the leading-order energy $E_0(g)$ (48) and the highest corrections to it by perturbation calculations over the new interaction functional W_I (47).

Below, in the following sections of this paper, we apply the GER method to different problems of theoretical physics:

- the problem of the polaron in QS,
- the phase transition phenomenon in the QFT model,
- the solution of the wave differential equation.

Each of these subjects reflects a feature of the GER method. High accuracy is reached in calculation of the ground state energy of the d-dimensional Fröhlich polaron. One effective scheme of mass renormalization in the $g\varphi^4_{2,3}$ theory, suggested within the GER method, leads to the correct prediction of the nature of phase transitions in this theory. Finally an estimate of non-Hermitean path integral arising in the theory of wave propagation in media with Gaussian noise is obtained. The reduction of the initial PI to the new representation generates a certain constraint equation determining this state and one should give preference to the representation, that is efficient for solving a given task.

3. THE POLARON PROBLEM

The study of the physical properties of a particle interacting with a quantum medium is common to many branches of physics. A classic example of this kind is the Fröhlich model of the polaron, — an electron moving with the polarization distortion of ions in a crystal. The polaron's popularity as a model is due to its similarity to many field-theoretical constructions where bosons couple linearly to fermions (the meson-nucleon interactions inside nuclei, the «dressing» of quarks in the nonperturbative vacuum of QCD, etc.). The polaron problem is treated most straightforwardly in the FI formalism which allows one to reduce this problem to an effective one-particle task and, leads to new results

not given by other conventional techniques. However, despite its long history and importance, the exact solution of the Fröhlich Hamiltonian is still lacking due to a high nonlocality (in time) and a Coulomb-like singularity in the polaron action. The application of the GER method to the d-dimensional polaron in this chapter results in highly accurate estimations of the main quasi-particle characteristic of the polaron — its ground-state energy.

The polaron problem embraces a wide range of questions concerning the behaviour of the electron of conductance in polar crystals [35—37]. The first field-theoretical formulation of polaron theory was proposed by Fröhlich [38] to describe the interaction of a single band electron with phonons, quanta associated with the longitudinal optical branch of lattice vibrations. Since that time, the Fröhlich polaron model has attracted interest as a testing ground of various nonperturbative methods in quantum physics. One of the main quasi-particle characteristics of the polaron is its ground-state energy (GSE) $E_0(\alpha)$.

Historically, the GSE of the polaron has been investigated in the weak [38], intermediate [39] and strong coupling regimes [40,41] using different methods. The first attempt to build the polaron theory, valid for arbitrary values of α, was made by Feynman [35] within the path integral (PI) formalism using variational estimations. As a result, Feynman's PI approach gives good upper bounds of $E_0(\alpha)$ in the entire range of α in a unified way.

There arises the question, whether the Feynman's estimations of the polaron GSE can be improved by introducing some trial actions, more general than the quadratic action with two variational parameters used in [35]. This question, in particular, has been studied within different variational approaches [42,43]. But giving variational answers, it could not estimate the next corrections to the obtained values.

Traditionally, the polaron problem has been investigated in three-dimensional space ($d=3$) [44,45]. In recent years, however, polaron effects have been observed in low-dimensional systems [46], and certain physical problems have been mapped into a two-dimensional ($d=2$) polaron theory [47]. The possibility that an electron may be trapped on the surface of a dielectric material has attracted much interest [48]. The GSE of the polaron for $d=2$ is discussed in [49,50].

In the Section, we investigate the GSE of the polaron in the case of arbitrary space dimensions ($d>1$) and try not only to improve Feynamn's result, but also to estimate the next corrections that allow one to test the accuracy and reliability of the obtained values.

3.1. Polaron Path Integral in d Dimensions. The Fröhlich longitudinal-optical (LO) polaron model for $d=3$ is determined by the Hamiltonian

$$H=\frac{1}{2m}\,\mathbf{p}^2+\hbar\,\omega\sum_{\mathbf{k}}\,a_{\mathbf{k}}^{\dagger}a_{\mathbf{k}}+\frac{1}{\sqrt{\Omega}}\sum_{\mathbf{k}}\,g_{\mathbf{k}}(a_{\mathbf{k}}^{\dagger}\,\mathrm{e}^{-i\,\mathbf{kx}}-a_{k}\,\mathrm{e}^{i\,\mathbf{kx}}), \tag{49}$$

which describes the interaction of an electron (position and momentum vectors $\mathbf{x}$ and $\mathbf{p}$, band mass m) with the phonon field (creation and annihilation operators $a_{\mathbf{k}}^{\dagger}$, $a_{\mathbf{k}}$, quantization volume Ω, Plank constant $\hbar$) associated with a LO branch of lattice vibrations (wave vector $\mathbf{k}$ and frequency ω) in a polar crystal. The electron-phonon interaction coefficient for coupling with the wave vector $\mathbf{k}$ in (49) is defined as follows:

$$g_{\mathbf{k}} = \frac{i\hbar\,\omega(\hbar/2m\omega)^{1/4}(4\pi\,\alpha)^{1/2}}{|\mathbf{k}|}, \tag{50}$$

where the dimensionless Fröhlich coupling constant α takes the value $\alpha \sim 1 + 20$ in most of the real ionic crystals (e.g.,$\alpha \simeq 5$ for sodium chloride). In the following, units will be chosen such that $\hbar = m = \omega = 1$.

Until now, no nontrivial solution of $H\Psi_n = E_n\Psi_n$ was known. It has been shown [51] for generalized Fröhlich models that the function $E_0(\alpha)$ has no points of nonanalyticity for an arbitrary $\alpha \geq 0$. Various methods [35,40,52,39] have been used to approximately calculate the spectrum of H, especially to obtain its GSE E_0 for selected (weak, intermediate or strong) regions of α.

To extend the Fröhlich Hamiltonian (49) written for $d = 3$ to arbitrary spatial dimensions $d > 1$, we follow a physical approach [53,54] inspired by the formulation of a lower-dimensional polaron problem obtained from the Fröhlich Hamiltonian of a higher-dimensional system by integrating out one or more dimensions. Following [54] we assume that the form of the Fröhlich Hamiltonian in d-dimensions is the same as in (49) except that now all vectors and operators are d-dimensional and the electron-phonon interaction coefficient $g_{\mathbf{k}}$ is redefined as follows:

$$|g_{\mathbf{k}}|^2 = \frac{\lambda_d^2}{|\mathbf{k}|^{d-1}}, \quad \lambda_d^2 = \Gamma\left(\frac{d-1}{2}\right)2^{d-3/2}\pi^{(d-1)/2}\alpha. \tag{51}$$

Accordingly, we write the FI representation of the free-energy $F(\beta)$ of a polaron with a given temperature $\Theta = 1/\beta$ as follows:

$$\exp(-\beta F) = \mathrm{Tr}\,[\exp(-\beta H)], \tag{52}$$

where the Hamiltonian H in (49) should be written in terms of the coordinates and momenta. The «Trace» $\mathrm{Tr} = \mathrm{Tr}_{el}\mathrm{Tr}_{ph}$ here is assumed to be taken over the whole space of states of the «electron + phonon» system.

It is well known from the famous paper by Feynman [35] that the path integral approach to the polaron has an advantage because the phonon trace Tr_{ph} in (52) can be adequately eliminated and as a consequence, the polaron

problem is reduced to an effective one-particle problem with retarded interaction. The result reads

$$Z_\beta(\alpha) = \exp(-\beta F) = \int_{\mathbf{x}(0)=\mathbf{x}(\beta)} \delta\,\mathbf{x}\,\exp(S[\mathbf{x}]), \tag{53}$$

where the action $S[\mathbf{x}]$ is

$$S[\mathbf{x}] = -\frac{1}{2}\int_0^\beta dt\,\dot{\mathbf{x}}^2(t) + \frac{\lambda_d^2}{8\pi}\iint_0^\beta dtds\,\frac{1}{|\mathbf{x}(t)-\mathbf{x}(s)|}\,\frac{e^{|t-s|}+e^{\beta-|t-s|}}{e^\beta-1}. \tag{54}$$

The free energy $F(\beta)$ tends to the GSE as $\beta \to \infty$ (zero temperature case)

$$E_0 = -\lim_{\beta\to\infty}\frac{1}{\beta}\ln Z_\beta(\alpha). \tag{55}$$

The path integral in (53) is not explicitly solvable due to the non-Gaussian character of S. For its variational estimation for $d = 3$, Feynman proposed [35] a quadratic two-body trial action S_F instead of S:

$$S[\mathbf{x}] \to S_F[\mathbf{x}] = -\frac{1}{2}\int_0^\beta dt\,\dot{\mathbf{x}}^2(t) +$$

$$+\frac{C}{2}\iint_0^\beta dtds[\mathbf{x}(t)-\mathbf{x}(s)]^2\exp\{-\omega|t-s|\}, \tag{56}$$

where constants C and ω are variational parameters. With the trial action S_F one gets an exact solution for path integral in (53). A variation for finding of the absolute minimum of $E_0^F(\alpha) = F_F(\alpha)$ for $\beta \to \infty$ over parameters C and ω leads to a rigorous upper bound of the polaron GSE at arbitrary α, that is Feynman's known result [35].

Here we will show that the application of the GER method improves Feynman's estimation. We consider the polaron GSE in the case of arbitrary space dimension $d > 1$ and start again from the FI in (53)—(54).

For further convenience, to get a symmetrical region over t [56], we change the variable of FI in (53) to

$$\mathbf{x}(t) \to \mathbf{r}(t-T), \quad T = \beta/2 \tag{57}$$

with the electron motion $\mathbf{r}(t)$ embedded in d-dimensional space. Accordingly, the GSE of the Fröhlich polaron $E_0(\alpha)$ (it will hereafter be denoted by $E(\alpha)$) can be defined as follows:

$$E(\alpha) = -\lim_{T\to\infty}\frac{1}{2T}\ln Z_T(\alpha), \tag{58}$$

where a FI is introduced [55]

$$Z_T(\alpha) = C_0 \int\limits_{\mathbf{r}(-T)=\mathbf{r}(T)} \delta\,\mathbf{r} \exp\left\{-\frac{1}{2}(\mathbf{r}D_0^{-1}\mathbf{r}) + \frac{\alpha}{2}\int\limits_{-T}^{T} dtds\, V[\mathbf{r}(t)-\mathbf{r}(s); t-s]\right\},$$

$$C_0 = \sqrt{\det D_0^{-1}}, \quad (\mathbf{r}, D_0^{-1}\mathbf{r}) = \int\limits_{-T}^{T} dtds\, \mathbf{r}(t)D_0^{-1}(t, s)\,\mathbf{r}(s). \tag{59}$$

The standard normalization $E(0) = 0$ in (58) is satisfied under the condition $Z_T(0) = 1$.

The free-electron system is described by the kinetic term $(\mathbf{r}D_0^{-1}\mathbf{r})$, where the differential operator D_0^{-1} and its Green function D_0 are given by (30)—(31) in the previous Section as $T \to \infty$.

The Coulomb-like interaction part, the electron self-interaction, is given by the retarded potential

$$V[\mathbf{R}; t-s] = \frac{\Gamma(d/2 - 1/2)}{4\sqrt{2}\pi^{(d+1)/2}} e^{-|t-s|} \int \frac{d\,\mathbf{k}}{|\mathbf{k}|^{d-1}} \exp(i\,\mathbf{kR}),$$

$$\mathbf{R} = \mathbf{r}(t) - \mathbf{r}(s). \tag{60}$$

with the electron position vector $\mathbf{r}(t)$ embedded into d-dimensions.

The path integral in (59) is not explicitly solvable due to the non-Gaussian character of $V[\mathbf{R}; t-s]$ in (60).

3.2. Bounds for the Polaron Ground-State Energy in d Dimensions. For α not too large, the PI in the initial presentation (59) may be estimated by using a perturbation expansion in α. The problem is to estimate $Z_T(\alpha)$ beyond the weak coupling regime. Accordingly, we can apply the GER method to this problem.

Our key steps will be the same as those in the previous Section. We remember that these are:

(i) *the introduction of new Gaussian measure $d\sigma$ (20) standing for the kinetic part of the FI,* which forms a new representation of the initial FI, and

(ii) *the requirements of the «normal-ordered» and «correct» form of the interaction part of the FI in this representation,* that is reached by introducing constraint equations (41), (42). This scheme results in a new representation of the initial FI: an exponential with the leading term of energy factorized out as a free multiplicand (48) and all the corrections to it are defined by another FI (47).

Performing this scheme and using formulae (41), (42) and (46)—(48), we obtain the new representation (47) of the GSE of optical polaron within the GER method as follows:

$$E(\alpha) = E_0(\alpha) + \Delta E(\alpha), \tag{61}$$

where the function $E_0(\alpha)$ being the «leading-order energy», or the GSE in the zeroth approximation, is (see Eq. (48))

$$E_0(\alpha) = -d\left\{ \frac{1}{2\pi} \int_0^\infty dk[\ln(k^2\tilde{D}(k)) - k^2\tilde{D}(k)+1] + \frac{\alpha_d}{3\sqrt{2\pi}} \int_0^\infty dt\, \frac{\exp(-t)}{F^{1/2}(t)} \right\}. \tag{62}$$

The function $F(t)$ in (62) is defined by the equations (see Eqs. (41) and (42))

$$F(t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \tilde{D}(k)(1 - e^{ikt}) = \frac{1}{\pi} \int_0^\infty dk\, \frac{1 - \cos(kt)}{k^2 + \alpha_d \tilde{\Sigma}(k)}, \tag{63}$$

$$\tilde{\Sigma}(k) = \int_{-\infty}^{\infty} dt\, e^{-ikt}\, \Sigma(t) = \frac{1}{3\sqrt{2\pi}} \int_0^\infty dt \exp(-t)\, \frac{1 - \cos(kt)}{F^{3/2}(t)}. \tag{64}$$

Here we have introduced the «effective coupling constant»

$$\alpha_d = \alpha \cdot R_d, \quad R_d = \frac{3\sqrt{\pi}\,\Gamma(d/2 - 1/2)}{2d\,\Gamma(d/2)}. \tag{65}$$

Our leading term (the zero-order approximation) $E_0(\alpha)$ gives an upper bound to the exact GSE of a polaron $E(\alpha)$. Actually, applying the Jensen's inequality to (61) one gets

$$\exp\{-2T \cdot E(\alpha)\} \geq \exp\{-2T \cdot E_0(\alpha)\}. \tag{66}$$

Consequently,

$$E_0(\alpha) \geq E(\alpha). \tag{67}$$

The high-order corrections $\Delta E(\alpha)$ in (61) can be obtained by evaluating the PI

$$\exp\{-2T \cdot \Delta E(\alpha)\} = C \int_{\mathbf{r}(-T) = \mathbf{r}(T)} \delta\mathbf{r} \exp\left\{ -\frac{1}{2} \iint_{-T}^{T} dtds\, \mathbf{r}(t) D^{-1}(t,s)\, \mathbf{r}(s) + W[\mathbf{r}] \right\}. \tag{68}$$

Here, the interaction functional written in the new representation is

$$W[\mathbf{r}] = \alpha_d \cdot \frac{\Gamma(d/2)d}{6\sqrt{2\pi}^{\,d/2+1}} \iint_{-T}^{T} dtds\, e^{-|t-s|} \times$$

$$\times \int \frac{d\,\mathbf{k}}{|\mathbf{k}|^{d-1}} \exp\{-\mathbf{k}^2 F(t-s)\} :\, e_2^{i\,\mathbf{k}[\mathbf{r}(t) - \mathbf{r}(s)]} :, \tag{69}$$

where $e_2^x = e^x - 1 - x - x^2/2$.

Due to equations (64) and (63) in the new representation, all the quadratic terms in the polaron action functional are concentrated only in the new Gaussian measure $d\sigma$ and do not enter $W[\mathbf{r}]$.

It should be stressed that representation (61) is completely equivalent to the initial representation (58) for asymptotically large $T \to \infty$. The Gaussian equivalent representation (61) gives the origin of various approximations differing from each other in the accuracy of deriving equations (63)—(64).

As a simple approximation of $\tilde{\Sigma}(k)$ obeying the necessary asymptotics, one can take the function:

$$\tilde{\Sigma}(k) = \frac{\mu^2}{\alpha_d} \cdot \frac{k^2}{\xi^2 + k^2}, \tag{70}$$

where μ and ξ are parameters. Then, (62) becomes

$$E_0(\alpha) = -\frac{d}{2}\left[\xi - \lambda + \frac{\mu^2}{2\lambda}\right] - \frac{\alpha_d \lambda^{3/2} d}{3\mu\sqrt{\pi}} \int_0^\infty \frac{dt \exp(-t)}{\sqrt{1 - \exp(-\lambda t) + \lambda t \xi^2/\mu^2}},$$

$$\lambda = \sqrt{\mu^2 + \xi^2}. \tag{71}$$

Minimizing the obtained energy over the parameters μ and ξ, one easily finds a variational upper bound in d dimensions. For $d = 3$ ($\alpha_3 = \alpha$) it explicitly reproduces the well-known Feynman's variational upper bound to the polaron GSE [35]:

$$E^F(\alpha) = \min_{\mu} \min_{\xi} E_0(\alpha, d = 3). \tag{72}$$

We stress that the extremal conditions on parameters μ, ξ in (72) are equivalent to a particular choice of the function $\tilde{\Sigma}(k)$ in (70). However, the function in (70) is not an exact solution of (64) and (63). It means, that Feynman's trial quadratic action does not represent entirely the Gaussian part of the polaron action for $d = 3$. Exact numerical solution of equations (64), (63) by the iteration procedure allows us to obtain $E_0(\alpha)$ more exactly, which improves Feynman's result $E^F(\alpha)$ in the entire range of α. The obtained numerical results $E_0(\alpha)$ for $d = 2$ and $d = 3$ as compared with Feynman's variational estimation are displayed in Tables I—VI.

The correction $\Delta E(\alpha)$ should be evaluated from the functional integral in (69) by expanding e^W in (68) in a series

$$\Delta E(\alpha) = \sum_{n=1}^{\infty} \Delta E_n(\alpha) = -\lim_{T\to\infty} \frac{1}{2T} \sum_{n=1}^{\infty} \frac{1}{n!} \int d\sigma \{W[\mathbf{r}]\}^n_{\text{connected}}. \tag{73}$$

We stress that (73) is not a standard perturbation series in the coupling constant α_d as α_d enters into W not only explicitly as a factor, but also implicitly through the function $F(t)$. The first term in (73) with $n = 1$ equals zero due to normal ordering. Nontrivial corrections are given by terms with $n \geq 2$. For the second order correction to $E_0(\alpha)$ we get

$$\Delta E_2(\alpha) = -\alpha_d^2 \cdot \frac{\Gamma(d/2)d^2}{18\pi^{3/2}} \sum_{n=2}^{\infty} Q_n R_n(\alpha), \tag{74}$$

where

$$Q_n = \frac{(2n)!\Gamma(n+1/2)}{16^n (n!)^2 \Gamma(n+d/2)},$$

$$R_n = \iiint_0^{\infty} dadbdc \left\{ e^{-a-c} \frac{[F(a+b)+F(b+c)-F(a+b+c)-F(b)]^{2n}}{[F(a)F(c)]^{n+1/2}} + \right.$$

$$+ e^{-a-2b-c} \frac{[F(a)+F(c)-F(a+b+c)-F(b)]^{2n}}{[F(a+b)F(b+c)]^{n+1/2}} +$$

$$\left. + e^{-a-2b-c} \frac{[F(a+b)+F(b+c)-F(a)-F(c)]^{2n}}{[F(a+b+c)F(b)]^{n+1/2}} \right\}.$$

We stress that expression (74) can further be simplified, but we keep this form for clarity.

Finally, we get the following expression for the GSE of the polaron

$$E^{(2)}(\alpha) = E_0(\alpha) + \Delta E_2(\alpha), \tag{75}$$

which can be evaluated numerically for arbitrary α and different space dimensions d.

Notice that $E_0(\alpha)$ in (62) is of an order of α^i, $(i = 0, 1, 2, \ldots)$ while $\Delta E_2(\alpha)$ in (74) is only of an order of α^j, $(j = 2, 3, \ldots)$.

The theory under consideration has two parameters α and d. In general, all our expressions should depend on both of them. Notice that key expressions in (64) and (63), completely defining the functions $F(t)$ and $\tilde{\Sigma}(k)$, depend only on the effective coupling constant α_d. This means that the following relations

$$F^{[n]}(\alpha_m, t) = F^{[m]}(\alpha_n, t), \quad \tilde{\Sigma}^{[n]}(\alpha_m, k) = \tilde{\Sigma}^{[m]}(\alpha_n, k) \quad n,m > 1 \tag{76}$$

take place, where the numbers of space-dimensions n and m are in square brackets [...]. In the particular case of $d = 2$ and $d = 3$, we found

$$F^{[2]}(\alpha, t) = F^{[3]}\left(\frac{3\pi\alpha}{4}, t\right), \quad \tilde{\Sigma}^{[2]}(\alpha, k) = \tilde{\Sigma}^{[3]}\left(\frac{3\pi\alpha}{4}, k\right). \tag{77}$$

Table I. Comparison of known weak coupling results for the polaron ground state energy $E(\alpha) = \alpha \cdot C_{\omega 1} + \alpha^2 \cdot C_{\omega 2} + O(\alpha^3)$ in two-dimensions

Authors	$C_{\omega 1}$	$C_{\omega 2}$
S.Das Sarma, B.Mason [58]	$-\pi/2$	–0.062
R.Feynman's theory [59]	$-\pi/2$	–0.04569
4th, 6th order pert. theory [59]	$-\pi/2$	–0.06397
O.Hipolito [60]	$-\pi/2$	–0.0245
Present $E_0(\alpha)$	$-\pi/2$	–0.046626
Present $E_0(\alpha) + \Delta E_2$	$-\pi/2$	–0.063974

Table II. Comparison of known weak coupling results for the polaron ground state energy $E(\alpha) = \alpha \cdot C_{\omega 1} + \alpha^2 \cdot C_{\omega 2} + O(\alpha^3)$ in three-dimensions

Authors	$C_{\omega 1}$	$C_{\omega 2}$
S.Das Sarma, B.Mason [58]	–1	–0.016
R.Feynman's theory [59]	–1	–0.012347
J.Röseler [61]	–1	–0.0159196*
T.Lee,... [52]	–1	–0.014
D.Larsen [39]	–1	–0.016
Present $E_0(\alpha)$	–1	–0.012598
Present $E_0(\alpha) + \Delta E_2$	–1	–0.015919

*The exact value

Then, considering (62) one easily finds that this scaling relation is also valid for $\frac{1}{d} E_0(\alpha_d)$. We have

$$E_0^{[2]}(\alpha) = \frac{2}{3} E_0^{[3]}\left(\frac{3\pi\,\alpha}{4}\right). \tag{78}$$

Note that the relation (78) was obtained earlier in [54,50]. But this scaling is not valid beyond E_0 because the interaction functional $W[\mathbf{r}]$ depends not only on α_d, but also on d in a complicated way.

Let us consider the asymptotic limits of spatial dimensions d at fixed finite α. We get

$$\lim_{d\to 1} \alpha_d = \frac{3\alpha}{d-1} \to \infty, \quad \lim_{d\to\infty} \alpha_d = \frac{3\alpha\sqrt{\pi e}}{\sqrt{2}\,d^{3/2}} \to 0. \tag{79}$$

Table III. Comparison of obtained estimations of the coefficient C_s of the polaron ground state energy $E(\alpha) = \alpha^2 \cdot C_s + O(1)$ for $d = 2$ as $\alpha \to \infty$

Authors	C_s
S.Das Sarma, B.Mason [58]	–0.392699
R.Feynman's theory [59]	–0.392699[1]
W.Xiaoguang, ... [59]	–0.4047[2]
O.Hipolito [60]	–0.392699
Present $E_0(\alpha)$	–0.392699
Present $E_0(\alpha) + \Delta E_2$	–0.400538

[1]Estimated in [59]
[2]Adiabatic approximation

Table IV. Comparison of obtained estimations of the coefficient C_s of the polaron ground state energy $E(\alpha) = \alpha^2 \cdot C_s + O(1)$ for $d = 3$ as $\alpha \to \infty$

Authors	C_s
Feynman, Schultz [65]	–0.1061
Pekar (by Miyake) [41]	–0.108504[1]
Miyake [41]	–0.108513[2]
Luttinger, Lu [62]	–0.1066
Marshall, Mills [67]	–0.1078
Sheng, Dow [68]	–0.1065
Adamowski, ... [57]	–0.1085128
Feranchuk, Komarov [69]	–0.1078
Efimov, Ganbold [56]	–0.10843

[1]Estimated in [41]
[2]The exact value

Taking into account (79) we can conclude that as d becomes larger, α_d decreases rapidly and in fact we deal with the effective weak-coupling regime $\alpha_d << 1$ even for α not too small. For example, the second-order corrections $\Delta E_2(\alpha)$ behave as follows:

$$\Delta E_2(\alpha) \underset{d \to \infty}{\to} -\frac{1}{8\pi} \alpha_d^2 \to 0. \tag{80}$$

In other words, our leading-order energy term $E_0(\alpha)$ tends to the exact GSE $E(\alpha)$ as d grows because the role of $\Delta E(\alpha)$ becomes insignificant.

Table V. The obtained estimations of the poralon ground state energy $E_0(\alpha)$ and $E^{(2)}(\alpha)$ for $d=2$ in the intermediate range of α compared with known results obtained in [60,70,58]

α	Feynman*	Hipolito [60]	Huybrecht [70]	Das Sarma [58]	Present	
					E_0	E_0+E_2
0.6364	−1.0198	−1.0266	−1.0201	−1.0405	−1.020	−1.028
1.909	−3.2247	−3.2263	−3.2263	−3.5690	−3.231	−3.250
3.183	−5.9191	−6.0902	−5.9193	−6.9688	−5.928	−6.039
4.450	−9.6935	−9.8723	−9.7154	−11.388	−9.710	−9.871

*Our estimation by Feynman's variational method

Table VI. The obtained estimations of the poralon ground state energy $E_0(\alpha)$ and $E^{(2)}(\alpha)$ for $d=3$ in the intermediate range of α compared with known results obtained in [57,65,64,39]

α	Osc. [57]	Feynman [65]	Smondyrev [64]		Larsen [39]		Present	
	upper	upper	upper	lower	upper	lower	E_0	E_0+E_2
0.5	−0.5	−0.5032	−0.5041	−0.5041	−0.5040	−0.5052	−0.504	−0.5041
1.0	−1.0	−1.0130	1.0167	−1.0175	−1.0160	−1.0270	−1.014	−1.017
1.5	−1.5	−1.5302	—	—	−1.5361	−1.576	−1.532	−1.539
2.0	−2.0	−2.0554	—	—	−2.0640	−2.172	−2.058	−2.071
2.5	−2.5	−2.5894	—	—	−2.5995	−2.872	−2.593	−2.614
3.0	−3.0	−3.1333	−3.1645	−3.2122	−3.1421	—	−3.138	−3.167
4.0	−4.0	−4.2565	—	—	−4.2771	—	−4.265	−4.305
5.0	−5.0	−5.4401	−5.4945	−5.7767	—	—	−5.452	−5.528
7.0	−7.356	−8.1127	−8.0406	−8.8832	—	—	−8.137	−8.255
9.0	−10.72	−11.486	−10.834	−12.654	—	—	−11.54	−11.69
11.0	−14.94	−15.710	−13.905	−17.165	—	—	−15.83	−16.04
20.0	−44.53	−45.283	—	—	—	—	−45.33	−45.99
30.0	−97.58	−98.328	—	—	—	—	−98.52	−99.86
40.0	−171.9	−172.60	—	—	—	—	−173.4	−175.1

3.3. Numerical Results. In this Section, we present numerical values of $E_0(\alpha)$ and $E^{(2)}(\alpha)$ estimated within the GER method and compare them with known results obtained in various (weak, strong and intermediate) ranges of α. Obtained results are given in Tables I-IV.

A. *Weak Coupling Limit.* Among known numerical results, concerning the GSE of the polaron, the more accurate are those obtained for $\alpha \to 0$. Below, we calculate the exact GSE of the d-dimensional polaron for the order α^2 in the

weak coupling limit and compare the accuracy of the obtained results with exact perturbation estimations presented in [52,58,59,49,60,54] for $d=2$ and $d=3$.

For α not too large, the polaron self-energy $E(\alpha)$ has the form

$$E(\alpha)=\alpha\cdot C_{\omega 1}+\alpha^2\cdot C_{\omega 2}+O(\alpha^3). \tag{81}$$

The coefficients $C_{\omega 1}$ and $C_{\omega 2}$ are known with a good accuracy for $d=2$ [54] and $d=3$ [58,54]. In our approach, the coefficient $C_{\omega 1}$ arises only from $E_0(\alpha)$ in (62); whereas the $C_{\omega 2}$, from both $E_0(\alpha)$ and $\Delta E_2(\alpha)$ in (74). We get the coefficients $C_{\omega 1}$ and $C_{\omega 2}$ exactly as follows

$$C_{\omega 1}=-\frac{R_d}{3}d \tag{82}$$

and

$$C_{\omega 2}=-\frac{R_d^2 d}{36}\left(1-\frac{8}{3\pi}\right)-\frac{R_d^2\Gamma(d/2)d^2}{9\pi^{3/2}}\sum_{n=2}^{\infty}\frac{(2n!)\Gamma(n+1/2)}{4^n(n!)^2\Gamma(n+d/2)}B_n,$$

$$B_n=\int_1^{\infty}\int dxdy\,\frac{1}{(x+y)^2}\left[\frac{1}{(x\cdot y)^{n+1/2}}+\frac{1}{(x+y-1)^{n+1/2}}\right]. \tag{83}$$

The behaviour of these coefficients with respect to the space-dimension number d is shown in Fig.1.

For comparison, in Table I we give the known results for $d=2$ as $\alpha\to 0$. One can see from Table I that our $C_{\omega 2}$ obtained only from $E_0(\alpha)$ improves Feynman's estimate about 2 per cent. Adding the next correction calculated from ΔE_2 results in $C_{\omega 2}=-0.063974$ which is in good agreement with the exact value in [54]. Note that ΔE_2 contributes about 40 per cent to the total value of $C_{\omega 2}$.

For three dimensions, obtained results are displayed in Table II together with the known results of the polaron GSE for the weak coupling limit. Our leading term of energy $E_0(\alpha)$ improves the Feynman variational estimation of $C_{\omega 2}$ by 2 per cent. Next correction results in $C_{\omega 2}=-0.015919$ which is in good agreement with the exact value in [54]. Note, for $d=3$ our ΔE_2 contributes about 29 per cent (smaller than for $d=2$) to the total value of $C_{\omega 2}$. Comparing the obtained results for $d=2$ and $d=3$, we conclude that higher-order corrections (the second-order one in our case) coming from $J_T(\alpha)$ are substantially more important for $d=2$ than for $d=3$. In other words, the polaron effect is stronger in low space dimensions (see Eq. (80)). This effect was noted earlier in [54,50].

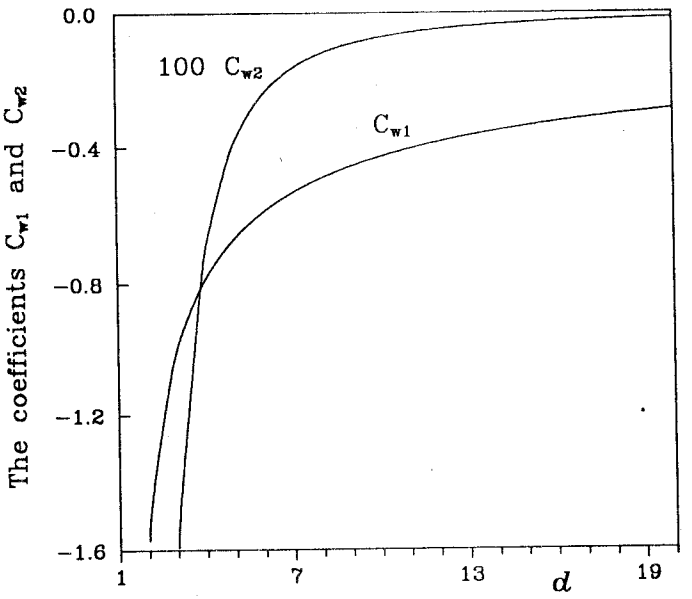

Fig. 1. The behaviour of the coefficients $C_{\omega 1}$ and $C_{\omega 2}$ of the polaron ground state energy $E(\alpha) = \alpha \cdot C_{\omega 1} + \alpha^2 \cdot C_{\omega 2} + O(\alpha^3)$ at the weak coupling limit $\alpha \to 0$ in dependence on the number of space-dimensions d

B. *Strong Coupling Regime.* The GSE of the polaron in the strong electron-phonon coupling regime has been considered in [41,58,59,57,56].

It is well known that at this limit

$$E(\alpha) = \alpha^2 \cdot C_s + O(1). \tag{84}$$

For large α (75) becomes

$$E^{(2)}(\alpha) = -\alpha_d^2 \left\{ \frac{d}{9\pi} + \frac{2\Gamma(d/2)d^2}{9\pi^{3/2}} \cdot \sum_{n=2}^{\infty} \frac{(2n)!\Gamma(n+1/2)}{16^n (n!)^2 n\Gamma(n+d/2)} \right\} + O(1). \tag{85}$$

For comparison, in Table III we give our result with the known results of the polaron GSE for $d = 2$ in the strong coupling regime $\alpha \to \infty$.

For three-dimensions the estimation of the next higher-order corrections for the coefficient C_s was obtained by the authors earlier in [56]:

$$C_s \leq -0.108431. \tag{86}$$

A comparison of the known results for the coefficient C_s for $d = 3$ is displayed in Table IV.

C. *Intermediate Coupling Range.* In the intermediate-coupling regime the main tool for obtaining polaron properties is the variational approach [35,52]. For $d = 3$, the Feynman variational method based on a trial oscillator-type action gives an upper bound of the polaron free energy, valid for arbitrary α. Generalizations of the Feynman action for $d = 3$ to the arbitrary density function [42] and arbitrary quadratic action [43] have improved this upper bound. In our opinion, the result [43] obtained for $d = 3$ is the best variational upper bound in the whole range α. But this variational method does not give the next corrections to this bound. Other numerical methods dealing with this problem [62,63] require specific complicated schemes of calculations which may introduce statistical errors. Estimations of both the upper and lower bounds for the polaron self-energy obtained in [39,64] should be improved.

Considering intermediate values of α, we have derived equations (64) and (63) numerically, by the following iteration scheme:

$$F_{n+1}(t) = \Phi_t[\tilde{\Sigma}_n],$$

$$\tilde{\Sigma}_n(k) = \Omega_k[F_n], \quad n \geq 0, \tag{87}$$

starting from reasonable assumed functions $F_0(t)$ and $\tilde{\Sigma}_0(k)$ (see (70)). Both the series $F_n(t)$ and $\tilde{\Sigma}_n(k)$ turn out to be rapidly convergent and the value of the leading term $E_0(\alpha)$ does not change after $n \geq 6$. The results for $E_0(\alpha)$ and $E^{(2)}(\alpha)$ in two dimensions are presented in Table V.

The values of $E_0(\alpha)$ and $E^{(2)}(\alpha)$ for $d = 3$ are given in Table VI (and displayed in Fig.2) in comparison with the known data [39,65,43,64]. Our $E_0(\alpha)$ for $d = 3$ coincides with the upper bound obtained in [43] and improves the variational results calculated in [71].

We have made preliminary estimations which indicate that the decreasing series in (73) is alternating. Then one can expect that the third-order correction $\Delta E_3(\alpha)$ may slightly increase the value of $E^{(2)}(\alpha)$ and inclusion of higher-order corrections $\Delta E_{n>2}(\alpha)$ might result in an insignificant oscillation of $E^{(n>2)}(\alpha)$ between $E_0(\alpha)$ and $E^{(2)}(\alpha)$. In other words, the obtained $E^{(2)}(\alpha)$ may be accepted as a lower bound of the ground state energy of the polaron. Note that numerical results obtained in [66] at three points ($\alpha = 1,3,5$) by the method of «partial averaging» lie exactly between our curves for $E_0(\alpha)$ and $E^{(2)}(\alpha)$. Recent exact Monte-Carlo calculations [72] are in good agreement with our results for $d = 3$.

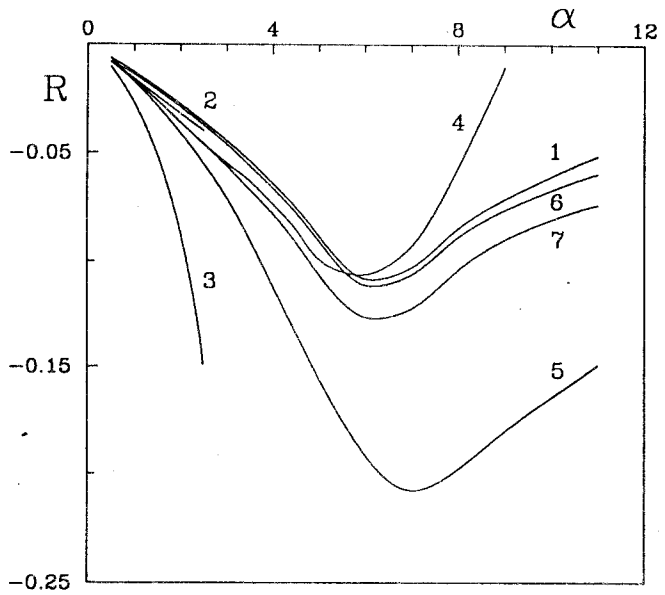

Fig. 2. Some known results of the polaron ground state energy E (in three-dimensional space) displayed as a function of the electron-phonon coupling constant α. For clarity, the ratio $R = (E_* - E_{\text{harm}})/|E_{\text{harm}}$ is shown, where E_* are estimations obtained in [86,52,89,97] and E_{harm} is the «harmonic-oscillator» approximation [86]. In these units the curve for E_{harm} coincides with the abscissa axis. Curves correspond to estimations: 1 — Feynmans's upper; 2/3 — Larsen's upper/lower; 4/5 — Smondyrev's upper/lower; 6 — our $E_0(\alpha)$ and 7 — our $E^{(2)}(\alpha)$

Our results obtained with the proposed method provide a reasonable description of both two- and three-dimensional polarons at an arbitrary coupling α. The consideration could be extended to computing the other characteristics of the polaron, the effective mass and the average number of phonons, as well as to estimating the energy of the polaron in the presence of a magnetic field due to the validity of the proposed method for the complex functionals.

4. CHARACTER OF PHASE TRANSITION IN TWO- AND THREE-DIMENSIONAL φ^4-THEORY

The phenomenon of spontaneous symmetry breaking, or in other words, the vacuum structure rearrangement is an important part of many quantum field

constructions. In this Section, we will investigate this phenomenon within the GER method. The problem, of course, can also be studied within the canonical quantization method. However, the functional representation has an advantage of calculating the whole effective potential (EP) in this theory, which allows one to get more information about phase transitions in the system under consideration.

4.1. Statement of the Problem. The scalar φ^4 theory in two- and three-dimensions has been intensively investigated [73,74] as a simple, but nontrivial example, on which the problem of spontaneous symmetry breaking or, in other words, the phase structure of quantum field models is studied. It has been found [75] that the highest order quantum corrections can give rise to the instability of the classical symmetric vacuum. There are two phases in this system and PT phenomena take place at certain coupling strengths. The most difficult problem here is to determine the order of the PT.

The simplest example, where the vacuum exhibits a nontrivial structure, is the φ_2^4 theory. Many papers [73—76] are devoted to investigation of the nature of PT in this model. We shortly treat some nonperturbative methods that seem to be basic among the investigations on this subject. An original approximation [73] using the Hartree-type renormalization exhibits the first order PT in this theory. A similar result was obtained [77] within the Gaussian EP approach. The dimensionless critical coupling constant, for which the first order phase transition takes place is $G = 1.62$ in both papers. These conclusions disagree with the mathematical theorems [83,84] proving that the second order PT should occur in the φ_2^4 model. There are papers [76—85], where different variational methods have been used for solving this problem and the second order PT has been observed in the region $G \sim 1$. In the previous studies [88,89], we have shown that the critical coupling constant leading to a second order PT cannot exceed the value $G_0 = 1.4392$ and may be found near $G_{crit} \sim 0.53$.

We study this problem using the method of the EP. The absolute minimum of the EP $V(\varphi_0)$ at the point $\varphi_0 = \varphi_c$ determines the true ground state (vacuum) of the theory. If a PT takes place at a certain coupling $g = g_c$, then for $g < g_c$ the system is still in the original unbroken symmetry phase with $\varphi_c = 0$. At reaching $g = g_c$ the origin $\varphi_0 = 0$ is no longer the absolute minimum of $V(\varphi_0)$ and the system goes to a new state with $\varphi_c \neq 0$ corresponding to the lower energy. The first-order PT means that the point $\varphi = 0$ remains local, but is not the absolute minimum of $V(\varphi_0)$. In other words, the first derivative of $V(\varphi_0)$ is zero and the second one is positive at the origin $\varphi_0 = 0$. In the case of the second-order transition, the point $\varphi_0 = 0$ is a local maximum of EP at $g > g_c$. The second

derivative of $V(\varphi_0)$ at $\varphi_0 = 0$ becomes negative. Thus, the coefficient $\alpha(g)$ in the representation of $V(\varphi_0)$ for small φ_0

$$V(\varphi_0) = E(g) + \alpha(g)\cdot \varphi_0^2 + O(\varphi_0^4) \tag{88}$$

plays an important role in determination of the character of phase transition. If $\alpha(g)$ is zero at certain $g = g_c$ and negative for $g > g_c$ up to $g \to \infty$, then one can say that the second-order PT appears here. On the contrary, the positiveness of $\alpha(g)$ for any g excludes the second-order transition. Rigorous calculation of $\alpha(g)$ at an arbitrary coupling constant is a complicated problem. However, we know that at large g, the coefficient $\alpha(g)$ remains negative in case of the second-order PT and is positive if the transition is of the first-order.

We study this problem qualitatively by using the GER method, described in Section 2.1. We will show the possibility of the second order PT in $g\varphi_2^4$ and give an estimation for the corresponding critical coupling constant g_c. For the model $g\varphi_3^4$ our result excludes the occurrence of the second order phase transition.

4.2. Renormalized Lagrangian of the $\varphi^4_{2,3}$-Model. We consider the $g\varphi^4$ scalar field model in two- and three-dimensions. We will use throughout this Section the Euclidean form of the model*. This theory contains ultraviolet divergences, but it is superrenormalizable, i.e., it has only a finite number of divergent Feynman diagrams. In order to remove these divergences we should introduce appropriate counter-terms into the Lagrangian. In this section we consider the superrenormalized scalar field theory with the Lagrangian:

$$L = \frac{1}{2}\varphi(x)[\partial^2 - m^2]\varphi(x) - \frac{g}{4}N_m\{\varphi^4(x)\} - R_m, \tag{89}$$

where we have introduced a «normal-ordered» form of interaction as follows:

$$N_m\{\varphi^4(x)\} = \varphi^4(x) - 6\varphi^2(x)D_m(0) + 3D_m^2(0),$$

$$D_m(x) = \int \frac{d^d k}{(2\pi)^d}\frac{\exp\{ikx\}}{m^2 + k^2}. \tag{90}$$

Here $x \in \Omega$, Ω is a large but finite volume in R^d, $(d = 2,3)$ and m and g are the mass and the self-coupling constants, respectively. In two-dimensions $(d = 2)$ all divergences are only of the «tadpole»-type and are readily removed by

*In the case of the Euclidean metrics a separation of the coordinates into space and time is unimportant, so the accepted notation for the «space-time» is R^d, where d relates to number of space coordinates plus Euclidean (imaginary) time as well.

introducing the normal product N_m of the fields $\varphi(x)$ into (89). In this case $R_m = 0$. In the three-dimensional theory there arise additional divergences which are cancelled by counter-terms

$$R_m = \frac{1}{2} A_m N_m\{\varphi^2(x)\} + \delta E_m, \tag{91}$$

where

$$A_m = 6g^2 \int d^3x D_m^3(x),$$

$$\delta E_m = \frac{3}{4} g^2 \int d^3x D_m^4(x) - \frac{3}{2} g^3 \int \frac{d^3k}{(2\pi)^3} \left\{ \int d^3x e^{ikx} D_m^2(x) \right\}^3. \tag{92}$$

At small g the Lagrangian (89) describes a system invariant with respect to the transformation $\varphi \leftrightarrow -\varphi$. The question is whether this symmetry remains for increasing g.

4.3. Effective Potential in the $\varphi^4_{2,3}$-Theory. The EP is defined as

$$V(\varphi_0) = -\lim_{\Omega \to \infty} \frac{1}{\Omega} \ln I_\Omega(\varphi_0),$$

$$I_\Omega(\varphi) = C_m \int \delta\varphi\, \delta\left\{ \varphi - \frac{1}{\Omega} \int d^dx \varphi(x) \right\} \exp \int_\Omega d^dx L[\varphi(x)], \tag{93}$$

$$C_m = \sqrt{\det\{-\partial^2 + m^2\}}.$$

All integrations are performed in Euclidean metrics.

According to the GER method, we transform the field variable as:

$$\varphi(x) = \phi_0 + b(x) + \phi(x), \tag{94}$$

where the new field variable $\phi(x)$ corresponding to the new mass μ and the function $b(x)$ satisfy the conditions:

$$\int_\Omega d^dx \phi(x) = 0, \quad \int_\Omega d^dx b(x) = 0, \quad b^2(x) = b^2 = \text{const}. \tag{95}$$

Let us go over to the normal ordering in the new fields $\phi(x)$ using the well-known formula [75]

$$N_m\{\exp\{\beta\varphi(x)\}\} = N_\mu \left\{ \exp\left\{ \beta(\phi_0 + b(x) + \phi(x) + \frac{\beta^2}{2} \Delta(m,\mu) \right\} \right\},$$

$$\Delta = \Delta(m,\mu) = D_m(0) - D_\mu(0), \tag{96}$$

$$D_\mu(x) = \int \frac{d^dk}{(2\pi)^d} \frac{\exp\{ikx\}}{\mu^2 + k^2} - \frac{1}{\mu^2\Omega}.$$

First we substitute (94) and (96) into (93) and perform integration over $d\phi_0$. Then, following the key steps of the GER method, we obtain

$$I_\Omega(\varphi_0) = e^{-\Omega V_0(\varphi_0)} \int d\sigma_\mu \times$$

$$\times \exp\left\{ \int_\Omega d^d x N_\mu \left\{ \frac{g}{4} [\phi^4(x) + 4\phi^3(x)(\varphi_0 + b(x)) + 12\varphi_0 b(x)\phi^2(x)] - \right.\right.$$

$$\left.\left. - \left[\frac{1}{2} A_\mu \phi^2(x) + A_\mu b(x)\phi(x) + \delta E_\mu + \frac{1}{2}(b^2 + \varphi_0^2) A_\mu \right] \right\} \right\},$$

$$\int d\sigma_\mu \cdot 1 = C_\mu \int \delta\phi \exp\left\{ -\frac{1}{2} \int_\Omega d^d x \phi(x)(-\partial^2 + \mu^2)\phi(x) \right\} = 1, \tag{97}$$

where the new counter-terms concentrated in the second square brackets in (97) coincide with (92) if we substitute $m \to \mu$. The leading order term of the EP is obtained as the «cactus»-type part $V_0(\varphi_0)$ of the EP as follows:

$$V_0(\varphi_0) = -\frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \left[\ln\left(1 + \frac{m^2 - \mu^2}{\mu^2 + k^2} \right) - \frac{m^2 - \mu^2}{\mu^2 + k^2} \right] + \frac{m^2}{2}(\varphi_0^2 + b^2) +$$

$$+ \frac{g}{4}(\varphi_0^4 + 6\varphi_0^2 b^2 + b^4 - 6\Delta(\varphi_0^2 + b^2) + 3\Delta^2) +$$

$$+ \frac{\varphi_0^2 + b^2}{2}(A_m - A_\mu) + \left(\delta E_m - \delta E_\mu - \frac{1}{2} A_m \Delta \right). \tag{98}$$

The requirement that the linear term $N_\mu\{\phi\}$ must not arise in the interaction and the quadratic field configurations be concentrated in the Gaussian measure $d\sigma_\mu$ leads to the following constraint equations for the parameters $b(x)$ and μ:

$$b(x)[-m^2 + 3g(\Delta - \varphi_0^2) - gb^2 - A_m + A_\mu] = 0,$$

$$\mu^2 - m^2 + 3g(\Delta - \varphi_0^2 - b^2) - A_m + A_\mu = 0. \tag{99}$$

Thus, we finally obtain the formula for the effective potential

$$V(\varphi_0) = V_0(\varphi_0) + V_{sc}(\varphi_0),$$

$$V_{sc}(\varphi_0) = -\lim_{\Omega \to \infty} \frac{1}{\Omega} \ln J_\Omega(\varphi_0), \tag{100}$$

where the new path integral is introduced:

$$J_\Omega(\varphi_0) = e^{-\Omega V_{sc}(\varphi_0)} =$$

$$= \int d\sigma_\mu \exp\left\{ \int_\Omega d^d x \times N_\mu \left\{ -\frac{g}{4} [\varphi^4(x) + 4\varphi^3(x)(\varphi_0 + b(x)) + \right.\right.$$

$$\left.\left. + 12\varphi_0 b(x)\, \phi^2(x)] - \left[\frac{1}{2} A_\mu \phi^2(x) + A_\mu b(x)\phi(x) + \delta E_\mu + \frac{1}{2}(b^2 + \varphi_0^2) A_\mu \right] \right\} \right\}. \quad (101)$$

Equations (98) and (99)—(101) define completely the EP at an arbitrary coupling g. Below we will investigate the EP in (100), whose parameters $b(x)$ and μ are limited by the constraints (96).

For further consideration, it will be convenient to work in units of m dealing with numerical results. We define

$$\xi = (\mu/m)^{4-d}, \quad \Phi_0^2 = 4\pi m^{2-d}\varphi_0^2 \quad \text{and} \quad B^2 = 4\pi m^{2-d} b^2. \quad (102)$$

4.4. The «Cactus»-Type Potential as the Leading-Order Term of the Effective Potential. In two-dimensions, the «cactus-type» part of the EP becomes as follows

$$V_0(\Phi_0) = \frac{m^2}{8\pi} \{\xi - 1 - \ln \xi + \Phi_0^2 + B^2$$

$$+ \frac{G}{6} [\Phi_0^4 + B^4 + 3\ln^2 \xi + 6(B^2\Phi_0^2 - B^2 \ln \xi - \Phi_0^2 \ln \xi)]\}. \quad (103)$$

We note that the potential (103) is invariant for $\Phi_0 \leftrightarrow B$.

The parameters ξ and B in (103) are limited by the following equations:

$$\begin{cases} B^2(\xi - GB^2) = 0, \\ 2\xi - 2 + 3G(\ln \xi - \Phi_0^2 - B^2) = 0. \end{cases} \quad (104)$$

Let us consider the constraint (104). A pair of «trivial» solutions:

$$B = 0 \quad \text{and} \quad \xi = 1 - \frac{3G}{2}(\ln \xi - \Phi_0^2) \quad (105)$$

can be found for an arbitrary coupling constant G. Since $G > G_0 = 1.4392$ an additional pair of «nontrivial» solutions

$$B = \frac{\xi}{G} \quad \text{and} \quad \xi = -2 + \frac{3G}{2}(\ln \xi - \Phi_0^2) \quad (106)$$

appears here, too. So for $G < G_0$ the only solution to be substituted into (103) is the «trivial» one, but since $G > G_0$ there is an alternative: one can choose

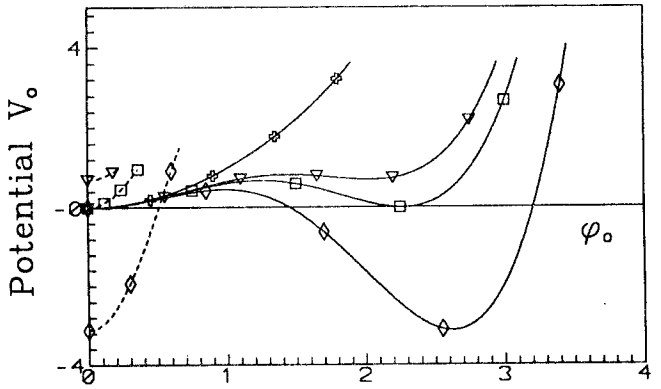

Fig. 3. The Gaussian part $V_0(\Phi_0)$ (in units of $m^2/8\pi$) of the effective potential as a function of Φ_0 for different values of the constant: crosses, $G = 0.5$; triangles, $G = 1.5$; squares, $G = 1.6251$ and rhombs, $G = 2.0$. The dashed lines represent the «nontrivial» branches. The «trivial» branches are denoted by the solid lines

either (105) or (106). We choose the pair obeying the lowest value of $V_0(\Phi_0)$ for certain fixed Φ_0.

All necessary calculations can be performed numerically. The obtained potential $V_0(\Phi_0)$ is plotted in Fig.3. Near the origin $\Phi_0 = 0$ the potential $V_0(\Phi_0)$ is presented by the «nontrivial» branch (if $G > G_0$) $B \neq 0$ as it is situated lower than the «trivial» one. But for larger values of Φ_0 the «trivial» solution $B = 0$ provides the lowest value of the potential. This picture leads to an interesting result. Let us consider the local minima of both branches. For $B = 0$ the minimum point $\Phi_0 = A$ in Fig.3 is given by the equations

$$\begin{cases} B = 0, \\ 2 - 3G \ln \xi + G\Phi_0^2 = 0. \end{cases} \tag{107}$$

On the other hand, the minimum of the «nontrivial» branch $B \neq 0$ is fixed at the origin $\Phi_0 = 0$ for any $G > G_0$ and (104) becomes

$$\begin{cases} \Phi_0 = 0, \\ 2 - 3G \ln \xi + GB^2 = 0. \end{cases} \tag{108}$$

Due to the invariance of the potential $V_0(\Phi_0, B)$ in (103) for $\Phi_0 \leftrightarrow B$ Eqs.(107) and (108) are identical. In other words, the minima of the potential (103) corresponding to different solutions of (104) are equal. The vacuum with $\langle\Phi(x)\rangle = \Phi_0 \neq 0$ is not lower than the initial one located at the point $\langle\Phi(x)\rangle = \Phi_0 = 0$. There is no reason for occurrence of the first order phase transition.

4.5. Non-Gaussian Correction in the φ_2^4-Model. In the previous Section, we derived the expression for the EP consisting of two parts. Considering only the «leading» term $V_0(\varphi_0)$ one can say nothing about the nature of the PT in the theory. To answer this question one should also consider the remaining part $V_{sc}(\varphi_0)$ of the effective potential, defined in (100). In the weak coupling limit one can estimate it expanding the exponential in (97) in perturbative series. But explicit calculation of the non-Gaussian functional integral $J_\Omega(\varphi_0)$ in (97) at arbitrary values of the coupling constant g and φ_0 is a complicated problem. However, we are able to estimate it for infinitesimal values of φ_0 at arbitrary g.

We rewrite (97) in the form correct for infinitesimal φ_0:

$$J_\Omega(\varphi_0) = \int d\sigma_\mu \exp\left\{ -\frac{g}{4}\int_\Omega d^d x N_\mu [\phi^4(x) + 4b(x)\phi^3(x)] + \right.$$

$$\left. + \frac{g^2\varphi_0^2}{2}\left[\int_\Omega d^d x N_\mu (\phi^3(x) + 3b(x)\phi^2(x))\right]^2 \right\}. \qquad (109)$$

This representation can easily be obtained due to the validity of the following transformation in the integrand of (97):

$$\exp(-\varphi_0 W) = \cosh(\varphi_0 W) \simeq \exp\left\{\frac{1}{2}\varphi_o^2 W^2 + O(\varphi_0^4)\right\}$$

for infinitesimal φ_0 and a finite functional W.

Applying to (109) the Jensen's inequality we get upper bound

$$V_{sc}(\varphi_0) \le V_{sc}^{+}(\varphi_0) = -\frac{g^2\varphi_0^2}{2\Omega}\int_\Omega d^d x \int_\Omega d^d y \int d\sigma_\mu \times$$

$$\times \{N_\mu \phi^3(x) N_\mu \phi^3(y) + 9b(x)b(y) N_\mu \phi^2(x) N_\mu \phi^2(y)\}. \qquad (110)$$

It is easy to show that

$$\int d\sigma_\mu N_\mu \phi^3(x) N_\mu \phi^3(y) = 6D_\mu^3(x-y),$$

$$\int d\sigma_\mu N_\mu \phi^2(x) N_\mu \phi^2(y) = 2D_\mu^3(x-y). \tag{111}$$

Then, we rewrite (110) in the form

$$V_{sc}^+(\Phi_0) = -\frac{m^2}{8\pi}\frac{3G^2\Phi_0^2}{2\xi}(Q+3B^2),$$

$$Q = \frac{4\pi \ln 2}{3\sqrt{3}} - 4\int_0^1 \frac{du}{u^2+3}\ln(1-u^2) = 2.3439\ldots. \tag{112}$$

Substituting the parameters ξ and B in either (105) or (106) into (112) one gets the behaviour of $V_{sc}^+(\Phi_0)$ for small values $\Phi_0 \sim 0$. Omitting the details of calculations we write the results

$$V_{sc}^+(\Phi_0) = -\frac{m^2}{8\pi}\left\{-\frac{3Q}{2}G^2\Phi_0^2 + O(\Phi_0^2)\right\} \quad \text{for} \quad G < G_* \tag{113}$$

and

$$V_{sc}^+(\Phi_0) = -\frac{m^2}{8\pi}\left\{-\left[\frac{3QG^2}{2\xi}+\frac{9G}{2}\right]\Phi_0^2 + O(\Phi_0^4)\right\} \quad \text{for} \quad G > G_*,$$

$$3G\ln\xi - \xi - 2 = 0. \tag{114}$$

From (103) we get the following asymptotic behaviour:

$$V_0(\Phi_0) = \frac{m^2}{8\pi}\{\Phi_0^2 + O(\Phi_0^4)\} \tag{115}$$

as $\Phi_0 \to 0$ at any G.

Finally, taking into account (100) we obtain the following behaviour of an upper bound of the EP in the region of small $\Phi_0 \sim 0$:

$$V^+[\Phi_0] = V_0[\Phi_0] + V_{sc}^+[\Phi_0] = \frac{m^2}{8\pi}[\alpha(G)\Phi_0^2 + O(\Phi_0^4)], \tag{116}$$

where

$$\alpha(G) = \begin{cases} \alpha_1(G) = 1 - 3QG^2/2, & G \le 1.6251, \\ \alpha_2(G) = 1 - 3QG^2/(2\xi) - 9G/2, & G > 1.6251, \end{cases}$$

$$3G\ln\xi - \xi - 2 = 0. \tag{117}$$

One can easily check that the coefficient $\alpha_1(G)$ in (117) becomes negative as $G > G_{crit} = 0.5333$ and remains negative for increasing G. But $\alpha_2(G)$ is negative at arbitrary $G > 1.4392$. In our opinion, it indicates an occurrence of the second order PT in the model under consideration.

4.6. Strong Coupling Regime in the φ_3^4-Model. In the three-dimensional case the counter-terms defined by (92) play an important role in the behaviour of the EP in the strong coupling regime. We have

$$D_\mu(x) = \frac{\exp\{-\mu|x|\}}{4\pi|x|}, \quad \Delta = \frac{m}{4\pi}(\xi - 1). \tag{118}$$

Substituting (118) into (99) we get

$$B(x)[2 + 3G(\Phi_0^2 - \xi + 1) + GB^2 + 3G^2 \ln \xi] = 0,$$

$$-2\xi^2 + 2 + 3G(\Phi_0^2 - \xi + 1) + GB^2 + 3G^2 \ln \xi = 0. \tag{119}$$

A non-trivial solution $B \neq 0$ exists only for $0 < \xi < 1$. Let us consider the solution $B = 0$. In the strong coupling regime we obtain

$$\xi = G\sqrt{\frac{3}{2}\ln G} + O(G\sqrt{\ln \ln G}). \tag{120}$$

In other words, the effective coupling constant

$$G_{\text{eff}} = \frac{g}{2\pi\mu} = \frac{G}{\xi} = \sqrt{\frac{2}{3\ln G}}\left\{1 + O\left(\frac{\ln \ln G}{\ln G}\right)\right\} \tag{121}$$

becomes small as $G \to \infty$ and one can successfully develop a perturbation expansion in G_{eff} series for the functional integral (101):

$$V_{\text{sc}}(\varphi_0) = \sum_{n=1}^{\infty} G_{\text{eff}}^n V_{\text{sc}}^{(n)}(\varphi_0). \tag{122}$$

Here $V_{\text{sc}}^{(1)} = 0$ due to normal ordering in the exponential in (101). After some calculations we obtain:

$$V_{\text{sc}}^{(1)}(\varphi_0) = V_{\text{sc}}^{(2)}(\varphi_0) = 0,$$

$$V_{\text{sc}}^{(3)}(\varphi_0) = \frac{m^3}{8\pi}\frac{18C_1}{\xi}G^3\Phi_0^2, \tag{123}$$

where the constant is

$$C_1 = \frac{1}{2\pi^6} \iiint \frac{d^3k d^3p d^3q}{(1+k^2)(1+p^2)(1+q^2)(1+(k+p)^2)(1+(k+q)^2)} =$$

$$= \frac{16}{\pi} \int_0^\infty \frac{du}{1+4u^2} (\arctan u)^2 = 1.7593\ldots .$$

Taking into account the «cactus»-type potential

$$V_0(\Phi_0) = \frac{m^3}{8\pi} \left\{ E_0(G) + \frac{3G}{2} (G \ln \xi - \xi)\Phi_0^2 + O(\Phi_0^4) \right\}, \tag{124}$$

we finally obtain the effective potential

$$V(\Phi_0) = V_0(\Phi_0) + V_{sc}(\Phi_0) = \frac{m^3}{8\pi} \{E(G) + \alpha(G)\cdot\Phi_0^2 + O(\Phi_0^4)\}, \tag{125}$$

where the desired coefficient

$$\alpha(G) = \frac{3G^2}{2} \ln G \left\{ 1 + \frac{\sqrt{96}C_1}{(\ln G)^{3/2}} + O\left(\frac{1}{(\ln G)^{5/2}} \right) \right\}, \tag{126}$$

is positive. This result excludes the second-order phase transitions in the ϕ_3^4 model. It can be accepted as an argument in favour of either existence of the only first-order transition or absence of any PT in the three-dimensional case.

Comparing the results (117) and (126) for $d = 2$ and $d = 3$ we find that the effective mass renormalization is crucial for this problem. In two-dimensions the mass renormalization includes the «tadpole» divergences only and the behaviour of $\alpha(G)$ in (117) indicates a favour of the second-order PT in φ_2^4. For $d = 3$ the mass renormalization contains an additional term of the second perturbative expansion's order which has the opposite sign comparable with a «tadpole» contribution. As a result, the function $\alpha(G)$ in (126) remains positive for all $G > 0$.

5. WAVE PROPAGATION IN RANDOMLY DISTRIBUTED MEDIA

Theoretical investigation of the propagation properties of waves in a randomly distributed environment reflects certain interest due to its many practical applications, including calculation of electronic conductance in crystals [90], wave localization [91] and dumping of signals in the atmosphere or water [92]. A series of different methods has been applied to this problem, among which path integral techniques [93]—[95] reflect considerable interest.

In this Section we investigate wave transmission in a randomly distributed media using the GER method.

5.1. The Green Function of the Wave Equation. The propagation of a wave $u(\mathbf{x})$ (e.g., electromagnetic) in a time-independent environment can be described by the wave equation given in real 3-dimensional space $\mathbf{x} \in R^3$:

$$[\Delta + \omega^2(1 + \varepsilon(\mathbf{x}))]u(\mathbf{x}|\varepsilon) = J(\mathbf{x}), \quad \omega \neq 0. \tag{127}$$

The constant ω is the «dielectric constant» and defines the frequency of unperturbed waves. $J(\mathbf{x})$ is the source function.

The random noise is described by a random stationary field $\varepsilon(\mathbf{x})$, which is assumed to vary stochastically with a certain correlation function $\langle\varepsilon(\mathbf{x})\varepsilon(\mathbf{y})\rangle$. For simplicity we shall consider Gaussian noise

$$\langle\varepsilon(\mathbf{x})\varepsilon(\mathbf{y})\rangle_\varepsilon = \lambda P(\mathbf{x} - \mathbf{y}) =$$

$$= \lambda \exp\left(-\frac{(\mathbf{x}-\mathbf{y})^2}{4l^2}\right) = \lambda \int \frac{d\,\mathbf{k}}{\pi^{3/2}} \exp\left\{-\mathbf{k}^2 + i\,\mathbf{k}\,\frac{\mathbf{x}-\mathbf{y}}{l}\right\}, \tag{128}$$

where the interaction coefficient λ shows the intensity of noise described by the distribution function $P(\mathbf{x}-\mathbf{y})$ with a correlation length l. These two constants define the influence of the Gaussian noise on the propagation of waves in media.

The solution of (127) can be represented in the form

$$u(\mathbf{x}|\varepsilon) = \int d\,\mathbf{y} G(\mathbf{x},\mathbf{y}|\varepsilon) J(\mathbf{y}),$$

where $G(\mathbf{x},\mathbf{y}|\varepsilon)$ is the Green function of wave equation:

$$[\Delta + \omega^2(1 + \varepsilon(\mathbf{x}))]G(\mathbf{x},\mathbf{y}|\varepsilon) = \delta(\mathbf{x} - \mathbf{y}). \tag{129}$$

The problem is to find the solution of (127) and then average it over random fields $\varepsilon(\mathbf{x})$ to find the wave amplitude:

$$u(\mathbf{x}) = \langle u(\mathbf{x}|\varepsilon)\rangle_\varepsilon.$$

For this the Green function should be averaged over random fields $\varepsilon(\mathbf{x})$:

$$G(\mathbf{x}-\mathbf{y}) = \langle G(\mathbf{x},\mathbf{y}|\varepsilon\,\rangle_\varepsilon.$$

Thus we consider this problem solved if the averaged Green function $G(\mathbf{x})$ is found and its asymptotic behaviour for large distances $|\mathbf{x}| \to \infty$ can be calculated.

Let use proceed to solve the equation (129) for the Green function. It is essential that the operator

$$K = \Delta + \omega^2(1 + \varepsilon(\mathbf{x}))$$

is not definitely positive. We shall consider the solution

$$G(\mathbf{x},\mathbf{y}|\varepsilon) = \frac{1}{K + i0}\,\delta(\mathbf{x} - \mathbf{y}),$$

corresponding to the so-called causal Green function. This solution can be written in integral representation like (29) as follows:

$$G(\mathbf{x},\mathbf{y}|\varepsilon) = -\frac{i}{2}\int_0^\infty du\, e^{\frac{i}{2}(K+i0)u}\delta(\mathbf{x} - \mathbf{y}) =$$

$$= -\frac{i}{2}\int_0^\infty du T_\tau \exp\left\{\frac{i}{2}\int_0^u d\tau\left[\left(\frac{\partial}{\partial \mathbf{x}_\tau}\right)^2 + \omega^2(1+\varepsilon(\mathbf{x}_\tau))\right]\right\}\delta(\mathbf{x} - \mathbf{y}).$$

Here we have used the «time-ordering» operator T_τ. Omitting details of calculations we display the results:

$$G(\mathbf{x},\mathbf{y}|\varepsilon) = -\frac{1}{2}\int_0^\infty \frac{du}{(2\pi\, iu)^{3/2}} \exp\left[-\frac{i}{2}\left(\omega^2 u + \frac{(\mathbf{x}-\mathbf{y})^2}{u}\right)\right] I_u(\mathbf{x},\mathbf{y}|\varepsilon),$$

where a FI is introduced:

$$I_u(\mathbf{x},\mathbf{y}|\varepsilon) = \int d\sigma_0 \exp\left\{\frac{i}{2}\omega^2\int_0^u d\tau\varepsilon\left(\mathbf{x}\frac{\tau}{u} + \mathbf{y}\left(1 - \frac{\tau}{u}\right) + \nu(\tau)\right)\right\}, \qquad (130)$$

with the measure defined as

$$d\sigma_0 = C_0\delta\,\nu \exp\left\{\frac{i}{2}\int_0^u d\,\tau\dot{\nu}^2(\tau)\right\}.$$

The integration in (130) is taken over «paths» ν obeying the condition

$$\nu(0) = \nu(u) = 0.$$

Here the normalization is chosen as

$$\int d\sigma_0 = 1 \quad \text{or,} \quad I_u(\mathbf{x},\mathbf{y}|\varepsilon)|_{\varepsilon=0} = 1.$$

Now we can average the functional $I_u(\mathbf{x},\mathbf{y}|\varepsilon)$ over the random fields $\varepsilon(\mathbf{x})$:

$$I_u(\mathbf{x} - \mathbf{y}) = \langle I_u(\mathbf{x},\mathbf{y}|\varepsilon)\rangle_\varepsilon =$$

$$= \int d\sigma_0 \exp\left\{ -\lambda \frac{\omega^4}{8} \iint_0^u d\tau d\tau' P\left(\mathbf{v}(\tau) - \mathbf{v}(\tau') + (\mathbf{x}-\mathbf{y}) \frac{\tau - \tau'}{u} \right) \right\}.$$

The averaged Green function is

$$G(\mathbf{x}) = -\frac{1}{2} \int_0^\infty \frac{du}{(2\pi i u)^{3/2}} \exp\left[\frac{i}{2} \left(\omega^2 u + \frac{\mathbf{x}^2}{u} \right) \right] I_u(\mathbf{x}),$$

where

$$I_u(\mathbf{x}) = C_0 \int_{\mathbf{v}(0)=\mathbf{v}(u)=0} \delta\mathbf{v} \exp\left\{ \frac{i}{2} \int_0^u d\tau \dot{\mathbf{v}}^2(\tau) - \right.$$

$$\left. - \lambda \frac{\omega^4}{8} \iint_0^u d\tau d\tau' P\left(\mathbf{v}(\tau) - \mathbf{v}(\tau') + \mathbf{x} \frac{\tau - \tau'}{u} \right) \right\}. \tag{131}$$

For further convenience we introduce the following notation:

$$r = |\mathbf{x}|, \quad u = \frac{r}{\omega} z, \quad \beta = r\omega, \quad \tau = \frac{z}{\omega^2} t, \quad \tau' = \frac{z}{\omega^2} s, \quad g = \frac{\lambda z^2}{4}$$

and change the variable of the FI:

$$\mathbf{v}(\tau) = \frac{\sqrt{z}}{\omega} \boldsymbol{\rho}(t).$$

Then we have

$$G(\beta) = -\frac{\omega}{2\sqrt{\beta}} \int_0^\infty \frac{dz}{(2\pi i z)^{3/2}} \exp\left[i \frac{\beta}{2} \left(z + \frac{1}{z} \right) \right] I(\beta, z),$$

where

$$I(\beta, z) = C_0 \int_{\boldsymbol{\rho}(0)=\boldsymbol{\rho}(\beta)} \delta\boldsymbol{\rho} \exp\left\{ \frac{i}{2} \int_0^\beta dt \dot{\boldsymbol{\rho}}^2(t) - \right.$$

$$\left. - \frac{g}{2} \iint_0^\beta dt ds P\left(\frac{\sqrt{z}}{\omega} (\boldsymbol{\rho}(t) - \boldsymbol{\rho}(s)) + \mathbf{n} \frac{t-s}{\omega} \right) \right\}, \tag{132}$$

where

$$\mathbf{n} = \frac{\mathbf{x}}{|\mathbf{x}|}, \quad (\mathbf{nn}) = 1.$$

5.2. Calculation of the FI by the GER Method. In order to apply the GER method to this problem, let us introduce into (132) symmetrical limits by redefining

$$2T = \beta, \quad t \to t - T, \quad s \to s - T, \quad \rho(t) \to \rho(t - T).$$

Then we rewrite (132):

$$I_T(z) = C_0 \int\limits_{\rho(-T)=\rho(T)=0} \delta\rho \exp\left\{ \frac{i}{2} \int\limits_{-T}^{T} dt\, \dot{\rho}^2(t) - \right.$$

$$\left. - \frac{g}{2} \int\limits_{-T}^{T} \int dtds P\left(\frac{\sqrt{z}}{\omega} [\rho(t) - \rho(s)] + \mathbf{n} \frac{t-s}{\omega} \right) \right\}.$$

Let us introduce the operator

$$D_0^{-1}(t-s) = i \frac{\partial^2}{\partial t^2} \delta(t-s).$$

Note that it differs from (30) by the factor $-i$. The Green function $D_0(t,s)$ corresponding to this operator satisfies some periodic conditions and reads

$$D_0(t,s) = -\frac{i}{2} |t-s| - \frac{ts}{2T} \to -\frac{i}{2} |t-s|.$$

Its Fourier transform is

$$\tilde{D}_0(p) = \frac{i}{p^2}.$$

Then we rewrite

$$I_T(z) = C_0 \int\limits_{\rho(-T)=\rho(T)=0} \delta\rho \exp\left\{ -\frac{1}{2} \iint\limits_{-T}^{T} dtds (\rho(t) \mathbf{D}_0^{-1}(t-s) \rho(s)) - gW[\rho] \right\},$$

$$C_0 = (\det \mathbf{D}_0)^{-1/2}.$$

The free «kinetic» term is diagonal:

$$(\rho(t) \mathbf{D}_0^{-1}(t-s)\, \rho(s)) = (\rho_i(t)\, \delta_{ij}\, D_0^{-1}(t-s)\, \rho_j(s)).$$

The interaction is given by

$$gW[\rho] = \frac{g}{2} \iint_{-T}^{T} dtds P\left(\frac{\sqrt{z}}{\omega} (\boldsymbol{\rho}(t) - \boldsymbol{\rho}(s)) + \mathbf{n} \frac{T-s}{\omega} \right) =$$

$$= \frac{g}{2} \iint_{-T}^{T} dtds \int \frac{d\,\mathbf{k}}{\pi^{3/2}} \exp\left\{ -\mathbf{k}^2 + i \frac{\mathbf{k}}{l\omega} (\sqrt{z}\,(\boldsymbol{\rho}(t) - \boldsymbol{\rho}(s)) + \mathbf{n}(t-s)) \right\}.$$

We find that the measure $d\mathcal{K}$ of momentum integration now becomes

$$d\mathcal{K}_{\mathbf{n}}(\mathbf{k}, t-s) = \frac{d\,\mathbf{k}}{\pi^{3/2}} \exp\left\{ -\mathbf{k}^2 + \frac{i}{l\,\omega} (\mathbf{kn})(t-s) \right\}.$$

Following the GER method, we define the new measure

$$d\sigma = C\delta\,\boldsymbol{\rho} \exp\left\{ -\frac{1}{2} \int \int_{-T}^{T} dtds\, (\boldsymbol{\rho}(t)\, \mathbf{D}^{-1}(t-s)\, \boldsymbol{\rho}(s)) \right\},$$

where

$$(\boldsymbol{\rho}(t)\, \mathbf{D}^{-1}(t-s)\, \boldsymbol{\rho}(s)) = (\rho_i(t) D_{ij}^{-1}(t-s)\, \rho_j(s)).$$

Notice that, the operator D_{ij}^{-1} has nondiagonal elements owing to the presence of the vector $\mathbf{n}$ in $W[\boldsymbol{\rho}]$.

In the following we will use the notations

$$\int d\sigma \exp\left\{ i \frac{\sqrt{z}}{l\,\omega} \mathbf{k}(\boldsymbol{\rho}(t) - \boldsymbol{\rho}(s)) \right\} = \exp\left\{ -\frac{z}{(l\,\omega)^2} (\mathbf{kF}(t-s)\mathbf{k}) \right\},$$

$$(\mathbf{kF}(t-s)\mathbf{k}) = (k_i F_{ij}(t-s) k_j),$$

$$\mathbf{F}(t-s) = \mathbf{D}(0) - \mathbf{D}(t-s) = \int_0^{\infty} \frac{dp}{\pi} [1 - \cos p(t-s)]\, \tilde{\mathbf{D}}(p^2),$$

$$g\hat{W}[\mathbf{b}] = \frac{g}{2} \int \int_{-T}^{T} dtds \int \frac{d\,\mathbf{k}}{\pi^{3/2}} \exp\left\{ -\left(\mathbf{k}[\mathbf{I} + \frac{z}{(l\,\omega)^2} \mathbf{F}(t-s)]\mathbf{k} \right) \right\} \times \tag{133}$$

$$\times \exp\left\{ i \frac{\sqrt{z}}{\omega l} \mathbf{k}(\mathbf{b}(t) - \mathbf{b}(s)) + i\,\mathbf{kn} \frac{t-s}{\omega l} \right\},$$

$$(|\mathbf{q}\rangle\langle\mathbf{q}|)_{ij} = q_i q_j,$$

$$\Phi(t-s)=\frac{gz^3}{(\omega l)^4}\int\frac{d\,\mathbf{q}}{\pi^{3/2}}\,|\mathbf{q}\rangle\langle\,\mathbf{q}|\times$$

$$\times\exp\left\{-\left(\mathbf{q}\left[\mathbf{I}+\frac{z}{(l\omega)^2}\,\mathbf{F}(t-s)\right]\mathbf{q}\right)+i\,\mathbf{qn}\,\frac{t-s}{\omega l}\right\}=$$

$$=\mathbf{\Phi}_0(t-s)+|\mathbf{n}\rangle\langle\,\mathbf{n}|\mathbf{\Phi}_1(t-s),$$

$$\Phi_{ij}(t-s)=\delta_{ij}\cdot\Phi_0(t-s)+n_i\,n_j\,\Phi_1(t-s).$$

Then we get

$$\omega_{ij}(t-s)=g\,\frac{\delta^2\hat{W}[\mathbf{b}]}{\delta b_i(t)\,\delta b_j(s)}\,|_{\mathbf{b}=0}=-\,[\tilde{\Phi}_{ij}(0)-\Phi_{ij}(t-s)].$$

Following all the steps described in section 2 we finally obtain

$$I_T(z)=\mathrm{e}^{-2TE_0(z)}J_T(z),$$

$$J_T(z)=C\int\delta\rho\,\exp\left\{-\frac{1}{2}\int\limits_{-T}^{T}dtds(\rho\,\mathbf{D}^{-1}\rho)-g:\tilde{W}[\rho]:\right\},\qquad(134)$$

where the leading-order term (or the zeroth approximation of the GER method) is

$$E_0(z)=\frac{3}{2\pi}\int\limits_0^\infty dp\left[\ln\left(1+\frac{i}{p^2}\,\mathbf{\Sigma}(p)\right)\right]-$$

$$-\frac{g}{2}\int\limits_{-\infty}^{\infty}dt\int\frac{d\,\mathbf{q}}{\pi^{3/2}}\exp\left\{-\left(\mathbf{q}\left[\mathbf{I}+\frac{z}{(l\,\omega)^2}\,\mathbf{F}(t)\right]\mathbf{q}\right)+i\,\mathbf{qn}\,\frac{t}{\omega l}\right\}.$$

The interaction functional in the new representation is

$$\tilde{W}[\rho]=-\frac{g}{2}\int\limits_{-T}^{T}dtds\int d\mathcal{K}_{\mathbf{n}}(\mathbf{q},t-s)\times$$

$$\times\exp\left\{-\left(\mathbf{q}\left[\mathbf{I}+\frac{z}{(l\omega)^2}\,\mathbf{F}(t-s)\right]\mathbf{q}\right)+i\,\mathbf{qn}\,\frac{t-s}{\omega l}\right\}:\mathrm{e}_2^{\,i\frac{\sqrt{z}}{l\omega}[\mathbf{q}(\rho(t)-\rho(s))]}:\qquad(135)$$

where $:e_2^z:=e^z-1-z-\frac{z^2}{2}$.

The requirement of the «*correct*» form for the interaction functional is held if we put

$$\tilde{\mathbf{D}}(p) = \tilde{\mathbf{D}}_0(p) + \tilde{\mathbf{D}}_0(p)\boldsymbol{\Sigma}(p)\tilde{\mathbf{D}}_0(p),$$

or

$$\tilde{\mathbf{D}}_0(p) = \frac{\tilde{\mathbf{D}}_0(p)}{\mathbf{I} + \tilde{\mathbf{D}}_0(p)\boldsymbol{\Sigma}(p)} = \frac{i\,\mathbf{I}}{p^2 + i\,\boldsymbol{\Sigma}(p)}.$$

Then (41) and (42) become

$$\boldsymbol{\Sigma}(p) = -\tilde{\mathbf{w}}(p) = \int_{-\infty}^{\infty} dt[1 - \cos(pt)]\boldsymbol{\Phi}(t) \tag{136}$$

and

$$\mathbf{F}(t) = i \int_0^{\infty} \frac{dp}{\pi} \frac{1 - \cos p(t)}{p^2 + i\,\boldsymbol{\Sigma}(p)}. \tag{137}$$

5.3. The Green Function for Large Distances. The initial (131) and the new (134) representations are equivalent. The next step is to solve (136) and (137) which allows one to calculate the function $E_0(z)$. The explicit form of the interaction functional (135) allows the highest corrections to be calculated. In principle, these calculations are similar to those in the polaron problem except that now all functionals are complex. Nevertheless, all transformations of the GER method applied here are valid. In the future we plan to solve these equations and investigate the behaviour of the Green function $G(\mathbf{x})$ for different values of the parameters λ and l.

So the main problem is to solve the integral equations (136) and (137). However this represents a laborious task and one can by-pass this difficulty considering the large distance's behaviour of the Green function $G(\beta)$.

We now consider wave propagation for large distances $\beta \to \infty$. Then by analogy with the polaron problem, where the similar asymptotics have been studied, we can expect that the following behaviour of the FI occurs

$$I(\beta, z) \sim \frac{1}{\beta^{0(1)}} \exp\{-\beta E(z; \lambda, \omega l)\}.$$

Consequently,

$$G(\beta) \sim \frac{1}{\beta^{0(1)}} \int_0^{\infty} \frac{dz}{z^{3/2}} \exp\left\{\beta\left[\frac{i}{2}\left(z + \frac{1}{z}\right) - E(z; \lambda, \omega l)\right]\right\} \sim$$

$$\sim \frac{1}{\beta^{0(1)}} \int_0^{\infty} \frac{dz}{z^{3/2}} \exp\{\beta S(z)\}, \tag{138}$$

where

$$S(z) = \frac{i}{2}\left(z + \frac{1}{z}\right) - E(z;\, \lambda,\, \omega l).$$

The main contribution to the FI in (138) for large β can be obtained by using the saddle-point method:

$$S(z) = S(z_0) - \frac{1}{2} S''(z_0)(z - z_0)^2 + O(z - z_0)^3,$$

with the conditions

$$S'(z_0) = 0, \quad S''(z_0) > 0.$$

Finally, one gets

$$G(\beta) \sim \frac{1}{\beta^{0(1)}} \exp\{\beta S(z_0)\}.$$

CONCLUSION

We have formulated a regular method for calculating a wide class of functional integrals beyond the region of perturbative expansion. Providing a good accuracy of the lowest approximation, this method has the following advantages compared to the variational approach: the possibility for obtaining higher-order corrections in a regular way and the validity for complex functionals and theories with divergencies.

We have applied this method to different problems of theoretical physics, namely:

(i) the polaron problem in QS,

(ii) the PT phenomenon in the QFT model,

(iii) the solution of the wave's differential equation.

These subjects show the efficiency of the GER method. High accuracy is achieved in calculation of the ground state energy of the d-dimensional polaron. An effective scheme of mass renormalization in the $g\varphi^4_{2,3}$ theory, suggested within the GER method, leads to the correct prediction of the nature of the PT in this theory. At last, an estimation of the nonhermitean path integral for the Green function in the theory of wave propagation in media with a Gaussian distribution is performed.

The developed approach opens up new possibilities for estimating, with high accuracy, the bound states of few-body systems under any potential as well as for investigating static characteristics of the polaron in magnetic fields, when the action of the system is complex and any variational method becomes inapplicable.

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