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PATH INTEGRAL APPROACH FOR SUPERINTEGRABLE POTENTIALS ON THE THREE-DIMENSIONAL HYPERBOLOID

C.Grosche

II. Institut für Theoretische Physik, Universität Hamburg,
Luruper Chaussee 149, 22761 Hamburg, Germany

G.S.Pogosyan, A.N.Sissakian

Joint Institute for Nuclear Research,
141980 Dubna, Moscow Region, Russia

In the present paper on superintegrable potentials on spaces of constant curvature we discuss the case of the three-dimensional hyperboloid. Whereas in many coordinate systems an explicit path integral solution for the corresponding potential is impossible, we list in the soluble cases the path integral solutions explicitly in terms of the propagators and the spectral expansions into the wave-functions. We find the analogues of the maximally and minimally superintegrable potentials of $\mathbb{R}^3$ on the hyperboloid and many minimally superintegrable potentials which emerge from the subgroup chains corresponding to $SO(3,1)$. Some special care is taken for the proper generalization of the harmonic oscillator and the Kepler problem.

В настоящей статье, посвященной задачам с суперинтегрируемыми потенциалами в пространствах постоянной кривизны, обсуждается случай трехмерного гиперболоида. Несмотря на невозможность точного решения соответствующих задач с помощью метода континуального интегрирования во многих разделяющих системах координат в работе приведены все случаи, когда это удается сделать точно для пропагаторов и спектральных разложений по волновым функциям. Построены аналоги максимально и минимально суперинтегрируемых потенциалов в $\mathbb{R}^3$ на трехмерном гиперболоиде и большое число минимально суперинтегрируемых потенциалов, возникающих из цепочек подгрупп, соответствующих группе $SO(3,1)$. Подробно рассматривается обобщение задач о гармоническом осцилляторе и кеплеровской проблемы.

1. INTRODUCTION

Motivation and Symmetry Methods in Physics

The present paper is the fourth in a sequel concerning superintegrable potentials in spaces of constant curvature. It continues our studies which started from the investigation in two- and three-dimensional Euclidean space, i.e., in $\mathbb{R}^2$ and $\mathbb{R}^3$, on the two- and three-dimensional sphere $S^{(2)}$ and $S^{(3)}$, and on the two-dimensional hyperboloid $\Lambda^{(2)}$. Our goal is devoted to the study of physical systems in spaces of constant curvature which have *accidental degeneracies*, i.e., systems which have due to their peculiar features the so-called *hidden symmetry* or *dynamical group structure*, thus giving rise to degeneracies in the energy spectrum, and additional integrals of motion, respectively observables.

The most well-known of these kinds of potential systems in three-dimensional flat space are the harmonic oscillator with quantum energy spectrum

$$E_N = \hbar\omega\big(N + \tfrac{3}{2}\big) \ , \qquad N \in \mathbb{N}_0 \ , \tag{1.1}$$

and the Kepler-Coulomb problem with the quantum energy spectrum

$$E_N = -\frac{Me^4}{2\hbar^2(N+1)^2} \ , \qquad N \in \mathbb{N}_0 \ . \tag{1.2}$$

Here, N denotes the principal quantum number, and for fixed N each level E_N for the oscillator is $(N+1)(N+2)/2$-fold degenerate, and in the Coulomb problem $(N+1)^2$-fold degenerate.

The particular symmetry features have the consequence that there are additional constants of motion in classical mechanics, respectively observables in quantum mechanics. In comparison, the orbits of a simple integrable system, e.g., a three-dimensional anharmonic oscillator, are generally only periodic with respect to each coordinate, but not globally*. For a physical system in D dimensions just to be integrable, a number of D constants of motion is required, with one of them the energy E. In classical mechanics these constants of motion have vanishing Poisson brackets with the Hamiltonian and with each other; in quantum mechanics they are operators which commute with the quantum Hamiltonian and with each other. For instance, for a spherical symmetric system, the constants of motion are the energy E, the square of the total angular momentum $\mathbf{L}^2$, and the square of the (usually chosen) z-component of the angular momentum L_z^2, in classical mechanics as well as in quantum mechanics.

In systems like the isotropic harmonic oscillator or the Kepler-Coulomb problem in three dimensions, there are two more functionally independent constants of motion. In the case of the harmonic oscillator the additional constants of motion correspond to the conservation of the quadrupole moment, the so-called Demkov tensor $T_{ik} = p_i p_k + \omega^2 x_i x_k$ [12], and in the case of the Kepler-Coulomb problem they correspond to the conservation of the square of another component of the angular momentum and the third component of the Pauli-Lenz-Runge vector $\mathbf{A} = \frac{1}{2M}(\mathbf{L}\times\mathbf{P} - \mathbf{P}\times\mathbf{L}) - e^2\mathbf{x}/|\mathbf{x}|$, and both systems have five constants of motion, respectively observables.

*However, they are periodic globally if the frequencies $\omega_{1,2,3}$ are commensurable, i.e., if their respective quotients are rational numbers.

A more careful investigation shows that the highly spherical symmetric systems of the isotropic harmonic oscillator and the Kepler-Coulomb problem can be perturbed in various ways by the incorporation of additional potential terms: First, this does not spoil the degeneracy of the energy levels at all, i.e., there are still five observables*; second, one of the observables is removed, i.e., they are four left; and third, only the minimum number of three observables for integrability remains. The first possibility is described by the notion of a *maximally superintegrable* system; the second possibility, by the notion of a *minimally superintegrable* systems, and the last possibility just describes an *integrable* system.

In this respect, the physical significance of the consideration of separation of variables in several coordinate systems is as follows. The free motion in some space is, of course, the most symmetric one, and the search for the number of coordinate systems which allow the separation of the Hamiltonian is equivalent to the investigation how many inequivalent sets of observables can be found. In particular, the free motion in various coordinate systems on the hyperboloid has been studied in Refs. 24,28,31,38. The incorporation of potentials usually removes at least some of the symmetry properties of the space. Well-known examples are spherically symmetric systems, and they are most conveniently studied in spherical coordinates.

All the superintegrable systems have the particular property that all the energy-levels of the system are organized in representations of the non-invariance group which contains representations of the dynamical subgroup realized in terms of the wave-functions of these energy-levels [20]. The additional integrals of motion also have the consequence that in the case of the superintegrable systems in two dimensions and maximally superintegrable systems in three dimensions all finite trajectories are found to be periodic; in the case of minimally superintegrable systems in three dimensions all finite trajectories are found to be quasi-periodic** [56]. Of course, in the case of the pure Kepler or the isotropic harmonic oscillator all finite trajectories are periodic.

Generally, a physical system in D dimensions is called *minimally superintegrable* if it has $2D-2$ integrals of motion, and it is called *maximally superintegrable* if it has $2D-1$ integrals of motion, respectively observables. Therefore we are led to the search for more (potential-) systems which have similar features concerning degeneracy and number of observables as the radial harmonic oscillator and the Coulomb problem.

A systematic study to classify separable potentials was undertaken by Smoro-

*In the sequel we use the notions "energy levels" and "periodicity of closed orbits", "observables" and "constants of motion", "Coulomb-" or "Kepler-problem", referring to quantum mechanical or classical mechanical properties, respectively, as synonymous.

**The notion quasi-periodic means that they are periodic in each coordinate, but not necessarily periodic in a global way. They are periodic globally if the respective periods are commensurable.

dinsky, Winternitz and co-workers [20, 71, 91], i.e., they looked for potentials which are separable in more than one coordinate systems. The separation of a quantum mechanical potential problem in more than one coordinate systems has the consequence that there are additional integrals of motion and that the spectrum is degenerate. The choice of a coordinate system then emphasizes which observables are considered to be the most appropriate for a particular investigation.

Superintegrable Systems

The harmonic oscillator in spaces of constant curvature has been discussed by, e.g., Bonatsos et al. [8], Higgs [43], Lemon [64], Granovsky et al. [22] and in [34, 35], as well as the Coulomb-Kepler problem in spaces of constant curvature – by Higgs [43] and Lemon [64], and in the general context of the SO(4, 2) dynamical algebra – by Barut et al. [5], and Granovsky et al. [23], Katayama [55], Pogosyan et al. [79], Otchik and Red'kov [77], Schrödinger [80], Stevenson [85], and Vinitsky et al. [88].

The notion of "superintegrability" [16, 53, 92] can now be introduced in spaces of constant curvature [34, 35]. Whereas the general form of potentials which are "superintegrable" in some kind is not clear until now, one knows that the corresponding Higgs-oscillator (c.f. Bonatsos et al. [8], Granovsky et al. [22], Higgs [43], Ikeda and Katayama [45], Katayama [55]), Leemon [64], Nishino [74], and Pogosyan et al. [79]) and Kepler problems (c.f. Granovsky et al. [23], Infeld [46], Infeld and Schild [47], Kalnins et al. [54], Kibler et al. [57], Kurochkin and Otchik [62], Nishino [74], Otchik and Red'kov [77], Vinitsky et al. [88, 89], Zhedanov [93]) in spaces of constant curvature do have additional constants of motion: the analogues of the flat space. For the Higgs-oscillator this is the Demkov-tensor [12,21,74], and in the Kepler problem a Pauli–Runge–Lenz vector on spaces of constant curvature can be defined, c.f. [23,43,62,64,74]. Corresponding path integral considerations are due to Barut et al. [3,4], Otchik and Red'kov [77], and [25] (D-dimensional case), and [34] (superintegrable aspects).

Disturbing the spherical symmetry usually spoils them. The first step consists in deforming the ring-shaped feature of the (maximally superintegrable) modified oscillator and Coulomb potential. One gets in the former a ring-shaped oscillator and in the latter the Hartmann potential, two minimally superintegrable systems. The number of coordinate systems which allow a separation of variables drops from eight to four (namely spherical, circular polar, oblate spheroidal and prolate spheroidal coordinates [57, 58]), and from four to three, namely spherical, parabolic, and prolate spheroidal II coordinates.

Disturbing the system further, one is left with, say, one-coordinate systems which still allow separation of variables. A constant electric field (Stark effect) allows only the separation in parabolic coordinates [29]. Here it is interesting to remark that in the momentum representation of the hydrogen atom the bound state spectrum is described by the free motion on the sphere $S^{(3)}$. To be more precise, the dynamical group $O(4)$ describes the discrete spectrum; and the Lorentz group $O(3,1)$, the continuous spectrum [2]. Now, there are six coordinate systems on $S^{(3)}$ which separate the corresponding Laplacian. The solution in spherical and cylindrical coordinates corresponds to the spherical and parabolic solution in the coordinate space representation. The elliptic cylindrical system is of special interest because it enables one to set up a complete classification for the energy-levels of the quadratic Zeeman effect (c.f. Solov'ev [83], Brown and Solov'ev [9], Herrick [42], Lakshmann and Hasegawa [63]).

The separation in parabolic coordinates is also possible in the case of a perturbation of the pure Coulomb field with a potential force $\propto z/r$ which allows an exact solution [27, 30]. The two-center Coulomb problem turns out to be separable only in spheroidal coordinates (Coulson and Josephson [10], Coulson and Robinson [11], Morse [72]) as has been studied first in the connection with the hydrogen-molecule ion by Teller [86].

Another possibility to disturb the spherical symmetry is to remove the invariance to rotations with respect to some axis, e.g., about a uniform magnetic field. Usually this invariance is used to illustrate the azimuthal quantum number m of the L_z operator. The physical meaning of this quantum number then is that there exists a preferred axis in space. This symmetry can be broken if one considers a Hamiltonian of a nucleus with an electric quadrupole moment Q and spin J in a spatially varying electric field [66, 84]. Here sphero-conical coordinates are most convenient, and the projection of the terminus of the angular momentum vector traces out a cone of elliptic cross section about the z-axis [84]. Also the problem of the asymmetric top (Kramers and Ittmann [61], Lukač [65], Smorodinsky et al. [67,82,90]), the symmetric oblate top [65], or the case of tensor-like potentials (Lukač and Smorodinsky [68]) can be treated best in sphero-conical coordinates. Therefore sphero-conical coordinates are most suitable for problems which have spherical symmetry but not a sphero-axial symmetry.

In order that a potential problem can be separated in ellipsoidal coordinates it is necessary that the shape of the potential resemble the shape of an ellipsoid. Of course, the anisotropic harmonic oscillator belongs to this class. Introduction of quartic and sextic [87] interaction terms then eventually allows only a separation of variables in ellipsoidal coordinates. Another example is the Neumann model [73], which describes a particle moving on a sphere subject to anisotropic harmonic forces (Babelon and Talon [1], and MacFarlane [69]).

Our first paper [33] dealt with superintegrable potentials in two- and three-dimensional flat space, where we distinguished minimally and maximally superintegrable systems. In two-dimensional Euclidean space they are four (maximally) superintegrable systems [16], i.e., the (generalized) harmonic oscillator $V_1(\mathrm{x})$, the

Holt-potential $V_2(\mathbf{x})$, the (generalized) Coulomb potential $V_3(\mathbf{x})$, and a modified Coulomb potential $V_4(\mathbf{x})$.*

In three-dimensional Euclidean space we found five maximally and nine minimally superintegrable systems. Among the maximally superintegrable ones are the (generalized) harmonic oscillator $V_1(\mathbf{x})$, the Holt-potential in IR^3, $V_2(\mathbf{x})$, and the (generalized) Coulomb potential $V_3(\mathbf{x})$; among the minimally superintegrable systems were a double-ring shaped oscillator $V_6(\mathbf{x})$, the Hartmann potential $V_7(\mathbf{x})$, a three-dimensional analogue of the Holt-potential $V_6(\mathbf{x})$, four potentials $V_2(\mathbf{x}), V_3(\mathbf{x}), V_4(\mathbf{x}), V_8(\mathbf{x})$ which emerged from the group chain $E(3) \supset E(2)$, i.e., they are superintegrable in IR^2, and the two potentials $V_1(\mathbf{x}), V_9(\mathbf{x})$ which emerged from the group chain $E(3) \supset \mathrm{SO}(3)$, i.e., they are superintegrable on the two-dimensional sphere $S^{(2)}$.

In our second paper [34] we continued our study on the two- and three-dimensional sphere. On $S^{(2)}$ we found only two potentials with the required properties, i.e., the (generalized) Higgs oscillator $V_1(\mathrm{s})$ and the (generalized) Coulomb potential $V_2(\mathrm{s})$. We have not been able to find the superintegrable analogues of the Holt potential and the modified Coulomb potential. On the three-dimensional sphere $S^{(3)}$ we have found three maximally superintegrable and four minimally superintegrable potentials, respectively. Among the maximally superintegrable potentials were the (generalized) Higgs oscillator $V_1(\mathrm{s})$, the Coulomb potential $V_2(\mathrm{s})$, and as a third potential – $V_3(\mathrm{s})$ a pure scattering potential which corresponds to $V_4(\mathbf{x})$ in IR^3. Among the minimally superintegrable systems there have been the analogues of the double ring-shaped oscillator $V_4(\mathrm{s})$ and the Hartmann potential $V_5(\mathrm{s})$ on $S^{(3)}$, and the two remaining potentials $V_6(\mathrm{s}), V_7(\mathrm{s})$ emerged from the group chain $\mathrm{SO}(4) \supset \mathrm{SO}(3)$.

In [35] we considered the superintegrable potentials on the two-dimensional hyperboloid $\Lambda^{(2)}$. We have found the analogues of the (generalized) harmonic oscillator $V_1(\mathrm{u})$, i.e., the Higgs oscillator in a space of constant negative curvature, the (generalized) Coulomb potential $V_2(\mathrm{u})$, and the Holt potential $V_3(\mathrm{u})$ on $\Lambda^{(2)}$. We also found two more systems $V_3(\mathrm{u}), V_4(\mathrm{u})$, which are due to the peculiarity of the hyperboloid that in spaces of constant negative curvature there are generally more orthogonal coordinate systems which separate the Schrödinger, respectively Boltzmann equation, in comparison to flat or constant positive curvature spaces. However, we have not been able to find a superintegrable version of the modified Coulomb potential, c.f. $V_4(\mathbf{x})$ in IR^2.

*The notion of minimally superintegrable systems in two dimensions does not make sense, because the number of integrals of motions equals two, and is thus equal to the number of integrals of motion which are required that the system is separable at all.

Interbasis Expansions

An important aspect of group path integration (see below) in quantum mechanics is the so-called interbasis expansion technique for problems which allow the representation of the wave-functions in various coordinate space representations. The basic formula is quite simple being

$$|\mathbf{k}>= \int dE_{\mathbf{l}} C_{\mathbf{p},\mathbf{k}}|\mathbf{p}> + \sum_{\mathbf{n}} C_{\mathbf{n},\mathbf{k}}|\mathbf{n}> \ , \tag{1.3}$$

where $|\mathbf{k}>$ stands for a basis of eigenfunctions of the Hamiltonian in the coordinate space representation $\mathbf{k}$, and $\int dE_{\mathbf{l}}$ is the spectral-expansion with respect to the coordinate space representation $\mathbf{l}$ with coefficients $C_{\mathbf{p},\mathbf{k}}, C_{\mathbf{n},\mathbf{k}}$ which can be discrete, continuous or both. The main difficulty is, in case one has two-coordinate space representations in the quantum numbers $\mathbf{k}$ and $\mathbf{p}, \mathbf{n}$, respectively, to find the expansion coefficients $C_{\mathbf{p},\mathbf{k}}$ and $C_{\mathbf{n},\mathbf{k}}$. Well known are the expansions which involve cartesian coordinates and polar coordinates. In the simple case of free quantum motion in Euclidean space, this means that exponentials representing plane waves are expanded in terms of Bessel functions and spherical waves, a discrete interbasis expansion, i.e., $e^{z\cos\psi} = \sum_{\nu\in\mathbb{Z}} e^{i\nu\psi} I_\nu(z)$.

This general method of changing a coordinate basis in quantum mechanics can now be used in the path integral. We assume that we can expand the short-time kernel, respectively the exponential $e^{z\mathbf{x}_j - 1 \cdot \mathbf{x}_j}$ in terms of matrix elements of a group [31] by choosing a specific coordinate basis. We then can change the coordinate basis by means of (1.3). Due to the unitarity of the expansion coefficients $C_{\mathbf{l},\mathbf{k}}$ the short-time kernel is expanded in the new coordinate basis, and the orthonormality of the basis allows one to perform explicitly the path integral, exactly in the same way as in the original coordinate basis. However, to find the dynamical group and its corresponding coordinate space representation in a superintegrable system, one of the principal problems, is not always very easy. From the two (or more) different equivalent coordinate space representations, formulae and path integral identities can be derived, and at the same time, yield interbasis coefficients. These identities actually correspond to integral and summation identities between special functions. The case of the expansion from cartesian coordinates to polar coordinates has been studied by Peak and Inomata [78], and they obtained the solution of the isotropic harmonic oscillator as well. The path integral solution of the isotropic harmonic oscillator in turn enables one to calculate numerous path integral problems related to the radial harmonic oscillator, actually problems which are of the so-called Besselian type, including the Coulomb problem. Furthermore, a (path integral) solution in a particular coordinate space representation can serve as a starting point for a perturbative analysis in cases where a system separates, say, in only one coordinate system, but is not exactly solvable. Then, the knowledege of the wave functions and interbasis coefficients of the corresponding exactly solvable model is of paramount importance, e.g., [32,70].

Path Integral Approach

In our investigations the path integral turns out to be a very convenient tool to formulate and solve the superintegrable potentials on spaces of constant curvature, in particular on the hyperboloid. The subsequent separation of variables in each problem can be performed in a straightforward and easy way. We start by considering the classical Lagrangian corresponding to the line element $ds^2 = g_{ab}dq^a dq^b$ of the classical motion in some D-dimensional Riemannian space, e.g., [13,17,39,59,81] and references therein

$$\mathcal{L}_{Cl}(\mathbf{q},\dot{\mathbf{q}}) = \frac{M}{2}\left(\frac{ds}{dt}\right)^2 - V(\mathbf{q}) = \frac{M}{2} g_{ab}(\mathbf{q})\dot{q}^a\dot{q}^b - V(\mathbf{q}) \ . \tag{1.4}$$

The quantum Hamiltonian is *constructed* by means of the Laplace-Beltrami operator

$$H = -\frac{\hbar^2}{2M}\Delta_{LB} + V(\mathbf{q}) = -\frac{\hbar^2}{2M}\frac{1}{\sqrt{g}}\frac{\partial}{\partial q^a} g^{ab}\sqrt{g}\frac{\partial}{\partial q^b} + V(\mathbf{q}) \tag{1.5}$$

as a *definition* of the quantum theory on a curved space. Here are $g = \det(g_{ab})$, $(g^{ab}) = (g_{ab})^{-1}$, and $\Delta_{LB} = g^{-1/2}\partial_a g^{ab} g^{1/2}\partial_b$. The scalar product for wave-functions on the manifold reads $(f,g) = \int d\mathbf{q}\sqrt{g} f^*(\mathbf{q})g(\mathbf{q})$, and the momentum operators which are hermitian with respect to this scalar product are given by

$$p_a = \frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial q^a} + \frac{\Gamma_a}{2}\right) , \qquad \Gamma_a = \frac{\partial \ln\sqrt{g}}{\partial q^a} \ . \tag{1.6}$$

In terms of the momentum operators (1.6) we can rewrite H by using an ordering prescription called product according, where we assume $g_{ab} = h_{ac}h_{cb}$ (other lattice formulations like the important midpoint prescription (MP) which corresponds to the Weyl ordering in the Hamiltonian, we do not discuss). Then we obtain for the Hamiltonian (1.5)

$$H = -\frac{\hbar^2}{2M}\Delta_{LB} + V(\mathbf{q}) = \frac{1}{2M} h^{ac} p_a p_b h^{cb} + \Delta V(\mathbf{q}) + V(\mathbf{q}) \ , \tag{1.7}$$

and for the path integral we obtain

$$K(\mathbf{q}'',\mathbf{q}';T) = \int\limits_{\mathbf{q}(t')=\mathbf{q}'}^{\mathbf{q}(t'')=\mathbf{q}''} \mathcal{D}\mathbf{q}(t)\sqrt{g(\mathbf{q})}$$
$$\times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}h_{ac}(\mathbf{q})h_{cb}(\mathbf{q})\dot{q}^a\dot{q}^b - V(\mathbf{q}) - \Delta V(\mathbf{q})\right]dt\right\}$$

$$\equiv \lim_{N\to\infty}\left(\frac{M}{2\pi i\epsilon\hbar}\right)^{ND/2}\prod_{k=1}^{N-1}\int d\mathbf{q}_k\sqrt{g(\mathbf{q}_k)}$$

$$\times\exp\left\{\frac{i}{\hbar}\sum_{j=1}^{N}\left[\frac{m}{2\epsilon}h_{bc}(\mathbf{q}_j)h_{ac}(\mathbf{q}_{j-1})\Delta q_j^a\Delta q_j^b-\epsilon V(\mathbf{q}_j)-\epsilon\Delta V(\mathbf{q}_j)\right]\right\}. \quad (1.8)$$

ΔV_{PF} denotes the well-defined quantum potential

$$\Delta V_{PF}(\mathbf{q})=\frac{\hbar^2}{8M}\Big[g^{ab}\Gamma_a\Gamma_b+2(g^{ab}\Gamma_b)_{,b}+g^{ab}{}_{,ab}$$

$$+2h^{ac}h^{bc}{}_{,ab}-h^{ac}{}_{,a}h^{bc}{}_{,b}-h^{ac}{}_{,b}h^{bc}{}_{,a}\Big]\ . \quad (1.9)$$

Here we have used the abbreviations $\epsilon=(t''-t')/N\equiv T/N$, $\Delta\mathbf{q}_j=\mathbf{q}_j-\mathbf{q}_{j-1}$, $\bar{q}_j=\frac{1}{2}(\mathbf{q}_j+\mathbf{q}_{j-1})$ for $\mathbf{q}_j=\mathbf{q}(t'+j\epsilon)$ $(t_j=t'+\epsilon j, j=0,\dots,N)$ and we interpret the limit $N\to\infty$ as equivalent to $\epsilon\to 0$, T fixed. The lattice representation can be obtained by exploiting the composition law of the time-evolution operator $U=\exp(-iHT/\hbar)$, respectively its semi-group property, and the discretized path integral emerges in a natural way. The classical Lagrangian is modified into an effective Lagrangian via $\mathcal{L}_{eff}=\mathcal{L}_{Cl}-\Delta V$. We use this path integral formulation throughout the paper. For the technique of space-time transformations we refer to [15, 18, 39, 59] and references therein.

Presentation of Results

The contents of this paper is as follows. In the next section we give an introduction into the formulation and construction of coordinate systems on the three-dimensional hyperboloid. This includes an enumeration of the coordinate systems according to [31, 50, 52, 75]. The enumeration includes the explicit statement of the quantity $\mathrm{u}=(u_0,u_1,u_2,u_3)$ in terms of the coordinate variables $\varrho=(\varrho_1,\varrho_2,\varrho_3)$, the line element $ds^2=ds^2(\varrho)$, the momentum operators P_{ϱ_i}, the Hamiltonian H_0, the form a potential $V(\mathrm{u})$ must have in order that the Schrödinger equation $H\Psi=(H_0+V)\Psi=E\Psi$ is separable, and the corresponding integrals of motion, respectively observables.

In Section III we present the three found maximally superintegrable potentials on the three-dimensional hyperboloid, including an analogue of a Stark effect potential which is, however, in comparison to $\mathbb{R}^3$ only minimally superintegrable. The maximally superintegrable systems have five integrals of motion. For instance, in the pure Coulomb problem in $\mathbb{R}^3$ they are the energy E, the square of the absolute value of the angular momentum $\mathbf{L}^2$, L_z^2, an observable corresponding to the semi-hyperbolic system, and the third component of the Pauli–Lenz–Runge vector $\mathbf{A}$ (the whole set of $E,\mathbf{L}^2,L_z^2,\mathbf{A}$ is not functionally independent). Actually, the first three of these constants of motion are typical for each radial problem, and the minimum number of three observables is required in order that a three-dimensional system is separable at all (in [16] a systematic listing of these constants of motion has been presented). We treat the first two potentials, i.e., the Higgs oscillator and the Coulomb potential on $\Lambda^{(3)}$, in some detail. The relevant observables are listed in the tables.

In Section IV we discuss the minimally superintegrable potentials on $\Lambda^{(3)}$. We find four potentials which have their counterparts in three-dimensional Euclidean space. The remaining potentials emerge from the subgroup structure of $SO(3,1)$, i.e., we find four potentials corresponding to the chain $SO(3,1) \supset E(2)$, two potentials corresponding to the chain $SO(3,1) \supset SO(3)$, where one of them is however equivalent to a previous one, and five potentials corresponding to the chain $SO(3,1) \supset SO(2,1)$, respectively. This yields 15 minimally superintegrable potentials on $\Lambda^{(3)}$. We do not explicitly list each solution again, because this would blow up our paper to much, and refer instead to our previous work concerning the superintegrable potentials in flat space [33], on the sphere [34], and on the two-dimensional hyperboloid [35]. In Sections III and IV we make frequently use of the path integral formulations of the Pöschl–Teller, the modified Pöschl–Teller, and the Rosen–Morse potential, whose solutions can be found in [33]– [35], and references therein, c.f., e.g., Böhm and Junker [7], [31,39,40], Fischer et al. [18], Inomata et al. [48], Kleinert and Mustapic [60].

In the fifth Section we summarize and discuss our results. Here we also establish a correspondence of maximally and minimally superintegrable potentials in two and three dimensions in the three spaces of constant curvature, i.e., Euclidean space, the sphere, and the hyperboloid. In addition we suggest analogues of the Holt potential on the two- and three-dimensional sphere and on the two- and three-dimensional hyperboloid, respectively. However, these potentials turn out to be only integrable. On the sphere the corresponding separating coordinate systems are the $k = k' = 1/\sqrt{2}$ particular case of the rotated elliptic, respectively rotated prolate spheroidal systems. On the hyperboloid the separating coordinate systems are the semi-hyperbolic systems. The flat space limits of these systems are parabolic coordinates in two and three dimensions.

2. COORDINATE SYSTEMS ON HYPERBOLOIDS

In this Section we construct the coordinate systems on the three-dimensional hyperboloid. However, first we cite some useful information concerning the construction of coordinate systems on the most important spaces of constant curvature. These are Euclidean spaces, spheres and hyperboloids.

For the classification of coordinate systems in an homogeneous space, and hence for sets of inequivalent observables, we need second-order differential operators I_i ($i \in J$, J an index set) which are at most quadratic in the derivatives.

In order that they can characterize a coordinate system which separates the Hamiltonian we must require that they commute with the Hamiltonian and with each other, i.e., $[H, I_i] = [I_i, I_j] = 0$. This property characterizes them as observables (in classical mechanics as constants of motion). In two-dimensional spaces we have *one characteristic operator* I which corresponds to the additional observable, and in three-dimensional spaces there are *two characteristic operators* I_1, I_2 which correspond to the two separation constants appearing for each coordinate system. Finding all inequivalent sets of $\{I\}$, respectively $\{I_1, I_2\}$, is equivalent in finding all inequivalent sets of observables for the Hamiltonian of the free motion. Because the operators $I_{1,2}$ commute with the Hamiltonian and with each other one can find simultaneously eigenfunctions of H, I, respectively H, I_1, I_2.

Before we begin discussing the coordinate systems on the three-dimensional hyperboloid in some detail, let us start with some remarks concerning harmonic analysis on $\Lambda^{(3)}$, and we cite some results from [52,90].

The homogeneous Lorentz group SO(3, 1) consists of those proper real linear transformations which leave the hyperboloid ($u_0 > 0$)

$$\mathrm{u} \cdot \mathrm{u} = \mathrm{u}^2 = u_0^2 - (u_1^2 + u_2^2 + u_3^2) = u_0^2 - \mathbf{u}^2 = R^2 \tag{2.1}$$

invariant. The Lie algebra is six-dimensional, and is generated by the spatial rotation generators

$$\begin{aligned} L_1 &= \frac{\hbar}{\mathrm{i}}\left(u_2\frac{\partial}{\partial u_3} - u_3\frac{\partial}{\partial u_2}\right), \\ L_2 &= \frac{\hbar}{\mathrm{i}}\left(u_1\frac{\partial}{\partial u_3} - u_3\frac{\partial}{\partial u_1}\right), \\ L_3 &= \frac{\hbar}{\mathrm{i}}\left(u_1\frac{\partial}{\partial u_2} - u_2\frac{\partial}{\partial u_1}\right), \end{aligned} \tag{2.2}$$

(note the sign convention in comparison to the sphere) and the Lorentz transformation generators

$$\begin{aligned} K_1 &= \frac{\hbar}{\mathrm{i}}\left(u_0\frac{\partial}{\partial u_1} + u_1\frac{\partial}{\partial u_0}\right), \\ K_2 &= \frac{\hbar}{\mathrm{i}}\left(u_0\frac{\partial}{\partial u_2} + u_2\frac{\partial}{\partial u_0}\right), \\ K_3 &= \frac{\hbar}{\mathrm{i}}\left(u_0\frac{\partial}{\partial u_3} + u_3\frac{\partial}{\partial u_0}\right). \end{aligned} \tag{2.3}$$

The commutation relations are

$$[L_i, L_j] = -\mathrm{i}\hbar\epsilon_{ijk}L_k\,, \quad [L_i, K_j] = -\mathrm{i}\hbar\epsilon_{ijk}K_k\,, \quad [K_i, K_j] = \mathrm{i}\hbar\epsilon_{ijk}K_k\,. \tag{2.4}$$

The Hamiltonian on $\Lambda^{(3)}$ can then be written as ($V(\mathrm{u})$ a potential on $\Lambda^{(3)}$)

$$H = H_0 + V(\mathrm{u}) \ , \qquad H_0 = -\frac{\hbar^2}{2MR^2}\Delta_{LB} = \frac{1}{2MR^2}(\mathbf{K}^2 - \mathbf{L}^2) \ . \qquad (2.5)$$

The irreducible representations of the identity component of SO(3, 1) are labeled by two numbers (j_0, σ), where j_0 is an integer or half-integer, and σ is complex. The eigenvalues of the Schrödinger operator H_0 are found to have the following form

$$E_{\sigma,j_0} = -\frac{\hbar^2}{2MR^2}[j_0^2 + \sigma(\sigma+2)],$$

$$\begin{aligned} &\text{continuous spectrum:} \quad j_0 = 0, \sigma = -1 + ip, \\ &\text{discrete spectrum:} \quad j_0 = 2n \ (n \in \mathbb{N}), \sigma = -1. \end{aligned} \qquad (2.6)$$

Actually, the discrete spectrum is not present in our case; for instance, it must be taken into account for the quantum motion on the single sheeted hyperboloid [31], on the SU(1,1) [7] and on the O(2,2) group manifold [31,51]. We have for the energy-spectrum of the free quantum motion on $\Lambda^{(3)}$

$$E_p = \frac{\hbar^2}{2MR^2}(p^2 + 1) \ , \qquad p > 0 \ . \qquad (2.7)$$

In the following enumeration we list in each case the definition of the coordinate systems, the metric, the momentum operators, the Hamiltonian and the observables I_1, I_2, respectively.

In the sequel we only consider *orthogonal* coordinate systems on the three-dimensional hyperboloid. $\mathrm{u} \in \Lambda^{(3)}$ is expressed as $\mathrm{u} = \mathrm{u}(\varrho)$, where $\varrho = (\varrho_1, \varrho_2, \varrho_3)$ are three-dimensional coordinates on $\Lambda^{(3)}$. Following Olevskiĭ [75] the line element is found to have the form

$$\begin{aligned} ds^2 = \epsilon_a g_{aa} d\varrho_a^2 \ &= \ \frac{1}{4}\Bigg[\frac{(\varrho_1-\varrho_2)(\varrho_1-\varrho_3)}{P(\varrho_1)}\mathrm{d}\varrho_1^2 + \frac{(\varrho_2-\varrho_3)(\varrho_2-\varrho_1)}{P(\varrho_2)}\mathrm{d}\varrho_2^2 \\ &+ \ \frac{(\varrho_3-\varrho_1)(\varrho_2-\varrho_1)}{P(\varrho_3)}\mathrm{d}\varrho_3^2\Bigg], \end{aligned} \qquad (2.8)$$

which must be a positive-definite quantity, hence $\epsilon_a = -1$, $a = 1, 2, 3$, and where $P(\varrho)$ is the *characteristic polynomial* corresponding to the coordinate system.

Table 2.1. Coordinate systems on the three-dimensional hyperboloid

Coordinate System Observables I_1, I_2	Coordinates	Separates Potential
I. Cylindrical $\tau_{1,2} \in \mathbb{R}, \varphi \in [0, 2\pi)$ $I_1 = K_3^2$ $I_2 = L_3^2$	$u_0 = R\cosh\tau_1\cosh\tau_2$ $u_1 = R\sinh\tau_1\cos\varphi$ $u_2 = R\sinh\tau_1\sin\varphi$ $u_3 = R\cosh\tau_1\sinh\tau_2$	V_1
II. Horicyclic $x_{1,2} \in \mathbb{R}, y > 0$ $I_1 = (K_1 + L_2)^2$ $I_2 = (K_2 - L_1)^2$	$u_0 = R\big[y + (x_1^2 + x_2^2)/y + 1/y\big]/2$ $u_1 = Rx_1/2y$ $u_2 = Rx_2/2y$ $u_3 = R\big[y + (x_1^2 + x_2^2)/y - 1/y\big]/2$	V_8, V_9, V_{10}
III. Sphero-Elliptic $\tau > 0, \tilde\alpha \in [-K, K]$ $\tilde\beta \in [-2K', 2K']$ $I_1 = \mathbf{L}^2, I_2 = L_1^2 + k'^2 L_2^2$	$u_0 = R\cosh\tau$ $u_1 = R\sinh\tau\,\mathrm{sn}\tilde\alpha\,\mathrm{dn}\tilde\beta$ $u_2 = R\sinh\tau\,\mathrm{cn}\tilde\alpha\,\mathrm{cn}\tilde\beta$ $u_3 = R\sinh\tau\,\mathrm{dn}\tilde\alpha\,\mathrm{sn}\tilde\beta$	V_1, V_2, V_7 $V_7'^{*}, V_{13}^{*}$
IV. Equidistant-Elliptic $\tau \in \mathbb{R}, \alpha \in (\mathrm{i}K', \mathrm{i}K' + 2K)$ $\beta \in [0, 4K')$ $I_1 = K_1^2 + K_2^2 - L_3^2$ $I_2 = L_3^2 + \sinh^2 f K_1^2$	$u_0 = R\cosh\tau\,\mathrm{sn}\alpha\,\mathrm{dn}\beta$ $u_1 = \mathrm{i}R\cosh\tau\,\mathrm{cn}\alpha\,\mathrm{cn}\beta$ $u_2 = \mathrm{i}R\cosh\tau\,\mathrm{dn}\alpha\,\mathrm{sn}\beta$ $u_3 = R\sinh\tau$	V_1, V_4^+, V_{14} V_{15}^+
V. Equidistant-Hyperbolic $\tau \in \mathbb{R}, \mu \in (\mathrm{i}K', \mathrm{i}K' + 2K)$ $\eta \in [0, 4K')$ $I_1 = K_1^2 + K_2^2 - L_3^2$ $I_2 = K_1^2 - \sin^2\alpha L_3^2$	$u_0 = -R\cosh\tau\,\mathrm{cn}\mu\,\mathrm{cn}\eta$ $u_1 = \mathrm{i}R\cosh\tau\,\mathrm{sn}\mu\,\mathrm{dn}\eta$ $u_2 = \mathrm{i}R\cosh\tau\,\mathrm{dn}\mu\,\mathrm{sn}\eta$ $u_3 = R\sinh\tau$	V_1, V_{14}
VI. Equidistant-Semi-Hyperbolic $\tau \in \mathbb{R}, \mu_{1,2} > 0$ $I_1 = K_1^2 + K_2^2 - L_3^2$ $I_2 = \{L_3, K_2\}$	$u_0 = \frac{R}{\sqrt{2}}\cosh\tau\Big(\sqrt{(1+\mu_1^2)(1+\mu_2^2)} + \mu_1\mu_2 + 1\Big)^{1/2}$ $u_1 = \frac{R}{\sqrt{2}}\cosh\tau\Big(\sqrt{(1+\mu_1^2)(1+\mu_2^2)} - \mu_1\mu_2 - 1\Big)^{1/2}$ $u_2 = R\cosh\tau\sqrt{\mu_1\mu_2}$ $u_3 = R\sinh\tau$	V_4, V_{15}
VII. Equidistant- Elliptic-Parabolic $\tau, a \in \mathbb{R}, \vartheta \in (-\pi/2, \pi/2)$ $I_1 = K_1^2 + K_2^2 - L_3^2$ $I_2 = (K_2 - L_3)^2 + K_1^2$	$u_0 = R\cosh\tau\frac{\cosh^2 a + \cos^2\vartheta}{2\cosh a\cos\vartheta}$ $u_1 = R\cosh\tau\frac{\sinh^2 a - \sin^2\vartheta}{2\cosh a\cos\vartheta}$ $u_2 = R\cosh\tau\tan\vartheta\tanh a$ $u_3 = R\sinh\tau$	V_4, V_{15}, V_{17}
VIII. Equidistant- Hyperbolic-Parabolic $\tau \in \mathbb{R}, b > 0, \vartheta \in (0, \pi)$ $I_1 = K_1^2 + K_2^2 - L_3^2$ $I_2 = (K_2 - L_3)^2 - K_1^2$	$u_0 = R\cosh\tau\frac{\cosh^2 b + \cos^2\vartheta}{2\sinh b\sin\vartheta}$ $u_2 = R\cosh\tau\frac{\sinh^2 b - \sin^2\vartheta}{2\sinh b\sin\vartheta}$ $u_1 = R\cosh\tau\cot\vartheta\coth b$ $u_3 = R\sinh\tau$	V_{17}
IX. Equidistant-Semi-Circular- Parabolic ($\tau \in \mathbb{R}, \xi, \eta > 0$) $I_1 = K_1^2 + K_2^2 - L_3^2$ $I_2 = \{K_1, K_2\} - \{K_1, L_3\}$	$u_0 = R\cosh\tau\frac{(\xi^2 + \eta^2)^2 + 4}{8\xi\eta}$ $u_1 = R\cosh\tau\frac{(\xi^2 + \eta^2)^2 - 4}{8\xi\eta}$ $u_2 = R\cosh\tau\frac{\eta^2 - \xi^2}{2\xi\eta}$ $u_3 = R\sinh\tau$	V_{16}, V_{17}, V_{18}

* after rotation with $I_2' = \cos 2f L_3^2 - \frac{1}{2}\sin 2f\{L_1, L_3\}$; $^+$ after rotation with $I_2' = \cosh 2f L_3^2 - \frac{1}{2}\sinh 2f\{K_2, L_3\}$

Table 2.1 (cont.)

Table 2 (cont.) Coordinate System	Coordinates	Separates Potential
X. Spherical $\tau > 0, \vartheta \in (0,\pi), \varphi \in [0,2\pi)$ $I_1 = \mathbf{L}^2$ $I_2 = L_3^2$	$u_0 = R\cosh\tau$ $u_1 = R\sinh\tau\sin\vartheta\cos\varphi$ $u_2 = R\sinh\tau\sin\vartheta\sin\varphi$ $u_3 = R\sinh\tau\cos\vartheta$	V_1, V_2, V_3 V_5, V_6, V_7 V_7', V_{13}
XI. Equidistant-Cylindrical $\tau_{1,2} \in \mathbb{R}, \varphi \in [0,2\pi)$ $I_1 = K_1^2 + K_2^2 - L_3^2$ $I_2 = L_3^2$	$u_0 = R\cosh\tau_1\cosh\tau_2$ $u_1 = R\cosh\tau_1\sinh\tau_2\cos\varphi$ $u_2 = R\cosh\tau_1\sinh\tau_2\sin\varphi$ $u_3 = R\sinh\tau_1$	V_1, V_3, V_4 V_5, V_{14}, V_{15}
XII. Equidistant $\tau_{1,2,3} \in \mathbb{R}$ $I_1 = K_1^2 + K_2^2 - L_3^2$ $I_2 = K_1^2$	$u_0 = R\cosh\tau_1\cosh\tau_2\cosh\tau_3$ $u_1 = R\cosh\tau_1\cosh\tau_2\sinh\tau_3$ $u_2 = R\cosh\tau_1\sinh\tau_2$ $u_3 = R\sinh\tau_1$	V_1, V_{14}, V_{17} V_{18}
XIII. Equidistant-Horicyclic $\tau, x \in \mathbb{R}, y > 0$ $I_1 = K_1^2 + K_2^2 - L_3^2$ $I_2 = (K_2 - L_3)^2$	$u_0 = R\cosh\tau(y + x^2/y + 1/y)/2$ $u_1 = R\sinh\tau$ $u_2 = R\cosh\tau x/y$ $u_3 = R\cosh\tau(y + x^2/y - 1/y)/2$	V_{16}, V_{17}
XIV. Horicyclic-Cylindrical $y, \varrho > 0, \varphi \in [0,2\pi)$ $I_1 = (K_1 + L_2)^2 + (K_2 - L_1)^2$ $I_2 = L_3^2$	$u_0 = R(y + \varrho^2/y + 1/y)/2$ $u_1 = R\varrho\cos\varphi/y$ $u_2 = R\varrho\sin\varphi/y$ $u_3 = R(y + \varrho^2/y - 1/y)/2$	V_9, V_{11}
XV. Horicyclic-Elliptic $y, \mu > 0, \nu \in (-\pi,\pi)$ $I_1 = (K_1 + L_2)^2 + (K_2 - L_1)^2$ $I_2 = L_3^2 + (K_1 + L_2)^2$	$u_0 = R[y + (\cosh^2\mu - \sin^2\nu)/y + 1/y]/2$ $u_1 = R\cosh\mu\cos\nu/y$ $u_2 = R\sinh\mu\sin\nu/y$ $u_3 = R[y + (\cosh^2\mu - \sin^2\nu)/y - 1/y]/2$	V_9, V_{11}^*
XVI. Horicyclic-Parabolic $y, \eta > 0, \xi \in \mathbb{R}$ $I_1 = (K_1 + L_2)^2 + (K_2 - L_1)^2$ $I_2 = \{L_3, K_1 + L_2\}$	$u_0 = R[y + (\xi^2 + \eta^2)^2/y + 1/y]/2$ $u_1 = R(\eta^2 - \xi^2)/2y$ $u_2 = R\xi\eta/y$ $u_3 = R[y + (\xi^2 + \eta^2)^2/y - 1/y]/2$	V_{10}, V_{11}, V_{12}
XVII. Prolate Elliptic $\alpha \in (iK', iK' + 2K)$ $\beta \in [0, 4K'), \varphi \in [0,2\pi)$ $I_1 = L_3^2$ $I_2 = \mathbf{L}^2 - (k'^2/k^2)(K_3^2 + L_3^2)$	$u_0 = R\,\mathrm{sn}\alpha\,\mathrm{dn}\beta$ $u_1 = iR\,\mathrm{dn}\alpha\,\mathrm{sn}\beta\cos\varphi$ $u_2 = iR\,\mathrm{dn}\alpha\,\mathrm{sn}\beta\sin\varphi$ $u_3 = iR\,\mathrm{cn}\alpha\,\mathrm{cn}\beta$	V_1, V_2^*, V_3 V_5, V_6^*
XVIII. Oblate Elliptic $\alpha \in (iK', iK' + 2K)$ $\beta \in [0, 4K'), \varphi \in [0,2\pi)$ $I_1 = L_3^2$ $I_2 = \mathbf{L}^2 + k'^2(L_3^2 - K_1^2 - K_2^2)$	$u_0 = R\,\mathrm{sn}\alpha\,\mathrm{dn}\beta$ $u_1 = iR\,\mathrm{cn}\alpha\,\mathrm{cn}\beta\cos\varphi$ $u_2 = iR\,\mathrm{cn}\alpha\,\mathrm{cn}\beta\sin\varphi$ $u_3 = iR\,\mathrm{dn}\alpha\,\mathrm{sn}\beta$	V_1, V_3, V_5
XIX. Elliptic-Cylindrical $\tau \in \mathbb{R}, \alpha \in (iK', iK' + 2K), \beta \in [0, 4K')$ $I_1 = K_1^2$ $I_2 = L_1^2 + k'^2(L_2^2 - K_3^2)$	$u_0 = R\,\mathrm{sn}\alpha\,\mathrm{dn}\beta\cosh\tau$ $u_1 = R\,\mathrm{sn}\alpha\,\mathrm{dn}\beta\sinh\tau$ $u_2 = iR\,\mathrm{dn}\alpha\,\mathrm{sn}\beta$ $u_3 = iR\,\mathrm{cn}\alpha\,\mathrm{cn}\beta$	V_1
XX. Hyperbolic-Cylindrical 1 $\tau \in \mathbb{R}, \mu \in (iK', iK' + 2K)$ $\eta \in [0, 4K')$ $I_1 = K_1^2, I_2 = K_3^2 - L_2^2 + k^2(K_1^2 - L_1^2)$	$u_0 = -R\,\mathrm{cn}\mu\,\mathrm{cn}\eta\cosh\tau$ $u_1 = -R\,\mathrm{cn}\mu\,\mathrm{cn}\eta\sinh\tau$ $u_2 = iR\,\mathrm{sn}\mu\,\mathrm{dn}\eta$ $u_3 = iR\,\mathrm{dn}\mu\,\mathrm{sn}\eta$	V_1

* after rotation with $I_2' = \cosh 2f\mathbf{L}^2 - \frac{1}{2}\sinh 2f(\{K_2, L_1\} - \{K_1, L_2\})$

Table 2.1 (cont.)

Table 2.1 (cont.) Coordinate System	Coordinates	Separates Potential
XXI. Hyperbolic-Cylindrical 2 $\mu\in(iK',iK'+2K)$ $\eta\in[0,4K'),\varphi\in[0,2\pi)$ $I_1=L_3^2, I_2=K_3^2+L_3^2-k^2(L_1^2+L_2^2)$	$u_0=-R\,\mathrm{cn}\mu\,\mathrm{cn}\eta$ $u_1=iR\,\mathrm{sn}\mu\,\mathrm{dn}\eta\cos\varphi$ $u_2=iR\,\mathrm{sn}\mu\,\mathrm{dn}\eta\sin\varphi$ $u_3=iR\,\mathrm{dn}\mu\,\mathrm{sn}\eta$	V_1,V_3,V_5
XXII. Semi-Hyperbolic $\mu_{1,2}>0,\varphi\in[0,2\pi)$ $I_1=L_3^2$ $I_2=\{K_1,L_2\}+\{K_2,L_1\}$	$u_0=\frac{R}{\sqrt{2}}\big(\sqrt{(1+\mu_1^2)(1+\mu_2^2)}+\mu_1\mu_2+1\big)^{1/2}$ $u_1=R\sqrt{\mu_1\mu_2}\cos\varphi$ $u_2=R\sqrt{\mu_1\mu_2}\sin\varphi$ $u_3=\frac{R}{\sqrt{2}}\big(\sqrt{(1+\mu_1^2)(1+\mu_2^2)}-\mu_1\mu_2-1\big)^{1/2}$	V_2,V_6 V_{19},V_{20}
XXIII. Elliptic-Parabolic 1 $a,\varrho\in\mathbb{R},\vartheta\in(-\pi/2,\pi/2)$ $I_1=(K_1+L_2)^2$ $I_2=2K_1^2+K_2^2+K_3^2+L_1^2$ $-\{K_1,L_2\}-\{K_2,L_1\}$	$u_0=R\frac{\cosh^2a+\cos^2\vartheta+\varrho^2}{2\cosh a\cos\vartheta}$ $u_1=R\varrho/\cosh a\cos\vartheta$ $u_2=R\tanh a\tan\vartheta$ $u_3=R\frac{\cosh^2a+\cos^2\vartheta-\varrho^2-2}{2\cosh a\cos\vartheta}$	$V_9',V_9^{(\omega=0)}$
XXIV. Hyperbolic-Parabolic 1 $b>0,\varrho\in\mathbb{R},\vartheta\in(0,\pi)$ $I_1=(K_1+L_2)^2$ $I_2=2L_2^2+L_1^2+K_2^2-K_3^2$ $-\{K_2,L_1\}-\{K_1,L_2\}$	$u_0=R\frac{\sinh^2b-\sin^2\vartheta+\varrho^2+2}{2\sinh b\sin\vartheta}$ $u_1=R\varrho/\sinh b\sin\vartheta$ $u_2=R\coth b\cot\vartheta$ $u_3=R\frac{\sinh^2b-\sin^2\vartheta-\varrho^2}{2\sinh b\sin\vartheta}$	$V_9',V_9^{(\omega=0)}$
XXV. Elliptic-Parabolic 2 $a>0,\vartheta\in(0,\pi/2),\varphi\in[0,2\pi)$ $I_1=L_3^2$ $I_2=2\mathbf{L}^2-\{L_2,K_1\}-\{L_1,K_2\}$	$u_0=R\frac{\cos^2\vartheta+\cosh^2a}{2\cosh a\cos\vartheta}$ $u_1=R\tanh a\tan\vartheta\cos\varphi$ $u_2=R\tanh a\tan\vartheta\sin\varphi$ $u_3=R\frac{\sinh^2a-\sin^2\vartheta}{2\cosh a\cos\vartheta}$	$V_2,V_9',V_9^{(\omega=0)}$
XXVI. Hyperbolic-Parabolic 2 $b>0,\vartheta\in(0,\pi/2),\varphi\in[0,2\pi)$ $I_1=L_3^2$ $I_2=\{K_1,L_2\}+\{K_2,L_1\}-K_1^2-K_2^2$	$u_0=R\frac{\cosh^2b+\cos^2\vartheta}{2\sinh b\sin\vartheta}$ $u_1=R\coth b\cot\vartheta\cos\varphi$ $u_2=R\coth b\cot\vartheta\sin\varphi$ $u_3=R\frac{\sin^2\vartheta-\sinh^2b}{2\sinh b\sin\vartheta}$	$V_9',V_9^{(\omega=0)}$
XXVII. Semi-Circular-Parabolic $\varrho\in\mathbb{R},\xi,\eta>0$ $I_1=(K_1+L_2)^2$ $I_2=\{L_3,K_1+L_2\}+\{K_3,K_2-L_1\}$	$u_0=R\frac{(\eta^2-\xi^2)^2+4\varrho^2+4}{8\xi\eta}$ $u_1=R\frac{\eta^2-\xi^2}{2\xi\eta}$ $u_2=R\frac{\varrho}{\xi\eta}$ $u_3=R\frac{(\eta^2-\xi^2)^2+4\varrho^2-4}{8\xi\eta}$	V_8
XXVIII. Ellipsoidal $0<1<\varrho_3<b<\varrho_2<a<\varrho_1$ $I_1=abK_1^2+aK_2^2+bK_3^2$ $I_2=(a+b)K_1^2+(a+1)K_2^2+(b+1)K_3^2$ $-aL_3^2-bL_2^2-L_1^2$	$u_0^2=R^2\frac{\varrho_1\varrho_2\varrho_3}{ab}$ $u_1^2=R^2\frac{(\varrho_1-1)(\varrho_2-1)(\varrho_3-1)}{(a-1)(b-1)}$ $u_2^2=-R^2\frac{(\varrho_1-b)(\varrho_2-b)(\varrho_3-b)}{(a-b)(b-1)b}$ $u_3^2=R^2\frac{(\varrho_1-a)(\varrho_2-a)(\varrho_3-a)}{(a-b)(a-1)a}$	V_1

Table 2.1 (cont.)

Table 2 (cont.) Coordinate System	Coordinates	Separates Potential
XXIX. Hyperboloidal $\varrho_3 < 0 < 1 < b < \varrho_2 < a < \varrho_1$ $I_1 = abK_1^2 - aL_3^3 - bL_2^2$ $I_2 = (a+b)K_1^2 - (a+1)L_3^2 - (b+1)L_2^2$ $+aK_2^2 + bK_3^2 - L_1^2$	$u_0^2 = -R^2 \frac{(\varrho_1 - 1)(\varrho_2 - 1)(\varrho_3 - 1)}{(a-1)(b-1)}$ $u_1^2 = -R^2 \frac{\varrho_1\varrho_2\varrho_3}{ab}$ $u_2^2 = -R^2 \frac{(\varrho_1 - b)(\varrho_2 - b)(\varrho_3 - b)}{(a-b)(b-1)b}$ $u_3^2 = R^2 \frac{(\varrho_1 - a)(\varrho_2 - a)(\varrho_3 - a)}{(a-b)(a-1)a}$	V_1
XXX. Paraboloidal $\varrho_3 < 0 < \varrho_2 < 1 < \varrho_1$ $a = b^* = \alpha + i\beta, \alpha, \beta \in \mathbb{R}$ $I_1 = -\|a\|^2 L_1^2 + \alpha(K_3^2 - L_2^2) - \beta\{K_3, L_2\}$ $I_2 = -2\alpha L_1^2 + (\alpha+1)(K_3^2 - L_2^2)$ $+\alpha(K_2^2 - L_3^2) + \beta(\{K_2, L_3\} - \{K_3, L_2\})$	$(u_1 + iu_0)^2 = 2R^2 \frac{(\varrho_1 - a)(\varrho_2 - a)(\varrho_3 - a)}{(a-b)(b-1)b}$ $u_2^2 = R^2 \frac{(\varrho_1 - 1)(\varrho_2 - 1)(\varrho_3 - 1)}{(a-1)(b-1)}$ $u_3^2 = -R^2 \frac{\varrho_1\varrho_2\varrho_3}{ab}$	–
XXXI. $0 < \varrho_3 < 1 < \varrho_2 < a < \varrho_1$ $I_1 = (K_3 + L_2)^2 - a(K_2 + L_3)^2 + aK_1^2$ $I_2 = (a+1)K_1^2 + K_3^2 - L_2^2 + a(L_3^2 - K_2^2)$ $+(K_2 + L_3)^2 + (K_3 + L_2)^2$	$(u_0 + u_1)^2 = R^2 \frac{\varrho_1\varrho_2\varrho_3}{a}$ $(u_0^2 - u_1^2)$ $= R^2 \frac{a(\varrho_1\varrho_2 + \varrho_1\varrho_3 + \varrho_2\varrho_3) - (a+1)\varrho_1\varrho_2\varrho_3}{a^2}$ $u_2^2 = -R^2 \frac{(\varrho_1 - 1)(\varrho_2 - 1)(\varrho_3 - 1)}{(a-1)}$ $u_3^2 = R^2 \frac{(\varrho_1 - a)(\varrho_2 - a)(\varrho_3 - a)}{a^2(a-1)}$	–
XXXII. $-\varrho_3 < 0 < 1 < \varrho_2 < a < \varrho_1$ $I_1 = -(K_3 + L_2)^2 + a(K_2 + L_3)^2 + aK_1^2$ $I_2 = (a+1)K_1^2 - K_3^2 + L_2^2 + a(K_2^2 - L_3^2)$ $-(K_2 + L_3)^2 - (K_3 + L_2)^2$	$(u_0 + u_1)^2 = -R^2 \frac{\varrho_1\varrho_2\varrho_3}{a}$ $(u_0^2 - u_1^2)$ $= R^2 \frac{a(\varrho_1\varrho_2 + \varrho_2\varrho_3 + \varrho_1\varrho_3) - (a+1)\varrho_1\varrho_2\varrho_3}{a^2}$ $u_2^2 = -R^2 \frac{(\varrho_1 - 1)(\varrho_2 - 1)(\varrho_3 - 1)}{(a-1)}$ $u_3^2 = R^2 \frac{(\varrho_1 - a)(\varrho_2 - a)(\varrho_3 - a)}{a^2(a-1)}$	–
XXXIII. $\varrho_3 < -1 < 0 < \varrho_2 < a < \varrho_1$ $I_1 = aK_1^2 - (K_2 + L_3)^2 + a(K_2 + L_3)^2$ $I_2 = (a-1)K_1^2 - K_2^2 + L_3^2 + a(L_2^2 - K_3^2)$ $-(K_2 + L_3)^2 + (K_3 + L_2)^2$	$(u_0 + u_1)^2 = -R^2 \frac{\varrho_1\varrho_2\varrho_3}{a}$ $(u_0^2 - u_1^2)$ $= R^2 \frac{a(\varrho_1\varrho_3 + \varrho_1\varrho_2 + \varrho_2\varrho_3) - (a-1)\varrho_1\varrho_2\varrho_3}{a^2}$ $u_2^2 = R^2 \frac{(\varrho_1 - a)(\varrho_2 - a)(\varrho_3 - a)}{a^2(a+1)}$ $u_3^2 = -R^2 \frac{(\varrho_1 + 1)(\varrho_2 + 1)(\varrho_3 + 1)}{(a+1)}$	–
XXXIV. $\varrho_3 < 0 < \varrho_2 < 1 < \varrho_1$ $I_1 = (L_2 - K_3)^2 - \{K_1, K_2 - L_3\}$ $I_2 = L_2^2 - K_3^2 - L_1^2$ $-(L_2 - K_3)^2 - \{L_1, L_2 - K_3\}$	$(u_0 - u_1)^2 = -R^2\varrho_1\varrho_2\varrho_3$ $2u_2(u_1 - u_0)$ $= R^2(\varrho_1\varrho_2 + \varrho_2\varrho_3 + \varrho_1\varrho_3 - \varrho_1\varrho_2\varrho_3)$ $u_1^2 + u_2^2 - u_0^2 = R^2(-\varrho_1\varrho_2\varrho_3$ $+\varrho_1\varrho_2 + \varrho_2\varrho_3 + \varrho_1\varrho_3 - \varrho_1 - \varrho_2 - \varrho_3)$ $u_3^2 = R^2(\varrho_1 - 1)(\varrho_2 - 1)(\varrho_3 - 1)$	–

In algebraic form a coordinate system on $\Lambda^{(3)}$ is described in the following way

$$\left.\begin{aligned}
u_0^2 &= R^2\frac{(\varrho_1-a_1)(\varrho_2-a_1)(\varrho_3-a_1)}{(a_2-a_1)(a_3-a_1)(a_4-a_1)} ,\\
u_1^2 &= -R^2\frac{(\varrho_1-a_2)(\varrho_2-a_2)(\varrho_3-a_2)}{(a_1-a_2)(a_3-a_2)(a_4-a_2)} ,\\
u_2^2 &= -R^2\frac{(\varrho_1-a_3)(\varrho_2-a_3)(\varrho_3-a_3)}{(a_1-a_3)(a_2-a_3)(a_4-a_3)} ,\\
u_3^2 &= -R^2\frac{(\varrho_1-a_4)(\varrho_2-a_4)(\varrho_3-a_4)}{(a_1-a_4)(a_2-a_4)(a_3-a_4)} ,
\end{aligned}\right\} \tag{2.9}$$

and we have for the characteristic polynomial

$$P(\varrho) = (\varrho-a_1)(\varrho-a_2)(\varrho-a_3)(\varrho-a_4) . \tag{2.10}$$

Fixing the numbers a_i, $i = 1,2,3,4$, and the range of the ϱ specifies a coordinate system. For the metric tensor then follows

$$g_{ab} = G_{ik}\frac{\partial u_i}{\partial \varrho_a}\frac{\partial u_k}{\partial \varrho_b} , \tag{2.11}$$

where G_{ik} is the metric tensor of the ambient space, which in the present case has the form $G_{ik} = \mathrm{diag}(1,-1,-1,-1)$, and in order that the line element $ds^2 = \sum_{ab}\epsilon_{ab}g_{ab}dq^a dq^b$ is positive definite appropriate $\epsilon_{aa} = \pm 1$ must be taken into account. Actually $\epsilon_{ab} = \epsilon_{aa} = -1, \forall_{a,b}$. In the following we state for convenience only the explicit form of ds^2. In Table 2.1 we summarize the results on the coordinate systems on $\Lambda^{(3)}$ according to [49,52,75]. The potentials $V_1,\dots,V_{20}$ refer to Sections 3, 4 and 5.

3. PATH INTEGRAL FORMULATION OF THE MAXIMALLY SUPERINTEGRABLE POTENTIALS ON $\Lambda^{(3)}$

In Table 3.1 we list the superintegrable potentials on the three-dimensional hyperboloid together with the separating coordinate systems. The cases where an explicit path integration is possible are underlined.

3.1. The Oscillator

We consider the generalized Higgs oscillator on the hyperboloid ($k_{1,2,3} > 0$)

$$V_1(\mathrm{u}) = \frac{M}{2}\omega^2R^2\frac{u_1^2+u_2^2+u_3^2}{u_0^2} + \frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac14}{u_1^2}+\frac{k_2^2-\frac14}{u_2^2}+\frac{k_3^2-\frac14}{u_3^2}\right) , \tag{3.1}$$

Table 3.1. Maximally superintegrable potentials on $\Lambda^{(3)}$

Potential $V(\mathbf{u})$	Coordinate Systems	Observables
$V_1(\mathbf{u}) = \frac{M}{2}\omega^2 R^2 \frac{u_1^2+u_2^2+u_3^2}{u_0^2} + \frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac14}{u_1^2} + \frac{k_2^2-\frac14}{u_2^2} + \frac{k_3^2-\frac14}{u_3^2}\right)$	Cylindrical Sphero-Elliptic Equidistant-Elliptic Equidistant-Hyperbolic Spherical Equidistant-Cylindrical Equidistant Prolate Elliptic Oblate Elliptic Elliptic-Cylindrical Hyperbolic-Cylindrical 1 Hyperbolic-Cylindrical 2 Ellipsoidal Hyperboloidal	$I_1 = \frac{1}{2MR^2}(\mathbf{L}^2 - \mathbf{K}^2) + V_1(\mathbf{u})$ $I_2 = \frac{1}{2M}L_3^2 + \frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac14}{\cos^2\varphi} + \frac{k_2^2-\frac14}{\sin^2\varphi}\right)$ $I_3 = \frac{1}{2M}\mathbf{L}^2 + \frac{\hbar^2}{2M}\left(\frac{1}{\sin^2\vartheta}\left(\frac{k_1^2-\frac14}{\cos^2\varphi} + \frac{k_2^2-\frac14}{\sin^2\varphi}\right) + \frac{k_3^2-\frac14}{\cos^2\vartheta}\right)$ $I_4 = \frac{1}{2M}K_3^2 - \frac{M}{2}\frac{\omega^2 R^4}{\cosh^2\tau_2} + \frac{\hbar^2}{2M}\frac{k_3^2-\frac14}{\sinh^2\tau_2}$ $I_5 = \frac{1}{2M}(L_1^2 + k'^2 L_2^2) + \frac{\hbar^2}{2MR^2\sinh^2\tau}\left(\frac{k_1^2-\frac14}{\mathrm{sn}^2\tilde\alpha\,\mathrm{dn}^2\tilde\beta} + \frac{k_2^2-\frac14}{\mathrm{cn}^2\tilde\alpha\,\mathrm{cn}^2\tilde\beta} + \frac{k_3^2-\frac14}{\mathrm{dn}^2\tilde\alpha\,\mathrm{sn}^2\tilde\beta}\right)$
$V_2(\mathbf{u}) = -\frac{\alpha}{R}\left(\frac{u_0}{\sqrt{u_1^2+u_2^2+u_3^2}} - 1\right) + \frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac14}{u_1^2} + \frac{k_2^2-\frac14}{u_2^2}\right)$ $\mathbf{A} = \frac{1}{2R}(\mathbf{L}\times\mathbf{K} - \mathbf{K}\times\mathbf{L}) - \frac{\alpha\mathbf{u}}{\lvert\mathbf{u}\rvert}, \quad \mathbf{u} = (u_1, u_2, u_3)$	Sphero-Elliptic Spherical Prolate Elliptic II Semi-Hyperbolic Elliptic Parabolic II	$I_1 = \frac{1}{2MR^2}(\mathbf{L}^2 - \mathbf{K}^2) + V_2(\mathbf{u})$ $I_2 = \frac{1}{2M}L_3^2 + \frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac14}{\cos^2\varphi} + \frac{k_2^2-\frac14}{\sin^2\varphi}\right)$ $I_3 = \frac{1}{2M}\mathbf{L}^2 + \frac{\hbar^2}{2M\sin^2\vartheta}\left(\frac{k_1^2-\frac14}{\cos^2\varphi} + \frac{k_2^2-\frac14}{\sin^2\varphi}\right)$ $I_4 = \frac{1}{2M}(L_1^2 + k'^2 L_2^2) + \frac{\hbar^2}{2MR^2\sinh^2\tau}\left(\frac{k_1^2-\frac14}{\mathrm{sn}^2\tilde\alpha\,\mathrm{dn}^2\tilde\beta} + \frac{k_1^2-\frac14}{\mathrm{cn}^2\tilde\alpha\,\mathrm{cn}^2\tilde\beta}\right)$ $I_5 = \frac{1}{2M}A_3 + \frac{\hbar^2}{2M\sinh^2\tau\sin^2\vartheta}\left(\frac{k_1^2-\frac14}{\cos^2\varphi} + \frac{k_2^2-\frac14}{\sin^2\varphi}\right)$

Table 3.1 (cont.)

Potential $V(\mathrm{u})$	Coordinate Systems	Observables
$V_3(\mathrm{u}) = \frac{\hbar^2}{2M}\left[\frac{k_3^2-\frac14}{u_3^2} + \frac{1}{\sqrt{u_1^2+u_2^2}}\left(\frac{k_1^2-\frac14}{\sqrt{u_1^2+u_2^2}+u_1} + \frac{k_2^2-\frac14}{\sqrt{u_1^2+u_2^2}-u_1}\right)\right]$	Spherical	$I_1 = \frac{1}{2MR^2}(\mathbf{L}^2 - \mathbf{K}^2) + V_3(\mathrm{u})$
	Equidistant-Cylindrcal	$I_2 = \frac{1}{2M}L_3^2 + \frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac14}{\cos^2\frac{\varphi}{2}} + \frac{k_2^2-\frac14}{\sin^2\frac{\varphi}{2}}\right)$
	Prolate Elliptic	$I_3 = \frac{1}{2M}\mathbf{L}^2 + \frac{\hbar^2}{2M}\left(\frac{1}{4\sin^2\vartheta}\left(\frac{k_1^2-\frac14}{\cos^2\frac{\varphi}{2}} + \frac{k_2^2-\frac14}{\sin^2\frac{\varphi}{2}}\right) + \frac{k_3^2-\frac14}{\cos^2\vartheta}\right)$
	Oblate Elliptic	$I_4 = \frac{1}{2M}(K_1^2 + K_2^2 - L_3^2) + \frac{\hbar^2}{8M\sinh^2\tau_2}\left(\frac{k_1^2-\frac14}{\cos^2\frac{\varphi}{2}} + \frac{k_2^2-\frac14}{\sin^2\frac{\varphi}{2}}\right)$
	Hypberbolic-Cylindrical 2	$I_5 = \frac{1}{2M}\left[K_3^2 + L_3^2 - k^2(L_1^2 + L_2^2)\right] - \frac{\hbar^2}{2M}\frac{1}{k^2}\left[\frac{1}{4\mathrm{sn}^2\mu\,\mathrm{dn}^2\nu}\left(\frac{k_1^2-\frac14}{\cos^2\frac{\varphi}{2}} + \frac{k_2^2-\frac14}{\sin^2\frac{\varphi}{2}}\right) + \frac{k_3^2-\frac14}{\mathrm{dn}^2\mu\,\mathrm{sn}^2\nu}\right].$
$V_4(\mathrm{u}) = \frac{\hbar^2}{4M\sqrt{u_1^2+u_2^2}}\left(\frac{k_1^2-\frac14}{\sqrt{u_1^2+u_2^2}+u_1} + \frac{k_2^2-\frac14}{\sqrt{u_1^2+u_2^2}-u_1}\right) + k_3u_3$	Equidistant-Elliptic*	$I_1 = \frac{1}{2MR^2}(\mathbf{L}^2 - \mathbf{K}^2) + V_4(\mathrm{u})$
	Equidistant Semi-Hyperbolic	$I_2 = \frac{1}{2M}(K_1^2 + K_2^2 - L_3^2) + \frac{\hbar^2}{8M\sinh^2\tau_2}\left(\frac{k_1^2-\frac14}{\sin^2\frac{\varphi}{2}} + \frac{k_2^2-\frac14}{\cos^2\frac{\varphi}{2}}\right)$
	Equidistant-Elliptic-Parabolic	$I_3 = \frac{1}{2M}L_3^2 + \frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac14}{\sin^2\frac{\varphi}{2}} + \frac{k_2^2-\frac14}{\cos^2\frac{\varphi}{2}}\right)$
	Equidistant-Cylindrcal	$I_4 = \frac{1}{2M}(\cosh 2f L_3^2 - \frac12\sinh 2f\{L_3, K_2\}) + \frac{\hbar^2}{4M}\left(\frac{k_1^2+k_2^2-\frac12}{k^2\mathrm{cn}^2\alpha + k'^2\mathrm{cn}^2\beta}\left(\frac{k'^2}{\mathrm{dn}^2\alpha} - \frac{1}{\mathrm{sn}^2\beta}\right) + (k_1^2 - k_2^2)\frac{k'}{k}\frac{k^2\mathrm{sn}\alpha\,\mathrm{cn}\beta}{+k'\mathrm{cn}\beta\,\mathrm{dn}\beta k^2\mathrm{cn}^2\alpha + k'^2\mathrm{cn}^2\beta}\right)$
		I_5 = Is there any fifth observable?

* after appropriate rotation, $\sin^2 f = k^2$.

which in the 14 separating coordinate systems has the form

Cylindrical $\big(\tau_{1,2} > 0, \varphi \in (0, \pi/2)\big)$:

$$V_1(\mathrm{u}) = \frac{M}{2}\omega^2 R^2\left(1 - \frac{1}{\cosh^2\tau_1 \cosh^2\tau_2}\right)$$
$$+\frac{\hbar^2}{2MR^2}\left(\frac{1}{\sinh^2\tau_1}\left(\frac{k_1^2-\frac{1}{4}}{\cos^2\varphi}+\frac{k_2^2-\frac{1}{4}}{\sin^2\varphi}\right)+\frac{k_3^2-\frac{1}{4}}{\cosh^2\tau_1\sinh^2\tau_2}\right) \tag{3.2}$$

Sphero-Elliptic $\big(\tau > 0, \tilde\alpha \in (0, K), \tilde\beta \in (0, K')\big)$:

$$= \frac{M}{2}\omega^2 R^2 \tanh^2\tau$$
$$+\frac{\hbar^2}{2MR^2\sinh^2\tau}\left(\frac{k_1^2-\frac{1}{4}}{\mathrm{sn}^2\tilde\alpha\,\mathrm{dn}^2\tilde\beta}+\frac{k_2^2-\frac{1}{4}}{\mathrm{cn}^2\tilde\alpha\,\mathrm{cn}^2\tilde\beta}+\frac{k_3^2-\frac{1}{4}}{\mathrm{dn}^2\tilde\alpha\,\mathrm{sn}^2\tilde\beta}\right) \tag{3.3}$$

Equidistant-Elliptic $\big(\tau > 0, \alpha \in (\mathrm{i}K', \mathrm{i}K' + K), \beta \in (0, K')\big)$:

$$= \frac{M}{2}\omega^2 R^2\left(1 - \frac{1}{\cosh^2\tau}\frac{1}{\mathrm{sn}^2\alpha\,\mathrm{dn}^2\beta}\right)$$
$$-\frac{\hbar^2}{2MR^2}\left[\frac{1}{\cosh^2\tau}\left(\frac{k_1^2-\frac{1}{4}}{\mathrm{cn}^2\alpha\,\mathrm{cn}^2\beta}+\frac{k_2^2-\frac{1}{4}}{\mathrm{dn}^2\alpha\,\mathrm{sn}^2\beta}\right)-\frac{k_3^2-\frac{1}{4}}{\sinh^2\tau}\right] \tag{3.4}$$

Equidistant-Hyperbolic $\big(\tau > 0, \mu \in (\mathrm{i}K', \mathrm{i}K' + K), \eta \in (0, K')\big)$:

$$= \frac{M}{2}\omega^2 R^2\left(1 - \frac{1}{\cosh^2\tau}\frac{1}{\mathrm{cn}^2\mu\,\mathrm{cn}^2\eta}\right)$$
$$-\frac{\hbar^2}{2MR^2}\left[\frac{1}{\cosh^2\tau}\left(\frac{k_1^2-\frac{1}{4}}{\mathrm{sn}^2\mu\,\mathrm{dn}^2\eta}+\frac{k_2^2-\frac{1}{4}}{\mathrm{dn}^2\mu\,\mathrm{sn}^2\eta}\right)-\frac{k_3^2-\frac{1}{4}}{\sinh^2\tau}\right] \tag{3.5}$$

Spherical $\big(\tau > 0, \vartheta \in (0, \pi/2), \varphi \in (0, \pi/2)\big)$:

$$= \frac{M}{2}\omega^2 R^2 \tanh^2\tau$$
$$+\frac{\hbar^2}{2MR^2\sinh^2\tau}\left(\frac{1}{\sin^2\vartheta}\left(\frac{k_1^2-\frac{1}{4}}{\cos^2\varphi}+\frac{k_2^2-\frac{1}{4}}{\sin^2\varphi}\right)+\frac{k_3^2-\frac{1}{4}}{\cos^2\vartheta}\right) \tag{3.6}$$

Equidistant-Cylindrical $\big(\tau_{1,2} > 0, \varphi \in (0, \pi/2)\big)$:

$$= \frac{M}{2}\omega^2 R^2\left(1 - \frac{1}{\cosh^2\tau_1 \cosh^2\tau_2}\right)$$
$$+\frac{\hbar^2}{2MR^2}\left(\frac{1}{\cosh^2\tau_1\sinh^2\tau_2}\left(\frac{k_1^2-\frac{1}{4}}{\cos^2\varphi}+\frac{k_2^2-\frac{1}{4}}{\sin^2\varphi}\right)+\frac{k_3^2-\frac{1}{4}}{\sinh^2\tau_1}\right) \tag{3.7}$$

Equidistant $(\tau_{1,2,3} > 0)$:

$$= \frac{M}{2}\omega^2 R^2 \left(1 - \frac{1}{\cosh^2\tau_1 \cosh^2\tau_2 \cosh^2\tau_3}\right)$$
$$+ \frac{\hbar^2}{2MR^2}\left[\frac{1}{\cosh^2\tau_1}\left(\frac{k_1^2 - \frac14}{\cosh^2\tau_2 \sinh^2\tau_3} + \frac{k_2^2 - \frac14}{\sinh^2\tau_2}\right) + \frac{k_3^2 - \frac14}{\sinh^2\tau_1}\right] \tag{3.8}$$

Prolate Elliptic $(\alpha \in (\mathrm{i}K', \mathrm{i}K' + K), \beta \in (0, K'), \varphi \in (0, \pi/2))$:

$$= \frac{M}{2}\omega^2 R^2 \left(1 - \frac{1}{\mathrm{sn}^2\alpha\, \mathrm{dn}^2\beta}\right)$$
$$- \frac{\hbar^2}{2MR^2}\left[\frac{1}{\mathrm{dn}^2\alpha\, \mathrm{sn}^2\beta}\left(\frac{k_1^2 - \frac14}{\cos^2\varphi} + \frac{k_2^2 - \frac14}{\sin^2\varphi}\right) + \frac{k_3^2 - \frac14}{\mathrm{cn}^2\alpha\, \mathrm{cn}^2\beta}\right] \tag{3.9}$$

Oblate Elliptic $(\alpha \in (\mathrm{i}K', \mathrm{i}K' + K), \beta \in (0, K'), \varphi \in (0, \pi/2))$:

$$= \frac{M}{2}\omega^2 R^2 \left(1 - \frac{1}{\mathrm{sn}^2\alpha\, \mathrm{dn}^2\beta}\right)$$
$$- \frac{\hbar^2}{2MR^2}\left[\frac{1}{\mathrm{cn}^2\alpha\, \mathrm{cn}^2\beta}\left(\frac{k_1^2 - \frac14}{\cos^2\varphi} + \frac{k_2^2 - \frac14}{\sin^2\varphi}\right) + \frac{k_3^2 - \frac14}{\mathrm{dn}^2\alpha\, \mathrm{sn}^2\beta}\right] \tag{3.10}$$

Elliptic-Cylindrical $(\alpha \in (\mathrm{i}K', \mathrm{i}K' + K), \beta \in (0, K'), \tau > 0)$:

$$= \frac{M}{2}\omega^2 R^2 \left(1 - \frac{1}{\mathrm{sn}^2\alpha\, \mathrm{dn}^2\beta \cosh^2\tau}\right)$$
$$+ \frac{\hbar^2}{2MR^2}\left(\frac{k_1^2 - \frac14}{\mathrm{sn}^2\alpha\, \mathrm{dn}^2\beta \sinh^2\tau} - \frac{k_2^2 - \frac14}{\mathrm{dn}^2\alpha\, \mathrm{sn}^2\beta} - \frac{k_3^2 - \frac14}{\mathrm{cn}^2\alpha\, \mathrm{cn}^2\beta}\right) \tag{3.11}$$

Hyperbolic-Cylindrical 1 $(\mu \in (\mathrm{i}K', \mathrm{i}K' + K), \eta \in (0, K'), \tau > 0)$:

$$= \frac{M}{2}\omega^2 R^2 \left(1 - \frac{1}{\mathrm{cn}^2\mu\, \mathrm{cn}^2\eta \cosh^2\tau}\right)$$
$$+ \frac{\hbar^2}{2MR^2}\left(\frac{k_1^2 - \frac14}{\mathrm{cn}^2\mu\, \mathrm{cn}^2\eta \sinh^2\tau} - \frac{k_2^2 - \frac14}{\mathrm{sn}^2\mu\, \mathrm{dn}^2\eta} - \frac{k_3^2 - \frac14}{\mathrm{dn}^2\mu\, \mathrm{sn}^2\eta}\right) \tag{3.12}$$

Hyperbolic-Cylindrical 2 $(\mu \in (\mathrm{i}K', \mathrm{i}K' + K), \eta \in (0, K'), \varphi \in (0, \pi/2))$:

$$= \frac{M}{2}\omega^2 R^2 \left(1 - \frac{1}{\mathrm{cn}^2\mu\, \mathrm{cn}^2\eta}\right)$$
$$- \frac{\hbar^2}{2MR^2}\left[\frac{1}{\mathrm{sn}^2\mu\, \mathrm{dn}^2\eta}\left(\frac{k_1^2 - \frac14}{\cos^2\varphi} + \frac{k_2^2 - \frac14}{\sin^2\varphi}\right) + \frac{k_3^2 - \frac14}{\mathrm{dn}^2\mu\, \mathrm{sn}^2\eta}\right] \tag{3.13}$$

Ellipsoidal $(a_{ij} = a_i - a_j, a_1 = 0, a_2 = 1, a_3 = b, a_4 = a)$:

$$
\begin{aligned}
&= \frac{M}{2}\omega^2 R^2 \Bigg[a_{14}a_{24}a_{34} \bigg(\frac{1}{(\varrho_1-\varrho_3)(\varrho_2-\varrho_3)} \frac{1}{\varrho_3-a_4} \\
&+ \frac{1}{(\varrho_1-\varrho_2)(\varrho_3-\varrho_2)} \frac{1}{\varrho_2-a_4} + \frac{1}{(\varrho_2-\varrho_1)(\varrho_3-\varrho_1)} \frac{1}{\varrho_1-a_4} \bigg) - 1 \Bigg] \\
&+ \frac{\hbar^2}{2MR^2} \Bigg\{ \frac{1}{(\varrho_1-\varrho_3)(\varrho_2-\varrho_3)} \\
&\times \left[a_{31}a_{21}a_{41} \frac{k_1^2-\frac{1}{4}}{\varrho_3-a_1} + a_{12}a_{32}a_{42} \frac{k_2^2-\frac{1}{4}}{\varrho_3-a_2} + a_{13}a_{23}a_{43} \frac{k_3^2-\frac{1}{4}}{\varrho_3-a_3} \right] \\
&+ \frac{1}{(\varrho_1-\varrho_2)(\varrho_3-\varrho_2)} \\
&\times \left[a_{31}a_{21}a_{41} \frac{k_1^2-\frac{1}{4}}{\varrho_2-a_1} + a_{12}a_{32}a_{42} \frac{k_2^2-\frac{1}{4}}{\varrho_2-a_2} + a_{13}a_{23}a_{43} \frac{k_3^2-\frac{1}{4}}{\varrho_2-a_3} \right] \\
&+ \frac{1}{(\varrho_2-\varrho_1)(\varrho_3-\varrho_1)} \\
&\times \left[a_{31}a_{21}a_{41} \frac{k_1^2-\frac{1}{4}}{\varrho_1-a_1} + a_{12}a_{32}a_{42} \frac{k_2^2-\frac{1}{4}}{\varrho_1-a_2} + a_{13}a_{23}a_{43} \frac{k_3^2-\frac{1}{4}}{\varrho_1-a_3} \right] \Bigg\} \qquad (3.14)
\end{aligned}
$$

Hyperboloidal $(a_{ij} = a_i - a_j, a_1 = 0, a_2 = 1, a_3 = b, a_4 = a)$:

$$
\begin{aligned}
&= \frac{M}{2}\omega^2 R^2 \Bigg[a_{14}a_{24}a_{34} \bigg(\frac{1}{(\varrho_1-\varrho_3)(\varrho_2-\varrho_3)} \frac{1}{\varrho_3-a_4} \\
&+ \frac{1}{(\varrho_1-\varrho_2)(\varrho_3-\varrho_2)} \frac{1}{\varrho_2-a_4} + \frac{1}{(\varrho_2-\varrho_1)(\varrho_3-\varrho_1)} \frac{1}{\varrho_1-a_4} \bigg) - 1 \Bigg] \\
&- \frac{\hbar^2}{2MR^2} \Bigg\{ \frac{1}{(\varrho_1-\varrho_3)(\varrho_2-\varrho_3)} \\
&\times \left[a_{31}a_{21}a_{41} \frac{k_2^2-\frac{1}{4}}{\varrho_3-a_1} + a_{12}a_{32}a_{42} \frac{k_1^2-\frac{1}{4}}{\varrho_3-a_2} - a_{13}a_{23}a_{43} \frac{k_3^2-\frac{1}{4}}{\varrho_3-a_3} \right] \\
&- \frac{1}{(\varrho_1-\varrho_2)(\varrho_3-\varrho_2)} \\
&\times \left[a_{31}a_{21}a_{41} \frac{k_2^2-\frac{1}{4}}{\varrho_2-a_1} + a_{12}a_{32}a_{42} \frac{k_1^2-\frac{1}{4}}{\varrho_2-a_2} - a_{13}a_{23}a_{43} \frac{k_3^2-\frac{1}{4}}{\varrho_2-a_3} \right] \\
&- \frac{1}{(\varrho_2-\varrho_1)(\varrho_3-\varrho_1)}
\end{aligned}
$$

$$\times\left[a_{31}a_{21}a_{41}\frac{k_2^2-\frac{1}{4}}{\varrho_1-a_1}+a_{12}a_{32}a_{42}\frac{k_1^2-\frac{1}{4}}{\varrho_1-a_2}-a_{13}a_{23}a_{43}\frac{k_3^2-\frac{1}{4}}{\varrho_1-a_3}\right]\Bigg\}. \tag{3.15}$$

In the following we do not display all path integral representations for all potentials in all separable coordinate systems. In order not to rouse the volume of the paper too much we display explicitly only those path integral representations, where an analytic solution is available.

Cylindrical Coordinates For the oscillator on $\Lambda^{(3)}$ we obtain the following path integral representation ($\lambda_1=2m\mp k_1\mp k_2+1, \lambda_2=2l\mp k_3-\nu+1, \nu^2=M^2\omega^2R^4/\hbar^2+1/4$):

$$\begin{aligned}
K^{(V_1)}(\mathrm{u}'',\mathrm{u}';T) &= R^{-3}\exp\left[-\frac{\mathrm{i}}{\hbar}T\left(\frac{\hbar^2}{2MR^2}+\frac{M}{2}R^2\omega^2\right)\right]\\
&\times\int\limits_{\tau_1(t')=\tau_1'}^{\tau_1(t'')=\tau_1''}\mathcal{D}\tau_1(t)\sinh\tau_1\cosh\tau_1\int\limits_{\tau_2(t')=\tau_2'}^{\tau_2(t'')=\tau_2''}\mathcal{D}\tau_2(t)\int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''}\mathcal{D}\varphi(t)\\
&\times\exp\Bigg\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\Bigg[\frac{M}{2}R^2\left(\dot\tau_1^2+\cosh^2\tau_1\dot\tau_2^2+\sinh^2\tau_1\dot\varphi^2+\frac{\omega^2}{\cosh^2\tau_1\cosh^2\tau_2}\right)\\
&-\frac{\hbar^2}{2MR^2}\left(\frac{1}{\sinh^2\tau_1}\left(\frac{k_1^2-\frac{1}{4}}{\cos^2\varphi}+\frac{k_2^2-\frac{1}{4}}{\sin^2\varphi}-\frac{1}{4}\right)\right)\\
&+\frac{1}{\cosh^2\tau_1}\left(\frac{k_3^2-\frac{1}{4}}{\sinh^2\tau_2}+\frac{1}{4}\right)\Bigg]dt\Bigg\}
\end{aligned}\tag{3.16}$$

$$\begin{aligned}
&=\sum_{m=0}^{N_m}\Bigg\{\sum_{l=0}^{N_l}\Bigg[\sum_{n=0}^{N_n}\mathrm{e}^{-\mathrm{i}E_NT/\hbar}\Psi_{nlm}^{(V_1)}(\tau_1'',\tau_2'',\varphi'';R)\Psi_{nlm}^{(V_1)}(\tau_1',\tau_2',\varphi';R)\\
&\quad+\int_0^\infty dp\,\mathrm{e}^{-\mathrm{i}E_pT/\hbar}\Psi_{plm}^{(V_1)}(\tau_1'',\tau_2'',\varphi'';R)\Psi_{plm}^{(V_1)\,*}(\tau_1',\tau_2',\varphi';R)\Bigg]\\
&\quad+\int_0^\infty dk\int_0^\infty dp\,\mathrm{e}^{-\mathrm{i}E_pT/\hbar}\Psi_{pkm}^{(V_1)}(\tau_1'',\tau_2'',\varphi'';R)\Psi_{pkm}^{(V_1)\,*}(\tau_1',\tau_2',\varphi';R)\Bigg\}.
\end{aligned}\tag{3.17}$$

The bound state wave-functions are given by, with $m=0,\dots,N_m=[\frac{1}{2}(2l\mp k_2-\nu)], l=0,\dots,N_l=[(\nu\mp k_3-1)/2], N=0,\dots,N_{Max}=[\frac{1}{2}(\nu\mp k_1\mp k_2\mp k_3)]$:

$$\begin{aligned}
&\Psi_{nlm}^{(V_1)}(\tau_1,\tau_2,\varphi;R)\\
&\quad=(\sinh\tau_1\cosh\tau_1)^{-1/2}S_n^{(\lambda_1,\lambda_2)}(\tau_1;R)\psi_l^{(\pm k_3,\nu)}(\tau_2)\phi_m^{(\pm k_2,\pm k_1)}(\varphi)\ ,
\end{aligned}\tag{3.18}$$

where

$$S_n^{(\lambda_1,\lambda_2)}(\tau_1;R) = \frac{1}{\Gamma(1+\lambda_1)}\left[\frac{2(\lambda_2-\lambda_1-2n-1)\Gamma(n+1+\lambda_1)\Gamma(\lambda_2-n)}{R^3\Gamma(\lambda_2-\lambda_1-n)n!}\right]^{1/2}$$
$$\times(\sinh\tau_1)^{1/2+\lambda_1}(\cosh\tau_1)^{2n+1/2-\lambda_2}{}_2$$
$$\times F_1(-n,\lambda_2-n;1+\lambda_1;\tanh^2\tau_1)\ , \qquad (3.19)$$

$$\psi_l^{(\pm k_3,\nu)}(\tau_2) = \frac{1}{\Gamma(1\pm k_3)}\left[\frac{2(\nu\mp k_3-2l-1)\Gamma(l+1\pm k_3)\Gamma(\nu-l)}{\Gamma(\nu\mp k_3-l)l!}\right]^{1/2}$$
$$\times(\sinh\tau_2)^{1/2\pm k_3}(\cosh\tau_2)^{2l+1/2-\nu}{}_2F_1(-l,\nu-l;1\pm k_3;\tanh^2\tau_2) \qquad (3.20)$$

$$\phi_m^{(\pm k_2,\pm k_1)}(\varphi) = \left[2(1+2m\pm k_1\pm k_2)\frac{m!\Gamma(m\pm k_1\pm k_2+1)}{\Gamma(1+m\pm k_1)\Gamma(1+m\pm k_2)}\right]^{1/2}$$
$$\times(\sin\varphi)^{1/2\pm k_2}(\cos\varphi)^{1/2\pm k_1}P_m^{(\pm k_2,\pm k_1)}(\cos 2\varphi)\ . \qquad (3.21)$$

The bound state energy spectrum is given by ($N = m+l+n+3$ is the principal quantum number)

$$E_N = -\frac{\hbar^2}{2MR^2}\Big[(2N\pm k_1\pm k_2\pm k_3-\nu)^2-1\Big]+\frac{M}{2}\omega^2R^2\ . \qquad (3.22)$$

In the limit $R\to\infty$ we obtain

$$E_N \simeq \hbar\omega(2N\pm k_1\pm k_2\pm k_3)\ ,\qquad N\in\mathbb{N}_0\ , \qquad (3.23)$$

which gives the correct spectrum for the corresponding superintegrable flat space oscillator, i.e., the generalized oscillator in $\mathbb{R}^3$ [33].
For the first set of continuous states we find

$$\Psi_{plm}^{(V_1)}(\tau_1,\tau_2,\varphi;R)$$
$$=(\sinh\tau_1\cosh\tau_1)^{-1/2}S_p^{(\lambda_1,\lambda_2)}(\tau_1;R)\psi_l^{(\pm k_3,\nu)}(\tau_2)\phi_m^{(\pm k_2,\pm k_1)}(\varphi)\ , \qquad (3.24)$$

where

$$S_p^{(\lambda_1,\lambda_2)}(\tau_1;R)$$
$$=\frac{1}{\Gamma(1+\lambda_1)}\sqrt{\frac{p\sinh\pi p}{2\pi^2R^3}}\,\Gamma\left(\frac{\lambda_2-\lambda_1+1-ip}{2}\right)\Gamma\left(\frac{\lambda_1-\lambda_2+1-ip}{2}\right)$$
$$\times(\tanh\tau_1)^{1/2+\lambda_1}(\cosh\tau_1)^{ip}{}_2F_1$$
$$\times\left(\frac{\lambda_2+\lambda_1+1-ip}{2},\frac{1+\lambda_1-\lambda_2-ip}{2};1+\lambda_1;\tanh^2\tau_1\right)\ , \qquad (3.25)$$

with the $\psi_l^{(\pm k_3,\nu)}(\tau_2)$ and $\phi_m^{(\pm k_2,\pm k_1)}(\varphi)$ as in (3.20, 3.21), and the continuous spectrum has the form

$$E_p = \frac{\hbar^2}{2MR^2}(p^2+1) + \frac{M}{2}\omega^2 R^2 \ . \tag{3.26}$$

In the limiting case $\omega \to 0$ we obtain for E_p

$$E_p = \frac{\hbar^2}{2MR^2}(p^2+1) \ , \tag{3.27}$$

which corresponds to the case where just a radial part is present and has the same feature as the spectrum of the free motion on $\Lambda^{(3)}$, i.e., there is no discrete spectrum in this case.

For the second set of continuous states we find

$$\Psi^{(V_1)}_{pkm}(\tau_1,\tau_2,\varphi;R) = (\sinh\tau_1\cosh\tau_1)^{-1/2} S_p^{(\lambda_1,ik)}(\tau_1;R)\psi_k^{(\pm k_3,\nu)}(\tau_2)\phi_m^{(\pm k_2,\pm k_1)}(\varphi) \ , \tag{3.28}$$

where

$$\begin{aligned}
S_p^{(\lambda_1,ik)}(\tau_1;R) &= \frac{1}{\Gamma(1+\lambda_1)}\sqrt{\frac{p\sinh\pi p}{2\pi^2R^3}}\,\Gamma\left(\frac{ik-\lambda_1+1-ip}{2}\right)\Gamma\left(\frac{\lambda_1-ik+1-ip}{2}\right)\\
&\quad\times(\tanh\tau_1)^{1/2+\lambda_1}(\cosh\tau_1)^{ip}{}_2F_1\\
&\quad\times\left(\frac{ik+\lambda_1+1-ip}{2},\frac{1+\lambda_1-ik-ip}{2};1+\lambda_1;\tanh^2\tau_1\right) \ ,
\end{aligned} \tag{3.29}$$

$$\begin{aligned}
\psi_k^{(\pm k_3,\nu)}(\tau_2) &= \frac{1}{\Gamma(1\pm k_3)}\sqrt{\frac{k\sinh\pi k}{2\pi^2}}\,\Gamma\left(\frac{\nu\mp k_3+1-ik}{2}\right)\Gamma\left(\frac{\pm k_3-\nu+1-ik}{2}\right)\\
&\quad\times(\tanh\tau_2)^{1/2\pm k_3}(\cosh\tau_2)^{ik}{}_2F_1\\
&\quad\times\left(\frac{\nu\pm k_3+1-ik}{2},\frac{1\pm k_3-\nu-ik}{2};1\pm k_3;\tanh^2\tau_2\right) \ ,
\end{aligned} \tag{3.30}$$

with the $\phi_m^{(\pm k_1,\pm k_2)}(\varphi)$ as in (3.21).

Spherical and Sphero-Elliptic Coordinates. In spherical coordinates we have the path integral representation ($\lambda_1 = 2m \mp k_1 \mp k_2 + 1, \lambda_2 = 2l \mp k_3 + \lambda_1 + 1, \nu^2 = M^2\omega^2R^4/\hbar^2 + 1/4$)

$$K^{(V_1)}(\mathrm{u}'',\mathrm{u}';T)$$

$$= \frac{e^{-i\hbar T/2MR^2}}{R^3} \int\limits_{\tau(t')=\tau'}^{\tau(t'')=\tau''} \mathcal{D}\tau(t)\sinh^2\tau \int\limits_{\vartheta(t')=\vartheta'}^{\vartheta(t'')=\vartheta''} \mathcal{D}\vartheta(t)\sin\vartheta \int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''} \mathcal{D}\varphi(t)$$

$$\times \exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}R^2\left(\dot\tau^2+\sinh^2\tau(\dot\vartheta^2+\sin^2\vartheta\dot\varphi^2)-\omega^2\tanh^2\tau\right)\right.\right.$$

$$-\frac{\hbar^2}{2MR^2}\frac{1}{\sinh^2\tau}\left(\frac{1}{\sin^2\vartheta}\left(\frac{k_1^2-\frac14}{\cos^2\varphi}+\frac{k_2^2-\frac14}{\sin^2\varphi}-\frac14\right)\right.$$

$$\left.\left.\left.+\frac{k_3^2-\frac14}{\cos^2\vartheta}-\frac14\right)\right]dt\right\} \tag{3.31}$$

$$= \sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\left\{\sum_{n=0}^{N_n} e^{-iE_N T/\hbar}\Psi_{nlm}^{(V_1)}(\tau'',\vartheta'',\varphi'';R)\Psi_{nlm}^{(V_1)}(\tau',\vartheta',\varphi';R)\right.$$

$$\left.+\int_0^\infty dp\, e^{-iE_p T/\hbar}\Psi_{plm}^{(V_1)}(\tau'',\vartheta'',\varphi'';R)\Psi_{plm}^{(V_1)\,*}(\tau',\vartheta',\varphi';R)\right\} \ . \tag{3.32}$$

The bound state wave-functions are given by $N=0,\ldots,N_{Max}=[\frac12(\nu\mp k_1\mp k_2\mp k_3)]$:

$$\Psi_{nlm}^{(V_1)}(\tau,\vartheta,\varphi;R)$$
$$= (\sinh^2\tau\sin\vartheta)^{-1/2}S_n^{(\lambda_2,\nu)}(\tau;R)\phi_l^{(\lambda_1,\pm k_3)}(\vartheta)\phi_m^{(\pm k_2,\pm k_1)}(\varphi)\ , \tag{3.33}$$

where

$$\phi_l^{(\lambda_1,\pm k_3)}(\vartheta) = \left[2(1+2l\pm k_3+\lambda_1)\frac{l!\Gamma(l+\lambda_1\pm k_3+1)}{\Gamma(1+l\pm k_3)\Gamma(1+l+\lambda_1)}\right]^{1/2}$$
$$\times(\sin\vartheta)^{1/2+\lambda_1}(\cos\vartheta)^{1/2\pm k_3}P_l^{(\lambda_1,\pm k_3)}(\cos 2\vartheta)\ , \tag{3.34}$$

where $(n=0,\ldots,N_n<(\nu-\lambda_2-1)/2)$

$$S_n^{(\lambda_2,\nu)}(\tau;R) = \frac{1}{\Gamma(1+\lambda_2)}\left[\frac{2(\nu-\lambda_2-2n-1)\Gamma(n+1+\lambda_2)\Gamma(\nu-n)}{R^3\Gamma(\nu-\lambda_2-n)n!}\right]^{1/2}$$
$$\times(\sinh\tau)^{\lambda_2+1/2}(\cosh\tau)^{2n+1/2-\nu}{}_2F_1(-n,\nu-n;1+\lambda_2;\tanh^2\tau). \tag{3.35}$$

The scattering states are

$$\Psi_{plm}^{(V_1)}(\tau,\vartheta,\varphi;R)$$
$$= (\sinh^2\tau\sin\vartheta)^{-1/2}S_p^{(\lambda_2,\nu)}(\tau;R)\phi_l^{(\lambda_1,\pm k_3)}(\vartheta)\phi_m^{(\pm k_2,\pm k_1)}(\varphi)\ , \tag{3.36}$$

where

$$
\begin{aligned}
&S_p^{(\lambda_2,\nu)}(\tau;R)\\
&\quad=\frac{1}{\Gamma(1+\lambda_2)}\sqrt{\frac{p\sinh\pi p}{2\pi^2R^3}}\,\Gamma\left(\frac{\nu-\lambda_2+1-\mathrm{i}p}{2}\right)\Gamma\left(\frac{\lambda_2-\nu+1-\mathrm{i}p}{2}\right)\\
&\quad\times(\tanh\tau)^{\lambda_2+1/2}(\cosh\tau)^{\mathrm{i}p}{}_2F_1\\
&\quad\times\left(\frac{\nu+\lambda_2+1-\mathrm{i}p}{2},\frac{\lambda_2-\nu+1-\mathrm{i}p}{2};1+\lambda_2;\tanh^2\tau\right)\ . \qquad (3.37)
\end{aligned}
$$

Here denote the wave-functions $\phi_m^{(\pm k_2,\pm k_1)}(\varphi)$ by the same wave-functions as in (3.21). The discrete and continuous energy spectra E_N and E_p are, of course, the same as in (3.22, 3.26), respectively.

The path integral solution in terms of the sphero-elliptic coordinates is very similar as for spherical coordinates, and the bound state wave-functions for the sphero-elliptic coordinates are given by

$$
\begin{aligned}
\Psi_{nlh}^{(V_1)}(\tau,\tilde\alpha,\tilde\beta;R)&=(\sinh^2\tau\mathrm{sn}\tilde\alpha\mathrm{cn}\tilde\alpha\mathrm{dn}\tilde\alpha\mathrm{sn}\tilde\beta\mathrm{cn}\tilde\beta\mathrm{dn}\tilde\beta)^{-1/2}\\
&\quad\times S_n^{(\lambda_2,\nu)}(\tau;R)\,\Xi_{lh}^{(\pm k_1,\pm k_2,\pm k_3)}(\tilde\alpha,\tilde\beta)\ , \qquad (3.38)
\end{aligned}
$$

with the same energy-spectrum as in the previous case. The wave-functions $\Xi_{lh}^{(\pm k_1,\pm k_2,\pm k_3)}(\tilde\alpha,\tilde\beta)$ have been determined in [34] and correspond to the free wave-function on the six-dimensional sphere in a cylindric-elliptic coordinate system. They are not explicitly known yet, and therefore the above solution in sphero-elliptic coordinates remains on a somewhat formal level. We present it for completeness, though. The continuous spectrum has the form

$$
\begin{aligned}
\Psi_{plh}^{(V_1)}(\tau,\tilde\alpha,\tilde\beta;R)&=(\sinh^2\tau\mathrm{sn}\tilde\alpha\mathrm{cn}\tilde\alpha\mathrm{dn}\tilde\alpha\mathrm{sn}\tilde\beta\mathrm{cn}\tilde\beta\mathrm{dn}\tilde\beta)^{-1/2}\\
&\quad\times S_p^{(\lambda_2,\nu)}(\tau;R)\,\Xi_{lh}^{(\pm k_1,\pm k_2,\pm k_3)}(\tilde\alpha,\tilde\beta)\ , \qquad (3.39)
\end{aligned}
$$

and E_p as in the previous case.

Equidistant-Cylindrical Coordinates. In equidistant cylindrical coordinates we obtain the path integral solution ($\lambda_1=2m\mp k_1\mp k_2+1,\lambda_2=2l+\lambda_1-\nu+1,\nu^2=M^2\omega^2R^4/\hbar^2+1/4$)

$$
\begin{aligned}
K^{(V_1)}(\mathrm{u}'',\mathrm{u}';T)&=R^{-3}\exp\left[-\frac{\mathrm{i}}{\hbar}T\left(\frac{\hbar^2}{2MR^2}+\frac{M}{2}R^2\omega^2\right)\right]\\
&\times\int\limits_{\tau_1(t')=\tau_1'}^{\tau_1(t'')=\tau_1''}\mathcal{D}\tau_1(t)\cosh^2\tau_1\int\limits_{\tau_2(t')=\tau_2'}^{\tau_2(t'')=\tau_2''}\mathcal{D}\tau_2(t)\sinh\tau_2\int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''}\mathcal{D}\varphi(t)
\end{aligned}
$$

$$\times \exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}R^2\left(\dot\tau_1^2+\cosh^2\tau_1(\dot\tau_2^2+\sinh^2\tau_2\dot\varphi^2)+\frac{\omega^2}{\cosh^2\tau_1\cosh^2\tau_2}\right)\right.\right.$$

$$-\frac{\hbar^2}{2MR^2}\left(\frac{1}{\cosh^2\tau_1}\left(\frac{1}{\sinh^2\tau_2}\left(\frac{k_1^2-\frac14}{\cos^2\varphi}+\frac{k_2^2-\frac14}{\sin^2\varphi}-\frac14\right)+\frac14\right)\right.$$

$$\left.\left.\left.+\frac{k_3^2-\frac14}{\sinh^2\tau_1}\right)\right]dt\right\} \tag{3.40}$$

$$=\sum_{m=0}^{\infty}\left\{\sum_{l=0}^{N_l}\left[\sum_{n=0}^{N_n}e^{-i\hbar E_N T/\hbar}\Psi_{nlm}^{(V_1)}(\tau_1'',\tau_2'',\varphi'';R)\Psi_{nlm}^{(V_1)}(\tau_1',\tau_2',\varphi';R)\right.\right.$$

$$\left.+\int_0^\infty dp\, e^{-iE_pT/\hbar}\Psi_{plm}^{(V_1)}(\tau_1'',\tau_2'',\varphi'';R)\Psi_{plm}^{(V_1)\,*}(\tau_1',\tau_2',\varphi';R)\right]$$

$$\left.+\int_0^\infty dk\int_0^\infty dp\, e^{-iE_pT/\hbar}\Psi_{pkm}^{(V_1)}(\tau_1'',\tau_2'',\varphi'';R)\Psi_{pkm}^{(V_1)\,*}(\tau_1',\tau_2',\varphi';R)\right\}\ . \tag{3.41}$$

We obtain one set of bound state wave-functions and two sets of scattering wave-functions. The bound state wave-functions are ($l+m=0,\dots,N_l=[\frac12[(\nu\mp k_1\mp k_2-2)],N=0,\dots,N_{Max}=[\frac12(\nu\mp k_1\mp k_2\mp k_3)]$:

$$\Psi_{nlm}^{(V_1)}(\tau_1,\tau_2,\varphi;R)$$
$$=(\cosh^2\tau_1\sinh\tau_2)^{-1/2}S_n^{(\pm k_3,\lambda_2)}(\tau_1;R)\psi_l^{(\lambda_1,\nu)}(\tau_2)\phi_m^{(\pm k_2,\pm k_1)}(\varphi)\,, \tag{3.42}$$

and E_N as in (3.22). The two sets of continuous states are

$$\Psi_{pkm}^{(V_1)}(\tau_1,\tau_2,\varphi;R)$$
$$=(\cosh^2\tau_1\sinh\tau_2)^{-1/2}S_p^{(\pm k_3,\lambda_2)}(\tau_1;R)\psi_l^{(\lambda_1,\nu)}(\tau_2)\phi_m^{(\pm k_2,\pm k_1)}(\varphi)\,, \tag{3.43}$$

$$\Psi_{plm}^{(V_1)}(\tau_1,\tau_2,\varphi;R)$$
$$=(\cosh^2\tau_1\sinh\tau_2)^{-1/2}S_p^{(\pm k_3,ik)}(\tau_1;R)\psi_k^{(\lambda_1,\nu)}(\tau_2)\phi_m^{(\pm k_2,\pm k_1)}(\varphi)\,. \tag{3.44}$$

The continuous spectrum has the form

$$E_p=\frac{\hbar^2}{2MR^2}(p^2+1)+\frac{M}{2}\omega^2R^2\ . \tag{3.45}$$

Here denote:

- the wave-functions $\phi_m^{(\pm k_2,\pm k_1)}(\varphi)$ are the same wave-functions as in (3.21),
- the wave-functions $\psi_{l,k}^{(\lambda_1,\nu)}(\tau_2)$ are the same wave-functions as in (3.20, 3.30) with $\pm k_3\to\lambda_1$,

- the wave-functions $S_{n,p}^{(\pm k_3,\lambda_2)}(\tau_1;R)$ are the same wave-functions as in (3.19, 3.25) with $\lambda_1 \to \pm k_3$, respectively.

Equidistant Coordinates. As the last system where an explicit solution is possible we consider the equidistant system and obtain the path integral solution ($\lambda_1 = 2m \mp k_1 - \nu + 1, \lambda_2 = 2l \mp k_2 - \lambda_1 + 1, \nu^2 = M^2\omega^2R^4/\hbar^2 + 1/4$)

$$\begin{aligned}
K^{(V_1)}(\mathrm{u}'',\mathrm{u}';T) &= R^{-3}\exp\left[-\frac{\mathrm{i}}{\hbar}T\left(\frac{\hbar^2}{2MR^2}+\frac{M}{2}R^2\omega^2\right)\right] \\
&\times \int\limits_{\tau_1(t')=\tau_1'}^{\tau_1(t'')=\tau_1''}\mathcal{D}\tau_1(t)\cosh^2\tau_1 \int\limits_{\tau_2(t')=\tau_2'}^{\tau_2(t'')=\tau_2''}\mathcal{D}\tau_2(t)\cosh\tau_2 \int\limits_{\tau_3(t')=\tau_3'}^{\tau_3(t'')=\tau_3''}\mathcal{D}\tau_3(t) \\
&\times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}R^2\left(\dot\tau_1^2+\cosh^2\tau_1(\dot\tau_2^2+\cosh^2\tau_1\dot\tau_3^2)\right.\right.\right. \\
&\left. + \frac{\omega^2}{\cosh^2\tau_1\cosh^2\tau_2\cosh^2\tau_3}\right) \\
&-\frac{\hbar^2}{2MR^2}\left(\frac{k_3^2-\frac14}{\sinh^2\tau_1}+\frac{1}{\cosh^2\tau_1}\left(\frac{1}{\cosh^2\tau_2}\left(\frac{k_1^2-\frac14}{\sinh^2\tau_3}+\frac14\right)\right.\right. \\
&\left.\left.\left.\left. +\frac{k_2^2-\frac14}{\sinh^2\tau_2}+\frac14\right)\right)\right]dt\right\} \qquad (3.46)\\
&= \sum_{m=0}^{N_m}\left\{\sum_{l=0}^{N_l}\left[\sum_{n=0}^{N_n}\mathrm{e}^{-\mathrm{i}E_NT/\hbar}\Psi_{nlm}^{(V_1)}(\tau_1'',\tau_2'',\tau_3'';R)\Psi_{nlm}^{(V_1)}(\tau_1',\tau_2',\tau_3';R)\right.\right. \\
&\left. +\int_0^\infty dp\,\mathrm{e}^{-\mathrm{i}E_pT/\hbar}\Psi_{plm}^{(V_1)}(\tau_1'',\tau_2'',\tau_3'';R)\Psi_{plm}^{(V_1)\,*}(\tau_1',\tau_2',\tau_3';R)\right] \\
&\left. +\int_0^\infty dk\int_0^\infty dp\,\mathrm{e}^{-\mathrm{i}E_pT/\hbar}\Psi_{pkm}^{(V_1)}(\tau_1'',\tau_2'',\tau_3'';R)\Psi_{pkm}^{(V_1)\,*}(\tau_1',\tau_2',\tau_3';R)\right\} \\
&+\int_0^\infty d\varrho\int_0^\infty dk\int_0^\infty dp\;\mathrm{e}^{-\mathrm{i}E_pT/\hbar}\Psi_{pk\varrho}^{(V_1)}(\tau_1'',\tau_2'',\tau_3'';R)\Psi_{pk\varrho}^{(V_1)\,*}(\tau_1',\tau_2',\tau_3';R)\ . \qquad (3.47)
\end{aligned}$$

We obtain one set of bound state wave-functions and three sets of scattering wave-functions. The bound state wave-functions are ($m = 0,\dots,N_m < (\nu \mp k_1 - 1)/2, l = 0,\dots,N_l < (\lambda_1 \mp k_2 - 1)/2, n = 0,\dots,N_n < (\lambda_2 \mp k_3 - 1)/2$)

$$\begin{aligned}
&\Psi_{nlm}^{(V_1)}(\tau_1,\tau_2,\tau_3;R) \\
&\quad = (\cosh^2\tau_1\cosh\tau_2)^{-1/2}S_n^{(\pm k_3,\lambda_2)}(\tau_1;R)\psi_l^{(\pm k_2,\lambda_1)}(\tau_2)\psi_m^{(\pm k_1,\nu)}(\tau_3)\ , \qquad (3.48)
\end{aligned}$$

and E_N as in (3.22). The three sets of continuous states are

$$\Psi^{(V_1)}_{plm}(\tau_1,\tau_2,\tau_3;R) = (\cosh^2\tau_1\cosh\tau_2)^{-1/2} S_p^{(\pm k_3,\lambda_2)}(\tau_1;R)\psi_l^{(\pm k_2,\lambda_1)}(\tau_2)\psi_m^{(\pm k_1,\nu)}(\tau_3)\,, \tag{3.49}$$

$$\Psi^{(V_1)}_{mkp}(\tau_1,\tau_2,\tau_3;R) = (\cosh^2\tau_1\cosh\tau_2)^{-1/2} S_p^{(\pm k_3,ik)}(\tau_1;R)\psi_k^{(\pm k_2,\lambda_1)}(\tau_2)\psi_m^{(\pm k_1,\nu)}(\tau_3)\,, \tag{3.50}$$

$$\Psi^{(V_1)}_{\varrho kp}(\tau_1,\tau_2,\tau_3;R) = (\cosh^2\tau_1\cosh\tau_2)^{-1/2} S_p^{(\pm k_3,ik)}(\tau_1;R)\psi_k^{(\pm k_2,i\varrho)}(\tau_2)\psi_\varrho^{(\pm k_1,\nu)}(\tau_3)\,, \tag{3.51}$$

and E_p as in (3.45). Here denote:

- the wave-functions $\psi_m^{(\pm k_1,\nu)}(\tau_3)$ are the same wave-functions as in (3.20) with $\tau_2\to\tau_3$, $l\to m$ and $\pm k_3\to\pm k_1$,
- the wave-functions $\psi_\varrho^{(\pm k_1,\nu)}(\tau_3)$ are the same wave-functions as in (3.30) with $\tau_2\to\tau_3$, $k\to\varrho$ and $\pm k_3\to\pm k_1$,
- the wave-functions $\psi_l^{(\pm k_2,\lambda_1)}(\tau_2)$ are the same wave-functions as in (3.19) with $\tau_1\to\tau_2$, $n\to l$, $(\lambda_1,\lambda_2)\to(\pm k_2,\lambda_1)$ and $R=1$,
- the wave-functions $\psi_k^{(\pm k_2,\lambda_1)}(\tau_2)$ are the same wave-functions as in (3.25) with $\tau_1\to\tau_2$, $p\to k$, $(\lambda_1,\lambda_2)\to(\pm k_2,\lambda_1)$ and $R=1$,
- the wave-functions $\psi_k^{(\pm k_2,i\varrho)}(\tau_2)$ are the same wave-functions as in (3.29) with $\tau_1\to\tau_2$, $(p,k)\to(k,\varrho)$, $\lambda_1\to\pm k_2$ and $R=1$,
- the wave-functions $S_{n,p}^{(\pm k_3,\lambda_2)}(\tau_1;R)$ and $S_p^{(\pm k_3,ik)}(\tau_1;R)$ are the same wave-functions as in (3.19, 3.25) and (3.29) with $\lambda_1\to\pm k_3$, respectively.

Let us remark that the wave-functions have been normalized in the domains $\varphi\in(0,\pi/2)$, $\vartheta\in(0,\pi/2)$ and $\tau>0$ in the spherical and in $\tau_{1,2,3}>0$ in the equidistant system. The positive sign for the k_i has to be taken whenever $k_i\geq\frac{1}{2}$ $(i=1,2,3)$, i.e., the potential term is repulsive at the origin, and the motion takes only place in the denoted domains. If $0<|k_i|<\frac{1}{2}$, i.e., the potential term is attractive at the origin, both the positive and the negative sign must be taken into account in the solution. This is indicated by the notion $\pm k_i$ in the formulae. It has also the consequence that for each k_i the motion can take place in the entire domains of the variables on $\Lambda^{(3)}$. In the present case this means that we must distinguish (e.g., in the equidistant system) eight cases: i) $\tau_{1,2,3}>0$, ii) $\tau_{1,2}>0$, $\tau_2\in\mathbb{R}$, iii) $\tau_1\in\mathbb{R}$, $\tau_{2,3}>0$, iv) $\tau_2\in\mathbb{R}$, $\tau_{1,3}>0$, v) $\tau_{1,2}\in\mathbb{R}$, $\tau_2>0$, vi) $\tau_1>0$, $\tau_{2,3}\in\mathbb{R}$, vii) $\tau_2>0$, $\tau_{1,3}\in\mathbb{R}$ and viii) $\tau_{1,2,3}\in\mathbb{R}$. In polar coordinates the same feature is recovered by the observation that the Pöschl-Teller barriers are absent for $|k_i| < \frac{1}{2}$.

In elliptic coordinates this feature is taken into account in the following way: Due to $\alpha \in (\mathrm{i}K', \mathrm{i}K' + K)$, we have $\mathrm{sn}(\alpha, k), \mathrm{i}\,\mathrm{cn}(\alpha, k) > k'/k$, $\mathrm{i}\,\mathrm{dn}(\alpha, k) \geq 0$, and we see that for $\alpha \in (\mathrm{i}K', \mathrm{i}K'+K)$ and $\beta \in (K', 4K')$ we get $u_0 \geq 0$, and the variables u_1, u_2, u_3 change their signs in the eight domains, i.e., $\beta \in (0, K'), \beta \in (K', 2K'), \beta \in (2K', 3K')$ and $\beta \in (3K', 4K')$. We then have for $\alpha \neq 0$

$$\begin{aligned} \mathrm{sn}(0, k') &= \mathrm{sn}(2K', k') = \mathrm{sn}(4K', k') = 0 \ , \\ \mathrm{cn}(K', k') &= \mathrm{cn}(3K', k') = 0 \ , \end{aligned} \tag{3.52}$$

and $\mathrm{dn}(\beta, k') > 0$, $\beta \in [0, 4K')$. For convenience, we have made the choice $\beta \in (0, K')$, and the same is true in all the following systems. The situation is similar in the hyperbolic system, where we choose $\mu \in (\mathrm{i}K', \mathrm{i}K' + K), \eta \in (0, K')$. In the sphero-elliptic system we must choose for the same reasons $\tilde{\alpha} \in (0, K)$ and $\tilde{\beta} \in (0, K')$.

3.2. The Coulomb Potential

We consider the Coulomb potential on the three-dimensional hyperboloid $(k_{1,2} > 0)$

$$V_2(\mathbf{u}) = -\frac{\alpha}{R}\left(\frac{u_0}{\sqrt{u_1^2 + u_2^2 + u_3^2}} - 1\right) + \frac{\hbar^2}{2M}\left(\frac{k_1^2 - \frac{1}{4}}{u_1^2} + \frac{k_2^2 - \frac{1}{4}}{u_2^2}\right) \ , \tag{3.53}$$

which in the five separating coordinate systems has the form

Sphero-Elliptic $(\tau > 0, \tilde{\alpha} \in (0, K), \tilde{\beta} \in (0, K'))$:

$$\begin{aligned} V_2(\mathbf{u}) &= -\frac{\alpha}{R}(\coth\tau - 1) \\ &+ \frac{\hbar^2}{2MR^2\sinh^2\tau}\left(\frac{k_1^2 - \frac{1}{4}}{\mathrm{sn}^2\tilde{\alpha}\,\mathrm{dn}^2\tilde{\beta}} + \frac{k_2^2 - \frac{1}{4}}{\mathrm{cn}^2\tilde{\alpha}\,\mathrm{cn}^2\tilde{\beta}}\right) \end{aligned} \tag{3.54}$$

Spherical $(\tau > 0, \vartheta \in (0, \pi), \varphi \in (0, \pi/2))$:

$$= -\frac{\alpha}{R}(\coth\tau - 1) + \frac{\hbar^2}{2MR^2\sinh^2\tau\sin^2\vartheta}\left(\frac{k_1^2 - \frac{1}{4}}{\cos^2\varphi} + \frac{k_2^2 - \frac{1}{4}}{\sin^2\varphi}\right) \tag{3.55}$$

Prolate Elliptic II $(\alpha \in (\mathrm{i}K', \mathrm{i}K' + K), \beta \in (0, K'), \varphi \in (0, \pi/2))$:

$$\begin{aligned} &= -\frac{\alpha}{R}\left(\frac{k^2\,\mathrm{sn}\alpha\,\mathrm{cn}\beta - k'\,\mathrm{cn}\beta\,\mathrm{dn}\beta}{k^2\,\mathrm{cn}^2\alpha + k'^2\,\mathrm{cn}^2\beta} - 1\right) \\ &+ \frac{\hbar^2}{2MR^2\,\mathrm{dn}^2\alpha\,\mathrm{sn}^2\beta}\left(\frac{k_1^2 - \frac{1}{4}}{\cos^2\varphi} + \frac{k_2^2 - \frac{1}{4}}{\sin^2\varphi}\right) \end{aligned} \tag{3.56}$$

Semi-Hyperbolic $\big(\mu_{1,2} > 0, \varphi \in (0, \pi/2)\big)$:

$$= -\frac{\alpha}{R}\left(\frac{\sqrt{1+\mu_1^2}+\sqrt{1+\mu_2^2}}{\mu_1+\mu_2} - 1\right) + \frac{\hbar^2}{2MR^2\mu_1\mu_2}\left(\frac{k_1^2-\frac{1}{4}}{\cos^2\varphi} + \frac{k_2^2-\frac{1}{4}}{\sin^2\varphi}\right) \tag{3.57}$$

Elliptic-Parabolic 2 $\big(a > 0, \vartheta \in (0, \pi), \varphi \in (0, \pi/2)\big)$:

$$= -\frac{\alpha}{R}\left(\frac{\cosh^2 a + \cos^2\vartheta}{\cosh^2 a - \cos^2\vartheta} - 1\right) + \frac{\hbar^2}{2MR^2}\coth^2 a \cot^2\vartheta\left(\frac{k_1^2-\frac{1}{4}}{\cos^2\varphi} + \frac{k_2^2-\frac{1}{4}}{\sin^2\varphi}\right) . \tag{3.58}$$

A_3 denotes the third component of the Pauli–Lenz–Runge vector on the hyperboloid [77], i.e.,

$$\mathbf{A} = \frac{1}{2R}(\mathbf{L}\times\mathbf{K} - \mathbf{K}\times\mathbf{L}) - \frac{\alpha\mathbf{u}}{\sqrt{u_1^2+u_2^2+u_3^2}} , \qquad \mathbf{u} = (u_1, u_2, u_3) . \tag{3.59}$$

The path integral for the Coulomb potential on $\Lambda^{(3)}$ can be explicitly evaluated in three coordinate systems which will be discussed in the following. In the prolate-elliptic II and the semi-hyperbolic system no explicit solution is known.

3.2.1. Spherical and Sphero-Elliptic Coordinates

The separation of the Coulomb problem in spherical coordinates is similarly done as for the sphero-elliptic one, and we have

$$K^{(V_2)}(\tau'', \tau', \vartheta'', \vartheta', \varphi'', \varphi'; T) = R^{-3}\exp\left[-\frac{\mathrm{i}}{\hbar}T\left(\frac{\hbar^2}{2MR^2} + \frac{\alpha}{R}\right)\right]$$
$$\times \int\limits_{\tau(t')=\tau'}^{\tau(t'')=\tau''} \mathcal{D}\tau(t)\sinh^2\tau \int\limits_{\vartheta(t')=\vartheta'}^{\vartheta(t'')=\vartheta''} \mathcal{D}\vartheta(t)\sin\vartheta \int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''} \mathcal{D}\varphi(t)$$
$$\times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}R^2\left(\dot\tau^2 + \sinh^2\tau(\dot\vartheta^2 + \sin^2\vartheta\dot\varphi^2)\right) + \frac{\alpha}{R}\coth\tau\right.\right.$$
$$\left.\left. - \frac{\hbar^2}{2MR^2\sinh^2\tau}\left(\frac{1}{\sin^2\vartheta}\left(\frac{k_1^2-\frac{1}{4}}{\cos^2\varphi} + \frac{k_2^2-\frac{1}{4}}{\sin^2\varphi} - \frac{1}{4}\right) - \frac{1}{4}\right)\right]dt\right\} \tag{3.60}$$
$$= \sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\left\{\sum_{N=0}^{N_{Max}} \mathrm{e}^{-\mathrm{i}E_N T/\hbar}\Psi_{Nlm}^{(V_2)}(\tau'', \vartheta'', \varphi''; R)\Psi_{Nlm}^{(V_2)\,*}(\tau', \vartheta', \varphi'; R)\right.$$

$$+\int_0^\infty dp\, \mathrm{e}^{-\mathrm{i}E_pT/\hbar}\Psi_{plm}^{(V_2)}(\tau'',\vartheta'',\varphi'';R)\Psi_{plm}^{(V_2)\,*}(\tau',\vartheta',\varphi';R)\Bigg\} \ . \qquad (3.61)$$

The bound and continuous wave-functions are given by

$$\Psi_{Nlm}^{(V_2)}(\tau,\vartheta,\varphi;R) = (\sinh\tau)^{-1}S_N^{(V_2)}(\tau;R) \times\sqrt{(l+\lambda_1+\tfrac{1}{2})\frac{\Gamma(l+\lambda_1+1)}{l!}}P_{l+\lambda_1}^{-\lambda_1}(\cos\vartheta)\phi_m^{(\pm k_2,\pm k_1)}(\varphi) \ , \qquad (3.62)$$

$$\Psi_{plm}^{(V_2)}(\tau,\vartheta,\varphi;R) = (\sinh\tau)^{-1}S_p^{(V_2)}(\tau;R) \times\sqrt{(l+\lambda_1+\tfrac{1}{2})\frac{\Gamma(l+\lambda_1+1)}{l!}}P_{l+\lambda_1}^{-\lambda_1}(\cos\vartheta)\phi_m^{(\pm k_2,\pm k_1)}(\varphi) \ , \qquad (3.63)$$

with $\lambda_1 = 2m\mp k_1\mp k_2+1$, $\lambda_2 = l+\lambda_1+\frac{1}{2}$, $N=n+l+2m\mp k_1\mp k_2+2$, the wave-functions (3.64, 3.66), with the wave-functions $\phi_m^{(\pm k_2,\pm k_1)}(\varphi)$ as in (3.21). The bound state spherical wave-functions and the energy-spectrum are given by

$$S_N^{(V_2)}(\tau;R) = \frac{2^{\lambda_2+\frac{1}{2}}}{\Gamma(2\lambda_2+\frac{1}{2})}\left[\frac{\sigma_N^2-\tilde N^2}{R^3\tilde N}\frac{\Gamma(\tilde N+\lambda_2+\frac{1}{2})\Gamma(\sigma_N+\lambda_2+\frac{1}{2})}{\Gamma(\tilde N-\lambda_2)\,\Gamma(\sigma_N-\lambda_2)}\right]^{1/2} \times(\sinh\tau)^{\lambda_2+1/2}\mathrm{e}^{-\tau(\sigma_N-n)}{}_2 \times F_1\bigg(-n,\lambda_2+\frac{1}{2}+\sigma_N;2\lambda_2+1;\frac{2}{1+\coth\tau}\bigg) \ , \qquad (3.64)$$

$$E_N = \frac{\alpha}{R}-\hbar^2\frac{\tilde N^2-1}{2MR^2}-\frac{M\alpha^2}{2\hbar^2\tilde N^2} \ . \qquad (3.65)$$

Here we have abbreviated $a=\hbar^2/M\alpha$ (the Bohr radius), $\sigma_N = aR/\tilde N$, $\tilde N = N\mp k_1\mp k_2+2$, $N=n+l+2m$, $N=0,1,2,\dots,N_{Max}<\sqrt{R/a}$. The continuous spectrum has the form

$$S_p^{(V_2)}(\tau;R) = \frac{2^{(\mathrm{i}/2)(p-\tilde p)+\lambda_2+\frac{1}{2}}}{\pi\Gamma(2\lambda_2+1)} \times\sqrt{\frac{p\sinh\pi p}{2R^3}}\Gamma\bigg(\lambda_2+\frac{1}{2}+\frac{\mathrm{i}}{2}(\tilde p-p)\bigg)\Gamma\bigg(\lambda_2+\frac{1}{2}-\frac{\mathrm{i}}{2}(\tilde p+p)\bigg) \times(\sinh\tau)^{\lambda_2+1/2}\exp\bigg[\tau\bigg(\frac{\mathrm{i}}{2}(\tilde p+p)-\lambda_2-\frac{1}{2}\bigg)\bigg]{}_2F_1 \times\bigg(\lambda_2+\frac{1}{2}+\frac{\mathrm{i}}{2}(\tilde p-p),\lambda_2+\frac{1}{2}-\frac{\mathrm{i}}{2}(\tilde p+p);2\lambda_2+1;\frac{2}{1+\coth\tau}\bigg) \ , \qquad (3.66)$$

$\tilde p = \sqrt{2MR^2(E_p-\alpha\hbar^2/R)}\Big/\hbar$, and E_p as in (3.27).

In the case of the pure Coulomb problem the angular wave-functions are just the spherical harmonics Y_l^m on $S^{(2)}$, i.e., we obtain for wave-functions in this case

$$\Psi_{nlm}^{(V_2)}(\tau,\vartheta,\varphi;R) = (\sinh\tau)^{-1} S_N^{(V_2)}(\tau;R) Y_l^m(\vartheta,\varphi) \ , \tag{3.67}$$

$$\Psi_{plm}^{(V_2)}(\tau,\vartheta,\varphi;R) = (\sinh\tau)^{-1} S_p^{(V_2)}(\tau;R) Y_l^m(\vartheta,\varphi) \ , \tag{3.68}$$

together with the principal quantum number $N = n + l + 1 = 0, \dots, N_{Max} < \sqrt{R/a} - \lambda_1 - 1/2$. In [25] it was shown that the $S_N^{(V_2)}(\tau;R)$ and $S_p^{(V_2)}(\tau;R)$ yield the correct radial wave-functions in $\mathbb{R}^3$, as $R \to \infty$.

The complete wave functions of the generalized Coulomb problem on the three-dimensional pseudosphere in sphero-elliptic coordinates are given by

$$\begin{aligned}\Psi_{nlh}^{(V_2)}(\tau,\tilde\alpha,\tilde\beta;R) &= (\sinh^2\tau\,\mathrm{sn}\tilde\alpha\,\mathrm{cn}\tilde\alpha\,\mathrm{dn}\tilde\alpha\,\mathrm{sn}\tilde\beta\,\mathrm{cn}\tilde\beta\,\mathrm{dn}\tilde\beta)^{-1/2} \\ &\quad\times S_N^{(V_2)}(\tau;R)\,\Xi_{lm}^{(\pm k_1,\pm k_2,\pm\frac{1}{2})}(\tilde\alpha,\tilde\beta) \ ,\end{aligned} \tag{3.69}$$

$$\begin{aligned}\Psi_{plh}^{(V_2)}(\tau,\tilde\alpha,\tilde\beta;R) &= (\sinh^2\tau\,\mathrm{sn}\tilde\alpha\,\mathrm{cn}\tilde\alpha\,\mathrm{dn}\tilde\alpha\,\mathrm{sn}\tilde\beta\,\mathrm{cn}\tilde\beta\,\mathrm{dn}\tilde\beta)^{-1/2} \\ &\quad\times S_p^{(V_2)}(\tau;R)\,\Xi_{lm}^{(\pm k_1,\pm k_2,\pm\frac{1}{2})}(\tilde\alpha,\tilde\beta) \ .\end{aligned} \tag{3.70}$$

Let us note that in the pure Coulomb case, the path integral evaluation is almost the same with only minor differences. The wave-functions $\Xi_{lm}^{(\pm k_1,\pm k_2,\pm\frac{1}{2})}$ are replaced by the wave-functions of the free motion on the sphere $S^{(2)}$, i.e., together with the notation $k, q = \pm 1, h + \tilde h = l(l+1)$

$$\Xi_{lm}^{(\pm k_1,\pm k_2,\pm\frac{1}{2})}(\tilde\alpha,\tilde\beta) \to \Lambda_{lh}^{k}(\tilde\alpha)\Lambda_{l\tilde h}^{q}(\tilde\beta) \ . \tag{3.71}$$

The quantum number λ_2 yields the usual angular momentum number $l \in \mathbb{N}_0$. The discrete spectrum has the same form as (3.65), however with the principal quantum number N now given by $N = n + l + 1$, therefore giving degeneracies with respect to the quantum number m. Everything else remains the same.

3.2.2. Elliptic-Parabolic 2 Coordinates

In order to evaluate the path integral in elliptic-parabolic 2 coordinates, one first separates off the φ-path integration, and then performs a time transformation. This gives ($\lambda_1 = 2m \mp k_1 \mp k_2 + 1$)

$$K^{(V_2)}(a'',a',\vartheta'',\vartheta',\varphi'',\varphi';T) = R^{-3}\exp\left[-\frac{\mathrm{i}}{\hbar}T\left(\frac{\hbar^2}{2MR^2}+\frac{\alpha}{R}\right)\right]$$

$$\times\int\limits_{a(t')=a'}^{a(t'')=a''}\mathcal{D}a(t)\int\limits_{\vartheta(t')=\vartheta'}^{\vartheta(t'')=\vartheta''}\mathcal{D}\vartheta(t)\tanh a\tan\vartheta\frac{\cosh^2 a-\cos^2\vartheta}{\cosh^2 a\cos^2\vartheta}\int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''}\mathcal{D}\varphi(t)$$

$$\times \exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}R^2\frac{(\cosh^2 a-\cos^2\vartheta)(\dot a^2+\dot\vartheta^2)+\sinh^2 a\sin^2\vartheta\dot\varphi^2}{\cosh^2 a\cos^2\vartheta}\right.\right.$$

$$-\frac{\alpha}{R}\frac{\cosh^2 a+\cos^2\vartheta}{\cosh^2 a-\cos^2\vartheta}$$

$$+\frac{\hbar^2}{2MR^2}\coth^2 a\cot^2\vartheta\left(\frac{k_1^2-\frac{1}{4}}{\cos^2\varphi}+\frac{k_2^2-\frac{1}{4}}{\sin^2\varphi}\right)$$

$$\left.\left.+\frac{\hbar^2}{8MR^2}\frac{\cosh^2 a+\cos^2\vartheta-1}{\sinh^2 a\sin^2\vartheta}\right]dt\right\} \tag{3.72}$$

$$=\frac{e^{-i\hbar T/2MR^2}}{R^3}(\coth a'\coth a''\cot\vartheta'\cot\vartheta'')^{1/2}$$

$$\times\sum_{m=0}^{\infty}\phi_m^{(\pm k_2,\pm k_1)}(\varphi'')\phi_m^{(\pm k_2,\pm k_1)}(\varphi')$$

$$\times\int\limits_{a(t')=a'}^{a(t'')=a''}\mathcal{D}a(t)\int\limits_{\vartheta(t')=\vartheta'}^{\vartheta(t'')=\vartheta''}\mathcal{D}\vartheta(t)\frac{\cosh^2 a-\cos^2\vartheta}{\cosh^2 a\cos^2\vartheta}$$

$$\times\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2R^2}\frac{\cosh^2 a-\cos^2\vartheta}{\cosh^2 a\cos^2\vartheta}(\dot a^2+\dot\vartheta^2)-\frac{\alpha}{R}\frac{\cosh^2 a+\cos^2\vartheta}{\cosh^2 a-\cos^2\vartheta}\right.\right.$$

$$\left.\left.+\frac{\hbar^2\lambda_1^2}{2MR^2}\coth^2 a\cot^2\vartheta+\frac{\hbar^2}{8MR^2}\frac{\cosh^2 a+\cos^2\vartheta-1}{\cosh^2 a-\cos^2\vartheta}\right]dt\right\} \tag{3.73}$$

$$=\frac{e^{-i\hbar T/2MR^2}}{R^3}(\coth a'\coth a''\cot\vartheta'\cot\vartheta'')^{1/2}$$

$$\times\sum_{m=0}^{\infty}\phi_m^{(\pm k_2,\pm k_1)}(\varphi'')\phi_m^{(\pm k_2,\pm k_1)}(\varphi')$$

$$\times\int_{\mathbb{R}}\frac{dE}{2\pi\hbar}e^{-iET/\hbar}\int_0^\infty ds''\int\limits_{a(0)=a'}^{a(s'')=a''}\mathcal{D}a(s)\int\limits_{\vartheta(0)=\vartheta'}^{\vartheta(s'')=\vartheta''}\mathcal{D}\vartheta(s)$$

$$\times\exp\left\{\frac{i}{\hbar}\int_0^{s''}\left[\frac{M}{2}(\dot a^2+\dot\vartheta^2)\right.\right.$$

$$\left.\left.-\frac{\hbar^2}{2M}\left(\frac{\lambda_1^2-\frac{1}{4}}{\sin^2\vartheta}+\frac{\beta^2-\frac{1}{4}}{\cos^2\vartheta}+\frac{\lambda_1^2-\frac{1}{4}}{\sinh^2 a}-\frac{\nu^2-\frac{1}{4}}{\cosh^2 a}\right)\right]ds\right\}, \tag{3.74}$$

where $\beta^2=\frac{1}{4}-2MER^2/\hbar^2, \nu^2=\frac{1}{4}+2MR^2(2\alpha/R-E)/\hbar^2$. The analysis of this path integral is rather involved and requires the same Green function analysis as the corresponding two-dimensional case [31], [35], which will be not repeated here in all details.

Let us note first that in the case of the pure Coulomb problem we have to replace $\lambda_1 = |j|$, $j \in \mathbb{Z}$, and the wave-functions in φ are circular waves, i.e., $\phi_j(\varphi) = e^{ij\varphi}/\sqrt{2\pi}$, $\varphi \in [0, 2\pi)$. Everything else remains the same.

To analyze the general case we proceed exactly in an analogous way as in [35]. For the discrete spectrum we expand the ϑ-path integration into Pöschl–Teller potential wave functions $\Phi_{n_1}^{(\lambda_1,\beta)}(\vartheta)$, and the a-path integration into the bound state contribution of the modified Pöschl–Teller potential wave functions $\psi_{n_2}^{(\lambda_1,\nu)}(a)$ of (B.6). The emerging Green function representation $G_{disc.}^{(V_2)}(E)$ of $K_{disc.}^{(V_2)}(T)$ has poles which are determined by the equation

$$(2n_1 + \lambda_1 + \beta + 1) = -(2n_2 + \lambda_1 - \nu + 1) \ . \tag{3.75}$$

Solving this equation for $E_{n_1 n_2}$ yields exactly the energy-spectrum (3.65), with the principal quantum number $N = n_1 + n_2 + \lambda_1 + 1 = 1, \ldots, N_{Max}$ as before. Taking the residuum gives the bound state wave-functions which are therefore given by

$$\begin{aligned}
\Psi_{mn_1n_2}^{(V_2)}(a, \vartheta, \varphi; R) &= (\coth a \cot \vartheta)^{1/2} \\
&\times \sqrt{\frac{\sigma_N^2 - \tilde{N}^2}{2R^3 \tilde{N}}} \psi_{n_1}^{(\lambda_1,\nu)}(a) \phi_{n_2}^{(\lambda_1,\beta)}(\vartheta) \phi_m^{(\pm k_2, \pm k_1)}(\varphi) \ ,
\end{aligned} \tag{3.76}$$

where ($\phi_m^{(\pm k_2,\pm k_1)}(\varphi)$ as in (3.21)),

$$\begin{aligned}
\psi_{n_1}^{(\lambda_1,\nu)}(a) &= \frac{1}{\Gamma(1+\lambda_1)} \left[\frac{2(\nu - \lambda_1 - 2n_1 - 1)\Gamma(n_1 + 1 + \lambda_1)\Gamma(\nu - n_1)}{n_1! \Gamma(\nu \mp k_2 - n_1)} \right]^{1/2} \\
&\times (\sinh a)^{1/2+\lambda_1} (\cosh a)^{2n_1 + 1/2 - \nu} {}_2F_2(-n_1, \nu - n_1; 1 + \lambda_1; \tanh^2 a) \ ,
\end{aligned} \tag{3.77}$$

$$\begin{aligned}
\phi_{n_2}^{(\lambda_1,\beta)}(\vartheta) &= \left[2(\beta + \lambda_1 + 2n_2 + 1) \frac{n_2! \Gamma(\beta + \lambda_1 + n_2 + 1)}{\Gamma(n_2 + \lambda_1 + 1)\Gamma(n_2 + \beta + 1)} \right]^{1/2} \\
&\times (\sin \vartheta)^{1/2+\lambda_1} (\cos \vartheta)^{\beta + 1/2} P_{n_2}^{(\lambda_1,\beta)}(\cos 2\vartheta) \ .
\end{aligned} \tag{3.78}$$

For the analysis of the continuous spectrum we must insert the entire Green functions of the Pöschl–Teller (A.5) and modified Pöschl–Teller potential (B.11). We then find the Green function $G^{(V_2)}(E)$ in elliptic-parabolic coordinates by considering the ds''-integration in (3.74) with the solutions of the Pöschl-Teller and modified Pöschl–Teller potential, respectively, and the result can be put in the following form (c.f. also [31] for some more details concerning the proper

Green function analysis)

$$
\begin{aligned}
&G^{(V_2)}(a'',a',\vartheta'',\vartheta',\varphi'',\varphi';E)\\
&=(R^2\tanh a'\tanh a''\tan\vartheta'\tan\vartheta'')^{-1/2}\sum_{m=0}^{\infty}\phi_m^{(\pm k_2,\pm k_1)}(\varphi')\phi_m^{(\pm k_2,\pm k_1)}(\varphi'')\\
&\times\Bigg\{\frac{1}{2}\sum_{n_2}\psi_{n_2}^{(\lambda_1,\nu)}(a'')\psi_{n_2}^{(\lambda_1,\nu)}(a')G_{PT}^{(\lambda_1,\beta)}(\vartheta'',\vartheta';E')\bigg|_{E'=\hbar^2(2n_1+\lambda_1+\beta+1)^2/2MR^2}\\
&+\frac{1}{2}\int_0^\infty dk\,\psi_k^{(\lambda_1,\nu)}(a'')\psi_k^{(\lambda_1,\nu)\,*}(a')G_{PT}^{(\lambda_1,\beta)}(\vartheta'',\vartheta';E')\bigg|_{E'=-\hbar^2k^2/2MR^2}\\
&+\Big[\text{appropriate term with } a \text{ and } \vartheta \text{ interchanged}\Big]\Bigg\}\ ,
\end{aligned}
\tag{3.79}
$$

in the notation of (A.1, A.5, B.6) and (B.11), respectively. Equation (3.79) also represents the Green function corresponding to the path integral (3.72). Analyzing the cuts gives the continuous states which have the form ($\tilde p^2=-\frac{1}{4}-2MR^2(2\alpha/R-E)/\hbar^2$)

$$
\begin{aligned}
&\Psi_{pkm}^{(V_2)}(a,\vartheta,\varphi;R)\\
&\qquad=(R^3\tanh a\tan\vartheta)^{-1/2}\psi_k^{(\lambda_1,\tilde p)}(a)\phi_k^{(\lambda_1,p)}(\vartheta)\phi_m^{(\pm k_2,\pm k_1)}(\varphi)\ ,
\end{aligned}
\tag{3.80}
$$

where

$$
\begin{aligned}
\psi_k^{(\lambda_1,\tilde p)}(a)&=\frac{\Gamma[\frac{1}{2}(1+\lambda_1+i\tilde p+ik)]\Gamma[\frac{1}{2}(1+\lambda_1+i\tilde p-ik)]}{\Gamma(1+\lambda_1)}\\
&\times\sqrt{\frac{k\sinh\pi k}{2\pi^2}}(\tanh a)^{\lambda_1-1/2}(\cosh a)^{ik}{}_2F_1\\
&\times\bigg(\frac{1+\lambda_1+i\tilde p+ik}{2},\frac{1+\lambda_1-i\tilde p+ik}{2};1+\lambda_1;\tanh^2 a\bigg)\ ,
\end{aligned}
\tag{3.81}
$$

$$
\begin{aligned}
\phi_k^{(\lambda_1,p)}(\vartheta)&=\frac{\Gamma[\frac{1}{2}(1+\lambda_1+ip+ik)]\Gamma[\frac{1}{2}(1+\lambda_1+ip-ik)]}{\Gamma(1+\lambda_1)}\\
&\times\sqrt{\frac{k\sinh\pi k}{2\pi^2}}(\tan\vartheta)^{\lambda_1-1/2}(\cos\vartheta)^{ip+1+\lambda_1}{}_2F_1\\
&\times\bigg(\frac{1+\lambda_1+ip+ik}{2},\frac{1+\lambda_1-ip+ik}{2};1+\lambda_1;-\sin^2\vartheta\bigg)\ .
\end{aligned}
\tag{3.82}
$$

The energy-spectra are as in (3.65,3.27), respectively. Putting both results together we obtain the path integral solution of the Coulomb problem on $\Lambda^{(3)}$ in elliptic parabolic 2 coordinates in the following form

$$K^{(V_2)}(a'',a',\vartheta'',\vartheta',\varphi'',\varphi';T)$$
$$=\sum_{m=0}^{\infty}\left\{\sum_{n_1,n_2}\mathrm{e}^{-\mathrm{i}E_NT/\hbar}\Psi^{(V_2)}_{mn_1n_2}(a'',\vartheta'',\varphi'';R)\Psi^{(V_2)}_{mn_1n_2}(a',\vartheta',\varphi';R)\right.$$
$$\left.+\int_0^\infty dk\int_0^\infty dp\,\mathrm{e}^{-\mathrm{i}E_pT/\hbar}\Psi^{(V_2)}_{pkm}(a'',\vartheta'',\varphi'';R)\Psi^{(V_2)\,*}_{pkm}(a',\vartheta',\varphi';R)\right\}. \quad (3.83)$$

3.3. A Radial Scattering Potential

We consider the potential in its five separating coordinate systems ($k_{1,2,3}>0$)

$$V_3(\mathrm{u})=\frac{\hbar^2}{2MR^2}\Bigg[-\frac{k_0^2-\frac14}{u_0^2}$$
$$+\frac{1}{\sqrt{u_1^2+u_2^2}}\left(\frac{k_1^2-\frac14}{\sqrt{u_1^2+u_2^2}+u_1}+\frac{k_2^2-\frac14}{\sqrt{u_1^2+u_2^2}-u_1}\right)+\frac{k_3^2-\frac14}{u_3^2}\Bigg] \quad (3.84)$$

Spherical $\big(\tau>0,\vartheta\in(0,\pi/2),\varphi\in(0,\pi)\big)$:

$$=\frac{\hbar^2}{2MR^2}\Bigg[-\frac{k_0^2-\frac14}{\cosh^2\tau}$$
$$+\frac{1}{\sinh^2\tau}\left(\frac{1}{4\sin^2\vartheta}\left(\frac{k_1^2-\frac14}{\cos^2(\varphi/2)}+\frac{k_2^2-\frac14}{\sin^2(\varphi/2)}\right)+\frac{k_3^2-\frac14}{\cos^2\vartheta}\right) \quad (3.85)$$

Equidistant-Cylindrical $\big(\tau_{1,2}>0,\varphi\in(0,\pi)\big)$:

$$=\frac{\hbar^2}{2MR^2}\left(\frac{k_3^2-\frac14}{\sinh^2\tau_1}+\frac{1}{\cosh^2\tau_1}\left(-\frac{k_0^2-\frac14}{\cosh^2\tau_2}\right.\right.$$
$$\left.\left.+\frac{1}{4\sinh^2\tau_2}\left(\frac{k_1^2-\frac14}{\cos^2(\varphi/2)}+\frac{k_2^2-\frac14}{\sin^2(\varphi/2)}\right)\right)\right) \quad (3.86)$$

Prolate Elliptic $\big(\alpha\in(\mathrm{i}K',\mathrm{i}K'+K),\beta\in(0,K'),\varphi\in(0,\pi)\big)$:

$$=-\frac{\hbar^2}{2MR^2}\left(-\frac{k_0^2-\frac14}{\mathrm{sn}^2\alpha\,\mathrm{dn}^2\beta}\right.$$
$$\left.+\frac{1}{4\,\mathrm{dn}^2\alpha\,\mathrm{sn}^2\beta}\left(\frac{k_1^2-\frac14}{\cos^2(\varphi/2)}+\frac{k_2^2-\frac14}{\sin^2(\varphi/2)}\right)+\frac{k_3^2-\frac14}{\mathrm{cn}^2\alpha\,\mathrm{cn}^2\beta}\right). \quad (3.87)$$

Oblate Elliptic $\big(\alpha\in(\mathrm{i}K',\mathrm{i}K'+K),\beta\in(0,K'),\varphi\in(0,\pi)\big)$:

$$= -\frac{\hbar^2}{2MR^2}\Bigg(-\frac{k_0^2-\frac14}{\mathrm{sn}^2\alpha\,\mathrm{dn}^2\beta}$$

$$+\frac{1}{4\,\mathrm{cn}^2\alpha\,\mathrm{cn}^2\beta}\bigg(\frac{k_1^2-\frac14}{\cos^2(\varphi/2)}+\frac{k_2^2-\frac14}{\sin^2(\varphi/2)}\bigg)+\frac{k_3^2-\frac14}{\mathrm{dn}^2\alpha\,\mathrm{sn}^2\beta}\Bigg) \tag{3.88}$$

Hyperbolic-Cylindrical 2 $\big(\mu\in(\mathrm{i}K',\mathrm{i}K'+K),\eta\in(0,K'),\varphi\in(0,\pi)\big)$:

$$= -\frac{\hbar^2}{2MR^2}\Bigg(-\frac{k_0^2-\frac14}{\mathrm{cn}^2\mu\,\mathrm{cn}^2\nu}$$

$$+\frac{1}{4\,\mathrm{sn}^2\mu\,\mathrm{dn}^2\nu}\bigg(\frac{k_1^2-\frac14}{\cos^2(\varphi/2)}+\frac{k_2^2-\frac14}{\sin^2(\varphi/2)}\bigg)+\frac{k_3^2-\frac14}{\mathrm{dn}^2\mu\,\mathrm{sn}^2\nu}\Bigg)\ . \tag{3.89}$$

Only in the first two coordinate systems an explicit solution is possible.

Spherical Coordinates. In the first separating coordinate system we have the following path integral representation together with its solution ($\lambda_1 = m + \frac12(1\mp k_1\mp k_2), \lambda_2 = 2l\mp k_3+\lambda_1+1$)

$$K^{(V_3)}(\mathrm{u}'',\mathrm{u}';T)$$

$$= \frac{\mathrm{e}^{-\mathrm{i}\hbar T/2MR^2}}{R^3}\int\limits_{\tau(t')=\tau'}^{\tau(t'')=\tau''}\mathcal{D}\tau(t)\sinh^2\tau\int\limits_{\vartheta(t')=\vartheta'}^{\vartheta(t'')=\vartheta''}\mathcal{D}\vartheta(t)\sin\vartheta\int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''}\mathcal{D}\varphi(t)$$

$$\times\exp\Bigg\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\Bigg[\frac{M}{2}R^2\Big(\dot\tau^2+\sinh^2\tau(\dot\vartheta^2+\sin^2\vartheta\dot\varphi^2)\Big)$$

$$\times-\frac{\hbar^2}{2MR^2}\Bigg(-\frac{k_0^2-\frac14}{\cosh^2\tau}$$

$$+\frac{1}{\sinh^2\tau}\Bigg(\frac{1}{4\sin^2\vartheta}\bigg(\frac{k_1^2-\frac14}{\cos^2(\varphi/2)}+\frac{k_2^2-\frac14}{\sin^2(\varphi/2)}-1\bigg)$$

$$+\frac{k_3^2-\frac14}{\cos^2\vartheta}-\frac14\Bigg)\Bigg)\Bigg]dt\Bigg\} \tag{3.90}$$

$$= \sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\int_0^\infty dp\,\mathrm{e}^{-\mathrm{i}E_p/\hbar}\Psi_{plm}^{(V_3)}(\tau'',\vartheta'',\varphi'';R)\Psi_{plm}^{(V_3)\,*}(\tau',\vartheta',\varphi';R)\ . \tag{3.91}$$

The path integral evaluation is successively performed by applying the path integral solution of the Pöschl–Teller potential in φ and ϑ, and for the $1/\sinh^2 r$-potential in τ. The spectrum is purely continuous and the wave-functions have the form

$$\Psi^{(V_3)}_{plm}(\tau,\vartheta,\varphi;R)=S_p^{(\lambda_2,k_0)}(\tau;R)\phi_l^{(\lambda_1,\pm k_3)}(\vartheta)\phi_m^{(\pm k_2,\pm k_1)}\left(\frac{\varphi}{2}\right)\ , \tag{3.92}$$

and E_p as in (3.27). Here the wave-functions $S_p^{(\lambda_2,k_0)}$ are usual continuous modified Pöschl–Teller wave-functions analogous to (3.25) with the parameters (λ_2,k_0), the $\phi_l^{(\lambda_1,\pm k_3)}(\vartheta)$ are the same as in (3.34) and the wave-functions $\phi_m^{(\pm k_2,\pm k_1)}(\varphi/2)$ are the same as in (3.21) with $\varphi\to\varphi/2$ and an additional factor $1/\sqrt{2}$.

Equidistant-Cylindrical Coordinates The solution in the second coordinate system has the form ($\lambda_1=2m\mp k_1\mp k_2+1$)

$$\begin{aligned}
&K^{(V_3)}(\mathrm{u}'',\mathrm{u}';T)\\
&=\frac{\mathrm{e}^{-\mathrm{i}\hbar T/2MR^2}}{R^3}\int\limits_{\tau_1(t')=\tau_1'}^{\tau_1(t'')=\tau_1''}\mathcal{D}\tau_1(t)\cosh^2\tau_1\int\limits_{\tau_2(t')=\tau_2'}^{\tau_2(t'')=\tau_2''}\mathcal{D}\tau_2(t)\sinh\tau_2\int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''}\mathcal{D}\varphi(t)\\
&\quad\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}R^2\Big(\dot\tau_1^2+\cosh^2\tau_1(\dot\tau_2^2+\sinh^2\tau_2\dot\varphi^2)\Big)\right.\right.\\
&\quad-\frac{\hbar^2}{2MR^2}\left(\frac{k_3^2-\frac14}{\sinh^2\tau_1}+\frac{1}{\cosh^2\tau_1}\left(-\frac{k_0^2-\frac14}{\cosh^2\tau_2}\right.\right.\\
&\quad\left.\left.\left.\left.+\frac{1}{4\sinh^2\tau_2}\left(\frac{k_1^2-\frac14}{\cos^2\frac{\varphi}{2}}+\frac{k_2^2-\frac14}{\sin^2\frac{\varphi}{2}}-1\right)+1\right)\right)\right]dt\right\}
\end{aligned} \tag{3.93}$$

$$=\sum_{m=0}^{\infty}\int_0^\infty dk\int_0^\infty dp\,\mathrm{e}^{-\mathrm{i}E_p/\hbar}\Psi^{(V_3)}_{pkm}(\tau_1'',\tau_2'',\varphi'';R)\Psi^{(V_3)\,*}_{pkm}(\tau_1',\tau_2',\varphi';R)\ . \tag{3.94}$$

The path integral evaluation is successively performed by means of the path integral solution for the Pöschl–Teller potential in φ, for the $1/\sinh^2 r$-potential in τ_2 and for the modified Pöschl–Teller potential in τ_1. The continuous wave-functions have the form (λ_1 as before)

$$\Psi^{(V_3)}_{pkm}(\tau_1,\tau_2,\varphi;R)=S_p^{(\pm k_3,\mathrm{i}k)}(\tau_1;R)\psi_k^{(\lambda_1,\pm k_0)}(\tau_2)\phi_m^{(\pm k_2,\pm k_1)}\left(\frac{\varphi}{2}\right)\ , \tag{3.95}$$

with $\phi_m^{(\pm k_2,\pm k_1)}(\varphi/2)$ as before, and the $S_p^{(\pm k_3,\mathrm{i}k)}(\tau_1;R)$ the same as in (3.44), and the $\psi_p^{(\lambda_1,k_0)}$ the same as in (3.92) with $\lambda_2\to\lambda_1$, $\tau\to\tau_2$ and $R=1$.

3.4. A Stark-Effect Potential

We consider the potential ($k_{1,2} > 0$)

$$V_4(\mathrm{u}) = \frac{\hbar^2}{4M\sqrt{u_1^2+u_2^2}}\left(\frac{k_1^2-\frac{1}{4}}{\sqrt{u_1^2+u_2^2}+u_1}+\frac{k_2^2-\frac{1}{4}}{\sqrt{u_1^2+u_2^2}-u_1}\right)+k_3u_3 \ . \quad (3.96)$$

In its four separating coordinate systems it has the form

Equidistant-Elliptic II $\big(\alpha\in(\mathrm{i}K',\mathrm{i}K'+K),\beta\in(0,K'),\tau>0\big)$:

$$V_4(\mathrm{u}) = \frac{1}{\cosh^2\tau}\frac{\hbar^2}{4MR^2}\left[\frac{k_1^2+k_2^2-\frac{1}{2}}{k^2\,\mathrm{cn}^2\alpha+k'^2\,\mathrm{cn}^2\beta}\left(\frac{k'^2}{\mathrm{dn}^2\alpha}-\frac{1}{\mathrm{sn}^2\beta}\right)\right.$$
$$\left.+(k_1^2-k_2^2)\frac{k'}{k}\frac{k^2\,\mathrm{sn}\alpha\,\mathrm{cn}\alpha+k'\,\mathrm{cn}\beta\,\mathrm{dn}\beta}{k^2\,\mathrm{cn}^2\alpha+k'^2\,\mathrm{cn}^2\beta}\right]+k_3R\sinh\tau \quad (3.97)$$

Equidistant-Semi-Hyperbolic $\big(\tau\in\mathbb{R},\mu_{1,2}>0\big)$:

$$=\frac{\hbar^2}{4MR^2\cosh^2\tau}\frac{1}{\mu_1+\mu_2}\left[(k_1^2+k_2^2-\tfrac{1}{2})\left(\frac{1}{\mu_1}+\frac{1}{\mu_2}\right)\right.$$
$$\left.+(k_1^2-k_2^2)\left(\frac{\sqrt{1+\mu_1^2}}{\mu_1}-\frac{\sqrt{1+\mu_2^2}}{\mu_2}\right)\right]+k_3R\sinh\tau \quad (3.98)$$

Equidistant-Elliptic-Parabolic $\big(\tau\in\mathbb{R},a>0,\vartheta\in(0,\pi/2)\big)$:

$$=\frac{\hbar^2}{2MR^2\cosh^2\tau}\frac{\cosh^2a\cos^2\vartheta}{\cosh^2a-\cos^2\vartheta}\left(\frac{k_1^2-\frac{1}{4}}{\sin^2\vartheta}+\frac{k_2^2-\frac{1}{4}}{\sinh^2a}\right)+k_3R\sinh\tau \quad (3.99)$$

Equidistant-Cylindrical $\big(\tau_1\in\mathbb{R},\tau_2>0,\varphi\in(0,\pi)\big)$:

$$=\frac{\hbar^2}{8MR^2\cosh^2\tau_1\sinh^2\tau_2}\left(\frac{k_1^2-\frac{1}{4}}{\sin^2(\varphi/2)}+\frac{k_2^2-\frac{1}{4}}{\cos^2(\varphi/2)}\right)+k_3R\sinh\tau_1 \ . \quad (3.100)$$

Because there are only four observables, the potential V_4 is only minimally and not maximally superintegrable on the hyperboloid. However, its flat space analogue $V_5(\mathbf{x})$ is maximally superintegrable, and we have included V_4 in this section, though. Unfortunately, the path integrals in all coordinate systems are not solvable and we omit the further details.

4. PATH INTEGRAL FORMULATION OF THE MINIMALLY SUPERINTEGRABLE POTENTIALS ON $\Lambda^{(3)}$

In this Section we list our findings of the minimally superintegrable potentials on $\Lambda^{(3)}$. They include the following:

1. The class of potentials which are the analogues of the minimally superintegrable potentials in $\mathbb{R}^3$ [16, 33]. For instance, the potentials V_5 and V_6 correspond to the double-ring shaped oscillator and the Hartmann potential in $\mathbb{R}^3$, respectively. The four found potentials are discussed in some detail.

2. The class of potentials which correspond to the group reduction $SO(3,1) \supset E(2)$, i.e., which are superintegrable in $\mathbb{R}^2$ [33]. Here the results of [33] will be used, and the problem of self-adjoint extensions for Hamiltonians unbounded from below is briefly mentioned.

3. The class of potentials which correspond to the group reduction $SO(3,1) \supset SO(3)$, i.e., which are superintegrable on $S^{(2)}$ [34]. In our list we have chosen for convenience a dependence according to $1/u_0^2$, but any function $F = F(u_0)$ admits separation of variables.

4. The class of potentials which correspond to the group reduction $SO(3,1) \supset SO(2,1)$, i.e., which are superintegrable on $\Lambda^{(2)}$ [35]. In our list we have chosen for convenience a dependence according to $1/u_3^2$, but any function $F = F(u_3)$ admits separation of variables. Because the features are repeating themselves, all those potentials can be treated simultaneously.

4.1. Analogues of the Minimally Superintegrable Potentials of $\mathbb{R}^3$

4.1.1. Double Ring-Shaped Oscillator

We consider the minimally superintegrable double ring-shaped potential V_5 on $\Lambda^{(3)}$ $(k_3 > 0)$

$$V_5(\mathbf{u}) = \frac{M}{2}\omega^2 R^2 \frac{u_1^2+u_2^2+u_3^2}{u_0^2} + \frac{\hbar^2}{2M}\left(\frac{k_3^2-\frac{1}{4}}{u_3^2} + \frac{F(u_2/u_1)}{u_1^2+u_2^2}\right), \tag{4.1}$$

which in the five separating coordinate systems has the form (φ with appropriate range)

Spherical $(\tau > 0, \vartheta \in (0, \pi/2))$:

$$V_5(\mathbf{u}) = \frac{M}{2}\omega^2 R^2 \tanh^2\tau + \frac{\hbar^2}{2MR^2\sinh^2\tau}\left(\frac{k_3^2-\frac{1}{4}}{\cos^2\vartheta} + \frac{F(\tan\varphi)}{\sin^2\vartheta}\right) \tag{4.2}$$

Equidistant-Cylindrical $(\tau_{1,2} > 0)$:

$$= \frac{M}{2}\omega^2 R^2\left(1 - \frac{1}{\cosh^2\tau_1\cosh^2\tau_2}\right) + \frac{\hbar^2}{2MR^2}\left(\frac{k_3^2-\frac{1}{4}}{\sinh^2\tau_1} + \frac{F(\tan\varphi)}{\cosh^2\tau_1\sinh^2\tau_2}\right) \tag{4.3}$$

Prolate Elliptic $\left(\alpha \in (\mathrm{i}K', \mathrm{i}K'+K), \beta \in (0, K')\right)$:

$$= \frac{M}{2}\omega^2 R^2\left(1 - \frac{1}{\mathrm{sn}^2\alpha\,\mathrm{dn}^2\beta}\right) - \frac{\hbar^2}{2MR^2}\left(\frac{k_3^2 - \frac{1}{4}}{\mathrm{cn}^2\alpha\,\mathrm{cn}^2\beta} + \frac{F(\tan\varphi)}{\mathrm{dn}^2\alpha\,\mathrm{sn}^2\beta}\right) \tag{4.4}$$

Oblate Elliptic $\left(\alpha \in (\mathrm{i}K', \mathrm{i}K'+K), \beta \in (0, K')\right)$:

$$= \frac{M}{2}\omega^2 R^2\left(1 - \frac{1}{\mathrm{sn}^2\alpha\,\mathrm{dn}^2\beta}\right) - \frac{\hbar^2}{2MR^2}\left(\frac{k_3^2 - \frac{1}{4}}{\mathrm{dn}^2\alpha\,\mathrm{sn}^2\beta} + \frac{F(\tan\varphi)}{\mathrm{cn}^2\alpha\,\mathrm{cn}^2\beta}\right) \tag{4.5}$$

Hyperbolic-Cylindrical 2 $\left(\mu \in (\mathrm{i}K', \mathrm{i}K'+K), \eta \in (0, K')\right)$:

$$= \frac{M}{2}\omega^2 R^2\left(1 - \frac{1}{k^2\,\mathrm{cn}^2\mu\,\mathrm{cn}^2\nu}\right) - \frac{\hbar^2}{2MR^2}\left(\frac{k_3^2 - \frac{1}{4}}{\mathrm{dn}^2\mu\,\mathrm{sn}^2\nu} + \frac{F(\tan\varphi)}{\mathrm{sn}^2\mu\,\mathrm{dn}^2\nu}\right). \tag{4.6}$$

An explicit solution is available in two coordinate systems, and we have the following path integral representations together with their solutions.

Spherical Coordinates. In spherical coordinates the solution is not very different from the solution of the generalized oscillator on $\Lambda^{(3)}$, the only difference being the φ-dependence. Hence we obtain $[\lambda_2 = 2l \pm k_3 + \lambda_F + 1, \nu^2 = M^2\omega^2R^4/\hbar^2 + 1/4, n = 0, \ldots, N_n < (\nu - \lambda_2 - 1)/2]$

$$K^{(V_5)}(\mathbf{u}'', \mathbf{u}'; T)$$

$$= \frac{\mathrm{e}^{-\mathrm{i}\hbar T/2MR^2}}{R^3} \int\limits_{\tau(t')=\tau'}^{\tau(t'')=\tau''} \mathcal{D}\tau(t)\sinh^2\tau \int\limits_{\vartheta(t')=\vartheta'}^{\vartheta(t'')=\vartheta''} \mathcal{D}\vartheta(t)\sin\vartheta \int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''} \mathcal{D}\varphi(t)$$

$$\times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}R^2\left(\dot{\tau}^2 + \sinh^2\tau(\dot{\vartheta}^2 + \sin^2\vartheta\dot{\varphi}^2) - \omega^2\tanh^2\tau\right)\right.\right.$$

$$\left.\left. - \frac{\hbar^2}{2MR^2\sinh^2\tau}\left(\frac{k_3^2 - \frac{1}{4}}{\cos^2\vartheta} + \frac{F(\tan\varphi) - \frac{1}{4}}{\sin^2\vartheta} - \frac{1}{4}\right)\right]dt\right\} \tag{4.7}$$

$$= \int dE_\lambda \sum_{l=0}^{\infty}\left\{\sum_{n=0}^{N_n} \mathrm{e}^{-\mathrm{i}E_N T/\hbar}\Psi_{\lambda l n}^{(V_5)}(\tau'', \vartheta'', \varphi''; R)\Psi_{\lambda l n}^{(V_5)}(\tau', \vartheta', \varphi'; R)\right.$$

$$\left. + \int_0^\infty dp\,\mathrm{e}^{-\mathrm{i}E_p T/\hbar}\Psi_{p l \lambda}^{(V_5)}(\tau'', \vartheta'', \varphi''; R)\Psi_{p l \lambda}^{(V_5)\,*}(\tau', \vartheta', \varphi'; R)\right\}. \tag{4.8}$$

The bound state wave-functions and the energy-spectrum are given by $(N = l + n)$

$$\Psi_{\lambda l n}^{(V_5)}(\tau, \vartheta, \varphi; R) = (\sinh^2\tau\sin\vartheta)^{-1/2} S_n^{(\lambda_2, \nu)}(\tau; R)\phi_l^{(\lambda_F, \pm k_3)}(\vartheta)\phi_\lambda^{(F)}(\varphi), \tag{4.9}$$

$$E_N = -\frac{\hbar^2}{2MR^2}\left[\left(2(N+1) \pm k_3 + \lambda_F - \nu\right)^2 - 1\right] + \frac{M}{2}\omega^2 R^2. \tag{4.10}$$

Table 4.1. Minimally superintegrable potentials on $\Lambda^{(3)}$: analogous of three dimensional flat space

Potential $V(\mathbf{u})$	Coordinate Systems	Observables
$V_5(\mathbf{u}) = \frac{M}{2}\omega^2 R^2 \frac{u_1^2+u_2^2+u_3^2}{u_0^2} + \frac{\hbar^2}{2M}\left(\frac{k_3^2-\frac14}{u_3^2} + \frac{F(u_2/u_1)}{u_1^2+u_2^2}\right)$	<u>Spherical</u> <u>Equidistant-Cylindrical</u> Prolate Elliptic Oblate Elliptic Hyperbolic Cylindrical 2	$I_1 = \frac{1}{2MR^2}(\mathbf{L}^2 - \mathbf{K}^2) + V_5(\mathbf{u}), \qquad I_2 = \frac{1}{2M}L_3^2 + F(\tan\varphi)$ $I_3 = \frac{1}{2M}\mathbf{L}^2 + \frac{\hbar^2}{2M}\left(\frac{k_3^2-\frac14}{\cos^2\vartheta} + \frac{F(\tan\varphi)}{\sin^2\vartheta}\right)$ $I_4 = \frac{1}{2M}(K_1^2+K_2^2-L_3)^2 - \frac{M}{2}\frac{\omega^2}{\cosh^2\tau_2} + \frac{\hbar^2}{2M}\frac{F(\tan\varphi)}{\tau_1\sinh^2\tau_2}$
$V_6(\mathbf{u}) = -\frac{\alpha}{R}\left(\frac{u_0}{\sqrt{u_1^2+u_2^2+u_3^2}} - 1\right) + \frac{\hbar^2}{2M(u_1^2+u_2^2)}\left(\frac{\beta u_3}{\sqrt{u_1^2+u_2^2+u_3^2}} + F\left(\frac{u_2}{u_1}\right)\right)$ $\sinh^2 f = k'^2/k^2$	<u>Spherical</u> Prolate Elliptic II Semi-Hyperbolic	$I_1 = \frac{1}{2MR^2}(\mathbf{L}^2 - \mathbf{K}^2) + V_6(\mathbf{u}), \qquad I_2 = \frac{1}{2M}L_3^2 + F(\tan\varphi)$ $I_3 = \frac{1}{2M}\mathbf{L}^2 + \frac{\hbar^2}{2M}\frac{F(\tan\varphi)+\beta\cos\vartheta}{\sin^2\vartheta}$ $I_4 = \frac{1}{2M}[(\cosh 2f\mathbf{L}^2 - \frac12\sinh 2f(\{K_2,L_1\} - \{K_1,L_2\})] - \alpha R\frac{k^2\mathrm{sn}\alpha\mathrm{cn}\alpha - k'\mathrm{cn}\beta\mathrm{dn}}{k^2\mathrm{cn}^2\alpha + k'^2\mathrm{cn}^2\beta}$ $+\frac{\hbar^2}{4M}\left(\frac{F(\tan(\varphi)+1}{k^2\mathrm{cn}^2\alpha + k'^2\mathrm{cn}^2\beta}\left(\frac{k'^2}{\mathrm{dn}^2\alpha} - \frac{1}{\mathrm{sn}^2\beta}\right) - \beta\frac{k'}{k}\frac{k^2\mathrm{sn}\alpha\mathrm{cn}\alpha + k'\mathrm{cn}\beta\mathrm{dn}\beta}{k^2\mathrm{cn}^2\alpha + k'^2\mathrm{cn}^2\beta}\right)$
$V_7(\mathbf{u}) = F(u_1^2+u_2^2+u_3^2) + \frac{\hbar^2}{2M}\left(-\frac{k_0^2-\frac14}{u_0^2} + \frac{k_1^2-\frac14}{u_1^2} + \frac{k_2^2-\frac14}{u_2^2} + \frac{k_3^2-\frac14}{u_3^2}\right)$	Sphero-Elliptic <u>Spherical</u>	$I_1 = \frac{1}{2MR^2}(\mathbf{L}^2 - \mathbf{K}^2) + V_7(\mathbf{u})$ $I_2 = \frac{1}{2M}L_3^2 + \frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac14}{\cos^2\varphi} + \frac{k_2^2-\frac14}{\sin^2\varphi}\right)$ $I_3 = \frac{1}{2M}\mathbf{L}^2 + \frac{\hbar^2}{2M}\left(\frac{1}{\sin^2\vartheta}\left(\frac{k_1^2-\frac14}{\cos^2\varphi} + \frac{k_2^2-\frac14}{\sin^2\varphi}\right) + \frac{k_3^2-\frac14}{\cos^2\vartheta}\right)$ $I_4 = \frac{1}{4M}(L_1^2 + k'^2L_2^2) - \frac{\hbar^2}{2M(k^2\mathrm{cn}^2\tilde\alpha + k'^2\mathrm{cn}^2\tilde\beta)}\left((k_1^2-\frac14)\left(\frac{1}{\mathrm{sn}^2\tilde\alpha} - \frac{k^2}{\mathrm{dn}^2\tilde\beta}\right)\right.$ $\left.+(k_2^2-\frac14)\left(\frac{k'^2}{\mathrm{cn}^2\tilde\alpha} - \frac{k^2}{\mathrm{dn}^2\tilde\beta}\right) - (k_3^2-\frac14)\left(\frac{k'^2}{\mathrm{dn}^2\tilde\alpha} - \frac{1}{\mathrm{sn}^2\tilde\beta}\right)\right)$
$V_8(\mathbf{u}) = \frac{\hbar^2}{2M}\frac{k_2^2-\frac14}{u_2^2} + \frac{M}{2}\left(\frac{\alpha}{(u_0-u_3)^2} + \omega^2\frac{R^2+4u_1^2+u_2^2}{(u_0-u_3)^4} - \frac{\lambda u_1}{(u_0-u_3)^3}\right)$ $P_{x_i} = \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial x_i},\ i = 1,2$	<u>Horicyclic</u> Semi-Circular-Parabolic	$I_1 = \frac{1}{2MR^2}(\mathbf{L}^2 - \mathbf{K}^2) + V_8(\mathbf{u}), \qquad I_2 = \frac{1}{2M}P_{x_1}^2 + 2M\omega^2x_1^2 - \lambda x_1$ $I_3 = \frac{1}{2M}P_{x_2}^2 + \frac{M}{2}\omega^2x_2^2 + \frac{\hbar^2}{2M}\frac{k_2^2-\frac14}{x_2^2}$ $I_4 = \frac{1}{2M}(\{L_3, K_1+L_2\} + \{K_3, K_2-L_1\})$ $+\frac{\xi^2\eta^2}{2}\frac{2\alpha(\xi^2+\eta^2) + \lambda(\xi^4-\eta^4) + M\omega^2(\xi^6+\eta^6)}{\xi^2+\eta^2}$

The scattering states and continuous spectrum have the form

$$\Psi^{(V_5)}_{pl\lambda}(\tau,\vartheta,\varphi;R) = (\sinh^2\tau\sin\vartheta)^{-1/2} S_p^{(\lambda_2,\nu)}(\tau;R)\phi_l^{(\lambda_F,\pm k_3)}(\vartheta)\phi_\lambda^{(F)}(\varphi), \tag{4.11}$$

$$E_p = \frac{\hbar^2}{2MR^2}(p^2+1) + \frac{M}{2}\omega^2 R^2 . \tag{4.12}$$

Here the wave-functions $\phi_\lambda^{(F)}(\varphi)$ are the eigenfunctions corresponding to the potential term $F(\tan\varphi)$ with eigenvalues $E_\lambda = \hbar^2\lambda^2/2M$, and the $\phi_l^{(\lambda_F,\pm k_3)}(\vartheta)$ are the same as in (3.34) with $\lambda_1 \to \lambda_F$, and the wave-functions $S_n^{(\lambda_2,\nu)}(\tau;R)$ and $S_p^{(\lambda_2,\nu)}(\tau;R)$ are the same as in (3.35) and (3.37), respectively.

Equidistant-Cylindrical Coordinates. $(\lambda_1 = 2l + \lambda_F - \nu + 1, \nu^2 = M^2\omega^2 R^4/\hbar^2 + 1/4)$

$$K^{(V_5)}(\mathrm{u}'',\mathrm{u}';T) = R^{-3}\exp\left[-\frac{\mathrm{i}}{\hbar}T\left(\frac{\hbar^2}{2MR^2} + \frac{M}{2}R^2\omega^2\right)\right]$$

$$\times \int\limits_{\tau_1(t')=\tau_1'}^{\tau_1(t'')=\tau_1''} \mathcal{D}\tau_1(t)\cosh^2\tau_1 \int\limits_{\tau_2(t')=\tau_2'}^{\tau_2(t'')=\tau_2''} \mathcal{D}\tau_2(t)\sinh\tau_2 \int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''} \mathcal{D}\varphi(t)$$

$$\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}R^2\left(\dot\tau_1^2 + \cosh^2\tau_1(\dot\tau_2^2 + \sinh^2\tau_1\dot\varphi^2) + \frac{\omega^2}{\cosh^2\tau_1\cosh^2\tau_2}\right)\right.\right.$$

$$\left.\left.-\frac{\hbar^2}{2MR^2}\left(\frac{1}{\cosh^2\tau_1}\left(\frac{F(\tan\varphi)-\frac14}{\sinh^2\tau_2}+\frac14\right) + \frac{k_3^2-\frac14}{\sinh^2\tau_1}\right)\right]dt\right\} \tag{4.13}$$

$$= \int dE_\lambda\left\{\sum_{l=0}^{N_l}\left[\sum_{n=0}^{N_n} \mathrm{e}^{-\mathrm{i}\hbar E_N T/\hbar}\Psi^{(V_5)}_{\lambda ln}(\tau_1'',\tau_2'',\varphi'';R)\Psi^{(V_5)}_{\lambda ln}(\tau_1',\tau_2',\varphi';R)\right.\right.$$

$$\left.+\int_0^\infty dp\,\mathrm{e}^{-\mathrm{i}E_pT/\hbar}\Psi^{(V_5)}_{pl\lambda}(\tau_1'',\tau_2'',\varphi'';R)\Psi^{(V_5)\,*}_{pl\lambda}(\tau_1',\tau_2',\varphi';R)\right]$$

$$\left.+\int_0^\infty dk\int_0^\infty dp\,\mathrm{e}^{-\mathrm{i}E_pT/\hbar}\Psi^{(V_5)}_{pk\lambda}(\tau_1'',\tau_2'',\varphi'';R)\Psi^{(V_5)\,*}_{pk\lambda}(\tau_1',\tau_2',\varphi';R)\right\}. \tag{4.14}$$

Again we have similar features as for the general oscillator, hence we have one set of bound state wave-functions, and two sets of scattering states. They are given by $(n = 0,\dots,N_n < (\lambda_2 - k_3 - 1)/2, l = 0,\dots,N_l < (\nu - \lambda_F - 1)/2)$

$$\Psi^{(V_5)}_{\lambda ln}(\tau_1,\tau_2,\varphi;R)$$
$$= (\cosh^2\tau_1\sinh\tau_2)^{-1/2} S_n^{(\pm k_3,\lambda_2)}(\tau_1;R)\psi_l^{(\lambda_F,\nu)}(\tau_2)\phi_\lambda^{(F)}(\varphi)\ , \tag{4.15}$$

$$\Psi_{pk\lambda}^{(V_5)}(\tau_1,\tau_2,\varphi;R) = (\cosh^2\tau_1\sinh\tau_2)^{-1/2} S_p^{(\pm k_3,\lambda_2)}(\tau_1;R)\psi_l^{(\lambda_F,\nu)}(\tau_2)\phi_\lambda^{(F)}(\varphi) \;, \tag{4.16}$$

$$\Psi_{pl\lambda}^{(V_5)}(\tau_1,\tau_2,\varphi;R) = (\cosh^2\tau_1\sinh\tau_2)^{-1/2} S_p^{(\pm k_3,\mathrm{i}k)}(\tau_1;R)\psi_k^{(\lambda_F,\nu)}(\tau_2)\phi_\lambda^{(F)}(\varphi) \;. \tag{4.17}$$

The wave-functions $\psi_{l,k}^{(\lambda_F,\nu)}(\tau_2)$ are the same as in (3.20) and (3.30) with $\pm k_1 \to \lambda_F$ and $m \to l$, respectively. The energy-spectra E_N and E_p are the same as in the previous paragraph.

4.1.2. Hartmann Potential

The next potential represents the analogue of the Hartmann potential V_6 in $\mathrm{I\!R}^3$ [33]. We consider

$$V_6(\mathrm{u}) = -\frac{\alpha}{R}\left(\frac{u_0}{\sqrt{u_1^2+u_2^2+u_3^2}} - 1\right) + \frac{\hbar^2}{2M(u_1^2+u_2^2)}\left(\frac{\beta u_3}{\sqrt{u_1^2+u_2^2+u_3^2}} + F\left(\frac{u_2}{u_1}\right)\right), \tag{4.18}$$

which in the two separating coordinate systems has the form (φ with appropriate range)

Spherical $(\tau > 0, \vartheta \in (0,\pi/2))$:

$$V_6(\mathrm{u}) = -\frac{\alpha}{R}(\coth\tau - 1) + \frac{\hbar^2}{2MR^2\sinh^2\tau}\,\frac{F(\tan\varphi)+\beta\cos\vartheta}{\sin^2\vartheta} \tag{4.19}$$

Prolate Elliptic II $(\alpha \in (\mathrm{i}K', \mathrm{i}K'+K), \beta \in (0,K'))$:

$$-\frac{\alpha}{R}\left(\frac{k^2\,\mathrm{sn}\alpha\,\mathrm{cn}\alpha - k'\,\mathrm{cn}\beta\,\mathrm{dn}\beta}{k^2\,\mathrm{cn}^2\alpha + k'^2\,\mathrm{cn}^2\beta} - 1\right) + \frac{\hbar^2}{4MR^2}\left(\frac{F(\tan\varphi)+1}{k^2\,\mathrm{cn}^2\alpha + k'^2\,\mathrm{cn}^2\beta}\left(\frac{k'^2}{\mathrm{dn}^2\alpha} - \frac{1}{\mathrm{sn}^2\beta}\right) - \beta\frac{k'}{k}\,\frac{k^2\,\mathrm{sn}\alpha\,\mathrm{cn}\alpha + k'\,\mathrm{cn}\beta\,\mathrm{dn}\beta}{k^2\,\mathrm{cn}^2\alpha + k'^2\,\mathrm{cn}^2\beta}\right)\Bigg] \;. \tag{4.20}$$

We treat only the spherical case with $F(\tan\varphi) = \gamma$. Then we obtain together with $\gamma \geq |\beta|$, $\lambda_\pm^2 = n^2 + \gamma \pm \beta, \lambda_2 = m + (\lambda_+ + \lambda_- + 1)/2, \varphi \in [0,2\pi)$:

$$K^{(V_6)}(\tau'',\tau',\vartheta'',\vartheta',\varphi'',\varphi';T) = R^{-3}\exp\left[-\frac{\mathrm{i}}{\hbar}T\left(\frac{\hbar^2}{2MR^2}+\frac{\alpha}{R}\right)\right]$$

$$\times \int\limits_{\tau(t')=\tau'}^{\tau(t'')=\tau''} \mathcal{D}\tau(t)\sinh^2\tau \int\limits_{\vartheta(t')=\vartheta'}^{\vartheta(t'')=\vartheta''} \mathcal{D}\vartheta(t)\sin\vartheta \int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''} \mathcal{D}\varphi(t)$$

$$\times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}R^2\left(\dot\tau^2+\sinh^2\tau(\dot\vartheta^2+\sin^2\vartheta\dot\varphi^2)\right)+\frac{\alpha}{R}\coth\tau\right.\right.$$

$$\left.\left.-\frac{\hbar^2}{8MR^2\sinh^2\tau}\left(\frac{\gamma+\beta-\frac14}{\sin^2(\vartheta/2)}+\frac{\gamma-\beta-\frac14}{\cos^2(\vartheta/2)}-\frac14\right)\right]dt\right\} \tag{4.21}$$

$$=\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\left\{\sum_{n=0}^{N_n}\mathrm{e}^{-\mathrm{i}E_NT/\hbar}\Psi_{nlm}^{(V_6)}(\tau'',\vartheta'',\varphi'';R)\Psi_{nlm}^{(V_6)\,*}(\tau',\vartheta',\varphi';R)\right.$$

$$\left.+\int_0^\infty dp\,\mathrm{e}^{-\mathrm{i}E_pT/\hbar}\Psi_{plm}^{(V_6)}(\tau'',\vartheta'',\varphi'';R)\Psi_{plm}^{(V_6)\,*}(\tau',\vartheta',\varphi';R)\right\} . \tag{4.22}$$

The path-integration in ϑ is of the Pöschl–Teller type, whereas the path integration in τ is essentially the same as for the Coulomb potential. Therefore the wave-functions for the bound and continuous spectrum are given by ($n = 0, \dots, N_n < \sqrt{R/a} - \lambda_2 - 1/2$, $a = \hbar^2/M\alpha$ is the Bohr radius)

$$\Psi_{nlm}^{(V_6)}(\tau,\vartheta,\varphi;R) = (\sinh^2\tau\sin\vartheta)^{-1/2}S_N(\tau;R)\phi_l^{(\lambda_+,\lambda_-)}(\vartheta)\frac{\mathrm{e}^{\mathrm{i}m\varphi}}{\sqrt{2\pi}} , \tag{4.23}$$

$$\Psi_{plm}^{(V_6)}(\tau,\vartheta,\varphi;R) = (\sinh^2\tau\sin\vartheta)^{-1/2}S_p(\tau;R)\phi_l^{(\lambda_+,\lambda_-)}(\vartheta)\frac{\mathrm{e}^{\mathrm{i}m\varphi}}{\sqrt{2\pi}} , \tag{4.24}$$

$$\phi_l^{(\lambda_+,\lambda_-)}(\vartheta) = \left[(2l+\lambda_++\lambda_-+1)\frac{l!\Gamma(l+\lambda_++\lambda_-+1)}{\Gamma(l+\lambda_++1)\Gamma(l+\lambda_-+1)}\right]^{1/2}$$

$$\times\left(\sin\frac{\vartheta}{2}\right)^{1/2+\lambda_+}\left(\cos\frac{\vartheta}{2}\right)^{1/2+\lambda_-}P_l^{(\lambda_+,\lambda_-)}(\cos\vartheta) , \tag{4.25}$$

with the Coulomb wave-functions $S_N(\tau;R), S_p(\tau;R)$ as in (3.64, 3.66) and the energy-spectra (3.65, 3.27), respectively. In the case when $R\to\infty$ the flat space limit is recovered [33,56,57].

4.1.3. Generalized Radial Potential

We consider the potential ($k_{0,1,2,3} > 0$)

$$V_7(\mathrm{u}) = F(u_1^2+u_2^2+u_3^2)+\frac{\hbar^2}{2M}\left(-\frac{k_0^2-\frac14}{u_0^2}+\frac{k_1^2-\frac14}{u_1^2}+\frac{k_2^2-\frac14}{u_2^2}+\frac{k_3^2-\frac14}{u_3^2}\right) , \tag{4.26}$$

which in the two separating coordinate systems has the form

Spherical $\left(\tau > 0, \vartheta \in (0,\pi/2), \varphi \in (0,\pi/2)\right)$:

$$V_7(\mathrm{u}) = F(\sinh^2\tau) + \frac{\hbar^2}{2MR^2}\left[\frac{1}{\sinh^2\tau}\left(\frac{1}{\sin^2\vartheta}\left(\frac{k_1^2-\frac14}{\cos^2\varphi} + \frac{k_2^2-\frac14}{\sin^2\varphi}\right) + \frac{k_3^2-\frac14}{\cos^2\vartheta}\right) - \frac{k_0^2-\frac14}{\cosh^2\tau}\right] \tag{4.27}$$

Sphero-Elliptic $\left(\tau > 0, \tilde\alpha \in (0,K), \tilde\beta \in (0,K')\right)$:

$$= F(\sinh^2\tau) + \frac{\hbar^2}{2MR^2}\left(\frac{1}{\sinh^2\tau}\left(\frac{k_1^2-\frac14}{\mathrm{sn}^2\tilde\alpha\,\mathrm{dn}^2\tilde\beta} + \frac{k_2^2-\frac14}{\mathrm{cn}^2\tilde\alpha\,\mathrm{cn}^2\tilde\beta} + \frac{k_3^2-\frac14}{\mathrm{dn}^2\tilde\alpha\,\mathrm{sn}^2\tilde\beta}\right) - \frac{k_0^2-\frac14}{\cosh^2\tau}\right) . \tag{4.28}$$

This potential is the analogue of the minimally superintegrable potential $V_1(\mathbf{x})$ in $\mathrm{I\!R}^3$ [33]. We have the following two path integral representations

$K^{(V_7)}(\mathrm{u}'', \mathrm{u}'; T)$

Sphero-Elliptic, $\lambda_2 = 2(m+l) \pm k_3 + \lambda_1 + 2$ (l, h as in subsection 3.1.2):

$$= \frac{\mathrm{e}^{-\mathrm{i}\hbar T/2MR^2}}{R^3} \times \int\limits_{\tau(t')=\tau'}^{\tau(t'')=\tau''} \mathcal{D}\tau(t)\sinh^2\tau \int\limits_{\tilde\alpha(t')=\tilde\alpha'}^{\tilde\alpha(t'')=\tilde\alpha''} \mathcal{D}\tilde\alpha(t) \int\limits_{\tilde\beta(t')=\tilde\beta'}^{\tilde\beta(t'')=\tilde\beta''} \mathcal{D}\tilde\beta(t)(k^2\,\mathrm{cn}^2\tilde\alpha + k'^2\,\mathrm{cn}^2\tilde\beta)$$

$$\times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}R^2\left(\dot\tau^2 + \sinh^2\tau(k^2\,\mathrm{cn}^2\tilde\alpha + k'^2\,\mathrm{cn}^2\tilde\beta)(\dot{\tilde\alpha}^2 + \dot{\tilde\beta}^2)\right) - F(\tau) - \frac{\hbar^2}{2MR^2}\left(-\frac{k_0^2-\frac14}{\cosh^2\tau} + \frac{1}{\sinh^2\tau}\left(\frac{k_1^2-\frac14}{\mathrm{sn}^2\tilde\alpha\,\mathrm{dn}^2\tilde\beta} + \frac{k_2^2-\frac14}{\mathrm{cn}^2\tilde\alpha\,\mathrm{cn}^2\tilde\beta} + \frac{k_3^2-\frac14}{\mathrm{dn}^2\tilde\alpha\,\mathrm{sn}^2\tilde\beta}\right)\right)\right]dt\right\} \tag{4.29}$$

$$= (R^2\sinh^2\tau'\sinh^2\tau''\,\mathrm{sn}\tilde\alpha'\,\mathrm{cn}\tilde\alpha'\,\mathrm{dn}\tilde\alpha'\,\mathrm{sn}\tilde\beta'\,\mathrm{cn}\tilde\beta'\,\mathrm{dn}\tilde\beta' \times \mathrm{sn}\tilde\alpha''\,\mathrm{cn}\tilde\alpha''\,\mathrm{dn}\tilde\alpha''\,\mathrm{sn}\tilde\beta''\,\mathrm{cn}\tilde\beta''\,\mathrm{dn}\tilde\beta'')^{-1/2} \times \sum_{lm} \Xi_{lm}^{(\pm k_1,\pm k_2,\pm k_3)}(\tilde\alpha'',\tilde\beta')\Xi_{lm}^{(\pm k_1,\pm k_2,\pm k_3)\,*}(\tilde\alpha',\tilde\beta')K_{lm}^{(V_7)}(\tau'',\tau';T) \ , \tag{4.30}$$

Spherical, $\lambda_1 = 2m \mp k_1 \mp k_2 + 1, \lambda_2 = 2l \mp k_3 + \lambda_1 + 1$:

$$
\begin{aligned}
&= \frac{e^{-i\hbar T/2MR^2}}{R^3} \int\limits_{\tau(t')=\tau'}^{\tau(t'')=\tau''} \mathcal{D}\tau(t)\sinh^2\tau \int\limits_{\vartheta(t')=\vartheta'}^{\vartheta(t'')=\vartheta''} \mathcal{D}\vartheta(t)\sin\vartheta \int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''} \mathcal{D}\varphi(t) \\
&\times \exp\Bigg\{\frac{i}{\hbar}\int_{t'}^{t''}\Bigg[\frac{M}{2}R^2\Big(\dot\tau^2+\sinh^2\tau(\dot\vartheta^2+\sin^2\vartheta\dot\varphi^2)\Big)-F(\tau) \\
&\quad -\frac{\hbar^2}{2MR^2}\Bigg(\frac{1}{\sinh^2\tau}\Bigg(\frac{1}{\sin^2\vartheta}\Bigg(\frac{k_1^2-\frac14}{\cos^2\varphi}+\frac{k_2^2-\frac14}{\sin^2\varphi}-\frac14\Bigg)+\frac{k_3^2-\frac14}{\cos^2\vartheta}-\frac14\Bigg) \\
&\quad -\frac{k_0^2-\frac14}{\cosh^2\tau}\Bigg)\Bigg]dt\Bigg\} && (4.31)\\
&= \frac{e^{-i\hbar T/2MR^2}}{R}(\sinh^2\tau'\sinh^2\tau''\sin\vartheta'\sin\vartheta'')^{-1/2} \\
&\times \sum_{m=0}^{\infty}\phi_m^{(\pm k_2,\pm k_1)}(\varphi'')\phi_m^{(\pm k_2,\pm k_1)}(\varphi') \\
&\times \sum_{l=0}^{\infty}\phi_l^{(\lambda_1,\pm k_3)}(\vartheta'')\phi_l^{(\lambda_1,\pm k_3)}(\vartheta')K_{lm}^{(V_7)}(\tau'',\tau';T)\ , && (4.32)
\end{aligned}
$$

with the remaining path integral $K_{lm}^{(V_7)}(T)$

$$
\begin{aligned}
K_{lm}^{(V_7)}(\tau'',\tau';T) &= \int\limits_{\tau(t')=\tau'}^{\tau(t'')=\tau''} \mathcal{D}\tau(t) \\
&\times \exp\Bigg\{\frac{i}{\hbar}\int_{t'}^{t''}\Bigg[\frac{M}{2}R^2\dot\tau^2-F(\tau)-\frac{\hbar^2}{2MR^2}\Bigg(\frac{\lambda_2^2-\frac14}{\sinh^2\tau}-\frac{k_0^2-\frac14}{\cosh^2\tau}\Bigg)\Bigg]dt\Bigg\}. \quad (4.33)
\end{aligned}
$$

The wave-functions $\phi_m^{(\pm k_2,\pm k_1)}(\varphi)$ and $\phi_l^{(\lambda_1,\pm k_3)}(\vartheta)$ are the same as in (3.21) and (3.34), respectively. This path integral cannot be further specified until $F(\tau)\equiv F(\sinh^2\tau)$ is known. The special case $F\equiv 0$ is trivial.

4.1.4. Analogue of the Holt Potential

The potential V_8 can be considered as an analogue of the minimally superintegrable Holt potential $V_6(\mathbf{x})$ in $\mathbb{R}^3$ [33, 44] $(\alpha,\lambda,\omega>0)$

$$
V_8(\mathrm{u}) = \frac{\hbar^2}{2M}\frac{k_2^2-\frac14}{u_2^2}+\frac{\alpha}{(u_0-u_3)^2}+\frac{M}{2}\omega^2\frac{R^2+4u_1^2+u_2^2}{(u_0-u_3)^4}-\frac{\lambda u_1}{(u_0-u_3)^2}\ . \quad (4.34)
$$

In the two separating coordinate systems it has the form

Horicyclic $\left(x_2, y > 0, x_1 \in \mathbb{R}\right)$:

$$V_8(\mathrm{u}) = \frac{y^2}{R^2}\left[\alpha + \frac{M}{2}\omega^2(4x_1^2 + x_2^2 + y^2) - \lambda x_1\right] + y^2\frac{\hbar^2}{2MR^2}\frac{k_2^2 - \frac{1}{4}}{x_2^2} \tag{4.35}$$

Semi-Circular-Parabolic $\left(\xi, \eta, \varrho > 0\right)$:

$$= \frac{\xi^2\eta^2}{R^2}\left[\frac{\alpha(\xi^2+\eta^2) - \frac{\lambda}{2}(\eta^4 - \xi^4) + \frac{M}{2}\omega^2(\xi^6+\eta^6)}{\xi^2+\eta^2} + \left(\frac{M}{2}\omega^2\varrho^2 + \frac{\hbar^2}{2M}\frac{k_2^2-\frac{1}{4}}{\varrho^2}\right)\right] . \tag{4.36}$$

The effect of the x_2- and ϱ-path integration in both cases $(x_2, \varrho > 0)$ is that in separating off the corresponding variable, the quantity α is shifted by the additional quantum numbers. The resulting path integrals in the variables (y, x_1) and (ξ, η) separate, however, only the former can be evaluated. Indeed, almost the same path integral problem we have already solved in [35]. The solution in horicyclic coordinates then has the following structure $(z = x_1 - \lambda/4M\omega^2)$

$$K^{(V_8)}(\mathrm{u}'', \mathrm{u}'; T) = \frac{1}{R^3}\int\limits_{y(t')=y'}^{y(t'')=y''}\frac{\mathcal{D}y(t)}{y^3}\int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}x(t)$$
$$\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}R^2\frac{\dot{\mathbf{x}}^2+\dot{y}^2}{y^2} - \frac{y^2}{R^2}\left(\alpha + \frac{M}{2}\omega^2(4x_1^2+x_2^2+y^2) - \lambda x_1\right) - \frac{y^2\hbar^2}{2MR^2}\frac{k_2^2-\frac{1}{4}}{x_2^2}\right]dt\right\} \tag{4.37}$$
$$= \frac{2M\omega}{\hbar R}\sqrt{x_2'x_2''}\sum_{l\in\mathbb{N}_0}\frac{l!}{\Gamma(l\pm k_2+1)}\left(\frac{M\omega}{\hbar}x_2'x_2''\right)^{\pm k_2}$$
$$\times\exp\left(-\frac{M\omega}{2\hbar}(x_2'^2 + x_2''^2)\right)L_l^{(\pm k_2)}\left(\frac{M\omega}{\hbar}x_2'^2\right)L_n^{(\pm k_2)}\left(\frac{M\omega}{\hbar}x_2''^2\right)$$
$$\times\sum_{m\in\mathbb{N}_0}\left(\frac{2M\omega}{\pi\hbar}\right)^{1/2}\frac{1}{2^m m!}H_m\left(\sqrt{\frac{2M\omega}{\hbar}}z'\right)H_m\left(\sqrt{\frac{2M\omega}{\hbar}}z''\right)$$
$$\times\exp\left[-\frac{M\omega}{\hbar}(z'^2+z''^2)\right]$$

$$\times \int_{y(t')=y'}^{y(t'')=y''} \frac{\mathcal{D}y(t)}{y} \exp\left[\frac{iM}{2\hbar}\int_{t'}^{t''}\left(R^2\frac{\dot{x}^2+\dot{y}^2}{y^2}-\frac{y^2}{R^2}\Big(E_{\alpha,\omega,\lambda}+\omega^2y^2\Big)\right)dt\right] , \tag{4.38}$$

with the quantity $E_{\alpha,\omega,\lambda}$ given by

$$E_{\alpha,\omega,\lambda} = \alpha + \hbar\omega(2m + 2l \pm k_2 + 2) - \frac{\lambda^2}{8M\omega^2} . \tag{4.39}$$

A path integral like this was calculated in [26], and we must distinguish two cases, first where $E_{\alpha,\omega,\lambda} > 0$, and second $E_{\alpha,\omega,\lambda} < 0$. In the first case only a continuous spectrum occurs, whereas in the second one bound states can exist with the number of levels given by $n = 0, 1, \ldots, N_n = [E_{\alpha,\omega,\lambda}/2\hbar\omega - 1/2]$. According to [35] we obtain therefore the following path integral solution for $V_8(\mathrm{u})$ in horicyclic coordinates $\big(\nu = -\mathrm{i}\sqrt{2MR^2E/\hbar^2 - 1/4}\big)$

$$\begin{aligned} K^{(V_8)}(\mathrm{u}'',\mathrm{u}';T) &= \frac{1}{R}\sum_{n=0}^{\infty}\psi_n(x_2')\psi_n(x_2'')\sum_{m=0}^{\infty}\psi_m(x_1')\psi_m(x_1'')\int_{\mathbb{R}}\frac{dE}{2\pi\mathrm{i}}\mathrm{e}^{-\mathrm{i}ET/\hbar} \\ &\quad\times\frac{\Gamma[\frac{1}{2}(1+\nu+E_{\alpha,\omega,\lambda}/\hbar\omega)]}{\sqrt{y'y''}\,\hbar\omega\Gamma(1+\nu)} \\ &\quad\times W_{-E_{\alpha,\omega,\lambda}/2\hbar\omega,\nu/2}\left(\frac{M\omega}{\hbar}y_>^2\right)M_{-E_{\alpha,\omega,\lambda}/2\hbar\omega,\nu/2}\left(\frac{M\omega}{\hbar}y_<^2\right) \end{aligned} \tag{4.40}$$

$$\begin{aligned} &= \sum_{l,m=0}^{\infty}\left[\sum_{n=0}^{N_n}\Psi_{nlm}^{(V_8)}(x_1'',x_2'',y'';R)\Psi_{nlm}^{(V_8)}(x_1',x_2',y';R)\,\mathrm{e}^{-\mathrm{i}E_nT/\hbar}\right. \\ &\quad\left.+\int_0^\infty dp\Psi_{plm}^{(V_8)}(x_1'',x_2'',y'';R)\Psi_{plm}^{(V_8)\,*}(x_1',x_2',y';R)\,\mathrm{e}^{-\mathrm{i}E_pT/\hbar}\right] . \end{aligned} \tag{4.41}$$

The bound state wave-functions have the form

$$\Psi_{nmq}^{(V_8)}(x_1,x_2,y;R) = \psi_n(y;R)\psi_m(x_1)\psi_l(x_2), \tag{4.42}$$

where

$$\begin{aligned} \psi_n(y;R) &= \sqrt{\frac{2n!(|E_{\alpha,\omega,\lambda}|/\hbar\omega - 2n - 1)y}{R^3\Gamma(|E_{\alpha,\omega,\lambda}|/\hbar\omega - n)}}\left(\frac{M\omega}{\hbar}y^2\right)^{|E_{\alpha,\omega,\lambda}|/2\hbar\omega - n - 1/2} \\ &\quad\times\exp\left(-\frac{M\omega}{2\hbar}y^2\right)L_n^{(|E_{\alpha,\omega,\lambda}|/\hbar\omega - 2n - 1)}\left(\frac{M\omega}{\hbar}y^2\right) , \end{aligned} \tag{4.43}$$

$$\psi_m(x_1) = \left(\frac{2M\omega}{\pi\hbar 2^{2m}(m!)^2}\right)^{1/4} H_m\left(\sqrt{\frac{2M\omega}{\hbar}}\left(x_1 - \frac{\lambda}{8\omega^2}\right)\right)$$
$$\times \exp\left(-\frac{M\omega}{\hbar}\left(x_1 - \frac{\lambda}{8\omega^2}\right)^2\right) , \qquad (4.44)$$

$$\psi_l(x_2) = \sqrt{\frac{2M\omega}{\hbar}\frac{l!}{\Gamma(l \pm k_2 + 1)}} x_2 \left(\frac{M\omega}{\hbar}x_2^2\right)^{\pm k_2/2}$$
$$\times \exp\left(-\frac{M\omega}{2\hbar}x_2^2\right) L_l^{(\pm k_2)}\left(\frac{M\omega}{\hbar}x_2^2\right) , \qquad (4.45)$$

with the discrete energy-spectrum given by

$$E_n = \frac{\hbar^2}{8MR^2} - \frac{\hbar^2}{2MR^2}\left(\frac{|E_{\alpha,\omega,\lambda}|}{\hbar\omega} - 2n - 1\right)^2 . \qquad (4.46)$$

The continuous wave-functions and the energy-spectrum have the form

$$\Psi_{plm}^{(V_8)}(x_1, x_2, y; R) = \psi_p(y; R)\psi_m(x_1)\psi_l(x_2) \qquad (4.47)$$

$$\psi_p(y; R) = \sqrt{\frac{\hbar}{M\omega}\frac{p\sinh\pi p}{2\pi^2 R^3 y}}\,\Gamma\left[\frac{1}{2}\left(1 + \mathrm{i}p + \frac{E_{\alpha,\omega,\lambda}}{\hbar\omega}\right)\right]$$
$$\times W_{-E_{\alpha,\omega,\lambda}/2\hbar\omega, \mathrm{i}p/2}\left(\frac{M\omega}{\hbar}y^2\right) , \qquad (4.48)$$

$$E_p = \frac{\hbar^2}{2MR^2}\left(p^2 + \frac{1}{4}\right) , \qquad (4.49)$$

and the $\psi_m(x_1), \psi_l(x_2)$ as in (4.44,4.45).

4.2. Minimally Superintegrable Potentials from the Group Chains SO(3,1) ⊃ $E(2)$, SO(3,1) ⊃ SO(3) and SO(3,1) ⊃ SO(2,1)

Whereas in the previous section we have presented the minimally superintegrable potentials which are the analogues of the $\mathbb{R}^3$ case, there are several potentials which emerge from the group structure of the three-dimensional hyperboloid. They are:

1. There are four potentials which emerge from SO(3,1) ⊃ $E(2)$, i.e., the four maximally superintegrable potentials in $\mathbb{R}^2$ are contained as minimally superintegrable on $\Lambda^{(3)}$. The four potentials in $\mathbb{R}^2$ are the oscillator, the Holt potential, the Coulomb potential, and a modified Coulomb potential.

2. There are two potentials which emerge form the group chain SO(3,1) ⊃ SO(3), i.e., the Higgs oscillator and the Coulomb potential on the two-

Table 4.2. Minimally superintegrable sotentials on $\Lambda^{(3)}$ from the Group Chain $SO(3,1) \supset E(2)$

Potential $V(\mathbf{u})$	Coordinate Systems	Observables
$V_9(\mathbf{u}) = \frac{M}{2}\omega^2 \frac{u_1^2+u_2^2}{(u_0-u_3)^4} + F(u_0-u_3) + \frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac14}{u_1^2} + \frac{k_2^2-\frac14}{u_2^2}\right)$ $P_{x_i} = \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial x_i},\ i=1,2$	Horicyclic Horicyclic-Cylindrical Horicyclic-Elliptic	$I_1 = \frac{1}{2MR^2}(\mathbf{L}^2 - \mathbf{K}^2) + V_8(\mathbf{u})$ $I_2 = \frac{1}{2M}P_{x_1}^2 + \frac{M}{2}\omega^2 x_1^2 + \frac{\hbar^2}{2M}\frac{k_1^2-\frac14}{x_1^2}$ $I_3 = \frac{1}{2M}P_{x_2}^2 + \frac{M}{2}\omega^2 x_2^2 + \frac{\hbar^2}{2M}\frac{k_2^2-\frac14}{x_2^2}$ $I_4 = \frac{1}{2M}L_3^2 + \frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac14}{\cos^2\varphi} + \frac{k_2^2-\frac14}{\sin^2\varphi}\right)$
$V_{10}(\mathbf{u}) = \frac{M}{2}\omega^2 \frac{4u_1^2+u_2^2}{(u_0-u_3)^4} + F(u_0-u_3) + \frac{k_1 u_1}{(u_0-u_3)^3} + \frac{\hbar^2}{2M}\frac{k_2^2-\frac14}{u_2^2}$ $P_{x_i} = \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial x_i},\ i=1,2$	Horicyclic Horicyclic-Parabolic	$I_1 = \frac{1}{2MR^2}(\mathbf{L}^2 - \mathbf{K}^2) + V_9(\mathbf{u})$ $I_2 = \frac{1}{2M}P_{x_1}^2 + 2M\omega^2 x_1^2 + k_1 x_1$ $I_3 = \frac{1}{2M}P_{x_2}^2 + \frac{M}{2}\omega^2 x_2^2 + \frac{\hbar^2}{2M}\frac{k_2^2-\frac14}{x_2^2}$ $I_4 = \frac{1}{4M}\{L_3, K_1 + L_2\} + \frac{M\omega^2(\xi^6+\eta^6) + 2k_1(\xi^4-\eta^4) + \hbar^2(k_2^2-\frac14)(1/\xi^2+1/\eta^2)/M}{2(\xi^2+\eta^2)}$
$V_{11}(\mathbf{u}) = -\frac{\alpha}{\sqrt{u_1^2+u_2^2}}\frac{1}{u_0-u_3} + F(u_0-u_3) + \frac{R^2\hbar^2}{4M}\frac{1}{(u_0-u_3)^2}\frac{1}{\sqrt{u_1^2+u_2^2}} \times \left(\frac{k_1^2-\frac14}{\sqrt{u_1^2+u_2^2}+u_1} + \frac{k_2^2-\frac14}{\sqrt{u_1^2+u_2^2}-u_1}\right)$	Horicyclic-Cylindrical Horicyclic-Elliptic II Horicyclic-Parabolic	$I_1 = \frac{1}{2MR^2}(\mathbf{L}^2 - \mathbf{K}^2) + V_{10}(\mathbf{u})$ $I_2 = \frac{1}{2M}L_3^2 + \frac{\hbar^2}{8M}\left(\frac{k_1^2-\frac14}{\cos^2(\varphi/2)} + \frac{k_2^2-\frac14}{\sin^2(\varphi/2)}\right)$ $I_3 = \frac{1}{2M}\Big[(K_1+L_2)^2 + (K_2-L_1)^2\Big] - \frac{\alpha}{\varrho} + \frac{\hbar^2}{8M\varrho^2}\left(\frac{k_1^2-\frac14}{\cos^2(\varphi/2)} + \frac{k_2^2-\frac14}{\sin^2(\varphi/2)}\right)$ $I_4 = \frac{1}{4M}\{L_3, P_2\} + \frac{1}{\xi\eta}\Big[-\alpha(\xi-\eta) + (\tfrac12 - 2k_1^2)\frac{\eta}{\xi} + (2k_2^2 - \tfrac12)\frac{\xi}{\eta}\Big]$
$V_{12}(\mathbf{u}) = F(u_0-u_3) - \frac{\alpha}{\sqrt{u_1^2+u_2^2}}\frac{1}{u_0-u_3} + \frac{\beta_1\sqrt{\sqrt{u_1^2+u_2^2}+u_1} + \beta_2\sqrt{\sqrt{u_1^2+u_2^2}-u_1}}{2(u_0-u_3)^{3/2}\sqrt{u_1^2+u_2^2}}$ $P_y = \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial y}$	Horicyclic-Mutually-Parabolic	$I_1 = \frac{1}{2MR^2}(\mathbf{L}^2 - \mathbf{K}^2) + V_{11}(\mathbf{u})$ $I_2 = \frac{1}{2M}P_y^2 + F(1/y^2)$ $I_3 = \frac{1}{4M}\{L_3, P_1\} - \frac{\alpha(\lambda-\mu) + \beta_1\mu\sqrt{\lambda} - \beta_2\lambda\sqrt{\mu}}{\lambda+\mu}$ $I_4 = \frac{1}{4M}\{L_3, P_2\} - \frac{\alpha(\xi-\eta) + (\beta_1+\beta_2)\eta\sqrt{\xi/2} - (\beta_1+\beta_2)\xi\sqrt{\eta/2}}{\xi+\eta}$

Table 4.3. Minimally superintegrable potentials on $\Lambda^{(3)}$ from the Group Chain $SO(3,1) \supset SO(3)$

Potential $V(\mathbf{u})$	Coordinate Systems	Observables
$V_7'(\mathbf{u}) = \frac{M}{2}\frac{u_1^2+u_2^2}{u_1^2+u_2^2+u_3^2}\frac{\omega^2}{u_3^2} + \frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac14}{u_1^2} + \frac{k_2^2-\frac14}{u_2^2}\right) + F(u_0)$ $\lambda^2 = \frac{M^2\omega^2}{\hbar^2}R^4 + \frac14$	Sphero-Elliptic Spherical	$I_1 = \frac{1}{2MR^2}(\mathbf{L}^2 - \mathbf{K}^2) + V_7'(\mathbf{u})$ $I_2 = \frac{1}{2M}\mathbf{L}^2 + \frac{\hbar^2}{2M}\left(\frac{1}{\sin^2\vartheta}\left(\frac{k_1^2-\frac14}{\cos^2\varphi} + \frac{k_2^2-\frac14}{\sin^2\varphi}\right) + \frac{k_3^2-\frac14}{\cos^2\vartheta}\right)$ $I_3 = \frac{1}{2M}L_3^2 + \left(\frac{k_1^2-\frac14}{\cos^2\varphi} + \frac{k_2^2-\frac14}{\sin^2\varphi}\right)$ $I_4 = \frac{1}{2M}(L_1^2 + k'^2L_2^2) - \frac{\hbar^2}{2M(k^2\text{cn}^2\tilde\alpha + k'^2\text{cn}^2\tilde\beta)}\Bigg[(k_1^2-\frac14)\left(\frac{1}{\text{sn}^2\tilde\alpha} - \frac{k^2}{\text{dn}^2\tilde\beta}\right)$ $+(k_2^2-\frac14)\left(\frac{k'^2}{\text{cn}^2\tilde\alpha} - \frac{k^2}{\text{dn}^2\tilde\beta}\right) - (\lambda^2-\frac14)\left(\frac{k'^2}{\text{dn}^2\tilde\alpha} - \frac{1}{\text{sn}^2\tilde\beta}\right)\Bigg]$
$V_{13}(\mathbf{u}) = -\frac{\alpha}{u_1^2+u_2^2+u_3^2}\frac{u_3}{\sqrt{u_1^2+u_2^2}} + F(u_0)$ $+\frac{\hbar^2}{4M\sqrt{u_1^2+u_2^2}}\left(\frac{k_1^2-\frac14}{\sqrt{u_1^2+u_2^2}+u_1} + \frac{k_2^2-\frac14}{\sqrt{u_1^2+u_2^2}-u_1}\right)$	Sphero-Elliptic* Spherical	$I_1 = \frac{1}{2MR^2}(\mathbf{L}^2 - \mathbf{K}^2) + V_{13}(\mathbf{u})$ $I_2 = \frac{1}{2M}L_3^2 + \frac{1}{2M}\left(\frac{k_1^2-\frac14}{\sin^2(\varphi/2)} + \frac{k_2^2-\frac14}{\cos^2(\varphi/2)}\right)$ $I_3 = \frac{1}{2M}\mathbf{L}^2 - \alpha\cot\vartheta + \frac{\hbar^2}{8M\sin^2\vartheta}\left(\frac{k_1^2-\frac14}{\sin^2(\varphi/2)} + \frac{k_2^2-\frac14}{\cos^2(\varphi/2)}\right)$ $I_4 = \frac{1}{2M}(\frac12\sin 2f\{L_1,L_3\} - \cos 2f L_3^2)$ $-\alpha\frac{k^2k'\text{sn}^2\tilde\alpha\text{sn}\tilde\beta\text{dn}\tilde\beta - (k^2+k'^2\text{cn}^2\tilde\beta)k\text{sn}\tilde\alpha\text{dn}\tilde\alpha}{k^2\text{cn}^2\tilde\alpha + k'^2\text{cn}^2\tilde\beta}$ $+\frac{\hbar^2}{2M(k^2\text{cn}^2\tilde\alpha + k'^2\text{cn}^2\tilde\beta)}$ $\times\Bigg(\frac{(k_1^2+k_2^2-\frac12)k'^2 + (k_2^2-k_1^2)k'\text{sn}\tilde\alpha\text{dn}\tilde\alpha}{\text{cn}^2\tilde\alpha}$ $+\frac{(k_1^2+k_2^2-\frac12)k^2 + (k_2^2-k_1^2)k\text{sn}\tilde\beta\text{dn}\tilde\beta}{\text{cn}^2\tilde\beta}\Bigg)$

* after appropriate rotation, $\sin^2 f = k^2$.

Table 4.4. Minimally superintegrable potentials on $\Lambda^{(3)}$ from the Group Chain $SO(3,1) \supset SO(2,1)$

Potential $V(\mathbf{u})$	Coordinate Systems	Observables
$V_{14}(\mathbf{u}) = \frac{M}{2}\frac{\omega^2}{u_0^2-u_1^2-u_2^2}\frac{u_1^2+u_2^2}{u_0^2} + \frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac14}{u_1^2}+\frac{k_2^2-\frac14}{u_2^2}\right) + F(u_3)$	Equidistant-Elliptic Equidistant-Hyperbolic Equidistant-Cylindrical Equidistant	$I_1 = \frac{1}{2MR^2}(\mathbf{L}^2-\mathbf{K}^2) + V_{14}(\mathbf{u}),\quad I_2 = K_1^2+K_2^2-L_3^2$ $I_3 = \frac{1}{2M}L_3^2 + \frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac14}{\cos^2\varphi}+\frac{k_2^2-\frac14}{\sin^2\varphi}\right)$ $I_4 = \frac{1}{2M}K_2^2 - \frac{M}{2}\frac{\omega^2R^4}{\cosh^2\tau_3} + \frac{\hbar^2}{2M}\frac{k_1^2-\frac14}{\sinh^2\tau_3}$
$V_{15}(\mathbf{u}) = -\frac{\alpha}{u_0^2-u_1^2-u_2^2}\left(\frac{u_0}{\sqrt{u_1^2+u_2^2}}-1\right) + F(u_3) + \frac{\hbar^2}{4M\sqrt{u_1^2+u_2^2}}\left(\frac{k_1^2-\frac14}{\sqrt{u_1^2+u_2^2}+u_1}+\frac{k_2^2-\frac14}{\sqrt{u_1^2+u_2^2}-u_1}\right)$	Equidistant-Elliptic* Equidistant-Semi-Hyperbolic Equidistant-Elliptic-Parabolic Equidistant-Cylindrical	$I_1 = \frac{1}{2MR^2}(\mathbf{L}^2-\mathbf{K}^2) + V_{15}(\mathbf{u}),\quad I_2 = K_1^2+K_2^2-L_3^2$ $I_3 = \frac{1}{2M}L_3^2 + \frac{\hbar^2}{8M}\left(\frac{k_1^2-\frac14}{\sin^2(\varphi/2)}+\frac{k_2^2-\frac14}{\cos^2(\varphi/2)}\right)$ $I_4 = \frac{1}{4M}\{K_1,L_3\} - \alpha R\frac{\sqrt{1+\mu_1}+\sqrt{1+\mu_2}}{\mu_1+\mu_2} + \frac{\hbar^2}{4M}\frac{1}{\mu_1+\mu_2}$ $\times\left[(k_1^2+k_2^2-\tfrac12)\left(\frac{\mu_1}{\mu_2}+\frac{\mu_2}{\mu_1}\right)+(k_1^2-k_2^2)\left(\frac{\sqrt{1+\mu_1}}{\mu_1}-\frac{\sqrt{1+\mu_2}}{\mu_2}\right)\right]$
$V_{16}(\mathbf{u}) = F(u_3) + \frac{\alpha}{(u_0-u_1)^2} + \frac{M}{2}\omega^2\frac{u_0^2-u_1^2+3u_2^2}{(u_0-u_1)^4} - \lambda\frac{u_2}{(u_0-u_1)^3}$	Equidistant-Semi-Circular-Parabolic Equidistant-Horicyclic	$I_1 = \frac{1}{2MR^2}(\mathbf{L}^2-\mathbf{K}^2) + V_{16}(\mathbf{u}),\quad I_2 = K_1^2+K_2^2-L_3^2$ $I_3 = \frac{1}{2M}(K_1-L_3)^2 + \alpha + 2M\omega^2x^2 - \lambda x$ $I_4 = \frac{1}{4M}(\{K_1,K_2\}-\{K_2,L_3\}) + \frac{\xi^2\eta^2}{\xi^2+\eta^2}\left[\alpha(\xi^2+\eta^2)+\tfrac{\lambda}{2}(\xi^2-\eta^4)+\tfrac{M}{2}\omega^2(\xi^6+\eta^6)\right]$
$V_{17}(\mathbf{u}) = \frac{M}{2}\frac{\omega^2}{(u_0-u_1)^2} + \frac{\hbar^2}{2M}\frac{k_2^2-\frac14}{u_2^2} + F(u_3)$	Equidistant-Elliptic-Parabolic Equidistant-Hyperbolic-Parabolic Equidistant-Semi-Circular-Parabolic Equidistant Equidistant-Horicyclic	$I_1 = \frac{1}{2MR^2}(\mathbf{L}^2-\mathbf{K}^2) + V_{17}(\mathbf{u}),\quad I_2 = K_1^2+K_2^2-L_3^2$ $I_3 = \frac{1}{2M}(K_1-L_3)^2 + \frac{\hbar^2}{2M}\frac{\kappa^2-\frac14}{x^2}$ $I_4 = \frac{1}{2M}K_2^2 + \frac{M}{2}\omega^2e^{2\tau_3}$
$V_{18}(\mathbf{u}) = \frac{\alpha}{u_0^2-u_1^2-u_2^2}\frac{u_2}{\sqrt{u_0^2-u_1^2}} + F(u_3)$	Equidistant-Semi-Circular-Parabolic Equidistant	$I_1 = \frac{1}{2MR^2}(\mathbf{L}^2-\mathbf{K}^2) + V_{18}(\mathbf{u}),\quad I_2 = K_1^2+K_2^2-L_3^2$ $I_3 = \frac{1}{4M}(\{K_1,K_2\}-\{K_2,L_3\}) + \alpha R\frac{\xi^2\eta^2}{\xi^2+\eta^2}\left(\frac{1}{\xi^2}-\frac{1}{\eta^2}\right)$ $I_4 = K_2^2$

* after appropriate rotation, $\sin^2 f = k^2$.

dimensional sphere. However, the case of the Higgs oscillator is already contained in the potential V_7, i.e., a generalized radial potential.

3. There are five potentials which emerge form the group chain $SO(3,1) \supset \mathrm{SO}(2,1)$, and all superintegrable potentials on $\Lambda^{(2)}$ are minimally superintegrable on $\Lambda^{(3)}$, among them the Higgs oscillator and the Coulomb potential.

In all cases the (path integral) solution is very easy. First, one can separate off the underlying two-dimensional superintegrable potential term. In the first case, the remaining path integral is a path integral in the horicyclic variable y in the Poincaré upper half-plane $\mathcal{H}$, say, in the second and third cases one is left, for instance, with a modified Pöschl–Teller or Rosen–Morse potential path integral. The specific form depends, of course, on the remaining "hyperbolic radial" potential, which can be chosen arbitrarily. It must be noted that in the cases of the Coulomb potential on $S^{(2)}$ and $\Lambda^{(2)}$ the problem of self adjoint continuation arises due to the negative bound states, a problem which will be discussed in detail elsewhere.

The following Tables summarize our finding of the minimally superintegrable potentials on $\Lambda^{(3)}$ which are due to the group structure of $\mathrm{SO}(3,1)$. We omit any details concerning the solution, and the interested reader is invited to consult Refs. 33,34 and 35, respectively, to check for the solution of the corresponding two-dimensional systems, in order to obtain the solution on $\Lambda^{(3)}$.

5. SUMMARY AND DISCUSSION

The purpose of this paper has been to present a comprehensive discussion of superintegrable potentials on the three-dimensional hyperboloid $\Lambda^{(3)}$. It has included an enumeration of the coordinate systems on $\Lambda^{(3)}$ as known from the literature, a systematic search for maximally and minimally superintegrable potentials by appropriate generalizations from the Euclidean space, the statement of the constants of motions, respectively operators, and in the soluble cases the evaluation of the corresponding path integral representation in order to find the quantum mechanical propagators, the Green functions, the discrete and continuous wave-functions, and the energy spectra, respectively.

In the enumeration of the 34 coordinate systems in section 2 we have followed [52, 75], however, supplemented by the corresponding Hamiltonian and the form of a corresponding separable potential, several rotated coordinate systems, i.e., the sphero-elliptic rotated, the equidistant-elliptic rotated, and the prolate-elliptic rotated. These rotated systems correspond in their respective flat space limit to sphero-conical II, cylindrical elliptic II, and prolate-spheroidal II coordinate systems, which in turn contain as additional degenerate systems the respective parabolic systems. However, for the complicated two-parametric systems XXIX–XXXIV hardly any statement and usage could have been made.

In Section 3 we have presented our results concerning the maximally superintegrable potentials on $\Lambda^{(3)}$. These have been the (generalized) Higgs oscillator $V_1(\mathrm{u})$, the (generalized) Coulomb potential $V_2(\mathrm{u})$, and a specific scattering potential $V_3(\mathrm{u})$. The potential $V_4(\mathrm{u})$, which is only minimally superintegrable on $\Lambda^{(3)}$, has been included in this section due to the fact that its flat space analogue in $\mathbb{R}^3$ is maximally superintegrable.

The Higgs oscillator and the Coulomb potential have been discussed in some detail, first for the pure oscillator and the Coulomb case, second with the incorporation of additional centrifugal terms which do not spoil the property of maximally superintegrability, similarly as the corresponding cases in $\mathbb{R}^3$ and on $S^{(3)}$. The energy spectrum and degeneracy of levels was also discussed.

In section 4 we have discussed the minimally superintegrable potentials on $\Lambda^{(3)}$. We have found the four analogues of the flat space case, in particular the ring-shaped oscillator, the Hartmann potential, a radial potential, and a Holt potential. The remaining minimally superintegrable potentials have emerged from the subgroup structure of $\mathrm{SO}(3,1)$, i.e., we have had to take into account the group chains $\mathrm{SO}(3,1) \supset E(2)$, $\mathrm{SO}(3,1) \supset \mathrm{SO}(3)$, $\mathrm{SO}(3,1) \supset \mathrm{SO}(2,1)$, which have given rise to four, one and five new minimally superintegrable potentials, respectively. In total, we have found 15 minimally superintegrable potentials on $\Lambda^{(3)}$. Whereas we have treated the ring-shaped oscillator and the Hartmann potential in some detail, the discussion for the other potentials has been mostly rather sketchy because the underlying superintegrable two-dimensional systems have been already solved in previous publications.

We have therefore continued the study of superintegrable systems in spaces of constant curvature. Furthermore, we would like to draw ones attention to the following observations:

1. Let us consider the potential V_{19} in semi-hyperbolic coordinates

$$\begin{aligned}
V_{19}(\mathrm{u}) &= \frac{M}{2}\omega^2\left(\frac{4u_0^2u_3^2}{R^2} + u_1^2 + u_2^2\right) + 2k_3u_0u_3 \\
&\quad + \frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac{1}{4}}{u_1^2} + \frac{k_2^2-\frac{1}{4}}{u_2^2}\right) \\
&= \frac{R^2}{\mu_1+\mu_2}\left[\frac{M}{2}\omega^2(\mu_1^3+\mu_2^3) + k_3(\mu_1^2-\mu_2^2)\right] \\
&\quad + \frac{\hbar^2}{2MR^2}\frac{1}{\mu_1\mu_2}\left(\frac{k_1^2-\frac{1}{4}}{\cos^2\varphi} + \frac{k_2^2-\frac{1}{4}}{\sin^2\varphi}\right) ,
\end{aligned} \tag{5.1}$$

and this potential is separable in this system. It has two flat-space limits in its full range of parameters, i.e., circular polar and parabolic coordinates.

Whereas it was possible to just simply state the potential $V_4(\mathrm{u})$, no explicit solution could have been found. This Stark-effect like potential could be of some interest, in particular in comparison with the potential $V_{19}(\mathrm{u})$. The potential in this limit corresponds to the second maximally superintegrable potential $V_2(\mathbf{x})$ of [33], c.f. Table D.2, and the limiting case it separates in these two coordinate systems, and in addition in the cartesian and circular elliptic system.

2. Let us consider the potential V_{20} in semi-hyperbolic coordinates

$$\begin{aligned} V_{20}(\mathrm{u}) &= \frac{M}{2}\omega^2\left(\frac{4u_0^2u_3^2}{R^2}+u_1^2+u_2^2\right)+\frac{\hbar^2}{2M}\frac{F(u_2/u_1)}{u_1^2+u_2^2}\ , \\ &= \frac{R^2}{\mu_1+\mu_2}\frac{M}{2}\omega^2(\mu_1^3+\mu_2^3)+\frac{\hbar^2}{2MR^2}\frac{F(\tan\varphi)}{\mu_1\mu_2}\ , \end{aligned} \tag{5.2}$$

and this potential is separable in this system. The potential in the flat space limit corresponds to the sixth minimally superintegrable potential $V_6(\mathbf{x})$ of [33], c.f. Table D.3, and the limiting case it separates in circular polar and parabolic coordinates.

3. The previous observations allow the following statement: We have found all five analogues of the maximally superintegrable potentials in $\mathbb{R}^3$, where we have the following identification (where the enumeration of the potentials in $\mathbb{R}^3$, is according to [33]; and the enumeration of the potentials on $S^{(3)}$ according to [34]):

Table 5.1. Correspondence of maximally superintegrable potentials in three dimensions

$V_{\Lambda^{(3)}}(\mathrm{u})$	#Systems	$V_{\mathbb{R}^3}(\mathbf{x})$	#Systems	$V_{S^{(3)}}(\mathbf{s})$	#Systems
$V_1(\mathrm{u})$	14(8)	$V_1(\mathbf{x})$	8	$V_1(\mathbf{s})$	6(8)
$V_2(\mathrm{u})$	5(4)	$V_3(\mathbf{x})$	4	$V_2(\mathbf{s})$	3(4)
$V_3(\mathrm{u})$	5(4)	$V_4(\mathbf{x})$	4	$V_3(\mathbf{s})$	2(2) [3(4)]
$V_4(\mathrm{u})$	4(4)	$V_5(\mathbf{x})$	4	–	1(1)
$V_{19}(\mathrm{u})$	1(2)	$V_2(\mathbf{x})$	4	–	1(1)

In parenthesis we have indicated the number of limiting coordinate systems, as $R\to\infty$. Note that for $V_3(\mathbf{s})$ we have two separating coordinate systems. In $V_3(\mathbf{s})$ we have also indicated the additional coordinate system which emerges, and causes an additional observable, if $k_3^2-1/4=0$, c.f. Table D.5. From the rotated sphero-elliptic system on $S^{(3)}$ two coordinate systems on $\mathbb{R}^3$ can be obtained by means of contraction, as $R \to \infty$, the cylindrical elliptic II and the cylindrical parabolic. Note also that $V_4(\mathrm{u})$ is only minimally superintegrable, but not maximally superintegrable. Furthermore, in [34] several potential systems were overlooked. However, the additional systems turn out to be only integrable but not superintegrable.

4. The linear potential on the hyperboloid seems to have a structure according to $u_0 u_3$, which turns out to be separable in an appropriately chosen *parabolic coordinate system*. However, on spaces of constant (non-vanishing) curvature, there seems to be no analogue of a *cartesian coordinate system*, which separates these kinds of potentials as well.

5. The coordinate systems XXX–XXXIII separate a radial potential according to $V(u_2, u_3) \propto \alpha/u_2^2 + \beta/u_3^2$, and XXXIV – according to $V(u_3) \propto \beta/u_3^2$, which are, however, trivial and not very interesting.

6. Let us finally note another application of the prolate elliptic coordinate system. It has the property that it separates the two-center Coulomb problem on the hyperboloid, similarly as the prolate elliptic system on the sphere separates the two-center Coulomb problem on $S^{(3)}$ [6, 76]. Let us consider two point charges located at $u_{1,2} = (1, 0, 0, \pm k')/k$ on the hyperboloid. Then it is not difficult to show by means of the prolate elliptic coordinate system that one has in algebraic form ($Z_\pm = Z_1 \pm Z_2$)

$$\begin{aligned} V(u_1, u_2, u) &= -Z_1 \frac{u_1 \cdot u}{\sqrt{(u_1 \cdot u)^2 - 1}} - Z_2 \frac{u_2 \cdot u}{\sqrt{(u_2 \cdot u)^2 - 1}} \\ &= -\frac{Z_+ \sqrt{(\varrho_1 - a_2)(\varrho_1 - a_3)} - Z_- \sqrt{(\varrho_2 - a_2)(\varrho_2 - a_3)}}{\varrho_1 - \varrho_2} \ . \end{aligned} \tag{5.3}$$

A detailed investigation of this problem will be presented elsewhere [36].

We cannot say for sure whether we really have found all possible superintegrable potentials on the hyperboloid. For a systematic search one must solve differential equations which emerge from the general form of a potential separable in a particular coordinate system, and the following changing variables. Because there are 34 coordinate systems on the hyperboloid which separate the Schrödinger equation, there are $33! \simeq 8.7 \cdot 10^{36}$ of such differential equations. This is not tractable, and one has to look for alternative procedures, in particular physical arguments. In this respect, we have found the relevant potentials which matter from a physical point of view, and which are the analogues of the flat space limit $\mathbb{R}^3$, including the corresponding coordinate systems.

Table 5.2. Correspondence of minimally superintegrable potentials in three dimensions

$V_{\Lambda^{(3)}}(\mathbf{u})$	#Systems	$V_{\mathbb{R}^3}(\mathbf{x})$	#Systems	$V_{S^{(3)}}(\mathbf{s})$	#Systems
		Analogues of flat space			
$V_5(\mathbf{u})$	5(3)	$V_5(\mathbf{x})$	4	$V_4(\mathbf{s})$	4(4)
$V_6(\mathbf{u})$	3(4)	$V_7(\mathbf{x})$	3	$V_5(\mathbf{s})$	2(3)
$V_7(\mathbf{u})$	2(2)	$V_1(\mathbf{x})$	2	$V_6(\mathbf{s})$	2(2)
$V_8(\mathbf{u})$	2(1)	$V_3(\mathbf{x})$	2	-	1(1)
$V_{20}(\mathbf{u})$	1(2)	$V_6(\mathbf{x})$	2	-	1(2)
		Potentials emerging from $SO(3,1) \supset E(2)$			
$V_9(\mathbf{u})$	3(3)	$V_2(\mathbf{x})$	3	-	-
V_9'	7(3)	$\frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac{1}{4}}{x^2}+\frac{k_2^2-\frac{1}{4}}{y^2}\right)+F(z)$	3	-	-
$V_{10}(\mathbf{u})$	2(2)	$V_3(\mathbf{x})$	2	-	-
$V_{11}(\mathbf{u})$	3(3)	$V_4(\mathbf{x})$	3	-	-
$V_{12}(\mathbf{u})$	2(2)	$V_8(\mathbf{x})$	2	-	-
		Potentials emerging from $SO(3,1) \supset SO(3)$			
$V_7'(\mathbf{u})$	2(2)	$V_1(\mathbf{x})$	2	$V_6(\mathbf{s})$	2(2)
$V_{13}(\mathbf{u})$	2(2)	$V_9(\mathbf{x})$	2	$V_7(\mathbf{s})$	2(2)
		Potentials emerging from $SO(3,1) \supset SO(2,1)$			
$V_{14}(\mathbf{u})$	4(3)	$V_2(\mathbf{x})$	3	-	-
$V_{15}(\mathbf{u})$	4(3)	$V_4(\mathbf{x})$	3	-	-
$V_{16}(\mathbf{u})$	2(1)	$V_3(\mathbf{x})$	2	-	-
$V_{17}(\mathbf{u})$	5(2)	$\frac{\hbar^2}{2M}\frac{k_2^2-\frac{1}{4}}{x^2}+F(z)$	2	-	-
$V_{18}(\mathbf{u})$	2(1)	$\alpha x+F(z)$	2	-	-

Summarizing, we have achieved an enumeration and classification of superintegrable systems in spaces of constant (positive, zero, or negative) curvature. Further studies along these lines could include the investigation of the corresponding interbasis expansions, the contraction of the wave-functions in the curved spaces with respect to their Euclidean flat space limit, their pseudo-Euclidean flat space limit, and the solution of various superintegrable potentials in the generic, respectively parametric coordinate systems [37]. Among the latter, the most important cases are the Coulomb problems, for instance the Coulomb problem in $\Lambda^{(2)}$ or $\Lambda^{(3)}$ in semi-hyperbolic coordinates, and the investigation of the Stark-effect in spaces of constant curvature which includes the solution of the corresponding Schrödinger equations. We hope to return to these issues in future.

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A. PATH INTEGRAL IDENTITY FOR THE PÖSCHL–TELLER POTENTIAL

As we shall see, we encounter particularly in the case of the Higgs oscillator, the Pöschl–Teller and the modified Pöschl–Teller potential in our path integral problems. The path integral solution of the Pöschl–Teller potential reads as follows (Böhm and Junker [7], Duru [14], [31,39,40], Fischer et al. [18], Inomata et al. [48], Kleinert and Mustapic [60], $0 < x < \pi/2$)

$$\int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}\dot{x}^2 - \frac{\hbar^2}{2M}\left(\frac{\alpha^2-\frac{1}{4}}{\sin^2 x} + \frac{\beta^2-\frac{1}{4}}{\cos^2 x}\right)\right]dt\right\}$$

$$= \sum_{n\in\mathbb{N}_0} \mathrm{e}^{-\mathrm{i}E_nT/\hbar}\phi_n^{(\alpha,\beta)}(x')\phi_n^{(\alpha,\beta)}(x'') \tag{A.1}$$

$$= \int_{\mathbb{R}} \frac{dE}{2\pi\mathrm{i}}\mathrm{e}^{-\mathrm{i}ET/\hbar}G_{PT}^{(\alpha,\beta)}(x'',x';E) \ . \tag{A.2}$$

The bound state wave-functions and the energy spectrum are given by

$$\phi_n^{(\alpha,\beta)}(x) = \left[2(\alpha+\beta+2n+1)\frac{n!\Gamma(\alpha+\beta+n+1)}{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}\right]^{1/2} \times(\sin x)^{\alpha+1/2}(\cos x)^{\beta+1/2}P_n^{(\alpha,\beta)}(\cos 2x) \ , \tag{A.3}$$

$$E_n = \frac{\hbar^2}{2M}(2n+\alpha+\beta+1)^2 \ . \tag{A.4}$$

The $P_n^{(\alpha,\beta)}$ are Jacobi polynomials. The Pöschl–Teller wave-functions $\phi^{(\alpha,\beta)}(x)$ are normalized to unity with respect to the scalar product $\int_0^{\pi/2}\left|\phi_n^{(\alpha,\beta)}(x)\right|^2 dx = 1$. The Green function $G_{PT}^{(\alpha,\beta)}(E)$ has the form

$$G_{PT}^{(\alpha,\beta)}(x'',x';E) = \frac{M}{2\hbar^2}\sqrt{\sin x'\sin x''}\frac{\Gamma(m_1-L_E)\Gamma(L_E+m_1+1)}{\Gamma(m_1+m_2+1)\Gamma(m_1-m_2+1)}$$

$$\times\left(\frac{1-\cos 2x'}{2}\cdot\frac{1-\cos 2x''}{2}\right)^{(m_1-m_2)/2}$$
$$\times\left(\frac{1+\cos 2x'}{2}\cdot\frac{1+\cos 2x''}{2}\right)^{(m_1+m_2)/2}$$
$$\times{}_2F_1\left(-L_E+m_1,L_E+m_1+1;m_1-m_2+1;\frac{1-\cos 2x_<}{2}\right)$$
$$\times{}_2F_1\left(-L_E+m_1,L_E+m_1+1;m_1+m_2+1;\frac{1+\cos 2x_>}{2}\right), \quad (A.5)$$

where $m_{1,2}=\frac{1}{2}(\beta\pm\alpha), L_E=-\frac{1}{2}+\frac{1}{2}\sqrt{2ME}/\hbar$, ${}_2F_1(a,b;c;z)$ is the hypergeometric function, and $x_>, x_<$ denotes the larger, respectively smaller of x', x''.

B. PATH INTEGRAL IDENTITY FOR THE MODIFIED PÖSCHL–TELLER POTENTIAL

The case of the modified Pöschl–Teller potential is given in [7, 19, 31, 39, 40, 48, 60]

$$\int\limits_{r(t')=r'}^{r(t'')=r''}\mathcal{D}r(t)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}\dot{r}^2-\frac{\hbar^2}{2M}\left(\frac{\kappa^2-\frac{1}{4}}{\sinh^2 r}-\frac{\lambda^2-\frac{1}{4}}{\cosh^2 r}\right)\right]dt\right\}$$
$$=\sum_{n=0}^{N_{max}}\mathrm{e}^{-\mathrm{i}E_nT/\hbar}\psi_n^{(\kappa,\lambda)\,*}(r')\psi_n^{(\kappa,\lambda)}(r'')$$
$$+\int_0^\infty dp\,\mathrm{e}^{-\mathrm{i}E_pT/\hbar}\psi_p^{(\kappa,\lambda)\,*}(r')\psi_p^{(\kappa,\lambda)}(r'') \quad (B.6)$$
$$=\int_{\mathbb{R}}\frac{dE}{2\pi\mathrm{i}}\mathrm{e}^{-\mathrm{i}ET/\hbar}G_{mPT}^{(\kappa,\lambda)}(r'',r';E)\ . \quad (B.7)$$

The bound states have the form

$$\psi_n^{(\kappa,\lambda)}(r)=N_n^{(\kappa,\lambda)}(\sinh r)^{\kappa+1/2}(\cosh r)^{n-\lambda+1/2}{}_2$$
$$\times F_1(-n,\lambda-n;1+\kappa;\tanh^2 r)\ , \quad (B.8)$$
$$N_n^{(\kappa,\lambda)}=\frac{1}{\Gamma(1+\kappa)}\left[\frac{2(\lambda-\kappa-2n-1)\Gamma(n+1+\kappa)\Gamma(\lambda-n)}{\Gamma(\lambda-\kappa-n)n!}\right]^{1/2}$$
$$E_n=-\frac{\hbar^2}{2M}(2n+\kappa-\lambda+1)^2\ . \quad (B.9)$$

Here denote $n=0,1,\ldots,N_{max}=[\frac{1}{2}(\lambda-\kappa-1)]\geq 0$, and only a finite number of bound states can exist depending on the strength of the attractive potential through and the repulsive centrifugal term as well. Here $[x]$ denotes the integer part of the real number x. The continuous states are

$$\psi_p^{(\kappa,\lambda)}(r) = N_p^{(\kappa,\lambda)}(\cosh r)^{ip}(\tanh r)^{\kappa+1/2}{}_2F_1 \times\left(\frac{\lambda+\kappa+1-ip}{2},\frac{\kappa-\lambda+1-ip}{2};1+\kappa;\tanh^2 r\right)$$

$$N_p^{(\kappa,\lambda)} = \frac{1}{\Gamma(1+\kappa)}\sqrt{\frac{p\sinh\pi p}{2\pi^2}}\,\Gamma\left(\frac{\lambda+\kappa+1-ip}{2}\right)\Gamma\left(\frac{\kappa-\lambda+1-ip}{2}\right). \tag{B.10}$$

The Green function $G_{mPT}^{(\kappa,\lambda)}(E)$ has the form

$$\begin{aligned} G_{mPT}^{(\kappa,\lambda)}(r'',r';E) &= \frac{M}{2\hbar^2}\frac{\Gamma(m_1-L_\lambda)\Gamma(L_\lambda+m_1+1)}{\Gamma(m_1+m_2+1)\Gamma(m_1-m_2+1)} \\ &\times(\cosh r'\cosh r'')^{-(m_1-m_2)}(\tanh r'\tanh r'')^{m_1+m_2+1/2} \\ &\times{}_2F_1\left(-L_\lambda+m_1,L_\lambda+m_1+1;m_1-m_2+1;\frac{1}{\cosh^2 r_<}\right) \\ &\times{}_2F_1\left(-L_\lambda+m_1,L_\lambda+m_1+1;m_1+m_2+1;\tanh^2 r_>\right), \end{aligned} \tag{B.11}$$

where we have set $m_{1,2} = \frac{1}{2}(\kappa \pm \sqrt{-2ME}/\hbar), L_\lambda = \frac{1}{2}(\lambda-1)$. We make extensively use of the solutions of the Pöschl–Teller and the modified Pöschl–Teller potential, respectively.

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