«ФИЗИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ И АТОМНОГО ЯДРА»
2000, ТОМ 31, ВЫП. 7А

УДК 539.12.01

PERIODIC SOLUTIONS AND INTEGRALS OF MOTION FOR THE CLASSICAL EQUATION OF RELATIVISTIC STRING WITH MASSIVE ENDS IN 3-DIMENSIONAL MINKOWSKI SPACE

B.M. Barbashov

Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, 141980 Dubna, Russia

It is well known that a straight-line relativistic string is an exact solution of the equation of motion and boundary conditions, when its massive ends move along a circular orbit. In this report, we investigate the exact solution of string equations for periodic motions of massive string ends which move along an elliptic orbit in the x, y-plane (planar motion). We determine analytically the coordinates of the string in terms of the Weierstrass elliptic functions. In the considered case, the curved string has a transverse excitation, and its ends have a radial momentum, not present in a straight-line string. We determine the shape of the curved string.

1. PERIODIC SOLUTIONS AND INTEGRAL OF MOTION

The string action with masses attached to its ends has the form

$$S = -\gamma \int\limits_{\tau_1}^{\tau_2} d\tau \int\limits_{\sigma_1}^{\sigma_2} d\sigma \sqrt{(\dot{x}x')^2 - \dot{x}^2 x'^2} - \sum_{i=1}^{2} m_i \int\limits_{\tau_1}^{\tau_2} \sqrt{x'^2(\tau, \sigma_i)}, \tag{1}$$

where $\gamma = 1/(2\pi\alpha')$ is the string tension, $\dot{x}^\mu(\tau,\sigma) = \partial x^\mu/\partial\tau$, $x'^\mu(\tau,\sigma) = \partial x^\mu/\partial\sigma$. The general solution to the equation of motion

$$\ddot{x}^\mu(\tau,\sigma) - x''^\mu(\tau,\sigma) = 0$$

is

$$x^\mu(\tau,\sigma) = \frac{1}{2}\left[\Psi_+^\mu(\tau+\sigma) + \Psi_-^\mu(\tau-\sigma)\right].$$

The orthogonal gauge condition $(\dot{x}^\mu \pm x'^\mu)^2 = 0$ results in equations for vectors $\Psi'^\mu_\pm(\tau \pm \sigma)$

$$\Psi'^2_\pm = 0,$$

according to which $\Psi'^{\mu}_{\pm}$ should be isotropic vectors, and for further consideration, it is convenient to represent them as expansions over a constant basis in the 3-dimensional Minkowski space:

$$\begin{aligned}\Psi'^{\mu}_{+}(\tau+\sigma) &= \frac{A_{+}(\tau+\sigma)}{f'(\tau+\sigma)}\left\{a^{\mu}+b^{\mu}\,f(\tau+\sigma)+c^{\mu}\frac{f^{2}(\tau+\sigma)}{2}\right\},\\ \Psi'^{\mu}_{-}(\tau-\sigma) &= \frac{A_{+}(\tau-\sigma)}{g'(\tau+\sigma)}\left\{a^{\mu}+b^{\mu}\,g(\tau+\sigma)+c^{\mu}\frac{f^{2}(\tau+\sigma)}{2}\right\},\end{aligned} \tag{2}$$

where $a^{\mu},\ b^{\mu},\ c^{\mu}$ is a constant basis, consisting of two isotropic vectors a^{μ}, c^{μ}: $(ac)=1$, $a^{2}=c^{2}=0$, and orthonormal space-like vector b^{μ}: $b^{2}=-1$, $(ab)=(bc)=0$.

The orthonormal gauge does not determine the functions $A_{\pm}(\tau\pm\sigma)$ in (2), and consequently, there is a possibility of fixing them by imposing further gauge conditions, since expressions (2) are invariant under conformal transformations of the parameters $\bar{\tau}\pm\bar{\sigma}=V_{\pm}(\tau\pm\sigma)$. We fix them by two more gauge conditions:

$$[\dot{x}'^{\mu}\pm\ddot{x}^{\mu}]^{2}=-A^{2}=\text{const},$$

which in terms of the vectors $\Psi'^{\mu}_{\pm}$ mean that the space-like vectors $\Psi''^{\mu}_{\pm}$ are modulo constant

$$\Psi''^{2}_{\pm}=-A^{2}.$$

The boundary conditions for the string ends $\sigma_1=0$ and $\sigma_2=l$ are the following

$$m_1\frac{d}{d\tau}\left(\frac{\dot{x}^{\mu}(\tau,0)}{\sqrt{\dot{x}^{2}(\tau,0)}}\right)=\gamma\,x'^{\mu}(\tau,0),\qquad m_2\frac{d}{d\tau}\left(\frac{\dot{x}^{\mu}(\tau,l)}{\sqrt{\dot{x}^{2}(\tau,l)}}\right)=-\gamma\,x'^{\mu}(\tau,l). \tag{3}$$

Now let us calculate the curvature $K_i(\tau)$ and torsions $\kappa_i(\tau)$ of boundary curves along which masses m_i are moving. To this end, we compare the boundary Eq. (3) with the Serret–Frenet equations for boundary curves [2]

$$\frac{d}{d\tau}\left(\frac{\dot{x}^{\mu}(\tau)}{\sqrt{\dot{x}^{2}(\tau)}}\right)=(-1)^{i+1}K_i(\tau)\,x'^{\mu}_{i}(\tau),\quad \frac{d}{d\tau}n^{\mu}_{i}(\tau)=\kappa_i(\tau)\,x'^{\mu}_{i},\quad i=1,2, \tag{4}$$

where $x^{\mu}_{i}(\tau)=x^{\mu}(\tau,\sigma_i)$, $n^{\mu}_{i}(\tau)=n^{\mu}(\tau,\sigma_i)$ are binormals of the boundary curves. By comparing with (3), we can find that $K_i(\tau)=\gamma/m_i$ is constant.

Projecting the second equation (4) onto $x_i'^{\mu}(\tau)$ and taking into account that $n_i^{\mu} \perp \dot{x}_i^{\mu}, x_i'^{\mu},\ \ n_i^2 = -1$, we obtain

$$\kappa_i(\tau) = \frac{(\dot{n}_i\, x_i')}{x_i'^2} = \frac{(n_i\, \dot{x}_i')}{\dot{x}_i^2} = \frac{A}{\dot{x}^2(\tau, \sigma_i)}. \tag{5}$$

Thus, torsions κ_i are determined by $\dot{x}^2(\tau, \sigma_i)$ and the constant A that is a nonzero coefficient of the second quadratic form of 2-dimensional string surface

$$b_{kl} = \left(n_{\mu}\, \frac{\partial^2 x^{\mu}}{\partial u_k \partial u_l}\right), \qquad u_1 = \tau,\ \ u_2 = \sigma, \qquad b_{11} = b_{22} = 0,\ \ b_{12} = b_{21} = A.$$

By inserting $\Psi_{\pm}'^{\mu}(\tau \pm \sigma_i)$ from (2) into the boundary equations (3) and taking into account that $A_{\pm}^2(\tau \pm \sigma) = A^2$, we get

$$\begin{aligned} &m_1 \left[\frac{d}{d\tau} \ln\left(\frac{g'(\tau)}{f'(\tau)}\right) + 2\, \frac{f'(\tau) + g'(\tau)}{f(\tau) - g(\tau)}\right] = 2\gamma\, \sqrt{\dot{x}^2(\tau, 0)},\ \sigma_1 = 0, \\ &m_2 \left[\frac{d}{d\tau} \ln\left(\frac{g'(\tau{-}l)}{f'(\tau{+}l)}\right) {+}2\, \frac{f'(\tau{+}l){+}g'(\tau{-}l)}{f(\tau{+}l){-}g(\tau{-}l)}\right] = -\, 2\gamma\, \sqrt{\dot{x}^2(\tau, l)},\ \sigma_2{=}l, \end{aligned} \tag{6}$$

where

$$\dot{x}^2(\tau, \sigma) = A^2\, \frac{[f(\tau+\sigma) - g(\tau-\sigma)]^2}{4\, f'(\tau+\sigma)\, g'(\tau-\sigma)}. \tag{7}$$

As is known [1], expression (7) is the general solution to the Liouville Eq. for $\dot{x}^2(\tau, \sigma)$, i.e., the Gauss equation for a minimal 2-dimensional surface:

$$\frac{\partial^2 \ln \dot{x}^2(\tau, \sigma)}{\partial^2 \tau} - \frac{\partial^2 \ln \dot{x}^2(\tau, \sigma)}{\partial^2 \sigma} = \frac{A^2}{\dot{x}^2(\tau, \sigma)}.$$

In 3-dimensional Minkowski space, we can, by using the expressions for $\dot{x}^2(\tau, \sigma_i)$, $(\sigma_1 = 0,\ \sigma_2 = l)$

$$\begin{aligned} \dot{x}^2(\tau, 0) \quad &= \quad \dot{x}_1^2(\tau) = A^2\, \frac{[f(\tau) - g(\tau)]^2}{4\, f'(\tau)\, g'(\tau)}, \\ \dot{x}^2(\tau, l) \quad &= \quad \dot{x}_2^2(\tau) = A^2\, \frac{[f(\tau+l) - g(\tau-l)]^2}{4\, f'(\tau+l)\, g'(\tau-l)} \end{aligned} \tag{8}$$

and boundary Eq. (6), express the functions $f(\tau),\ g(\tau)$ in terms of $\dot{x}_i^2(\tau)$ and $K_i = \gamma/\mu_i$ [3].

The first boundary results in the equations:

$$\begin{aligned}
\mathcal{D}[f(\tau)] &= \mathcal{D}\left[\int^{\tau}\frac{d\eta}{\sqrt{\dot{x}_1^2(\eta)}}\right] + \frac{A^2}{\dot{x}_1^2(\tau)} - K_1^2\,\dot{x}_1^2(\tau) - 2K_1\,\frac{d}{d\tau}\sqrt{\dot{x}_1^2(\tau)}, \\
\mathcal{D}[g(\tau)] &= \mathcal{D}\left[\int^{\tau}\frac{d\eta}{\sqrt{\dot{x}_1^2(\eta)}}\right] + \frac{A^2}{\dot{x}_1^2(\tau)} - K_1^2\,\dot{x}_1^2(\tau) + 2K_1\,\frac{d}{d\tau}\sqrt{\dot{x}_1^2(\tau)},
\end{aligned} \tag{9}$$

where

$$\mathcal{D}[f(\tau)] = \frac{f'''(\tau)}{f'(\tau)} - \frac{3}{2}\left(\frac{f''(\tau)}{f'(\tau)}\right)^2$$

is the Schwarz derivative.

The second boundary results in the equations:

$$\begin{aligned}
\mathcal{D}[f(\tau+l)] &= \mathcal{D}\left[\int^{\tau}\frac{d\eta}{\sqrt{\dot{x}_2^2(\eta)}}\right] + \frac{A^2}{\dot{x}_2^2(\tau)} - K_2^2\,\dot{x}_2^2(\tau) + 2\,K_2\,\frac{d}{d\tau}\sqrt{\dot{x}_2^2(\tau)}, \\
\mathcal{D}[g(\tau-l)] &= \mathcal{D}\left[\int^{\tau}\frac{d\eta}{\sqrt{\dot{x}_2^2(\eta)}}\right] + \frac{A^2}{\dot{x}_2^2(\tau)} - K_2^2\,\dot{x}_2^2(\tau) - 2\,K_2\,\frac{d}{d\tau}\sqrt{\dot{x}_2^2(\tau)}.
\end{aligned} \tag{10}$$

Thus, the functions $f(\tau)$, $g(\tau)$ and therefore according to (2) the string coordinates $x^{\mu}(\tau,\sigma)$ are completely defined by K_i and boundary value of component of matric tensors $\dot{x}_i^2(\tau) = \dot{x}^2(\tau,\sigma_i)$.

Let us consider a simple example, where $\kappa_i(\tau) = A/\dot{x}^2(\tau,\sigma_i)$ is constant, then from (9), (10) we derive equations

$$\begin{aligned}
\mathcal{D}[f(\tau)] &= \mathcal{D}[g(\tau)] = \frac{A^2}{\dot{x}_{1,0}^2} - K_1^2\,\dot{x}_{1,0}^2 = 2\,\omega^2, \\
\mathcal{D}[f(\tau+l)] &= \mathcal{D}[g(\tau-l)] = \frac{A^2}{\dot{x}_{2,0}^2} - K_2^2\,\dot{x}_{2,0}^2 = 2\,\omega^2,
\end{aligned} \tag{11}$$

which have solutions:

$$\mathcal{D}[f(\tau)] = -2\sqrt{f'(\tau)}\frac{d^2}{d\tau^2}\left(\frac{1}{\sqrt{f'}}\right) = 2\,\omega^2 \implies \frac{1}{\sqrt{f'(\tau)}} = B\,\cos(\omega\tau+\theta_0),$$

and finally

$$f(\tau) = B^{-2}\tan(\omega\tau+\theta_0).$$

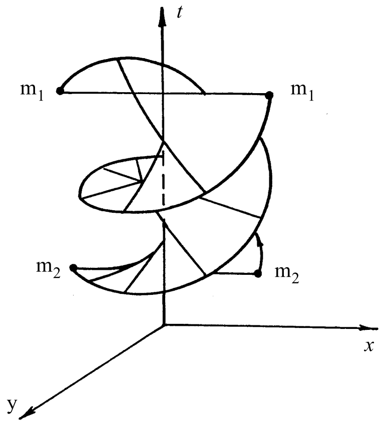

Fig. 1.

In this case, the string surface is a helicoid (see Fig. 1 and [5]) because the string coordinate has the form

$$x^{\mu}(\tau,\sigma) = A\left\{\tau, \frac{\sin(\omega\sigma-\theta_0)}{\omega}[\sin(\omega\tau+\phi_0), \cos(\omega\tau+\theta)]\right\}. \qquad (12)$$

Thus our approach is best described in terms of Schwarz derivatives because an important property of $\mathcal{D}[f(\tau)]$ is that it is invariant under Möbius transformations (linear-fractional transformations)

$$\phi(\tau) = \frac{a\,f(\tau)+b}{c\,f(\tau)+d}, \ \ (ad-b=1) \implies \mathcal{D}[\phi(\tau)] = \mathcal{D}[f(\tau)]. \qquad (13)$$

It is a remarkable fact that the system of boundary equations (9) and (10) possesses conserved quantities [3] and periodic solutions when $\dot{x}^2(\tau,\sigma_i)$ are periodic with a period $2l$: $\dot{x}^2(\tau+2l,\sigma_i) = \dot{x}^2(\tau,\sigma_i)$.

In the general case, we can represent equations (9) and (10) in the form

$$\begin{aligned} \mathcal{D}[f(\tau)] - \mathcal{D}[g(\tau)] &= -4\,K_1\frac{d}{d\tau}\sqrt{\dot{x}^2(\tau,0)}, \\ \mathcal{D}[f(\tau+l)] - \mathcal{D}[g(\tau-l)] &= 4\,K_2\frac{d}{d\tau}\sqrt{\dot{x}^2(\tau,l)}. \end{aligned}$$

Eliminating $\mathcal{D}[g(\tau)]$ from these two equations by changing τ to $\tau+l$ in the second Eq. and subtracting one from another, we get

$$\mathcal{D}[f(\tau+2l)] - \mathcal{D}[f(\tau)] = 4\,\frac{d}{d\tau}\left[K_1\sqrt{\dot{x}^2(\tau,0)} + K_2\sqrt{\dot{x}^2(\tau+l,l)}\right]. \qquad (14)$$

Eliminating $\mathcal{D}[(\tau)]$ by changing τ to $\tau - l$, we obtain the equation for $g(\tau)$

$$\mathcal{D}[g(\tau)] - \mathcal{D}[g(\tau - 2l)] = 4\,\frac{d}{d\tau}\left[K_1\sqrt{\dot{x}^2(\tau,0)} + K_2\sqrt{\dot{x}^2(\tau - l, l)}\right]. \qquad (15)$$

Now let us note that equations (14), (15) and the expressions

$$\dot{x}^2(\tau,\sigma_i) = A^2\,\frac{[f(\tau+\sigma_i) - g(\tau-\sigma_i)]^2}{4\,f'(\tau+\sigma_i)\,g'(\tau-\sigma_i)}$$

are invariant under Möbius transformations, and their being periodic $\dot{x}^2(\tau + 2l,\sigma_i) = \dot{x}^2(\tau,\sigma_i)$ leads to the transformation of the functions

$$f(\tau+2l) = \frac{a\,f(\tau)+b}{c\,f(\tau)+d}, \qquad g(\tau+2l) = \frac{a\,g(\tau)+b}{c\,g(\tau)+d}, \qquad (ad - bc = 1)$$

$$f'(\tau+2l) = \frac{f'(\tau)}{(c\,f(\tau)+d)^2}, \qquad g'(\tau+2l) = \frac{g'(\tau)}{(c\,g(\tau)+d)^2}. \qquad (16)$$

Thus, taking into account the property of the Schwarz derivative, from (13), (14), and (15), we obtain the integral of motion [4]

$$K_1\sqrt{\dot{x}^2(\tau,0)} + K_2\sqrt{\dot{x}^2(\tau \pm l, l)} = h, \qquad (17)$$

where h is a positive constant of integration. The equality (17) can be interpreted geometrically as follows. Since the length of a boundary curve L_i between points τ_1 and τ_2 is given by

$$L_i(\tau_1,\tau_2) = \int\limits_{\tau_1}^{\tau_2}\sqrt{\dot{x}^2(\tau,\sigma_i)}\,d\tau,$$

then integrating (17) in the interval $[\tau_1,\tau_2]$ and expressing the curvature K_i through the curvature radius $R_i = 1/K_i$, we arrive at the equality

$$\frac{L_1(\tau_1,\tau_2)}{R_1} + \frac{L_2(\tau_1,\tau_2)}{R_2} = h(\tau_2 - \tau_1).$$

From this expression it is seen that the sum of the lengths of boundary curves divided by constant radii R_i of their curvatures grows linearly with the parameter τ as though their element of the length were constant $\sqrt{\dot{x}^2_{i,0}}$. Consequently, we can set the constant h to be equal to

$$h = \frac{\sqrt{\dot{x}^2_{1,0}}}{R_1} + \frac{\sqrt{\dot{x}^2_{2,0}}}{R_2}.$$

In the Euclidean geometry, these curves are called the Bertrand curves [2]. When $K_1 = K_2,\ (m_1 = m_2)$, they are conjugate Bertrand curves, i.e., the centre of curvature of one curve lies always on the other curve.

2. DEFINITION OF THE STRING WORLD SURFACE

The representation of $\sqrt{\dot{x}^2(\tau,\sigma_i)}$ in the form

$$\sqrt{\dot{x}^2(\tau,0)} = \frac{h}{K_1 + K_2\, p(\tau)}, \qquad \sqrt{\dot{x}^2(\tau+l,l)} = \frac{h\, p(\tau)}{K_1 + K_2\, p(\tau)}, \tag{18}$$

where $p(\tau)$ is a positive and periodic function $p(\tau+2l) = p(\tau)$, makes the integral of motion (17) an identity. From (8) and (18) we obtain

$$p(\tau) = \sqrt{\frac{\dot{x}^2(\tau+l,l)}{\dot{x}(\tau,0)}} = \left|\frac{f(\tau+2l) - g(\tau)}{f(\tau) - g(\tau)}\right| \sqrt{\frac{f'(\tau)}{f'(\tau+2l)}}.$$

Taking into account equality (16) for $f'(\tau+2l)$, we can express $g(\tau)$ through functions $f(\tau)$ and $p(\tau)$

$$g(\tau) = \frac{[a + p(\tau)]f(\tau) + b}{c\, f(\tau) + d + p(\tau)}, \quad g'(\tau) = \frac{f'(\tau)\mathcal{Q}[p] + p'(\tau)\mathcal{F}[f]}{[c\, f(\tau) + d + p(\tau)]^2},$$

where $\mathcal{Q}[p] = p^2(\tau) + (a+d)\, p(\tau) + 1,\ \mathcal{F}[f] = cf^2(\tau) + (d-a)\, f(\tau) - b$ are positive valuated polynomials if one assumes that $|a+d| < 2$.

Now from (18) we can express the function $f(\tau)$ in terms of the function $p(\tau)$ and constants $A,\ h,\ K_1,\ K_2$

$$\frac{f'(\tau)}{\mathcal{F}[f]} = \frac{\sqrt{p'^2 + \left(\frac{A}{h}\right)^2\, [K_1 + K_2\, p(\tau)]^2 \mathcal{Q}[p]} - p'(\tau)}{2\mathcal{Q}[p]}.$$

As a result we obtain from (14), (15) the elliptic equation for a positive definite function $p(\tau)$

$$p'^2(\tau) = h^2\, p^2(\tau) - \frac{A^2}{h^2}[K_1 + K_2\, p(\tau)]^2[p^2(\tau) + (a+d)p(\tau) + 1]. \tag{19}$$

Indeed, at the point $p(\tau) = 0$, Eq. (19) results in $p'^2(\tau) = -A^2\, K_1^2/h^2 < 0$, which is inadmissible. Consequently, $p(\tau)$ takes values either on the half-line $p(\tau) > 0$ or on $p(\tau) < 0$. We fix the sign: $p(\tau) > 0$.

Now we consider the solution of Eq. (19) for equal masses $m_1 = m_2 = m$, $K_1 = K_2 = K = \gamma/m$. In this case after putting $a + d = 2\cos 2\alpha,\ h^2 = 4AK\sin\alpha$ from (19) we derive more simple elliptic equation

$$p'^2(\tau) = h^2\, p^2(\tau) - \left(\frac{A\, K}{h}\right)^2 [1 + p(\tau)]^2[p^2(\tau) + 2\cos 2\alpha\, p(\tau) + 1]. \tag{20}$$

Substituting into Eq. (20) the expression

$$p(\tau) = \frac{\sqrt{2} - s(u)}{\sqrt{2} + s(u)},$$

where the new function $s(u)$ satisfies the inequality $|s(u)| < \sqrt{2}$, and the new variable $u = \tau h/2^{3/2}$, we arrive at the following equation for $s(u)$

$$s'^2(u) = s^4(u) - 6\, s^2(u) + 4\,(1 - \cot^2\alpha), \quad \cot^2\alpha < 1. \tag{21}$$

The general solution of this equation has the form

$$s(u) = s_0\, \frac{\mathcal{P}(u) - e_1 - \sqrt{(e_1 - e_2)(e_1 - e_3)}}{\mathcal{P}(u) - e_1 + \sqrt{(e_1 - e_2)(e_1 - e_3)}}, \tag{22}$$

where $s_0 = \sqrt{3 - \sqrt{s + 4\cot^2\alpha}} < \sqrt{2}$ is the amplitude of oscillations, $\mathcal{P}(u)$ is the periodic Weierstrass elliptic function [6] with real roots e_i

$$e_1 = 1, \quad e_2 = \sqrt{1 - ctg^2\alpha} - 1/2, \quad e_3 = -\sqrt{1 - \cot^2\alpha} - 1/2, \quad (e_1 + e_2 + e_3 = 0).$$

The real period $2\omega_1$ of $\mathcal{P}(u)$ is given by the elliptic integral

$$2\omega_1 = \int\limits_{e_1}^{\infty} \frac{dt}{\sqrt{(t - e_1)(t - e_2)(t - e_3)}} = \frac{h}{2^{3/2}} l.$$

It is to be fixed at $2\omega_1 = l\,\sqrt{2AK\sin\alpha}$, which results in the constraint on arbitrary constants: A, α, l, because the left-hand side of this equation is the function of α. The behavior of functions $\mathcal{P}(u)$ and $s(u)$ is drawn in Figs. 2 and 3.

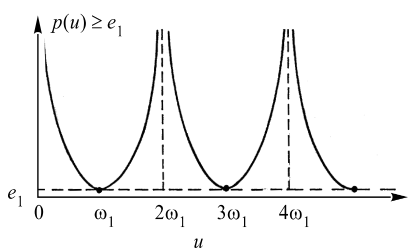

Fig. 2.

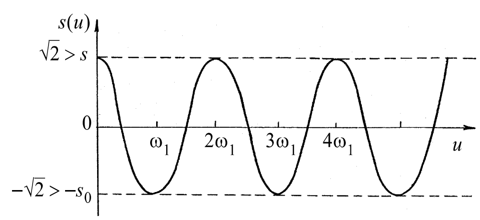

Fig. 3.

Thus, $s(u)$ defines $\dot{x}^2(\tau, \sigma_i)$ as a smooth periodic function

$$\sqrt{\dot{x}^2(\tau,0)} = \frac{h}{K}\,\frac{1}{1+p(\tau)} = \sqrt{\frac{A\sin\alpha}{K}}\left(1+\frac{s(u)}{\sqrt{2}}\right), \qquad \dot{x}^2(\tau,0) = \dot{x}^2(\tau,l),$$
$$\sqrt{\dot{x}^2(\tau+l,l)} = \frac{h}{K}\,\frac{p(\tau)}{1+p(\tau)} = \sqrt{\frac{A\sin\alpha}{K}}\left(1-\frac{s(u)}{\sqrt{2}}\right). \tag{23}$$

To compute the functions $f(\tau),\ g(\tau)$, and string coordinates, let us introduce the trigonometric representation for these functions through the angles $\phi(\tau)$ and $\theta(\tau)$

$$f(\tau) = \sqrt{2}\tan\left[\frac{\phi(\tau)-\theta(\tau)}{2}\right], \quad g(\tau) = -\sqrt{2}\cot\left[\frac{\phi(\tau)+\theta(\tau)}{2}\right].$$

In the frame of reference, where

$$a^\mu = \frac{1}{\sqrt{2}}\{1,0,1\}, \quad b^\mu = \{0,1,0\}, \quad c^\mu = \frac{1}{\sqrt{2}}\{1,0,-1\},$$

we get

$$\begin{aligned}
\psi_+^{'\mu}(\tau+\sigma) &= \frac{A}{\phi'(\tau+\sigma)-\theta'(\tau+\sigma)}\left\{1;\sin[\phi-\theta];\cos[\phi-\theta]\right\}, \\
\psi_-^{'\mu}(\tau-\sigma) &= \frac{A}{\phi'(\tau-\sigma)+\theta'(\tau-\sigma)}\left\{1;-\sin[\phi+\theta];\cos[\phi+\theta]\right\},
\end{aligned}$$

where the angles $\phi(\tau),\ \theta(\tau)$ are expressed through the elliptic functions $s(u)$ in the following manner:

$$\phi'(\tau) = \sqrt{A\,K\,\sin\alpha}\,\frac{2-s^2(u)}{2\cot^2\alpha+s^2(u)}; \quad \theta'(\tau) = -\sqrt{A\,K\,\sin\alpha}\,\frac{s'(u)}{2\cot^2\alpha+s^2(u)}. \tag{24}$$

In the case when $s(u) = \text{const}$ and, as a consequence, $\phi'(\tau) = \text{const} = \omega,\ \theta'(\tau) = 0,\ \theta(\tau) = \theta_0$, one gets a straight-line string with the angular velocity ω (compare (12))

$$\begin{aligned}
\phi_+^{'\mu}(\tau+\sigma) &= \frac{A}{\omega}\left\{1,\sin[\omega(\tau+\sigma)-\theta_0],\cos[\omega(\tau+\sigma)-\theta_0]\right\}, \\
\phi_-^{'\mu}(\tau-\sigma) &= \frac{A}{\omega}\left\{1,-\sin[\omega(\tau-\sigma)+\theta_0],-\cos[\omega(\tau-\sigma)+\theta_0]\right\}.
\end{aligned}$$

In general case, by integration of (24) we obtain for the angle $\phi(\tau)$ the expression

$$\phi(\tau) = \phi(0) + \phi'(0)\tau + i\left\{[J(u_1)-J(u_2^*)]\,u + \frac{1}{2}\,\ln\left[\frac{\sigma(u-u_1)\sigma(u+u_1^*)}{\sigma(u+u_1)\sigma(u-u_1^*)}\right]\right\},$$

where $\sigma(u)$ is the Weierstrass entire function; $J(u) = -\int \mathcal{P}(u)du$ is a quasi-periodic function; u_1 is a complex constant determined by the equation $s(u_1) = i\sqrt{2}\cot\alpha$. For the angle θ, one obtains:

$$\theta(\tau) = \text{arcctg}\left[\frac{s(u)}{\sqrt{2}}\tan\alpha\right] - \alpha.$$

Now one can determine the string vectors:

$$\begin{aligned} \dot{x}^\mu(\tau,0) = \dot{x}^\mu(\tau,l) &= \frac{A}{\phi'^2(\tau) - \dot{\theta}^2(\tau)}\left\{\dot{\phi}(\tau), \dot{\vec{x}}(\phi(\tau),\theta(\tau))\right\}, \\ x'^\mu(\tau,0) = -x'^\mu(\tau,l) &= \frac{A}{\phi'^2(\tau) - \theta'^2(\tau)}\left\{-\dot{\theta}(\tau), \vec{x}'(\phi(\tau),\theta(\tau))\right\}. \end{aligned}$$

For these solutions we cannot turn to the gauge $t = \tau$, because

$$\dot{t}(\tau,\sigma_i) = \frac{A\dot{\phi}(\tau)}{\dot{\phi}^2(\tau) - \dot{\theta}^2(\tau)}, \quad t'(\tau,\sigma_i) = \frac{-A\dot{\theta}(\tau)}{\dot{\phi}^2(\tau) - \dot{\theta}^2(\tau)}.$$

The string world surface is not a helicoid and does not belong to the class of developable surfaces (ruled surfaces), therefore, it describes transverse excitations of the string and radial motions of the masses m_i.

3. THE OSCILLATION WITH A SMALL AMPLITUDE: $s_0 = \sqrt{2}\,\varepsilon \ll 1$

If oscillation has a small amplitude $s_0 = \sqrt{3 - \sqrt{5 + 4\cot^2\alpha}} = \sqrt{2}\,\varepsilon$, then $\cot^2\alpha = 1 - 3\varepsilon^2 \sim 1$, and we arrive at the degenerate case of the elliptic function $\mathcal{P}(u)$, when $e_2 \sim e_3$, $e_1 \simeq -2e_2$, $\omega_1 = \pi/\sqrt{6}$. In this case, we have

$$\mathcal{P}(u) = -\frac{1}{2} + \frac{3}{2}\frac{1}{\sin^2\left(\pi\frac{\tau}{l}\right)} - \frac{\varepsilon^2}{4}\cos\left(\frac{\pi\tau}{l}\right) + \mathcal{O}(\varepsilon^3),$$

$$s(u) = \sqrt{2}\,\varepsilon\cos\left(\pi\frac{\tau}{l}\right) + \mathcal{O}(\varepsilon^2).$$

Then from (23) we obtain simple expression for $\dot{x}^2(\tau,0) = \dot{x}^2(\tau,l)$

$$\dot{x}^2(\tau,\sigma_i) = \frac{A\sin\alpha}{K}\left[1 + \varepsilon\cos\left(\pi\frac{\tau}{l}\right)\right]^2$$

which satisfies the integral of motion (17)

$$K\sqrt{\dot{x}^2(\tau,0)} + K\sqrt{\dot{x}^2(\tau \pm l, l)} = 2\sqrt{AK\sin\alpha}.$$

In this approximation, the angles $\theta(\tau)$ and ϕ take the form

$$\begin{aligned}\theta(\tau) &= \operatorname{arcctg}\left[\varepsilon \cot\alpha\, \cos(\pi\tau/l)\right] - \alpha, \\ \phi(\tau) &= \phi(0) + (\pi - 2\alpha)\frac{\tau}{l} - \frac{\varepsilon^2}{\sqrt{3}}\sin(2\pi\tau/l).\end{aligned} \tag{25}$$

Now we can consider a geometrical picture of the movement of massive string ends in the (x, y)-plane. The element of length of boundary curve is given by

$$d\mathcal{L}^2 = \cos^2\alpha\,[1 - 2\varepsilon\cos(\pi\tau/l)]\,d\tau^2.$$

It is an ellipse with semiaxes (see Fig. 4)

$$a = \frac{2l}{\pi}\,(1 + \varepsilon/2)\,\cos\alpha, \quad b = \frac{2l}{\pi}\,(1 - \varepsilon/2)\,\cos\alpha.$$

Then the shape of the curved string is an ellipsoid to leading order in the parameter ε.

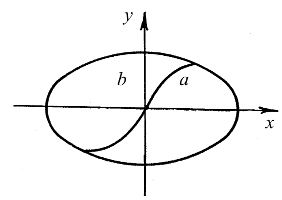

Fig. 4.

4. CONCLUSION

The geometrical method proposed here for solving the boundary problem in the theory of the relativistic string with massive ends is based on the torsions $\kappa_i(\tau)$ of world trajectories of the string ends, and the string world surface is completely determined by trajectories of massive ends. We investigated the shape of a confining string for periodic motion of its ends and showed that the shape of the curved string is an ellipsoid to the leading order in the parameter ε in deviation from straightness. It is possible to find that the angular momentum and energy are the same in this leading order as for a straight string, but the curved string has a small radial momentum $\sim \varepsilon^2$, not present in the straight string.

REFERENCES

1. **Barbashov B.M., Nesterenko V.V.** — Introduction to the Relativistic String Theory. World Scientific, Singapore, 1990.
2. **Norden A.P.** — Differential Geometry. M.: Uchpedgiz, 1949 (in Russian).
3. **Barbashov B.M., Chervyakov A.M.** — J. Phys. A: Math. Gen., 1991, v.24, p.11.
4. **Barbashov B.M.** — J. Phys. A: Math. Gen., 1997, v.30, p.4651.
5. **Barbashov B.M.** — Classical Dynamics of Rotating String with Massive Ends, in: Proceedings of the Workshop «Strong Interaction of Long Distances», Hadronic Press ISBN, 1995, p.257–275; JINR preprint E2-94-444, Dubna, 1994.
6. **Bateman H., Erdelyi A.** — Higher Transcendental Functions, v.3. Mc. Graw-Hill Book Company, New-York, 1955.