«ФИЗИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ И АТОМНОГО ЯДРА»
2000, ТОМ 31, ВЫП. 7А

УДК 539.12.01

NONLINEAR REPRESENTATIONS OF THE LORENTZ GROUP AND THE WIGNER FUNCTION FOR RELATIVISTIC FREE PARTICLES

O.I.Zavialov

V.A.Steklov Mathematical Institute, Moscow, Russia

The generalization of the Wigner function for the case of particles with relativistic Hamiltonian $H(\mathbf{p}) = \sqrt{\mathbf{p^2 + m^2}}$ is given; the transformation properties of the wave functions with respect to the Lorentz group are discussed.

1. INTRODUCTION

N.N.Bogoliubov was always interested in alternative, nonstandard approaches to the quantum theory, in particular, in the so-called Wigner approach [1].

The main idea of the Wigner approach was that in some cases the evolution of the quantum system of particles can be treated as the evolution of the classical ensemble, characterized by the (pseudo)probability distribution density $W(\mathbf{p}, \mathbf{q}; t)$ of particles in the phase (p, q)-space ($\mathbf{p}$ being the momentum of the particle, $\mathbf{q}$ being its coordinate). In such cases (for example, in case of the free nonrelativistic motion) the corresponding quantum dynamics, based on the Schrödinger equation, proves to be consistent with their classical dynamics, based on the Hamiltonian equations.

Thus the nice feature of the Wigner approach is that the quantum evolution and the evolution of the classical ensemble (1) are identical. However, such identity should not be understood too simple-mindedly: the Wigner function is not necessarily positive (that was what we really meant using the word «pseudo» in the neologism «pseudoprobability» previously).

The generalization of the Wigner function $W(\mathbf{p}, \mathbf{q}; t)$ for the free relativistic particles, that is for the Hamiltonian

$$H(\mathbf{p}) = \omega(\mathbf{p}) = \sqrt{\mathbf{p^2 + m^2}}, \tag{1}$$

was given by us (together with A.M.Malokostov) only quite recently [2]. We shall describe this generalization a bit later.

However, before this let us make several comments. It is well known that the straight-forward quantum mechanical theoretical scheme is compatible with the relativistic principles only for free particles (in fact, this was just the main reason to introduce quantum field theory). Difficulties arise even for free particles, for example, when trying to construct the coordinate operator $\widehat{\mathbf{x}}$ [3]. Nevertheless, it is generally assumed that free particles corresponding to the Hamiltonian (1) can possess the usual wave function in momenta representation (the function on the upper mass hyperboloid), let us say, $\psi(\mathbf{p})$. The scalar product of two such functions $\psi_1(\mathbf{p})$ and $\psi_2(\mathbf{p})$ is

$$\langle \psi_1(\mathbf{p}), \psi_{\mathbf{2}}(\mathbf{p})\rangle = \int \mathbf{d}\mu(\mathbf{p})\, \psi_{\mathbf{1}}^*(\mathbf{p})\psi_{\mathbf{2}}(\mathbf{p}),$$

where

$$d\mu(\mathbf{p}) = \frac{\mathbf{dp}}{\omega(\mathbf{p})}$$

is the invariant measure on the mass hyperboloid. One assumes that the transformation properties of the momentum wave functions $\psi(\mathbf{p})$ with respect to the Lorentz transformations Λ are as follows:

$$\psi_\Lambda(\mathbf{p}) = \psi(\Lambda\mathbf{p})$$

(this is the main assumption used, in particular, when constructing the Fock space). The standard momentum space $L_2(d\mathbf{p}, \mathbb{R}^{\mathbf{3}})$ of the square integrable wave functions $\psi'(\mathbf{p})$ with the Lebesque measure $d\mathbf{p}$ is achieved in result of the isomorphism

$$\psi'(\mathbf{p}) = \frac{\psi(\mathbf{p})}{\sqrt{\omega(\mathbf{p})}},$$

while the general transformation law of the wave function under the Lorentz transformations can take the form

$$\psi'_\Lambda(\mathbf{p}) = \sqrt{\frac{\omega(\Lambda\mathbf{p})}{\omega(\mathbf{p})}}\mathbf{e}^{\mathbf{i}\Omega(\Lambda\mathbf{p})-\mathbf{i}\Omega(\mathbf{p})}\psi'(\Lambda\mathbf{p}). \tag{2}$$

Here $\Omega(\mathbf{p})$ is an arbitrary real function.

In this space the momentum operator $\widehat{\mathbf{p}}$ is just the multiplication operator

$$\widehat{\mathbf{p}}\psi'(\mathbf{p}) = \mathbf{p}\psi'(\mathbf{p}).$$

It follows from the canonical commutation relations that the coordinate operator $\widehat{\mathbf{x}}$ (if it exists) should take the form $\widehat{\mathbf{x}} = i\frac{\partial}{\partial \mathbf{p}}$ [3]. The respective coordinate wave function $\widetilde{\psi}'(\mathbf{x})$ (if such a function exists) should be simply the Fourier transform of the wave function $\psi'(\mathbf{p})$:

$$\widetilde{\psi}'(\mathbf{x}, \mathbf{t}) = \frac{\mathbf{1}}{(\mathbf{2}\pi)^{\mathbf{3/2}}} \int \mathbf{dp}\, \mathbf{e}^{\mathbf{ipx}}\psi'(\mathbf{p})\mathbf{e}^{-\mathbf{i}\omega(\mathbf{p})\mathbf{t}}. \tag{3}$$

Now we enumerate the desired properties of the possible relativistic analog $W(\mathbf{p},\mathbf{x};\mathbf{t})$ of the Wigner function:

1) it should be bilinear with respect to the wave function $\psi(\mathbf{p})$;

2) the function W being integrated over the coordinate $\mathbf{x}$ gives the probability density in the momentum space

$$\psi^*(\mathbf{p},\mathbf{t})\psi(\mathbf{p},\mathbf{t}) = \int \mathbf{dx}\,\mathbf{W}(\mathbf{p},\mathbf{x},\mathbf{t}); \tag{4}$$

3) the function W being integrated over the momentum $\mathbf{p}$ gives the probability density in the coordinate space

$$\widetilde{\psi}^*(\mathbf{x},\mathbf{t})\widetilde{\psi}(\mathbf{x},\mathbf{t}) = \int \mathbf{dp}\,\mathbf{W}(\mathbf{p},\mathbf{x},\mathbf{t});$$

4) the Wigner function admits the classical evolution law, namely:

$$W(\mathbf{p},\mathbf{x},\mathbf{t}+\tau) = \mathbf{W}\Big(\mathbf{p},\mathbf{x}-\frac{\mathbf{p}}{\omega(\mathbf{p})}\tau,\mathbf{t}\Big). \tag{5}$$

Now we claim that the natural relativistic analog for the Wigner function can be given by the following relation

$$\begin{aligned} W(\mathbf{p},\mathbf{x},\mathbf{t}) \quad = \quad & \frac{1}{(2\pi)^3}\int d\mathbf{p_1}\,\mathbf{dp_2}\,\psi'^*(\mathbf{p_1})\psi'(\mathbf{p_2})\delta\big(\mathbf{p}-(\mathbf{p_1}\dotplus\mathbf{p_2})\big) \\ & \times\exp\Big(i\big(\omega(\mathbf{p_1})-\omega(\mathbf{p_2})\big)\mathbf{t}+\mathbf{i}(\mathbf{p_2}-\mathbf{p_1})\mathbf{x}\Big). \end{aligned} \tag{6}$$

Here the symbol $\dotplus$ serves to denote the special «sum on the mass shell»: if one introduces two 4-vectors $P_1 = (\omega(\mathbf{p_1}),\mathbf{p_1})$ and $P_2 = (\omega(\mathbf{p_2}),\mathbf{p_2})$, then by definition

$$P_1 \dotplus P_2 \equiv m\frac{P_1+P_2}{\sqrt{(P_1+P_2)^2}}.$$

The quantity $\mathbf{p_1}\dotplus\mathbf{p_2}$ is just the space part of the 4-vector $P_1 \dotplus P_2$. In other words,

$$\mathbf{p_1}\dotplus\mathbf{p_2} = \mathbf{m}\frac{\mathbf{p_1}+\mathbf{p_2}}{\sqrt{\mathbf{2}\big(\mathbf{m^2}+\omega(\mathbf{p_1})\omega(\mathbf{p_2})-\mathbf{p_1}\mathbf{p_2}\big)}}. \tag{7}$$

This function can be shown to satisfy all the necessary properties 1)–4).

Of course, the situation with the physical interpretation of the relativistic function W is not at all better than in nonrelativistic case. It can also be negative. Moreover, the general principles of the classical mechanics imply such transformation properties of the function $W(\mathbf{p},\mathbf{x},\mathbf{t})$ with respect to the Lorentz group which by no means follow from the transformation law (2). This shows once again that even for free particles incompatibility of the quantum mechanical principles and the special relativity might be deeper than it is usually assumed. However, in the present paper, we'll try to develop quite an opposite point of view: we'll make an attempt to find such transformation laws for the wave functions which lead to simple «natural» transformation properties for the function W (as if it really were the classical distribution density on the phase space). We shall see that such transformations (if they exist) are sure to be nonlinear, at least in two-dimensional space-time. However there is no doubt that, in their main features, our conclusions will remain valid also for the four-dimensional theory.

The described possibility (for nonlinearity of the transformation laws for the wave functions) seems interesting to us due to the following reasons. The combination of the wave functions constituting the function W gives rise to the family of trajectories of the Wigner (pseudo)particles quite independently of the naive «pseudoprobability» interpretation of the Wigner function $W(\mathbf{p}, \mathbf{x}; t)$. Every such trajectory generates a continuous sequence of some events in the space-time with any number of dimensions. If to return to four-dimensional reality, the four-dimensional coordinates describing these events must, in any case, be transformed (under a change of the reference frame) in agreement with the standard Einstein formulas no matter if the «straight-forward» Wigner interpretation makes any sense or not (if the combination $W(\mathbf{p}, \mathbf{x}; t)$, in principle, can be measured experimentally). Suppose that such nonlinear transformations can seriously compete with the usual linear transformations. In this case they have a chance to play an important, and probably a destructive role in the standard quantum field theory. In particular, the creation and annihilation operators $a^+(\mathbf{p})$ and $a(\mathbf{p})$ entering the standard scalar field

$$\varphi(x) = \frac{1}{\sqrt{2(2\pi)^3}} \int \frac{d\mathbf{k}}{\omega(\mathbf{k})} \left[a^+(\mathbf{k})\mathbf{e}^{\mathbf{i}\omega(\mathbf{k})\mathbf{x}^0 - \mathbf{ikx}} + \mathbf{a}(\mathbf{k})\mathbf{e}^{-\mathbf{i}\omega(\mathbf{k})\mathbf{x}^0 + \mathbf{ikx}}\right]$$

will change (so far, I don' know how) their transformation properties. In this case, will $\varphi(x)$ remain scalar? If not, which object will get the role of the scalar field? I would like to discuss this and other dissident questions in subsequent publications.

From now on let us be confined to the two-dimensional Minkovski space. In this case (see [2]) the Wigner function can be written as follows:

$$\begin{aligned} W(p, x, t) \quad &= \quad W(m\mathrm{sh}\gamma, x, t) = \frac{m}{2\pi\mathrm{ch}\gamma} \int d\gamma_1\, d\gamma_2 \mathrm{ch}\gamma_1 \mathrm{ch}\gamma_2 \\ &\times \delta\left(\gamma - \frac{\gamma_1 + \gamma_2}{2}\right) \Psi^*_{\gamma_1, x, t} \Psi_{\gamma_2, x, t}. \end{aligned} \tag{8}$$

Here, we changed the momenta variables p, p_1, and p_2 into the hyperbolic angles γ, γ_1, and γ_2:

$$m\mathrm{sh}\gamma = p, \qquad m\mathrm{sh}\gamma_1 = p_1, \qquad m\mathrm{sh}\gamma_2 = p_2.$$

Due to the fact that now the vectors $\mathbf{p}$, $\mathbf{p}_1$, $\mathbf{p}_2$ and $\mathbf{x}$ become one-dimensional they are denoted as p, p_1, p_2, and q, respectively. In (8) the following notations were also used

$$\Psi_{\gamma,x,t} = \Psi_\gamma e^{-itm\mathrm{ch}\gamma + ixm\mathrm{sh}\gamma}, \tag{9}$$

and

$$\Psi_\gamma \equiv \Psi(m\mathrm{sh}\gamma). \tag{10}$$

In what follows we set $m = 1$ and denote $W(p,0,0)$ like W_γ. We introduce also the Fourier-transforms related to all these functions. Namely,

$$\begin{aligned} \tilde{W}(\xi) &= \int d\gamma\, \mathrm{ch}\gamma W_\gamma e^{i\gamma\xi}, \\ \tilde{\Psi}(\xi) &= \int d\gamma\, \mathrm{ch}\gamma \Psi_\gamma e^{i\gamma\xi}. \end{aligned} \tag{11}$$

Next we calculate $\tilde{W}(\xi)$ and due to (12) arrive at the equation

$$\tilde{W}(\xi) = \frac{1}{2\pi}\tilde{\Psi}^*\left(-\frac{\xi}{2}\right)\tilde{\Psi}\left(\frac{\xi}{2}\right). \tag{12}$$

Now we consider an infinitely small Lorentz transformation characterized by a (small) hyperbolic angle γ_0. Let us call the result of the corresponding transformation of the initial Wigner ensemble to be «boosting» of the latter. It is not difficult to find out that *the Lorentz-invariance of the classical evolution of Wigner pseudoparticles implies that the «boosted» ensemble should correspond to a new Wigner function* $W'(p,x,0)$:

$$W(p,x,0) \to W'(p,x,0) = W(p',x'), \tag{13}$$

where

$$p' = p - \omega(p)\gamma_0, \tag{14}$$

$$x' = x\left[1 + \frac{p}{\omega(p)}\gamma_0\right]. \tag{15}$$

In order to guess how the transformation law (with respect to Lorentz boosts) for the wave functions $\psi(p)$ looks like let us use the formulas (13), (14) and (15) fixing the conditions of the Lorentz-invariance for the Wigner function. Here our speculations will be based only on partial information contained in relations

(13)–(15). Namely, let us now use these relations only for $x = 0$. One finds for infinitely small γ_0:

$$W(p,0;0) \to W'(p,0;0) = W(p',0;0), \tag{16}$$

where

$$p' = p - \gamma_0 \omega(p) = \mathrm{sh}\gamma - \gamma_0 \approx \mathrm{sh}(\gamma - \gamma_0), \tag{17}$$

and, for sure, in the right-hand side of (17) one has to keep only the terms of the first order with respect to γ_0. For the function $\tilde{W}(\xi)$ (given in (15)) the transformation (13) takes the form

$$\tilde{W}(\xi) \to \tilde{W}'(\xi) = W(\xi) + i\xi\gamma_0\tilde{W}(\xi) + \gamma_0 \int d\gamma\, e^{i\xi\gamma} W_\gamma(\mathrm{sh}\gamma) \tag{18}$$

$$= \tilde{W}(\xi)[1 + i\gamma_0\xi] + \gamma_0 \int d\eta\, \Delta(\xi - \eta)\tilde{W}(\eta), \tag{19}$$

where $\Delta(\xi)$ is the Fourier-transform (in the sense of distributions) of the function $(\mathrm{th}\gamma)$:

$$\Delta(\xi) = \int \frac{d\gamma}{2\pi} (\mathrm{th}\gamma)\, e^{i\xi\gamma}. \tag{20}$$

Now let us use the relation (12) in order to find the relation between the initial wave function $\tilde{\Psi}(\xi)$ and the transformed wave function $\tilde{\Psi}'(\xi)$. We claim that the «condition (13) of the Lorentz-invariance» of the Wigner function $\tilde{W}(\xi)$ implies that the wave function $\tilde{\Psi}(\xi)$ is transformed according to the law

$$\tilde{\Psi}(\xi) \to \tilde{\Psi}'(\xi) = \tilde{\Psi}(\xi) + i\gamma_0\xi\tilde{\Psi}(\xi) \tag{21}$$

$$+ \gamma_0 \int d\eta\, \Delta(2\xi - 2\eta) \frac{\tilde{\Psi}^*(-\eta)\tilde{\Psi}(\eta)}{\tilde{\Psi}^*(-\xi)} + \gamma_0 A(\xi)\tilde{\Psi}(\xi), \tag{22}$$

where $A(\xi)$ is an arbitrary function, satisfying the condition

$$A^*(-\xi) = -A(\xi).$$

Thus we came to the indication that, may be, the transformations of the momentum wave functions $\psi(p)$ (used, in particular, to construct the Fock space) are realized by nonlinear operators. The possible class of such transformations is given (infinitesimally) by the formula (20), which is valid, literally, only on the dense set of the Hilbert space consisting of functions $\tilde{\Psi}(\xi)$ that have no zeroes on the real axis.

Of course, one should try to take into account the complete condition of the Lorentz-invariance for the Wigner function beyond the plane $x = 0$. This is sure to impose additional restrictions on the possible functions $A(\xi)$ entering (20). May be, choosing the appropriate function $A(\xi)$, one can manage that the nonlinear operation (20) will conserve the Hilbert norm of the function $\psi(p)$.

REFERENCES

1. **Wigner E.** — Phys. Rev., 1932, v.40, p.749.
2. **Zavialov O.I., Malokostov A.M.** — Teor. Mat. Fiz., 1999, v.119, p.67.
3. **Newton T., Wigner E.** — Rev. Mod. Phys., 1949, v.21, p.400.