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TRIVIALITY OF THE LADDER APPROXIMATION TO QCD

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The validity of the ladder approximation (LA) in QCD in the context of the corresponding Schwinger–Dyson (SD) and Slavnov–Taylor (ST) equations is investigated. In contrast to QED, in QCD because of color degrees of freedom the summation of the ladder diagrams within the Bethe–Salpeter (BS) integral equation for the quark-gluon vertex at zero momentum transfer on the account of the corresponding ST identity does provide an addition constraint on the quark SD equation itself. Moreover, the solution of the constraint equation requires the full quark propagator should be almost trivial (free-type) one, i.e., only trivial quark propagator is allowed in the LA to QCD.

1. INTRODUCTION

It is well known that the full dynamical information of any quantum field gauge theory such as QCD is contained in the corresponding quantum equations of motion, the so-called SD equations for propagators (lower Green's functions) and vertices (higher Green's functions) [1]. The BS type integral equations for higher Green's functions and bound-state amplitudes should be also included into this system. It should be complemented by the corresponding ST identities [1-3], which are consequences of the exact gauge invariance and therefore *«are exact constraints on any solution to QCD»* [1]. Precisely this system of equations can serve as an adequate and effective tool for the nonperturbative approach to QCD [1]. It is important to understand however, that the above-mentioned system is an infinite chain of strongly coupled highly nonlinear integral equations, so there is no hope for an exact solutions. For this reason, some truncation scheme is always needed in order to make these equations tractable for getting physical information from them.

Because of its relative simplicity the most popular is the LA in various forms. In general, it consists of approximating the full vertices by their free perturbative (point-like) counterparts in the corresponding kernels of the above-mentioned integral equations. The main problem of the LA is, of course, its self-consistency since unlike QED, QCD is much more complicated gauge theory. It contains many different sectors. Making some truncation in one sector, it is necessary to be sure that nothing is going wrong in other sectors since the above-mentioned SD system of equations may remain strongly coupled even after truncation. The key elements relating different sectors in QCD are the above-mentioned ST identities, playing thus an important role in the investigation of the problem of self-consistency of any truncation scheme. Everybody knows that the LA is bad (for example, it is explicitly gauge-dependent truncation scheme), nevertheless, everybody continues to use it. The problem is that until now there was not an exact criterion to prove or disprove the LA in gauge theories. Here we precisely propose how to formulate this criterion.

2. DERIVATION OF THE CONSTRAINT EQUATION

It is convenient to begin our analysis from the quark SD equation in QCD which is just the same as in QED with only trivial replacement $g^2 \to g_F^2$. Here $g_F^2 = g^2 C_F$ and C_F is the eigenvalue of the quadratic Casimir operator in the fundamental representation (for SU(N), in general, $C_F = (N^2-1)/2N = 4/3, N = 3$). Thus the renormalized version of the quark SD equation is

$$\tilde{S}^{-1}(p) = Z_2\, S_0^{-1}(p) + i\tilde{\Sigma}(p) = Z_2\, S_0^{-1}(p) - \tilde{g}_F^2 \int \frac{d^n l}{(2\pi)^n} \gamma_\alpha\, \tilde{S}(l)\, \gamma_\beta\, \tilde{D}_{\alpha\beta}(q), \tag{1}$$

where the renormalized quark propagator is related to the unrenormalized one as $S(p) = Z_2 \tilde{S}(p)$ and the full gluon propagator is renormalized as follows $D_{\alpha\beta}(q) = Z_V \tilde{D}_{\alpha\beta}(q)$ with the renormalization of the coupling constant beeing determined by the corresponding combination of the above and below introduced renormalization constants, $Z's$ [3]. The differential form of the renormalized quark SD equation becomes

$$\partial_\mu \tilde{S}^{-1}(p) = -Z_2 i \gamma_\mu + \partial_\mu i\tilde{\Sigma}(p) = -Z_2\, i\, \gamma_\mu + \tilde{g}_F^2 \int \frac{d^n l}{(2\pi)^n} \gamma_\alpha [\partial_\mu \tilde{S}(l)] \gamma_\beta \tilde{D}_{\alpha\beta}(q), \tag{2}$$

up to unimportant total derivative (which is assumed as usual to vanish at the ends of integration). This is the differential form of the quark SD equation relevant for further discussion.

Let us introduce now the renormalized ST identity [1]

$$\begin{aligned} k_\mu \tilde{\Gamma}_\mu^a(p,k)\left[Z_g^{-1} + \tilde{b}(k^2;\xi)\right] &= \left[Z_B^{-1}\, T^a - \tilde{B}^a(p,k;\xi)\right] \tilde{S}^{-1}(p+k) \\ &- \tilde{S}^{-1}(p) \left[Z_B^{-1}\, T^a - \tilde{B}^a(p,k;\xi)\right], \end{aligned} \tag{3}$$

where the corresponding renormalized quark-gluon vertex, the ghost self-energy and the ghost-quark scattering kernel are related to their unrenormalized counterparts as follows: $\Gamma_\mu(p,k) = Z_1^{-1}\tilde{\Gamma}_\mu(p,k)$ and $b(k^2;\xi) = Z_g \tilde{b}(k^2;\xi),\ B(p,k;\xi) =$

$Z_B \tilde{B}(p,k;\xi)$. For future aim we have introduced an explicit dependence on a gauge fixing parameter ξ into the ghost degrees of freedom. Let us note that in the ST identity (3) the transfer momentum runs through the ghost degrees of freedom. The following important relation between renormalization constants holds, $Z_1 Z_B = Z_2 Z_g$.

Differentiation of the renormalized ST identity (3) with respect to k_μ and then setting $k = 0$, yields

$$\begin{aligned} \tilde{\Gamma}^a_\mu(p,0)\left[Z_g^{-1} + \tilde{b}(0;\xi)\right] &= Z_B^{-1} T^a \partial_\mu \tilde{S}^{-1}(p) - \tilde{\Pi}^a_\mu(p,0;\xi) - \tilde{\Psi}^a_\mu(p;\xi)\tilde{S}^{-1}(p) \\ &+ \tilde{S}^{-1}(p)\tilde{\Psi}^a_\mu(p;\xi), \end{aligned} \tag{4}$$

where we introduced the following notations

$$\tilde{\Psi}^a_\mu(p;\xi) = \left[\frac{\partial}{\partial k_\mu}\tilde{B}^a(p,k;\xi)\right]_{k=0} \tag{5}$$

and

$$\tilde{\Pi}^a_\mu(p,0;\xi) = \tilde{B}^a(p,0;\xi)\partial_\mu \tilde{S}^{-1}(p). \tag{6}$$

It is assumed that possible unphysical kinematic singularities have been already removed from the vertex by means of Ball and Chiu procedure [4]. Also a few remarks are in order. Though the regular dependence of the ghost-quark scattering kernel $\tilde{B}^a(p,k;\xi)$ on the ghost self-energy momentum k is preserved by the general Taylor's result [5]

$$\tilde{B}^a(p,0;\xi) = \xi F^a(p) = 0 \quad at \quad \xi = 0, \tag{7}$$

which shows that this kernel exists at small k in any covariant gauge and vanishes only in the Landau gauge $\xi = 0$, nevertheless, the regular dependence of the ghost self-energy itself on its momentum is not so obvious. However, here we treat the ghost self-energy as a regular function of its momentum since a singular dependence (which, in principle should not be excluded $a\ priori$) requires completely different investigation and is left for consideration elsewhere.

The renormalized version of the BS-type integral equation for the color nonsinglet vertex at zero momentum transfer is

$$\tilde{\Gamma}^a_\mu(p,0) = -Z_1 i\gamma_\mu T^a - \tilde{g}^2 \int \frac{d^n l}{(2\pi)^n}\gamma_\alpha T^b \tilde{S}(l)\tilde{\Gamma}^a_\mu(l,0)\tilde{S}(l)\gamma_\beta T^b \tilde{D}_{\alpha\beta}(q). \tag{8}$$

Substituting (4) into the BS-type integral equation (8), one obtains

$$
\begin{aligned}
Z_B^{-1}T^a\partial_\mu \tilde{S}^{-1}(p) - \tilde{\Pi}_\mu^a(p,0;\xi) - \tilde{\Psi}_\mu^a(p;\xi)\,\tilde{S}^{-1}(p) + \tilde{S}^{-1}(p)\,\tilde{\Psi}_\mu^a(p;\xi) \\
= -Z_1[Z_g^{-1} + \tilde{b}(0;\xi)]i\gamma_\mu T^a \\
+\tilde{g}^2\, Z_B^{-1}\, T^b\, T^a\, T^b \int \frac{d^n l}{(2\pi)^n}\gamma_\alpha[\partial_\mu\, \tilde{S}(l)]\gamma_\beta \tilde{D}_{\alpha\beta}(q) \\
+\tilde{g}^2 \int \frac{d^n l}{(2\pi)^n}\gamma_\alpha T^b \tilde{S}(l)\tilde{\Pi}_\mu^a(l,0;\xi)\tilde{S}(l)\gamma_\beta T^b \tilde{D}_{\alpha\beta}(q) \\
-\tilde{g}^2 \int \frac{d^n l}{(2\pi)^n}\gamma_\alpha T^b \tilde{\Psi}_\mu^a(l;\xi)\tilde{S}(l)\gamma_\beta T^b \tilde{D}_{\alpha\beta}(q) \\
+\tilde{g}^2 \int \frac{d^n l}{(2\pi)^n}\gamma_\alpha T^b \tilde{S}(l)\tilde{\Psi}_\mu^a(l;\xi)\gamma_\beta T^b \tilde{D}_{\alpha\beta}(q),
\end{aligned}
\tag{9}
$$

where the obvious identity, $\partial_\mu S^{-1}(p) = -S^{-1}(p)[\partial_\mu\, S(p)]\, S^{-1}(p)$, has been already used. Let us now use the commutation relation between color matrices $[T^a, T^b] = if_{abc}T^c$, where f_{abc} are the antisymmetric $SU(3)$ structure constants with nonzero values, given for example in Ref. 6. Then one obtains, $T^bT^aT^b = [C_F - \frac{1}{2}C_A]T^a$, where C_F is the above-mentioned eigenvalue of the quadratic Casimir operator in the fundamental representation while C_A is the same but in the adjoint representation, $C_A = N$ for $SU(N)$ ($N = 3$ for QCD). So from this relation and on account of the renormalized version of the differential form of the quark SD equation (2) (because of $\tilde{g}^2 C_F = \tilde{g}_F^2$) and using $Z_1 Z_B = Z_2 Z_g$, one arrives at

$$
\begin{aligned}
\tilde{\Pi}_\mu^a(p,0;\xi) \quad &- \quad \tilde{\Psi}_\mu^a(p;\xi)\,\tilde{S}^{-1}(p) + \tilde{S}^{-1}(p)\,\tilde{\Psi}_\mu^a(p;\xi) = \\
&- \quad Z_1\tilde{b}(0;\xi)\, i\,\gamma_\mu\, T^a \\
&- \quad \frac{1}{2}\, C_A\, T^a\, \tilde{g}^2 Z_B^{-1} \int \frac{d^n l}{(2\pi)^n}\,\gamma_\alpha\,[\partial_\mu \tilde{S}(l)]\,\gamma_\beta\,\tilde{D}_{\alpha\beta}(q) \\
&+ \quad \tilde{g}^2 \int \frac{d^n l}{(2\pi)^n}\gamma_\alpha\, T^b\, \tilde{S}(l)\,\tilde{\Pi}_\mu^a(l,0;\xi)\,\tilde{S}(l)\gamma_\beta T^b\,\tilde{D}_{\alpha\beta}(q) \\
&- \quad \tilde{g}^2 \int \frac{d^n l}{(2\pi)^n}\gamma_\alpha T^b\,\tilde{\Psi}_\mu^a(l;\xi)\,\tilde{S}(l)\gamma_\beta T^b\,\tilde{D}_{\alpha\beta}(q) \\
&+ \quad \tilde{g}^2 \int \frac{d^n l}{(2\pi)^n}\gamma_\alpha\, T^b\,\tilde{S}(l)\,\tilde{\Psi}_\mu^a(l;\xi)\gamma_\beta\, T^b\,\tilde{D}_{\alpha\beta}(q).
\end{aligned}
\tag{10}
$$

This is a renormalized version of the general constraint equation which relates ghost degrees of freedom to those of quark ones in a new manner within the LA. In principle, the solutions of the quark SD equation (1) should be compatible with the solutions (if any) of the above derived constraint equation (10). At first sight it is too complicated integral equation, but nevertheless its analysis is rather simple. Its solution without ghosts (omitted by «hand»), which is standard procedure in the LA to covariant gauge QCD, immediately leads to the almost trivial quark propagator [7]. Let us now treat ghosts degrees of freedom in more sophisticated fashion.

3. SOLUTION WITH GHOSTS

In order to investigate the general constraint equation (10) in more sophisticated way, let us write down formal BS-type integral equations for the compositions defined in (5) and (6). Like the quark-gluon vertex, these objects are vector quantities at zero momentum transfer. They satisfy formally absolutely the same BS-type integral equations in the LA as the above-mentioned quark-gluon vertex at zero momentum transfer (8). Thus they are the LA equations for (5) and (6) in the same way as (8) is the LA to the exact BS integral equation for the quark-gluon vertex at zero momentum transfer. The one of the differences is that there are no point-like counterparts of these quantites in QCD. But the main difference is that, in contrast to the BS-type integral equation for the quark-gluon vertex (8), these equations are completely auxiliary. They have no independent role, the main purpose to use (or to postulate) them is to show (as was mentioned above) that nothing explicitly depends on ghost degrees of freedom in QCD. So in complete analogy with (8), the formal homogeneous BS-type equation in the LA for the composition (6) looks like

$$\tilde{\Pi}^a_\mu(p,0;\xi) = -\tilde{g}^2 \int \frac{d^n l}{(2\pi)^n} \gamma_\alpha T^b \tilde{S}(l) \tilde{\Pi}^a_\mu(l,0;\xi) \tilde{S}(l) \gamma_\beta T^b \tilde{D}_{\alpha\beta}(q). \tag{11}$$

For the composition defined in (5) it is convenient to write down the formal BS-type integral equations for its left- and right-hand side combinations, $\tilde{\Psi}^{a(l)}_\mu(p;\xi) = \tilde{\Psi}^a_\mu(p;\xi)\tilde{S}^{-1}(p)$ and $\tilde{\Psi}^{a(r)}_\mu(p;\xi) = \tilde{S}^{-1}(p)\tilde{\Psi}^a_\mu(p;\xi)$, respectively

$$\begin{aligned} \tilde{\Psi}^{a(l)}_\mu(p;\xi) &= -\tilde{g}^2 \int \frac{d^n l}{(2\pi)^n} \gamma_\alpha T^b \tilde{S}(l) \tilde{\Psi}^{a(l)}_\mu(l;\xi) \tilde{S}(l) \gamma_\beta T^b \tilde{D}_{\alpha\beta}(q), \\ \tilde{\Psi}^{a(r)}_\mu(p;\xi) &= -\tilde{g}^2 \int \frac{d^n l}{(2\pi)^n} \gamma_\alpha T^b \tilde{S}(l) \tilde{\Psi}^{a(r)}_\mu(l;\xi) \tilde{S}(l) \gamma_\beta T^b \tilde{D}_{\alpha\beta}(q). \end{aligned} \tag{12}$$

Substituting (11) and (12) into the constraint equation (10), we are left with

$$-Z_1 \tilde{b}(0;\xi) i\gamma_\mu = \frac{1}{2} C_A \tilde{g}^2 Z_B^{-1} \int \frac{d^n l}{(2\pi)^n} \gamma_\alpha [\partial_\mu \tilde{S}(l)] \gamma_\beta \tilde{D}_{\alpha\beta}(q). \tag{13}$$

From Eq. (2), however it follows

$$\int \frac{d^n l}{(2\pi)^n} \gamma_\alpha [\partial_\mu \tilde{S}(l)] \gamma_\beta \tilde{D}_{\alpha\beta}(q) = \tilde{g}_F^{-2} \partial_\mu i \tilde{\Sigma}(p), \tag{14}$$

so Eq. (13) by denoting $\tilde{b}_1(0;\xi) = 2Z_1Z_B(\frac{C_F}{C_A})\tilde{b}(0;\xi) = 2Z_2Z_g(\frac{C_F}{C_A})\tilde{b}(0;\xi)$, becomes $-i\gamma_\mu\tilde{b}_1(0;\xi) = \partial_\mu i\tilde{\Sigma}(p)$. Its general solution is, $i\tilde{\Sigma}(p) = -i\hat{p}\tilde{b}_1(0;\xi) + im_c$, where m_c is the constant of integration of the dimension of mass. From the quark SD equation (1) it finally follows

$$\tilde{S}(p) = Z_2S_0^{-1}(p) + i\tilde{\Sigma}(p) = \frac{i\tilde{Z}_2}{\hat{p} - \tilde{m}}, \tag{15}$$

where now $\tilde{Z}_2^{-1} = Z_2\Big(1 + 2Z_g(C_F/C_A)\tilde{b}(0;\xi)\Big)$ and $\tilde{m} = \tilde{Z}_2(m_c + Z_2m_0)$. Thus the use of the auxiliarly BS-type integral equations (11) and (12) allows one to treat ghost degrees of freedom in more sophisticated way than simply omit them by «hand», but, nevertheless the quark propagator in the LA again remains trivial one (15) apart from the redefinitions of the quark mass and the quark wave function renormalization constant with the help of ghost self-energy at zero point. Nothing explicitly depends on ghosts as it should be indeed in QCD.

4. CONCLUSIONS

In summary, we have formulated the method how to prove or disprove the LA in gauge theories such as QED and QCD. It is easy to show [7] that in QED the summation of the ladder diagrams within the BS integral equation for the quark-photon vertex at zero momentum transfer on account of the corresponding WT identity does not provide an additional constraint on the solution to the quark SD equation itself. In other words, there is no criterion to prove or disprove the use of the LA in QED. In contrast to this, in QCD because of color degrees of freedom the summation of the ladder diagrams within the BS integral equation for the quark-gluon vertex at zero momentum transfer on account of the corresponding ST identity *does* provide an addition constraint on the solution to the quark SD equation itself. Moreover, the solution of the constraint equation (10) requires that the full quark propagator should be almost trivial (free-type) one (15), i.e., there is *no nontrivial* quark propagator in QCD in the LA. In other words, there is no running quark mass in the LA to QCD as well. This is our main result. It does not depend on how one treats ghost degrees of freedom in the LA to covariant gauge QCD, omitting them by «hand» in the general constraint equation (10) or in more sophisticated fashion by using auxiliarly BS-type integral equations (11) and (12) there. Let us underline once more, that the existence of the constraint equation (10) in the LA to QCD in any gauge is only due to color degrees of freedom and not ghosts. Let us make one thing perfectly clear. The quark SD equation, remaining a nonlinear integral equation even in the LA, may have formally a number of nontrivial, approximate solutions. However, none of them (analytical or numerical) will satisfy the constraint which comes from the exact ST identity. So all results based on the LA to QCD in any covariant or noncovariant gauges should be reconsidered (see also Ref. 7).

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