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EXACT RESULTS FOR 1D SIMPLE-EXCLUSION PROCESS WITH ORDERED-SEQUENTIAL DYNAMICS AND OPEN BOUNDARIES

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An exact and rigorous calculation of the current and density profile in the steady state of the one-dimensional fully asymmetric simple-exclusion process (FASEP) with open boundaries and forward-ordered sequential dynamics is presented. An interpretation of the phase transitions between the different phases is given in terms of eigenvalue splitting from a bounded quasi-continuous spectrum.

1. INTRODUCTION

One-dimensional (1D) systems of particles, hopping stochastically to the nearest neighbors (with hard-core exclusion), provide examples of systems far from thermal equilibrium, which exhibit boundary-induced phase transitions and steady state phases with long-range correlations. Here we consider the current and density profile in the steady state of a 1D fully asymmetric simple-exclusion process (FASEP) on a chain of L sites, with open boundaries and forward-ordered sequential dynamics. Each site can be empty or occupied by exactly one particle. At each time step a particle is injected with probability α at the left end. Then each pair of nearest-neighbor sites is updated sequentially from the left to the right: a particle hops with probability p one site to the right, provided that site is empty. Finally, a particle is removed with probability β at the right end.

In the case of random-sequential dynamics, a matrix-product representation of the steady state probability distribution has been found by Derrida, Evans, Hakim, and Pasquier [1]. The representation involves two infinite-dimensional square matrices D and E, which act on the vectors of an auxiliary vector space $\mathcal{S}$, and satisfy a quadratic algebra known as the DEHP algebra. The open boundary conditions are taken into account by the action of the above matrices on two vectors, $|V\rangle \in \mathcal{S}$ and $\langle W| \in \mathcal{S}^{\dagger}$, the dual of $\mathcal{S}$. We make use of the mapping of the algebra for the ordered-sequential dynamics onto the DEHP algebra, suggested in [2]. Starting from one of the matrix representations of the DEHP algebra given in [1], we obtain matrices D and E with nonzero elements only on the main and the upper (for D), or lower (for E) next-to-the-main diagonal. These matrices solve the bulk algebra for the ordered-sequential update, $pDE = D+(1-p)E$, and satisfy the left, $\langle W|E = \alpha^{-1}\langle W|$, and right, $D|V\rangle = (\beta^{-1} - 1)|V\rangle$, boundary conditions. Crucial points for our method are: (i) the choice of the vectors $\langle W| = |V\rangle^T = (1,\ 0,\ 0,\ \dots)$, and (ii) the representation of the 'lattice translation operator' $C \equiv E + D$ as a symmetric tri-diagonal matrix. By standard arguments, the expressions for the stationary current J_L and particle density $\rho_L(i)$ at site i are

$$J_L = Z_{L-1}/Z_L, \qquad \rho_L(i) = Z_L^{-1}\langle W|C^{i-1}DC^{L-i}|V\rangle, \tag{1}$$

where $Z_L = \langle W|C^L|V\rangle$. In our representation J_L and $\rho_L(i)$ depend on the elements of the matrices D and C only in the first $[L/2] + 1$ rows and columns ($[x]$ denotes the entire part of $x \geq 0$). Therefore, for any finite L and a sufficiently large integer $M \geq [L/2]+1$, we can use a truncated M-dimensional representation of the matrices and vectors involved. The truncated lattice propagator C_M is

$$C_M(\xi,\eta) = \frac{d}{p}\begin{pmatrix} a+\xi+\eta & \sqrt{1-\xi\eta} & 0 & 0 & \dots & \dots \\ \sqrt{1-\xi\eta} & a & 1 & 0 & \dots & \dots \\ 0 & 1 & a & 1 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & a & 1 \\ \dots & \dots & \dots & \dots & 1 & a \end{pmatrix}, \tag{2}$$

where

$$d = \sqrt{1-p}, \quad a = d + d^{-1}, \quad \xi = \frac{p-\alpha}{\alpha d}, \quad \eta = \frac{p-\beta}{\beta d}. \tag{3}$$

In the limit $M \to \infty$ the results become exact for any size of the chain. Since the matrix C_M is (real or complex) symmetric, and has, as we have shown, a real nondegenerate spectrum, it can be diagonalized by a similarity transformation with an orthogonal matrix U_M. This makes possible the explicit calculation of the relevant scalar products. For details we refer the reader to [3].

2. SPECTRAL PROPERTIES OF C_M

Let $\lambda_M(k)$, $k = 1, \dots, M$, be the eigenvalues of $C_M(\xi,\eta)$. For $p \neq 0, 1$ we set $\lambda = (d/p)(a + 2x)$ and write the secular equation in the form

$$(1-\xi\eta)U_M(x) + (2x\,\xi\eta - \xi - \eta)U_{M-1}(x) = 0, \tag{4}$$

where $U_n(x)$ is the Chebyshev polynomial of the second kind. After the substitution: $x = \cos\phi$, if $|x| \leq 1$, and $x = \cosh\phi$, if $|x| \geq 1$, by assuming first $|x| \leq 1$ and $\xi\eta \neq 1$, we rewrite (4) as an equation for ϕ

$$\sin[(M+1)\phi]/\sin(M\phi) = (\xi + \eta - 2\xi\eta\cos\phi)/(1-\xi\eta). \tag{5}$$

We need to consider only the roots $\phi \in [0, \pi]$. The case of $|x| \geq 1$ is obtained by analytical continuation to imaginary ϕ. The condition $\xi\eta = 1$, or $(1-\alpha)(1-\beta) = 1-p$, defines a line on which the mean-field approximation is exact. The analysis of Eq. (5) shows that there are four regions in the square $\alpha, \beta \in [0,1]^2$ with different spectral properties of C_M. Their boundaries involve the mean-field line, as well as the lines $\xi = 1$ ($\alpha = \alpha_c \equiv 1-d$) and $\eta = 1$ ($\beta = \beta_c \equiv 1-d$).

Region A: $\alpha_c < \alpha \leq 1$ and $\beta_c < \beta \leq 1$. For sufficiently large M Eq. (5) has exactly M simple real roots $\phi_M(k)$, $k = 1, \dots, M$, in the interval $(0, \pi)$. The eigenvalues of the matrix C_M are

$$\lambda_M(k) = (d/p)[a + 2\cos\phi_M(k)], \qquad k = 1, \dots, M. \tag{6}$$

A complete set of orthonormal eigenvectors of C_M is given by the column-vectors $|u_M(k)\rangle$, $k = 1, \dots, M$, with components

$$|u_M(k)\rangle_1 \equiv u_M(1,k) = b_M(k)\frac{\sin[M\phi_M(k)]}{\sqrt{1-\xi\eta}},$$

$$|u_M(k)\rangle_l \equiv u_M(l,k) = b_M(k)\sin[(M+1-l)\phi_M(k)], \text{ for } l = 2, \dots, M, \tag{7}$$

where $b_M(k)$ is the normalization constant.

Region B: $(1-\alpha)(1-\beta) < 1-p$ and $\alpha < \alpha_c$ or $\beta < \beta_c$. For sufficiently large M Eq. (5) has $M-1$ simple real roots $\phi_M(k)$, $k = 2, \dots, M$, in the interval $(0, \pi)$. The missing eigenvalue of C_M is provided by the pair of complex conjugate imaginary solutions $\phi = \pm \mathrm{i}\phi_M(1)$ which yield the largest eigenvalue

$$\lambda_M(1) = (d/p)[a + 2\cosh\phi_M(1)]. \tag{8}$$

The remaining $M-1$ eigenvalues have the form (6).

Region C: $(1-\alpha)(1-\beta) > 1-p$ and $\alpha > \alpha_c$ or $\beta > \beta_c$. Now the off-diagonal elements $(C_M)_{1,2} = (C_M)_{1,2} = \mathrm{i}\sqrt{\xi\eta - 1}$, see Eq. (2), are imaginary. The largest eigenvalue of C_M has the same analytical form (8) as in regin B; the remaining $M-1$ eigenvalues have the form (6). The diagonalization problem in regions C and D (see below) differs from the one in regions A and B in that the matrix C_M is complex symmetric, and not Hermitian (or real symmetric).

Region D: $\alpha < \alpha_c$ and $\beta < \beta_c$. The essential difference from the previous case is that for sufficiently large M there are two large eigenvalues of the matrix C_M, which have the form (8) and map onto one another under the transformation $\xi \leftrightarrow \eta$. The remaining $M-2$ eigenvalues have the form (6). The case $\xi = \eta > 1$ is a special one, since then the two large eigenvalues $\lambda_M(1,2) = (d/p)(a + 2\cosh\xi) \pm O(\xi^{-M})$ become degenerate in the limit $M \to \infty$.

In the thermodynamic limit region A corresponds to the *maximum current phase*; regions B, C and D for $\xi > \eta$ ($\alpha < \beta$) belong to the *low-density phase*, and for $\xi < \eta$ ($\alpha > \beta$) belong to the *high-density phase*. The distinction between the latter three regions within a single phase is expeced to affect more subtle characteristics like density profile, correlation functions, rate of approach to the thermodynamic limit.

3. CALCULATION OF THE CURRENT

In region A we obtain in the limit $M \to \infty$ the exact result ($\xi \neq \eta$)

$$Z_L^{\mathrm{A}}(\xi,\eta) = \left(\frac{d}{p}\right)^L \left[\frac{\xi}{\xi-\eta} I_L(\xi) + \frac{\eta}{\eta-\xi} I_L(\eta)\right], \tag{9}$$

where

$$I_L(\xi) = \frac{2}{\pi}\int_0^{\pi} \mathrm{d}\phi \frac{(a+2\cos\phi)^L \sin^2\phi}{1-2\xi\cos\phi+\xi^2}. \tag{10}$$

The expression for $Z_L^{\mathrm{A}}(\xi,\xi)$ can be obtained by taking the limit $\eta \to \xi$ in (9).

In regions B and C there is a contribution from the single largest eigenvalue:

$$Z_L^{\mathrm{B,C}}(\xi,\eta) = \left(\frac{d}{p}\right)^L \frac{\xi-\xi^{-1}}{\xi-\eta}(a+\xi+\xi^{-1})^L + Z_L^{\mathrm{A}}(\xi,\eta) \quad (\xi > \eta). \tag{11}$$

The case $\eta > \xi$ follows from the above by exchanging places of ξ and η. In region D ($\xi \neq \eta$) there are separate contributions from the two large eigenvalues:

$$Z_L^{\mathrm{D}}(\xi,\eta)=\left(\frac{d}{p}\right)^L \left[\frac{\xi-\xi^{-1}}{\xi-\eta}(a+\xi+\xi^{-1})^L+\frac{\eta-\eta^{-1}}{\eta-\xi}(a+\eta+\eta^{-1})^L\right]+Z_L^{\mathrm{A}}(\xi,\eta). \tag{12}$$

On the line $\xi = \eta$ in region D Eq. (12) yields

$$Z_L^{\mathrm{D}}(\xi,\xi)=\left(\frac{p}{d}\right)^L \left[\frac{L(\xi-\xi^{-1})^2}{\xi(a+\xi+\xi^{-1})}+1+\xi^{-2}\right](a+\xi+\xi^{-1})^L+Z_L^{\mathrm{A}}(\xi,\xi). \tag{13}$$

The exact results for the current follow from Eq. (1) and the above expressions.

Current in the Maximum-Current Phase. By substituting the leading-order asymptotic form of the Laplace integral (10) in the expression for $Z_L^{\mathrm{A}}(\xi,\eta)$, we obtain the large-$L$ asymptotic form of the current

$$J_L^{\mathrm{m.c.}} = \frac{1-\sqrt{1-p}}{1+\sqrt{1-p}}[1+O(L^{-1})] \tag{14}$$

independently of the parameters α and β.

Current in the Low- and High-Density Phases. Due to the dominant contribution of the largest eigenvalue, we obtain that up to exponentially small in L corrections

$$J_L^{\mathrm{l.d.}}(\xi, \eta) \simeq (p/d)(a + \xi + \xi^{-1})^{-1} = \frac{\alpha(p - \alpha)}{p(1 - \alpha)}. \tag{15}$$

The result for the high-density phase follows under the replacement $\xi \leftrightarrow \eta$ ($\alpha \leftrightarrow \beta$):

$$J_L^{\mathrm{h.d.}}(\xi, \eta) \simeq (p/d)(a + \eta + \eta^{-1})^{-1} = \frac{\beta(p - \beta)}{p(1 - \beta)}. \tag{16}$$

Only on the line $\xi = \eta > 1$ in region D the current $J_L^{\mathrm{D}}(\xi, \xi)$ has $O(L^{-1})$ corrections to the thermodynamic limit, see Eq. (13). The limiting expressions for the current coincide with the mean-field results [4].

4. CALCULATION OF THE LOCAL DENSITY PROFILE

Here we present the large-L asymptotic forms only (for the exact results see [3]).

Local Density in the Maximum-Current Phase. To obtain the particle density profile on the *macroscopic scale* $r = i/L$, as $L \to \infty$, we assume that $i \gg 1$ and $L - i \gg 1$. Then, by using the assymptotic form of $Z_n(\xi, \eta)$ for $n \gg 1$, we obtain the density profile

$$\rho_L^{\mathrm{m.c.}}(rL) \simeq \frac{\sqrt{1-p}}{1 + \sqrt{1-p}} + \frac{L^{-1/2}\sqrt{d}}{\sqrt{\pi}(1+d)} \frac{1 - 2r}{\sqrt{r(1-r)}} \quad (0 < r < 1) \tag{17}$$

independently of the parameters α and β; it has the same shape as in the case of random-sequential dynamics, see Eq. (53) in [5].

Local Density in the Low-Density Phase. By neglecting terms which are uniformly in $i = 1, \ldots, L$ exponentially small as $L \to \infty$, we obtain that the local density of the low-density phase in regions B and C is given by

$$\rho_L^{\mathrm{B,C}}(i) \simeq \frac{\alpha(1-p)}{p(1-\alpha)} - \frac{\xi I_{L-i}(\xi) - \eta I_{L-i}(\eta)}{(a + \xi + \xi^{-1})^{L-i+1}}. \tag{18}$$

One clearly sees that the shape of the density profile drastically changes on crossing the phase boundary. In the low-density phase the profile is constant (up to exponentially small in L terms) near the left end of the chain, and changes exponentially fast near the right end. The bending of the profile near the right end is downward in region B and upward in region C. In the part of region D occupied by the low-density phase ($\xi > \eta > 1$) we obtain

$$\rho_L^{\mathrm{D}}(i) \simeq \frac{\alpha(1-p)}{p(1-\alpha)} + \frac{\eta-\eta^{-1}}{a+\xi+\xi^{-1}}\left(\frac{a+\eta+\eta^{-1}}{a+\xi+\xi^{-1}}\right)^{L-i} - \\ -\frac{\xi I_{L-i}(\xi)-\eta I_{L-i}(\eta)}{(a+\xi+\xi^{-1})^{L-i+1}}. \tag{19}$$

A comparison with Eq. (18) reveals a *new feature:* the leading-order asymptotic form of the density profile changes on passing from region C to region D *within* the low-density phase.

Local Density in the High-Density Phase. By ignoring the uniformly in $i = 1, \ldots, L$ exponentially small as $L \to \infty$ corrections, we obtain that the local density of the high-density phase in regions B and C is

$$\rho_L^{\mathrm{B,C}}(i) \simeq 1 - \frac{\beta}{p} + \frac{\eta I_{i-1}(\eta) - \xi I_{i-1}(\xi)}{(a+\eta+\eta^{-1})^i}. \tag{20}$$

The profile bends near the left end of the chain: upward in region B and downward in region C. In the part of region D occupied by the high-density phase ($\eta > \xi > 1$)

$$\rho_L^{\mathrm{D}}(i) \simeq 1 - \frac{\beta}{p} - \frac{\xi-\xi^{-1}}{a+\eta+\eta^{-1}}\left(\frac{a+\xi+\xi^{-1}}{a+\eta+\eta^{-1}}\right)^{i-1} + \frac{\eta I_{i-1}(\eta)-\xi I_{i-1}(\xi)}{(a+\eta+\eta^{-1})^i}. \tag{21}$$

As in region C, the profile bends downward near the left end of the chain. Its asymptotic form changes on passing from region C to region D *within* the high-density phase.

The above asymptotic expressions are in excellent agreement with the results of computer simulations. The bulk densities coincide with the mean-field results [4].

Local Density on the Coexistence Line. The condition $\xi = \eta > 1$ defines the coexistence line between the low- and high-density phases in region D. On the *macroscopic scale* of distance, i.e., when $r \equiv i/L = O(1)$ as $L \to \infty$, by ignoring the $O(L^{-1})$ corrections, we obtain

$$\rho_L^{\mathrm{coex}}(rL;\xi,\xi) \simeq \frac{1}{a+\xi+\xi^{-1}}\left[d+\xi^{-1}+(\xi-\xi^{-1})r\right]. \tag{22}$$

The local density changes linearly between the bulk densities of the low- ($r = 0$) and high-density ($r = 1$) phase.

5. CONCLUSIONS

For the FASEP with ordered-sequential dynamics open boundary conditions we have calculated rigorously the current and the local particle density, both for finite chains and in the thermodynamic limit. For any finite L these quantities are real-analytic functions of the parameters; only in the thermodynamic limit different asymptotic forms appear. We have shown that the asymptotic form of the profile changes when α or β crosses the value $1-\sqrt{1-p}$ within the high- or low-density phase, respectively. This reflects the appearance of a second correlation length, related to the next-to-the-largest eigenvalue of the lattice propagator. A similar fact has been found in the case of random-sequential dynamics [6].

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