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DEFORMATIONS OF THE BOSON AND FERMION REPRESENTATIONS OF $sp(4,R)$ AND $sp(4)$

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With a view towards applications in nuclear physics, the boson and fermion realization of the compact $sp(4)$ and noncompact $sp(4,R)$ and their q-deformed versions are investigated and compared. The deformed realizations are based on distinct deformations of the boson and fermion creation and annihilation operators. In the boson case there is a simple transformation of the «classical» bosons to q-deformed ones. In the fermion case an additional index is introduced in order to satisfy the Pauli principle and in this case a simple transformation function between the «classical» and q-deformed operators is not known. Three important reduction chains of these algebras are explored in both the classical and deformed cases. For the primary reduction, the $su(2)$ substrusture can be interpreted in both cases as a pseudospin algebra. The other two reductions in the fermion case are $su(2)$ algebras, associated with pairing between identical fermions or coupling of two fermions of different kinds. In the boson case the infinite deformed ladder series $u_q^0(1,1)$ and two infinite deformed discrete series $u_q^{\pm}(1,1)$ are obtained. Each reduction provides for a complete classification of the basis states. In the boson case the initial as well as the deformed representations act in the same Fock space, but the deformation in the fermion case leads to basis state whose content is very different from the classical one. In a Hamiltonian theory this implies a dependance of the matrix elements on the deformation parameter, leading to the possibility of greater flexibility and richer structures within the framework of q-deformed algebraic descriptions.

INTRODUCTION

Symplectic algebras can be used to describe many-particle systems. The noncompact $sp(2n,R)$ and compact $sp(2n)$ versions of these algebras enter naturally when the number of particles or couplings between the particles change in a pairwise fashion from one configuration to the next [1]. The applications to nuclear structure [2, 3] are based on different interpretations of the quantum numbers of the bosons or fermions used to create the respective representations.

The boson case gives a description of collective vibrational excitations of a system of particles moving in an n-dimensional harmonic oscillator potential. The realization of $sp(2n)$ in terms of fermions has been used to explore pairing correlations in nuclei [4]. For the simplest two-dimensional cases ($n = 2$) the compact $sp(4)$ is isomorphic to ($\sim$) $o(5)$ and the noncompact $sp(4, R) \sim o(3, 2)$ [2]. Both these cases have interesting physical applications and are easily generalized to higher dimensions.

In the last decade a lot of effort, from a purely mathematical [5, 6] as well as physical point of view, has been concentrated on various deformations of the classical Lie algebras. Deformed algebras introduce a new degree of freedom that can give a better explanation of nonlinear effects. Their study can also lead to a deeper understanding of the physical significance of the deformation. The general feature of these deformations is that in the limit, when the deformation parameter $q \to 1$, the q-algebra reverts back to the classical Lie algebra. As a result, many similarities between the classical Lie algebras and their deformations, especially with respect to their representations, can be exploited, particularly in physical applications.

Based on the analogous realization of «classical» and q-deformed boson representations of $sp(4, R)$ [7] and fermion representations of $sp(4)$ [8], we will outline the similarities, as well as the important differences, with respect to their subalgebraic and action space structures. The deformations of the $sp(4, R)$ and $sp(4)$ algebras are obtained in terms of standard q-bosons and q-deformed fermions, and following the «classical» procedure, we investigate the enveloping algebras of $sp(4, R)$ and $sp(4)$ that are so obtained. All the deformed compact and noncompact subalgebras in both realizations are considered. Methods for specifying labels of the basis states via eigenvalues of the operators generating these subalgebras are also presented.

1. DEFORMED CREATION AND ANNIHILATION OPERATORS OF BOSONS AND FERMIONS

1.1. q-Bosons. The q-deformation in the boson case is obtained by means of the transformation of the classical bosons $b_i^\dagger$ and b_i, $i = \pm 1$ with $(b_i^\dagger)^* = b_i$ [9]:

$$a_i^\dagger = \sqrt{\frac{[N_i]}{N_i}} b_i^\dagger, \quad a_i = \sqrt{\frac{[N_i + 1]}{N_i + 1}} b_i, \tag{1}$$

where $[X]_k \equiv \dfrac{q^{kX} - q^{-kX}}{q^k - q^{-k}}$. Obviously $(a_i^\dagger)^* = a_i$ and $N_i = b_i^\dagger b_i$, $i = \pm 1$ are the classical operators of the number of bosons of each kind. It is possible to interpret the deformation of the classical boson creation and annihilation operators $b_i^\dagger$ and b_i, where $i = \pm 1$, by analyzing the expansion of the coefficients in (1) in terms of the deformation parameter τ, introduced as $q = \mathrm{e}^{\tau}$:

$$\frac{[N_i]}{N_i} = 1 + \frac{1}{6}\left(N_i^2 - 1\right)\tau^2 + \frac{1}{12}\left(\frac{1}{10}N_i^4 - \frac{1}{3}N_i^2 + \frac{7}{30}\right)\tau^4 + O\left(\tau^6\right). \quad (2)$$

This is an infinite expansion containing all the *even* powers of the deformation parameter and also all the *even* powers of each of the operators N_i, $i = \pm 1$.

The q-deformed commutation relations for the deformed oscillator that follow from (1) are:

$$a_i a_i^\dagger - q^1 a_i^\dagger a_i = q^{-N_i}, \quad a_i a_i^\dagger - q^{-1} a_i^\dagger a_i = q^{N_i}, \quad (3)$$

$$[a_i, a_k^\dagger] = 0, i \neq k, \quad [a_i^\dagger, a_k^\dagger] = [a_i, a_k] = 0. \quad (4)$$

The commutation relations of the number operator with the q-deformed bosons are the same as for the classical case:

$$[N_i, a_i] = -a_i, \quad [N_i, a_i^\dagger] = a_i^\dagger. \quad (5)$$

1.2. q-Deformed Fermions. In a «classical» realization, fermion creation $c_{m,\sigma}^\dagger$ and annihilation $c_{m,\sigma}$ operators, where $\sigma = \pm 1$, are introduced for a state of total angular momentum $j = \dfrac{2k+1}{2}$, where $k = 0, 1, 2...$ with projection m along the z axis ($-j \leq m \leq j$). These operators satisfy Fermi anticommutation relations:

$$\{c_{m',\sigma'}, c_{m,\sigma}^\dagger\} = \delta_{m',m}\delta_{\sigma',\sigma}, \quad \{c_{m',\sigma'}^\dagger, c_{m,\sigma}^\dagger\} = \{c_{m',\sigma'}, c_{m,\sigma}\} = 0, \quad (6)$$

and Hermitian conjugation is given by $(c_{m,\sigma}^\dagger)^* = c_{m,\sigma}$.

In analogy with the boson case, to deform these operators we introduce Hermitian conjugate operators $\alpha_{m,\sigma}^\dagger$ and $\alpha_{m,\sigma}, (\alpha_{m,\sigma}^\dagger)^* = \alpha_{m,\sigma}$, $m = -j, -j + 1, ..., j$, $\sigma = \pm 1$, with a q-deformed anticommutation relation that holds for every σ and m in the form [6]:

$$\alpha_{m,\sigma}\alpha_{m,\sigma}^\dagger + q^{\pm 1}\alpha_{m,\sigma}^\dagger\alpha_{m,\sigma} = q^{\pm\mathcal{N}_{m,\sigma}}, \quad (7)$$

where $\mathcal{N}_{m,\sigma} = c_{m,\sigma}^\dagger c_{m,\sigma}$ and $\mathcal{N}_\sigma = \sum_{m=-j}^{j}\mathcal{N}_{m,\sigma}$ are the classical fermion number operators. Their action on the deformed fermion operators is defined as in the classical and boson cases (5):

$$[\mathcal{N}_\sigma, \alpha_{m,\sigma'}^\dagger] = \delta_{\sigma,\sigma'}\alpha_{m,\sigma'}^\dagger, \quad [\mathcal{N}_\sigma, \alpha_{m,\sigma'}] = -\delta_{\sigma,\sigma'}\alpha_{m,\sigma'}, \quad \sigma, \sigma' = \pm 1. \quad (8)$$

In this case a simple transformation function of the deformed fermion operators which depends only on a single term $\mathcal{N}_{m,\sigma}$ as in (7) is not easy to obtain. To facilitate a further comparison with the boson case, we shall use (7) but for the total sum $\mathcal{N}_\sigma$, divided by the dimension of the fermion space $2\Omega_j = 2j+1$, along with the requirement that the deformation is performed only on the σ index:

$$\alpha_{m,\sigma}\alpha^\dagger_{m,\sigma} + q^{\pm 1}\alpha^\dagger_{m,\sigma}\alpha_{m,\sigma} = q^{\pm\frac{\mathcal{N}_\sigma}{2\Omega_j}}. \tag{9}$$

Using both anticommutation relations, it follows that $\alpha^\dagger_{m,\sigma}\alpha_{m,\sigma} = \left[\dfrac{\mathcal{N}_\sigma}{2\Omega_j}\right]$ which yields

$$\sum_m \alpha^\dagger_{m,\sigma}\alpha_{m,\sigma} = 2\Omega_j\left[\frac{\mathcal{N}_\sigma}{2\Omega_j}\right], \quad \sum_m \alpha_{m,\sigma}\alpha^\dagger_{m,\sigma} = 2\Omega_j\left[1-\frac{\mathcal{N}_\sigma}{2\Omega_j}\right]. \tag{10}$$

In the $q \to 1$ limit, assuming that $\alpha^\pm_{m,\sigma} \to c^\pm_{m,\sigma}$, (10) reverts back to the classical formula for $\mathcal{N}_\sigma$. This justifies the introduction of the weight coefficient $\omega \equiv 1/(2\Omega_j)$ in (9). In this case the analog of (2) is obviously the same, but with respect to the ratio $\dfrac{\mathcal{N}_\sigma}{2\Omega_j}$ and thus reveals the dependence of the deformation of the fermion operators on the dimension of the shell for which they were introduced. The remaining anticommutation relations for the q-deformed operators can be chosen from among various possibilities [10] to coincide with the respective classical ones.

2. q-DEFORMED $sp_q(4,R)$ and $sp_q(4)$ ALGEBRAS AND THEIR SUBALGEBRAIC STRUCTURE

In analogy with the «classical» case we can construct from the q-deformed bosons the following set of ten operators:

$$\begin{gathered} F^q_{i,j} = a^\dagger_i a^\dagger_j, \quad G^q_{i,j} = (F^q_{j,i})^* = a_i a_j, \quad i,j = \pm 1, \\ J_\pm = a^\dagger_{\pm 1} a_{\mp 1}, \quad J_0 = \frac{1}{2}(N_1 - N_{-1}), \quad N = N_1 + N_{-1}. \end{gathered} \tag{11}$$

In this representation the raising and lowering operators $F^q_{i,j}$, $G^q_{i,j}$ and J_+, $J_- = J_+^*$ are deformed. The centralizing set of operators N and J_0 expressed as linear combinations of the classical number operators N_1, N_{-1} are retained after the deformation.

The set of generators of the deformed $sp_q(4)$ algebra analogous to (11), but in terms of the q-deformed fermion creation and annihilation operators $\alpha^\dagger_{m,\sigma}(\alpha_{m,\sigma})$, that fulfill the anticommutation relations (9), is:

$$\begin{aligned} F_{\sigma,\sigma'} &= \xi_{\sigma,\sigma'} \sum_{m=-j}^{j} (-1)^{j-m} \alpha^{\dagger}_{m,\sigma} \alpha^{\dagger}_{-m,\sigma'}, \\ G_{\sigma,\sigma'} &= \xi_{\sigma,\sigma'} \sum_{m=-j}^{j} (-1)^{j-m} \alpha_{-m,\sigma} \alpha_{m,\sigma'}, \end{aligned} \tag{12}$$

and

$$\begin{aligned} \tau_+ &= E_{1,-1} = \eta \sum_{m=-j}^{j} \alpha^{\dagger}_{m,1} \alpha_{m,-1}, \\ \tau_- &= E_{-1,1} = \eta \sum_{m=-j}^{j} \alpha^{\dagger}_{m,-1} \alpha_{m,1}, \\ \tau_0 &= \frac{1}{2}(\mathcal{N}_1 - \mathcal{N}_{-1}), \quad \mathcal{N} = \mathcal{N}_1 + \mathcal{N}_{-1}, \end{aligned} \tag{13}$$

where the constants are defined as $\xi_{\sigma,\sigma'} = \dfrac{\eta}{\sqrt{(1+\delta_{\sigma,\sigma'})}}$ and $\eta = \dfrac{1}{\sqrt{2\Omega_j}}$. The operator $F_{\sigma\sigma}(G_{\sigma\sigma})$ creates (annihilates) a q-deformed pair of particles of the same kind and by construction $F_{\sigma,\sigma'} = F_{\sigma',\sigma} = (G_{\sigma,\sigma'})^{\dagger}$. The additional index $m \neq 0$ of the fermion creation and annihilation operators is introduced in order to construct nonzero operators $F_{\sigma,\sigma}$ and $G_{\sigma,\sigma}$, but for each particular j we have a sum over all the m values so the index $\sigma = \pm 1$ defines the algebraic properties of the generators $F_{\sigma,\sigma'}$, $G_{\sigma,\sigma'}$ and $E_{\sigma,\sigma'}$. The two Cartan generators $\mathcal{N}_{\sigma}$, $\sigma = \pm 1$, are not deformed.

It is easy to calculate the commutation relations between the operators from the two sets (11) and (12), (13) using the q-boson commutation relations (3), (4) and the q-fermion anticommutation relations (9). They are given in [7, 8] and in the limit $q \to 1$ revert to the respective commutation relations of the algebra generators of $Sp(4,R)$ and $Sp(4)$, respectively. So in this way we have generated the q-deformed boson and fermion representations of $Sp_q(4,R)$ and $Sp_q(4)$.

2.1. Reductions of $sp_q(4,R)$ and $sp_q(4)$ to Compact Subalgebras. The reduction to subalgebras in the boson and fermion representations is the same as in the nondeformed case. The most important of these is the reduction to $u_q(2)$ given in the boson case by N and the operators $J_0, J_{\pm}$ (11) which commute in the following way:

$$[J_+, J_-] = [2J_0], \quad [J_0, J_{\pm}] = \pm J_{\pm}, \quad [N, J_k] = 0, \quad k = 0, \pm. \tag{14}$$

Since the operator N acts also as a first order invariant of $u_q(2)$, the decomposition $U_q(2) = SU_q(2) \otimes U(1)$ is realized, where the $su_q(2) \sim so_q(3)$ is generated by

$J_0, J_\pm$. The second order Casimir operator in this case is given by the operator

$$\mathbf{J}^2 = J_- J_+ + [J_0][J_0+1] = \left[\frac{N}{2}\right]\left[\frac{N}{2}+1\right] \tag{15}$$

and is expressed in terms of the first-order invariant N of $U_q(2)$.

The fermion analog of the same subgroup $U_q(2)$ of $Sp_q(4)$ is generated by the set of the operators $\tau_{0,\pm1}$ and $\mathcal{N}$ (13) with commutation relations

$$[\tau_+, \tau_-] = \left[2\frac{\tau_0}{2\Omega_j}\right], \quad [\tau_0, \tau_\pm] = \pm\tau_\pm, \quad [\mathcal{N}, \tau_\sigma] = 0, \quad \sigma = 0, \pm. \tag{16}$$

In this case, in the first commutator of (16) the resulting operator τ_0 is rescaled by the factor ω. This is also reflected in the second order Casimir operator of the subgroup $SU_q^\tau(2)$:

$$\boldsymbol{\tau}^2 = 2\Omega_j(\tau_-\tau_+ + [\omega\tau_0]\,[\tau_0+1]_\omega). \tag{17}$$

The Casimir operators (15) and (17) coincide with the classical ones in the limit $q \to 1$, but in the fermion case it is not possible to express $\boldsymbol{\tau}^2$ as a function of $\mathcal{N}$. As a result of the appearance of the shell dependent factor ω indexing the q-bracket in (17), a different deformation is introduced for the different j-shells, which gives additional freedom in the physical applications.

In the fermion case three more distinct representations of $U_q(2)$ are realized. The first one, a $u_q^0(2)$ subalgebra, is generated by the operators

$$K_{+1}^0 \equiv F_{1,-1}, \quad K_{-1}^0 \equiv G_{1,-1}, \quad K_0^0 \equiv \frac{\mathcal{N}}{2} - \Omega_j, \tag{18}$$

which commute in the following way:

$$\begin{gathered}[K_{+1}^0, K_{-1}^0] = \left[2\frac{K_0^0}{2\Omega_j}\right], \quad [K_0^0, K_{\pm1}^0] = \pm K_{\pm1}^0, \\ [\tau_0, K_k^0] = 0, \text{ where } k = 0, \pm1.\end{gathered} \tag{19}$$

The operator τ_0 (13) commutes with the generators of $su_q^0(2)$ (19) and acts as a first order invariant of $u_q^0(2) = su_q^0(2) \oplus u(1)$. The operators K_k^0, $k = \pm1$ couple q-deformed particles of two different kinds. The second order Casimir operator of the subgroup $SU_q^0(2)$ is given by

$$C_2(SU_q^0(2)) = 2\Omega_j\left(K_{-1}^0 K_{+1}^0 + \left[\omega K_0^0\right]\left[K_0^0+1\right]_\omega\right), \tag{20}$$

which coincides with the classical invariant in the limit $q \to 1$. Note that again the factor ω appears in the deformed invariant.

Next, two mutually complementary subalgebras $su_q^+(2)$ and $su_q^-(2)$ of $sp_q(4)$ are given by the operators:

$$F_{+1}^{\pm} = F_{\pm 1, \pm 1}, \quad G_{-1}^{\pm} = G_{\pm 1, \pm 1}, \quad E_0^{\pm} = \frac{\mathcal{N}_{\pm}}{2} - \frac{\Omega_j}{2}. \tag{21}$$

They have the following commutation relations:

$$\left[F_{+1}^{\pm}, G_{-1}^{\pm}\right] = \rho_{\pm}[4\omega E_0^{\pm}], \quad \left[E_0^{\pm}, F_{+1}^{\pm}\right] = F_{+1}^{\pm}, \quad \left[E_0^{\pm}, G_{-1}^{\pm}\right] = -G_{-1}^{\pm}, \tag{22}$$

with $\rho_{\pm} = \dfrac{q^{\pm 1} + q^{\pm\omega}}{2}$. The first order invariants $\mathcal{N}_{\mp 1}$ of $u_q^{\pm}(2)$ give the extension of $su_q^{\pm}(2)$ to the subgroup $u_q^{\pm}(2) = su_q^{\pm}(2) \oplus u^{\mp}(1)$. The operator $F_{+1}^{\pm}$ $(G_{-1}^{\pm})$ creates (destroys) a q-deformed pair of particles of the same kind. The Casimir invariant of the subgroup $SU_q^{\pm}(2)$ is given by

$$C_2(SU_q^{\pm}(2)) = \Omega_j \left(G_{-1}^{\pm} F_{+1}^{\pm} + \rho_{\pm}\left[2\omega E_0^{\pm}\right]\left[E_0^{\pm} + 1\right]_{2\omega}\right). \tag{23}$$

The Casimir operator coincides with the classical one in the limit $q \to 1$. In this expression the deformation factor 2ω indicates that these two subalgebras give the short roots of $sp_q(4)$. In the q-deformed case the factors $\rho_{\pm}$ distinguish the eigenvalues of the second order invariants of the two mutually complementary subgroups, which is an important difference from the «classical» and boson cases.

2.2. Noncompact Subalgebras of $Sp_q(4, R)$. So far we have focused on compact structures; we now turn to a consideration of the noncompact cases that appear in the boson representation of $sp_q(4, R)$. The boson counterpart of $u_q^0(2)$ is a deformation $u_q^0(1,1)$ [9] generated by the boson analogs of (18):

$$K_+^0 = F_{1,-1}^q, \quad K_-^0 = G_{1,-1}^q, \quad K_0^0 = \frac{1}{2}(N+1), \quad J_0. \tag{24}$$

Their commutators differ from (19) by a sign and rescaling factor in the first one:

$$\left[K_+^0, K_-^0\right] = -\left[2K_0^0\right], \quad \left[K_0^0, K_{\pm}^0\right] = \pm K_{\pm}^0. \tag{25}$$

The operator J_0 (11) commutes with the generators K_i^0, $i = 0, \pm 1$ of $su_q^0(1,1)$ and acts as a first order invariant of $u_q^0(1,1) = su_q^0(1,1) \oplus u_{J_0}(1)$. The second order Casimir invariant of $SU_q^0(1,1)$ is given by

$$(K^0)^2 \equiv \left[K_0^0\right]\left[K_0^0 - 1\right] - K_+^0 K_-^0 = [J_0]^2 - \left[\frac{1}{2}\right]^2, \tag{26}$$

which is expressed finally in terms of the squared q-deformed brackets of the first order invariant J_0, a result that does not hold in the fermion case.

Finally, the two mutually complementary deformed representations $u_q^{\pm}(1,1)$ [9], each realized by only one kind of q-boson, are the respective analogs of $u_q^{\pm}(2)$. The operators

$$K_+^{\pm} = \frac{1}{[2]} F^q_{\pm 1, \pm 1}, \quad K_-^{\pm} = \frac{1}{[2]} G^q_{\pm 1, \pm 1}, \quad K_0^{\pm} = \frac{1}{2}\left(N_{\pm 1} + \frac{1}{2}\right), \quad N_{\mp 1} \quad (27)$$

commute among themselves in the following way:

$$\left[K_+^{\pm}, K_-^{\pm}\right] = -[2K_0^{\pm}]_2, \quad \left[K_0^{\pm}, K_{\pm}^{\pm}\right] = \pm K_{\pm}^{\pm}. \quad (28)$$

The nondeformed operators $N_{\mp 1}$ extend the $su_q^{\pm}(1,1)$ to $u_q^{\pm}(1,1)$ and act as first-order Casimir invariants. The second-order Casimir invariant in this case, as in the classical case, is a constant but a q-number:

$$(K^{\pm})^2 = [K_0^{\pm}]_2 [K_0^{\pm} + 1]_2 - K_-^{\pm} K_+^{\pm} = 1/([2]^2)\left([1/2]^2 - 1\right). \quad (29)$$

3. ACTION SPACE OF THE BOSON REALIZATION OF $sp_q(4,R)$

The action space of the deformed boson representation of $sp_q(4,R)$ can be written down directly since there exists a simple transformation between the classical and q-bosons (1) which implies that this action space is equivalent to the classical case. As is well known, the classical bosons act in a Hilbert space $\mathcal{H}$ with a vacuum $|0\rangle$ so that $b_i\ |0\rangle = 0$. The scalar product in $\mathcal{H}$ is chosen so that $b_i^{\dagger}$ is the Hermitian conjugate of b_i $[(b_i^{\dagger})^* = b_i]$ and $\langle 0|0\rangle = 1$. The vectors

$$|\nu_1, \nu_{-1}\rangle = \frac{(b_1^{\dagger})^{\nu_1} (b_{-1}^{\dagger})^{\nu_{-1}}}{\sqrt{\nu_1! \nu_{-1}!}} |0\rangle, \quad (30)$$

where ν_1, ν_{-1} run over all nonnegative integers, form an orthonormal basis in $\mathcal{H}$. In terms of the deformed boson operators, the basis vectors (30) are $(a_i|0\rangle = 0)$:

$$|\nu_1, \nu_{-1}\rangle = \frac{(a_1^{\dagger})^{\nu_1} (a_{-1}^{\dagger})^{\nu_{-1}}}{\sqrt{[\nu_1]! [\nu_{-1}]!}} |0\rangle \equiv \frac{(b_1^{\dagger})^{\nu_1} (b_{-1}^{\dagger})^{\nu_{-1}}}{\sqrt{\nu_1! \nu_{-1}!}} |0\rangle, \quad (31)$$

where $[X]! = [1][2][3]...[X]$. They are the common eigenvectors of the boson number operators $N_1 = b_1^{\dagger} b_1$, $N_{-1} = b_{-1}^{\dagger} b_{-1}$ and $N = N_1 + N_{-1}$:

$$\begin{gathered} N_1|\nu_1, \nu_{-1}\rangle = \nu_1|\nu_1, \nu_{-1}\rangle, \quad N_{-1}|\nu_1, \nu_{-1}\rangle = \nu_{-1}|\nu_1, \nu_{-1}\rangle, \\ N|\nu_1, \nu_{-1}\rangle = \nu|\nu_1, \nu_{-1}\rangle, J_0|\nu_1, \nu_{-1}\rangle = i_0|\nu_1, \nu_{-1}\rangle, \end{gathered} \quad (32)$$

where $\nu = \nu_1 + \nu_{-1}$ and $i_0 = (\nu_1 - \nu_{-1})/2$.

The deformed boson representation of $sp_q(4,R)$ is reducible and decomposes into two irreducible ones, each acting in the subspaces $\mathcal{H}_+$ and $\mathcal{H}_-$ of $\mathcal{H}$ labeled by the eigenvalue of the invariant operator $P=(-1)^N$, where $\mathcal{H}_+$ is spanned by the vectors (30) with $\nu=\nu_1+\nu_{-1}$ *even* and $\mathcal{H}_-$ with ν *odd*, respectively. It follows from the reduction of $sp_q(4,R)$ that an irreducible unitary representation (IUR) of $U_q(2)$ is realized in each $\mathcal{H}^{\nu}_{\pm}$ space and therefore into a direct sum of eigensubspaces of N defined by the condition that ν is fixed.

The decomposition of each $\mathcal{H}_{\pm}$ space into a direct sum of eigenspaces of J_0 with eigenvalue i_0, induces an irreducible (ladder) representation of the $u^0_q(1,1)$, $\mathcal{H}^{i_0}_{\pm}$. The operators $N=N_1+N_{-1}$ and $J_0=\dfrac{1}{2}(N_1-N_{-1})$ can be considered as another complete set of operators, both diagonal in the basis (31) and therefore uniquely specifying the states. This follows from the Howe duality for the quantum groups $U^0_q(1,1)$ and $U_q(2)$ as mutually centralizing subgroups of $Sp_q(4,R)$ [11]. The spaces $\mathcal{H}_+$ and $\mathcal{H}_-$ are also decomposed into direct sums of eigensubspaces of N_{-1} and N_1 so in each $\mathcal{H}^{\nu_{\mp 1}}_{\pm}$ an IUR of $u^{\pm}(1,1)$ is realized.

3.1. Even Space of the Bosons. In what follows we will consider the $\mathcal{H}_+$ space with $\nu=\nu_1+\nu_{-1}$ *even*. The states of $\mathcal{H}_-$ can be obtained from the $\mathcal{H}_+$ states with the help of the operators $a^{\dagger}_k$ and a_k $(k=\pm 1)$. The later can be considered to be the odd generators of the superalgebraic extension of the even $sp_q(4,R)$. Looking forward to future applications, we represent the basis states $|\nu_1,\nu_{-1}\rangle\in\mathcal{H}_+$ as

$$|n_1,n_0,n_{-1}\rangle=\eta(n_1,n_0,n_{-1})(T^1_1)^{n_1}(T^1_0)^{n_0}(T^1_{-1})^{n_{-1}}|0\rangle\equiv|\nu_1,\nu_{-1}\rangle. \tag{33}$$

The operators $T^1_k, k=0,\pm 1$ are components of a first rank tensor with respect to the $SU_q(2)$ subgroup of $Sp_q(4,R)$ obtained from its pair raising generators $F^q_{i,j}$ (11) in the following way for $k=1/2(i+j)$:

$$T^1_{k=1/2(i+j)}=\sqrt{[1+\delta_{i,-j}]}F^q_{i,j}q^{1/2(iN_j+jN_i)},\quad i,j=\pm 1. \tag{34}$$

The newly introduced quantum numbers n_i $(i=0,\pm 1)$, which are needed to specify the states, are integers restricted by the linkages:

$$\nu_1=2n_1+n_0,\quad \nu_{-1}=2n_{-1}+n_0, \tag{35}$$

where $\eta(n_1,n_0,n_{-1})$ is the normalization factor. This representation of the basis states is useful for a consideration of appropriate mapping procedures [12]. Note that

$$\nu=2n=2n_1+2n_0+2n_{-1},\quad j_0=\frac{1}{2}(\nu_1-\nu_{-1})=n_1-n_{-1}. \tag{36}$$

The basis states (33) are presented in Table 1, below where ν enumerates the rows and i_0 the columns:

Table 1.

ν/i_0	3	2	1	0	-1	-2	-3
0				$\|0,0,0\rangle$			
2			$\|1,0,0\rangle$	$\|0,1,0\rangle$	$\|0,0,1\rangle$		
4		$\|2,0,0\rangle$	$\|1,1,0\rangle$	$\|0,2,0\rangle$ $\|1,0,1\rangle$	$\|0,1,1\rangle$	$\|0,0,2\rangle$	
6	$\|3,0,0\rangle$	$\|2,1,0\rangle$	$\|1,2,0\rangle$ $\|2,0,1\rangle$	$\|0,3,0\rangle$ $\|1,1,1\rangle$	$\|0,2,1\rangle$ $\|1,0,2\rangle$	$\|0,1,2\rangle$	$\|0,0,3\rangle$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

We now can consider two extreme cases for possible values of the additional quantum number n_0:

1. n_0 takes on maximal values. In this case we have $n_{-1} = 0$ or $n_1 = 0$ at $\nu_1 \neq \nu_{-1}$ and $n_{-1} = n_1 = 0$ at $\nu_1 = \nu_{-1}$. There are two possibilities:

- $\nu_1 \geq \nu_{-1}$ and the coupling of T_0^1 is to the maximal degree for the states $|n_1, n_0, 0\rangle$, where $n_1 = j_0$, $\max n_0 = \nu_{-1}$, $n_{-1} = 0$.

- $\nu_1 \leq \nu_{-1}$, with states $|0, n_0, n_{-1}\rangle$ with $n_1 = 0$, $\max n_0 = \nu_1$, $n_{-1} = -j_0$. This case corresponds to a coupling to maximal degree for the operator T_0 (the upper states in Table 1). With it we move along the columns by acting on the minimal weight state an infinite number of times. So this basis is associated with the $u_q^0(1,1)$ subalgebra of $sp(4,R)$.

2. n_0 takes on minimal values. Since we are in the space $\mathcal{H}_+$, the integers ν_1 and ν_{-1} are simultaneously *even* or *odd*. For ν_1 and ν_{-1} *even*, the states $|n_1, 0, n_{-1}\rangle$ with $\left(n_1 = \dfrac{\nu_1}{2},\ \min n_0 = 0,\ n_{-1} = \dfrac{\nu_{-1}}{2}\right)$ are realized. For ν_1 and ν_{-1} *odd*, we have $|n_1, 1, n_{-1}\rangle$ with $\left(n_1 = \dfrac{\nu_1}{2} - \dfrac{1}{2},\ \min n_0 = 1,\ n_{-1} = \dfrac{\nu_{-1}}{2} - \dfrac{1}{2}\right)$. In this way the q-deformed oscillators are coupled to maximal degrees in n_1 and n_{-1} for the components T_1^1 and T_{-1}^1 (bosons of the same kind). The basis states in $\mathcal{H}_+$ represented as $|n_1, n_0, n_{-1}\rangle$ vectors, in the case of $\min n_0 = 0$ or 1 are the ones below the states corresponding to the first case.

Each state from the left (right) diagonals of the pyramid on Table 1 is obtained by the action on the minimal weight state with the raising operators $T_1(T_{-1})$, respectively. These basis states correspond to the reduction of $sp_q(4,R)$ to the two mutually complementary subalgebras $u_q^{\pm}(1,1)$. These two forms of the basis states are equivalent. The transition between Case 1 and Case 2 is realized by means of the relation

$$(T_0^1)^2 = q^{-1}[2]T_1^1 T_{-1}^1. \tag{37}$$

3.2. Even Space of the Deformed Fermions. In general, the classical fermion operators act in a finite space $\mathcal{E}_j$ for a particular j-level. The finite nature of the representation is due to the Pauli principle, $c^\dagger_{m,\sigma}c^\dagger_{m,\sigma}|0\rangle = 0$, that allows no more than $2\Omega_j$ identical fermions in a single j-shell. In $\mathcal{E}_j$ the vacuum $|0\rangle$ is defined by $c_{m,\sigma}|0\rangle = 0$ and the scalar product is chosen so that $\langle 0|0\rangle = 1$.

The states that span the $\mathcal{E}_j$ space consist of different numbers of fermion creation operators acting on the vacuum. They form an orthonormal basis in each space and are eigenvectors of the fermion number operators $\mathcal{N}_1$, $\mathcal{N}_{-1}$ and hence of the total number operator $\mathcal{N} = \mathcal{N}_1 + \mathcal{N}_{-1}$. In this way they span two subspaces $\mathcal{E}_j^\pm$ labeled by the eigenvalue of the invariant operator $P = (-1)^N$ of $Sp(4)$. To explore pairing correlations in nuclei, we will consider the *even* space $\mathcal{E}_j^+$, which contains states of fermions coupled pairwise. Usually representations of $Sp(4)$ are labeled by the largest eigenvalue of the number operator N and the reduced isospin of the uncoupled fermions in the corresponding state [1]. In each representation of $Sp(4)$ the maximum number of particles is $4\Omega_j$ and the respective state consists of no uncoupled fermions (reduced isospin zero). It follows that only one quantum number is needed, namely Ω_j. Within a representation, Ω_j is dropped from the labeling of the states. Thus the symmetric representation consists of states of a system with a total angular momentum $J = 0^+$.

In general, the q-deformed fermion operators act as in the classical case in a finite metric space $\mathcal{E}_j$ for each particular j-level, with a vacuum $|0\rangle$ defined by $\alpha_{m,\sigma}\ |0\rangle = 0$ and $\langle 0|0\rangle = 1$. The q-deformed states are different from the classical ones, but reduce to the classical ones in the limit $q \to 1$.

In analogy with the boson case, we can define the fermion representations in terms of q-deformed boson-like creation $F^\dagger_k$ (annihilation G_{-k}) operators related to the $Sp_q(4)$ generators (12) in the following way:

$$F^\dagger_k = F^\dagger_{\frac{1}{2}(\sigma+\sigma')} \equiv F_{\sigma,\sigma'} \quad (G_{-k} = G_{-\frac{1}{2}(\sigma+\sigma')} \equiv G_{\sigma,\sigma'}), \tag{38}$$

where $k = \dfrac{1}{2}(\sigma+\sigma') = 0, \pm 1$, $\sigma, \sigma' = \pm 1$. The operators (38) can be considered as components of a tensor of rank 1 with respect to the subgroup $SU^\tau_q(2)$. These operators create a pair of q-fermions coupled to a total angular momentum $J = 0$ and a total isospin $T = 1$. In analogy with the classical limit and the boson case (33), a set of vectors that span each space $\mathcal{E}_j^+$ in the q-deformed case can be chosen to be of the form:

$$|\mu_1, \mu_0, \mu_{-1}) = \left(F^\dagger_1\right)^{\mu_1} \left(F^\dagger_0\right)^{\mu_0} \left(F^\dagger_{-1}\right)^{\mu_{-1}} |0\rangle\,. \tag{39}$$

The basis is obtained by orthonormalization of (39). The basis states (39) are uniquely specified by the classification schemes which use the $su_q(2)$ subalgebras and the relevant Cartan generators — the nondeformed operators $\mathcal{N}_{\pm 1}$, or equiv-

Table 2.

μ/i	2	1	0	-1	-2
0			$\|0,0,0)$		
2		$\|1,0,0)$	$\|0,1,0)$	$\|0,0,1)$	
4	$\|2,0,0)$	$\|1,1,0)$	$\|1,0,1)$ $\|0,2,0)$	$\|0,1,1)$	$\|0,0,2)$
6		$\|2,0,1)$ $\|1,2,0)$	$\|1,1,1)$ $\|0,3,0)$	$\|1,0,2)$ $\|0,2,1)$	
8			$\|2,0,2)$ $\|1,2,1)$ $\|0,4,0)$		

alently the set $\mathcal{N}$ and τ_0. The eigenvalues of these operators that label the basis states coincide with the ones in the classical case.

States (39) are the eigenvectors of the fermion number operators $\mathcal{N}_1$, $\mathcal{N}_{-1}$:

$$\mathcal{N}_1\,|\mu_1,\mu_0,\mu_{-1}) = (2\mu_1+\mu_0)\,|\mu_1,\mu_0,\mu_{-1})\,, \tag{40}$$

$$\mathcal{N}_{-1}\,|\mu_1,\mu_0,\mu_{-1}) = (2\mu_{-1}+\mu_0)\,|\mu_1,\mu_0,\mu_{-1})\,, \tag{41}$$

or equivalently of the operators $\mathcal{N}=\mathcal{N}_1+\mathcal{N}_{-1}$ and $\tau_0=\dfrac{1}{2}(\mathcal{N}_1-\mathcal{N}_{-1})$ which are both diagonal in the basis (39):

$$\mathcal{N}\,|\mu_1,\mu_0,\mu_{-1}) = \mu\,|\mu_1,\mu_0,\mu_{-1})\,,\ \ \mu = 2(\mu_1+\mu_{-1}+\mu_0). \tag{42}$$

$$\tau_0\,|\mu_1,\mu_0,\mu_{-1}) = i\,|\mu_1,\mu_0,\mu_{-1})\,,\ \ i = \mu_1-\mu_{-1}. \tag{43}$$

Their eigenvalues can be used to classify the basis within a representation Ω_j. The basis states labeled by $|\mu_1,\mu_0,\mu_{-1})$ for $\Omega_{3/2}=2$ are shown in Table 2, where μ enumerates the rows; and i, the columns.

In the fermion representation the basis vectors are degenerate in the sense that more than one of the common eigenstates of the operators $\mathcal{N}$ and τ_0 have one and the same eigenvalues $\{\mu,i\}$ and thus belong to one and the same cell of Table 2. Here a relation analogous to (37) exists only for the maximum weight states; an additional quantum number is needed to specify them completely. In the fermion realization of $sp_q(4)$ the eigenvalue of the second order Casimir operator, which is related to the seniority quantum number, is used to label the states.

1. The reduction chain $Sp_q(4)\supset SU_q^0(2)\otimes U_q(1)_{\tau_0}$ describes pairing between fermions of different types and corresponds to Case 1 for bosons with maximum n_0 value. These states are the last ones in each of the cells in Table 2. The seniority quantum number $s=(\mu_1+\mu_{-1})_{\max}$ introduced in the basis state labeling scheme, $|i,2(\mu_1+\mu_{-1})_{\max},\mu)$, comes from the eigenvalue of the second

order Casimir operator for this q-deformed subalgebra:

$$C_2(SU_q^0(2))|\mu_1,\mu_0,\mu_{-1}) = \\ = 2\Omega_j\left[\frac{1}{2\Omega_j}\right][\Omega_j-s]_\omega[(\Omega_j-s+1)]_\omega|\mu_1,\mu_0,\mu_{-1}). \quad (44)$$

2. The other reduction, $Sp_q(4)\supset U_q(2)_{N_\pm}\supset SU_q^\pm(2)\supset U_q(1)_{N_\mp}$, introduces deformation in the model of coupled fermions of the same kind and corresponds to Case 2 for bosons with minimal n_0 value. The basis states are placed first in each cell in Table 2. In this limit the labeling is $|\mu_{\mp 1},\mu_{0\,\max},\mu_{\pm 1})$, where $\mu_{0\,\max}=\{0 \text{ or } 1\}$ is the seniority quantum number. The action of the Casimir operator on the states is given by

$$C_2(SU_q^\pm(2))\,|\mu_1,\mu_0,\mu_{-1}) = \\ = \rho_\pm\ \Omega_j\left[\frac{1}{\Omega_j}\right]\left[\frac{\Omega_j-\mu_{0\,\max}}{2}\right]_{2\omega}\left[\frac{\Omega_j-\mu_{0\,\max}}{2}+1\right]_{2\omega}|\mu_1,\mu_0,\mu_{-1})\,. \quad (45)$$

In the deformed case the action of the Casimir invariant of $SU_q^+(2)$ differs from that of the Casimir invariant of $SU_q^-(2)$ by the factor ρ_+/ρ_-.

The deformed basis states are labeled by the classical eigenvalues of the invariant operators of the reduction in each of the cases considered. The matrix elements, particularly of the raising and lowering generators of $sp_q(4)$ and the second order invariants, are also deformed which leads to different results in physical applications. The deformation may lead to basis states whose content is very different from the classical case, since there is no known simple function that transforms the classical fermion operators $c^\dagger_{m,\sigma}$ and $c_{m,\sigma}$ into the q-deformed ones, $\alpha^\dagger_{m,\sigma}$ and $\alpha_{m,\sigma}$. Indeed, a smooth function may not exist since the anticommutation relations (9) hold simultaneously with both signs for one and the same σ, as defined by (9).

CONCLUSIONS

Deformed boson and fermion realizations of the simplest two-dimensional algebra, $sp_q(4,R)$ and $sp_q(4)$, have been introduced. We have investigated their subalgebras as well as the action spaces of their representations. Without going into a detailed investigation of the relations between these algebraic constructions that would require mapping procedures or a consideration of equivalent tensor structures, we were able to study their similarities and differences based on their «classical» realization, which are important in considering physical applications of the theory.

First of all the construction of the generators of both the algebras is analogous in terms of bilinear combinations of deformed boson or fermion creation and annihilation operators. In the boson case there is a simple transformation of the «classical» bosons to the q-deformed ones. In the fermion case an additional index is needed to satisfy the Pauli principle, so in practice $sp_q(4)$ is realized as a subalgebra of the direct product $sp_q(4) \oplus sp(2j+1)$. A simple transformation function between the «classical» and q-deformed operators is not known.

For the two primary reductions, the $su_q(2)$ substructure can be interpreted as a quasi-spin algebra. The other two reductions, which apply in the fermion case, are $su_q(2)$ algebras that can be associated either with pairing between identical fermions or the coupling of two fermions of different kinds. In the boson case the infinite deformed ladder series $u_q^0(1,1)$ and two infinite deformed discrete series $u_q^{\pm}(1,1)$ correspond to the pairing limits of the compact case.

In the boson case the classical as well as the deformed representations act in the same Fock space, which is infinite dimensional. The basis states are nondegenerate. Although different coupling schemes for the boson creation operators can be considered, there is a relation between the algebraic generators which links them.

The deformation in the *even* fermion case leads to basis states whose content is very different from the classical one. In this case the basis states can be degenerate, but each reduction provides a complete classification of the basis states, including a resolution of the degeneracy. In limiting cases the respective subalgebras provide for a physical interpretation of the different kinds of pairing with an associated seniority quantum number which appears in the eigenvalues of the respective second-order Casimir invariants.

The matrix elements of the second-order invariants and the deformed generators of $sp_q(4, R)$ and $sp_q(4)$ were also deformed. This introduced a new parameter which leads to different results in physical applications. In particular, in the fermion case this deformation depends on the dimension of the shell.

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