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SEPARATION OF VARIABLES AND LIE-ALGEBRA CONTRACTIONS. APPLICATIONS TO SPECIAL FUNCTIONS

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A review is given of some recently obtained results on analytic contractions of Lie algebras and Lie groups and their application to special function theory. The contractions considered are from $O(3)$ to $E(2)$ and from $O(2,1)$ to $E(2)$, or $E(1,1)$. The analytic contractions provide relations between separable coordinate systems on various homogeneous manifolds. They lead to asymptotic relations between basis functions and overlap functions for the representations of different groups.

INTRODUCTION

Lie-algebra contractions were introduced into physics by Inönü and Wigner [1] in 1953 as a mathematical expression of a philosophical idea, namely the *correspondence principle*. This principle tells us that whenever a new physical theory surplants an old one, there should exist a well defined limit in which the results of the old theory are recovered. More specifically Inönü and Wigner established a relation between the Lorentz group and the Galilei one in which the former goes over into the latter as the speed of light satisfies $c \to \infty$.

The theory of Lie-algebra contractions (and deformations) has acquired a life of its own. It provides a framework in which large sets of Lie algebras can be embedded into families depending on parameters. All algebras in such a family have the same dimension, but they are not mutually isomorphic [2].

Two types of Lie-algebra contractions exist in the literature. The first are standard Inönü–Wigner contractions [1, 3, 4]. They can be interpreted as singular changes of basis in a given Lie algebra L. Indeed, consider a basis $\{e_1, \ldots, e_n\}$ of L and a transformation $f_i = U_{ik}(\varepsilon)e_k$, where the matrix U realizing the transformation depends on some parameters ε. For $\varepsilon \to 0$ (i. e., some, or all of the components of ε vanishing) the matrix $U(\varepsilon)$ is singular. In this limit the commutation relations of L change (continuously) into those of a different, nonisomorphic, Lie algebra L'.

More recently, «graded contractions» have been introduced [5–7]. They are more general than the Inönü–Wigner ones and can be obtained by introducing parameters modifying the structure constants of a Lie algebra L_1 in a manner respecting a certain grading and then taking limits when these parameters go to zero.

It is well known that there exists an intimate relationship between the theory of special functions and Lie group theory, well presented in the books of Vilenkin [8], Talman [9], and Miller [10]. In fact all properties of large classes of special functions can be obtained from the representation theory of Lie groups, making use of the fact that the special functions occur as basis functions of irreducible representations, as matrix elements of transformation matrices, as Clebsch–Gordon coefficients, or in some other guise. Recently, the class of functions treatable by group theoretical and algebraic methods has been extended to the so-called q-special functions that have been related to quantum groups [11–13].

One very fruitful application of Lie theory in this context is the algebraic approach to the separation of variables in partial differential equations [14–19]. In this approach separable coordinate systems (for Laplace–Beltrami, Hamilton–Jacobi and other invariant partial differential equations) are characterized by complete sets of commuting second order operators. These lie in the enveloping algebra of the Lie algebra of the isometry group, or in some cases of the conformal group, of the corresponding homogeneous space.

A question, that up to the last few years has received little attention in the literature, is that of connections between the separation of variables in different spaces, e. g., in homogeneous spaces of different Lie groups. In particular, it is of interest to study the behavior of separable coordinates, sets of commuting operators and the corresponding separated eigenfunctions under deformations and contractions of the underlying Lie algebras.

A recent series of papers [20–29] has been devoted to this new aspect of the theory of Lie-algebra and Lie-group contractions: the relation between the separation of variables in spaces of nonzero constant curvature and in flat spaces. The curved spaces were realized as spheres $S_n \sim O(n+1)/O(n)$, Lorentzian hyperboloids $H_n \sim O(n,1)/O(n)$, or $O(n,1)/O(n-1,1)$. The flat spaces where either Euclidean E_n, or pseudo-Euclidean $E(n-1,1)$ ones. The curved and flat spaces were related by a contraction of their isometry groups and the corresponding isotropy groups of the origin.

The essential point of these articles was the introduction of «analytic contractions». The contraction parameter is R which is either the radius of the sphere

S_n, or the corresponding quantity $x_0^2 - x_1^2 - ... - x_n^2 = R^2$ for the hyperboloid. The contractions are «analytic» because the parameter R figures not only in the structure constants of the original Lie algebra, but also in the coordinate systems, in the operators of the Lie algebra, in the invariant operators characterizing the coordinate systems, in the separated eigenfunctions and the eigenvalues.

Once the parametrization displaying the parameter R is established, it is possible to follow the contraction procedure $R \to \infty$ explicitly for all quantities: for the Lie algebra realized by vector fields, the Laplace–Beltrami operators, the second-order operators in the enveloping algebras, characterizing separable systems of coordinates, the separated ordinary differential equations, the eigenfunctions and the coefficients of the interbases expansions.

For two-dimensional spaces all types of coordinates were considered; for example, contractions of $O(3)$ to $E(2)$ relate elliptic coordinates on S_2 to elliptic and parabolic coordinates on E_2. They also relate spherical coordinates on S_2 to polar and Cartesian coordinates on E_2 [20, 22, 24]. Similarly, all 9 coordinate systems on the H_2 hyperboloid can be contracted to at least one of the four systems on E_2, or one of the 10 separable systems on $E_{1,1}$ [21, 23].

Contractions from S_n to E_n were considered for subgroup type coordinates in Refs. 25, 26, 28, for subgroup type coordinates and certain types of elliptic and parabolic ones.

The main application of analytic contractions in this context is to derive special function identities, specially asymptotic formulas. Among other possible applications we mention the theory of finite dimensional integrable and superintegrable systems [30, 31].

In this paper we restrict ourselves to two-dimensional spaces of constant curvature.

1. SEPARATION OF VARIABLES IN TWO-DIMENSIONAL SPACES OF CONSTANT CURVATURE

1.1. Operator Approach to the Separation of Variables. Let us first consider a quite general two-dimensional Riemannian, pseudo-Riemannian or complex Riemannian spaces with a metric

$$ds^2 = g_{ik}\, du^i du^k, \quad u = (u^1, u^2). \tag{1.1}$$

In this space we introduce a classical free Hamiltonian

$$H = g_{ik}(u) p_i p_k, \tag{1.2}$$

where $p_i = -\partial H/\partial u_i$ are the momenta classically conjugate to the coordinates u^i. We also introduce the corresponding Laplace–Beltrami operator

$$\Delta_{\mathrm{LB}} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial u^i} \sqrt{g} g^{ik} \frac{\partial}{\partial u^k}. \tag{1.3}$$

We will be interested in two related question:

1. What are the quadratic polynomials on phase space

$$Q = a^{ik}(u) p_i p_k \tag{1.4}$$

that Poisson commute with the Hamiltonian

$$\{Q, H\} = \sum_{i=1}^{2} \left(\frac{\partial Q}{\partial u_i} \frac{\partial H}{\partial p_i} - \frac{\partial Q}{\partial p_i} \frac{\partial H}{\partial u_i} \right) = 0? \tag{1.5}$$

In other words, when do quadratic (in the momenta) integrals of motion exist? Respectively what are the second-order Hermitian operators

$$Q = \{a^{ik}(u) \partial_{u_i} \partial_{u_k}\} \tag{1.6}$$

(where the bracket denotes symmetrization) that Lie commute with the Laplace–Beltrami operator

$$[Q, H] = QH - HQ = 0? \tag{1.7}$$

2. Do the Hamilton–Jacobi and Laplace–Beltrami equations in the considered space allow the separation of variables, and if so, how do we classify and construct separable coordinates? By separation of variables for the Hamilton–Jacobi equation we mean additive separation

$$g_{ik} \frac{\partial S}{\partial_{u^i}} \frac{\partial S}{\partial_{u^k}} = \lambda, \tag{1.8}$$

$$S = S_1(u^1, \lambda, \mu) + S_2(u^2, \lambda, \mu). \tag{1.9}$$

For the Laplace–Beltrami operator we have in mind multiplicative separation

$$\Delta \Psi = \lambda \Psi, \quad \Psi = \Psi_1(u^1, \lambda, \mu) \Psi_2(u^2, \lambda, \mu). \tag{1.10}$$

In both cases λ and μ are the separation constants.

In this review article we shall mainly be interested in the Laplace–Beltrami operators in different spaces. However, some aspects of separation are simpler to discuss for the Hamilton–Jacobi equation. In two-dimensional Riemannian space the two equations separate in the same coordinate systems.

The existence of integrals of motion that are either linear or quadratic in the momenta was analyzed by Darboux [32] and Koenigs [33] in a note published in Volume 4 of Darboux's lectures. In particular it was shown that a metric (1.1) can allow 0, 1, 2, 3 or 5 linearly independent quadratic integrals. The case of 5 quadratic integrals occurs if and only if the metric corresponds to a space of constant curvature. In this case the second-order integrals are «reducible». That means that the metric allows precisely three first-order integrals

$$L_i = a_i(u)p_1 + b_i(u)p_2, \quad u = (u_1, u_2), \quad i = 1, 2, 3 \tag{1.11}$$

and all second-order integrals are expressed as second-order polynomials (with constant coefficients) in terms of the first-order ones:

$$Q = \sum_{i,k=1}^{3} A_{ik} L_i L_k, \qquad A_{ik} = \text{const.} \tag{1.12}$$

If the polynomial Q is the square of a first-order operator L, then it will provide a subgroup type coordinate. This is best seen by considering the corresponding first-order operator

$$X = \xi(u_1, u_2)\partial_{u_1} + \eta(u_1, u_2)\partial_{u_1} \tag{1.13}$$

that generates a one-dimensional subgroup of the isometry group G. From (u_1, u_2) we transform to the new coordinates (v_1, v_2) «straightening out» the vector field (1.13) to the form

$$X = \partial_{v_1}. \tag{1.14}$$

Then v_1 will be an ignorable variable. The complementary variable $v_2 = \phi(u_1, u_2)$ can be replaced by an arbitrary function of v_2, the ignorable variable v_1 can be replaced by $f(v_1) + g(v_2)$, where both f and g are arbitrary. The separable coordinates are v_1 and v_2 (with the above-mentioned arbitrariness).

Now let us assume that an irreducible quadratic integral Q as in (1.4) is known for a considered metric (1.1) (that is, Q is not square of a linear integral). We can then impose that two equations be satisfied simultaneously. In the classical case they are

$$g_{ik}\frac{\partial S}{\partial_{u^i}}\frac{\partial S}{\partial_{u^k}} = \lambda, \qquad a_{ik}\frac{\partial S}{\partial_{u^i}}\frac{\partial S}{\partial_{u^k}} = \mu. \tag{1.15}$$

Similarly, we can consider the quantum mechanics of a free particle in such a space and write two simultaneous equations

$$\begin{aligned}\hat{H}\Psi &= \left(\frac{1}{\sqrt{g}}\frac{\partial}{\partial u^i}\sqrt{g}g^{ik}\frac{\partial}{\partial u^k}\right)\Psi = \lambda\Psi,\\ \hat{Q}\Psi &= \left(\frac{1}{\sqrt{g}}\frac{\partial}{\partial u^i}\sqrt{g}a^{ik}\frac{\partial}{\partial u^k}\right)\Psi = \mu\Psi.\end{aligned} \tag{1.16}$$

Separable coordinates for the two systems (1.15) and (1.16) are obtained by simultaneously transforming $\hat{H}$ and $\hat{Q}$ to a standard form, in which the matrices g_{ik} and a_{ik} of (1.15) (and (1.16)) are diagonal. This can be done by solving the characteristic equation

$$|a_{ik} - \rho\, g_{ik}| = 0. \tag{1.17}$$

If two distinct roots ρ_1 and ρ_2 exist, they will provide separable coordinates, at least over the field of complex number. If we are considering real spaces, then it may happen that ρ_1 and ρ_2 are real only in part of the space and do not parametrize the entire space. We will see below that this indeed happens for instance in the pseudo-Euclidean plane $E_{1,1}$.

The roots ρ_1 and ρ_2 can be replaced by any functions $u = u(\rho_1)$, $v = v(\rho_2)$. This freedom can be used to transform H and Q simultaneously to the form

$$\begin{aligned}H &= \frac{1}{\alpha(v)+\beta(u)}(p_u^2 + p_v^2) = \lambda,\\ Q &= \frac{1}{\alpha(v)+\beta(u)}(\beta(u)p_u^2 - \alpha(v)p_v^2) = \mu.\end{aligned} \tag{1.18}$$

The Hamiltonian H in (1.18) is in its Liouville form [34]. The separated equations are

$$\alpha H + Q = \alpha\lambda + \mu, \quad \beta H - Q = \beta\lambda - \mu \tag{1.19}$$

for the Hamilton–Jacobi equation and similarly

$$(\alpha\hat{H} + \hat{Q})\Psi = (\alpha\lambda + \mu)\Psi, \qquad (\beta\hat{H} - \hat{Q})\Psi = (\beta\lambda - \mu)\Psi \tag{1.20}$$

for the Laplace–Beltrami equation.

Let us now restrict ourselves to two-dimensional spaces M of constant curvature, that is to the Euclidean plane E_2, pseudo-Euclidean plane $E_{1,1}$, sphere S_2 and two-, or one-sheeted hyperboloid H_2. Each of these has a three-dimensional isometry group G. The Lie algebra L of G has in each case a standard basis which we denote $\{X_1, X_2, X_3\}$.

The Laplace–Beltrami operator $\hat{H} = \Delta_{\mathrm{LB}}$ (1.3) is in each case proportional to the Casimir operator $\hat{C}$ of the Lie algebra L. The operator Q commuting with H will have the form

$$\hat{Q} = \sum_{i,k=1}^{3} A_{ik} X_i X_k, \quad A_{ik} = A_{ki}, \tag{1.21}$$

where A is a constant matrix. Let $g \in G$ be an element of the Lie algebra of the isometry group of the considered space. Let us rewrite Eq. (1.21) in matrix form

$$\hat{Q} = u^T A u, \qquad u^T = \{X_1, X_2, X_3\}. \tag{1.22}$$

The transformation g acting on the space M induces a transformation $u' = gu$ on the Lie algebra L. The Casimir operator Δ_{LB} stays invariant, but $\hat{Q}$ transforms to

$$\hat{Q} = u^{'T} g^T A g u^{'}. \tag{1.23}$$

Thus, for spaces of constant curvature a classification of operators $\hat{Q}$ commuting with $\hat{H}$ reduces to a classification of symmetric matrices $A = A^T$ into equivalence classes under the congruence transformation

$$A' = g^T A g, \quad g \in G. \tag{1.24}$$

This problem, as we shall see below, can be reduced to that of classifying elements of Jordan algebras into equivalence classes.

Furthermore, the operator $\hat{C}$ of L can also be written in the form

$$\hat{C} = u^T C u, \qquad C = \begin{pmatrix} c_1 & & \\ & c_2 & \\ & & c_3 \end{pmatrix}. \tag{1.25}$$

Two matrices A and $\tilde{A}$ will give equivalent coordinate systems if they satisfy

$$\tilde{A} = \lambda g^T A g + \mu C, \qquad \lambda \neq 0, \tag{1.26}$$

where λ and μ are real constants.

1.2. Separable Coordinate Systems in the Euclidean Plane. The Lie algebra of the isometry group $E(2)$ is given by

$$L = x_2 \partial_{x_1} - x_1 \partial_{x_2}, \quad P_1 = \partial_{x_1}, \quad P_2 = \partial_{x_2}. \tag{1.27}$$

The operator $\hat{Q}$ of Eq. (1.21) will in this case be

$$\hat{Q} = aL^2 + b_1(LP_1 + P_1 L) + b_2(LP_2 + P_2 L) + \\ + c_1 P_1^2 + c_2 P_2^2 + 2c_3 P_1 P_2. \tag{1.28}$$

An $E(2)$ transformation matrix will be written as

$$g = \begin{pmatrix} 1 & \xi^T \\ 0 & R \end{pmatrix}, \qquad \xi^T = (\xi_1, \xi_2), \quad R \in \mathbb{R}^2, \quad R^T R = I. \tag{1.29}$$

The matrix A of Eq. (1.22) is

$$A = \begin{pmatrix} a & b^T \\ b & S \end{pmatrix}, \qquad S = \begin{pmatrix} c_1 & c_3 \\ c_3 & c_2 \end{pmatrix}, \qquad b^T = (b_1, b_2), \tag{1.30}$$

and C of (1.25) is

$$C = \begin{pmatrix} 0 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \tag{1.31}$$

since the Casimir operator is

$$C = \Delta = P_1^2 + P_2^2. \tag{1.32}$$

The transformation (1.26) with $\lambda = 1$ has two invariants in the space of symmetric matrices A, namely

$$I_1 = a, \quad I_2 = \{[a(c_1 - c_2) - (b_1^2 - b_2^2)]^2 + 4(ac_3 - b_1 b_2)^2\}^{1/2}. \tag{1.33}$$

Correspondingly, the operator $\hat{Q}$ can be transformed into one of four canonical forms

$$1)\ I_1 = 0, \quad I_2 = 0, \quad Q_C = P_1^2, \tag{1.34}$$

$$2)\ I_1 \neq 0, \quad I_2 = 0, \quad Q_R = L^2, \tag{1.35}$$

$$3)\ I_1 = 0, \quad I_2 \neq 0, \quad Q_P = LP_2 + P_2 L, \tag{1.36}$$

$$4)\ I_1 \neq 0, \quad I_2 \neq 0, \quad Q_E = L^2 - D^2 P_2^2, \quad D^2 = \frac{I_2}{I_1^2}. \tag{1.37}$$

The first two forms correspond to subgroup-type coordinates. Thus, Q_C of (1.34) corresponds to Cartesian coordinates, in which $P_1 = \partial_x$ (and also $P_2 = \partial_y$) is already straightened out. Both x and y are ignorable. The second, Q_R, corresponds to polar coordinates

$$x = r\cos\phi, \quad y = r\sin\phi \tag{1.38}$$

in which $L = \partial_\phi$ is straightened out so that ϕ is an ignorable variable.

The coordinates corresponding Q_P of (1.36) are the parabolic coordinates

$$x = \frac{1}{2}(u^2 - v^2), \qquad y = uv. \tag{1.39}$$

Equivalently, if we take $\tilde{Q}_P = LP_1 + P_1 L$, the prescription (1.17) leads to

$$x = uv, \qquad y = \frac{1}{2}(u^2 - v^2). \tag{1.40}$$

Finally, Q_E of Eq. (1.37) corresponds to elliptic coordinates in the plane. They can be written as

$$x = D\cosh\xi\cos\eta, \qquad y = D\sinh\xi\sin\eta. \tag{1.41}$$

The results are presented in Table 1.

1.3. Separable Coordinate Systems in the Pseudo-Euclidean Plane. The Lie algebra of the isometry group $E(1,1)$ can be represented by

$$K = (t\partial_x + x\partial_t)\,, \qquad P_0 = \partial_t, \qquad P_1 = \partial_x. \tag{1.42}$$

The second-order operator (1.21) is

$$\hat{Q} = aK^2 + b_0(KP_0 + P_0K) + b_1(KP_1 + P_1K) + \\ + c_0P_0^2 + c_1P_1^2 + 2c_2P_0P_1. \tag{1.43}$$

Equivalently, the matrix A of (1.24) is

$$A = \begin{pmatrix} a & b^T \\ b & c \end{pmatrix}, \qquad C = \begin{pmatrix} c_0 & c_2 \\ c_2 & c_1 \end{pmatrix}, \qquad b^T = (b_0, b_1). \tag{1.44}$$

We will classify the operators $\hat{Q}$ into conjugate classes and the action of the group $E(1,1)$, including the reflactions

$$\Pi_0 : (x,t) \to (x,-t), \quad \Pi_1 : (x,t) \to (-x,t). \tag{1.45}$$

An element of $E(1,1)$, acting on the Lie algebra (K, P_0, P_1) can be represented as

$$g = \begin{pmatrix} 1 & \xi^T \\ 0 & \Lambda \end{pmatrix}, \qquad \xi^T = (\xi_0, \xi_1), \quad \Lambda \in \mathbb{R}^2,\ \Lambda^T J \Lambda = J \tag{1.46}$$

with

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.47}$$

The matrix A of (1.44) is subject to the transformation (1.26) and in this case we have

$$A' = g^T A g = \begin{pmatrix} a & a\xi^T + \beta^T\Lambda \\ \xi a + \Lambda^T\beta & \Lambda^T C\Lambda + \Lambda^T\beta\xi^T + \xi\beta^T\Lambda + a\xi\xi^T \end{pmatrix}. \tag{1.48}$$

Table 1. **Orthogonal coordinate systems on two-dimensional Euclidean plane** E_2

Coordiante systems	Integrals of motion	Solution of Helmholtz equation
I. Cartesian x, y $-\infty < x, y < \infty$	$Q_C = P_1^2 - P_2^2$	Exponential functions
II. Polar $x = r\cos\varphi, \quad y = r\sin\varphi$ $0 \le r < \infty$, $0 \le \varphi \le 2\pi$	$Q_R = L^2$	Product of Bessel function and exponential
III. Parabolic $x = (u^2 - v^2)/2, \quad y = uv$ $0 \le u < \infty, -\infty < v < \infty$	$Q_P = LP_2 + P_2L$	Product of two parabolic cylinder functions
IV. Elliptic $x = D\cosh\xi\cos\eta,$ $y = D\sinh\xi\sin\eta$ $0 \le \xi < \infty$, $0 \le \eta < 2\pi$	$Q_E = L^2 - D^2P_2^2$	Product of periodic and nonperiodic Mathieu functions

One of the invariants of this transformation is the constant a which can be chosen to be $a = 1$, or is already $a = 0$.

Let us first consider $a \neq 0$. Choosing $\xi = -\beta^T\Lambda$ and putting $a = 1$, we obtain

$$A' = \begin{pmatrix} 1 & 0 \\ 0 & C' \end{pmatrix}, \qquad C' = J\Lambda^{-1}J(C - \beta\beta^T)\Lambda. \tag{1.49}$$

Notice that C' and C are symmetric matrices, but we have

$$X \equiv J(C - \beta\beta^T), \qquad JX^T = XJ, \tag{1.50}$$

that is, X is an element of the Jordan algebra $jo(1,1)$. Since Λ is an element of the Lie group $O(1,1)$, we are faced with a well-known problem: the classification of elements of a Jordan algebra with respect to conjugation under the corresponding Lie group. The results are known for all classical Lie and Jordan algebras [35], and for $jo(1,1)$ they are quite simple. The matrix X can be transformed into one of the following

$$\begin{gathered} X_1 = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}, \quad X_2 = \begin{pmatrix} p & q \\ -q & p \end{pmatrix}, \\ X_3 = \begin{pmatrix} p+\epsilon & \epsilon \\ -\epsilon & p-\epsilon \end{pmatrix}, \quad q > 0, \quad \epsilon = \pm 1, \end{gathered} \tag{1.51}$$

with $p \in \mathbb{R}$, $q \in \mathbb{R}$.

For $a = 0$, $(b_0, b_1) \neq (0,0)$ transformation (1.26) leads to Eq. (1.48) in which we set $a = 0$. We choose the matrix Λ contained in $O(1,1)$ to transform $\Lambda^T\beta$ to standard form and then choose ξ to simplify the matrix C. For $|b_0| > |b_1|$, $|b_0| < |b_1|$ and $|b_0| = |b_1|$ we can transform A into

$$A_1 = \begin{pmatrix} 0 & \epsilon\sqrt{b_0^2 - b_1^2} & 0 \\ \epsilon\sqrt{b_0^2 - b_1^2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \epsilon = \pm 1, \tag{1.52}$$

$$A_2 = \begin{pmatrix} 0 & 0 & \epsilon\sqrt{b_1^2 - b_0^2} \\ 0 & 0 & 0 \\ \epsilon\sqrt{b_1^2 - b_0^2} & 0 & 0 \end{pmatrix}, \qquad \epsilon = \pm 1, \tag{1.53}$$

$$A_3 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & \gamma & -\gamma \\ 1 & -\gamma & \gamma \end{pmatrix}, \qquad \gamma = 0, 1, \tag{1.54}$$

respectively.

Finally, for $a = b_0 = b_1 = 0$, $C \neq 0$ we can use Λ to transform C into one of its standard forms JX_i, $i = 1, 2, 3$ with X_i as in (1.51).

Thus, we have obtained a classification of matrices A that determine the operator $\hat{Q}$. Let us now list the corresponding operators. We first notice that $a = b_1 = b_0 = 0$. The corresponding operator $\hat{Q}$ is in the enveloping algebra of a maximal Abelian subalgebra of $e(1,1)$, namely (P_0, P_1). Similarly, for $a = 1$ and $X = X_1$ in Eq. (1.51) with $p = q$ we find that $Q = K^2$ is in the enveloping algebra of a different maximal Abelian subalgebra of $e(1,1)$, namely $o(1,1)$ (generated by K). These two cases correspond to subgroup type coordinates, the other ones, to nonsubgroup type.

The list of operators must be further simplifield by linear combinations with the Casimir operator

$$C = P_0^2 - P_1^2. \tag{1.55}$$

Finally, we obtain a representative list of 11 second-order operators in the enveloping algebra of the Lie algebra $e(1,1)$.

$$\begin{aligned} Q_1(a,b) &= a(P_0^2 + P_1^2) + 2bP_0P_1, \\ &\quad (a,b) = (1,0), (1,1), \text{or}\,(0,1), \\ Q_2 &= K^2, \\ Q_3 &= KP_1 + P_1K, \\ Q_4 &= KP_0 + P_0K, \end{aligned}$$

$$
\begin{aligned}
Q_5 &= K(P_0+P_1)+(P_0+P_1)K,\\
Q_6 &= K(P_0+P_1)+(P_0+P_1)K+(P_0-P_1)^2,\\
Q_7 &= K^2-l^2P_0P_1, \quad l>0,\\
Q_8 &= K^2-D^2P_1^2, \quad D>0,\\
Q_9 &= K^2-d^2P_1^2, \quad d>0,\\
Q_{10} &= K^2+(P_0+P_1)^2,\\
Q_{11} &= K^2-(P_0+P_1)^2.
\end{aligned}
\tag{1.56}
$$

To obtain separable coordinates we proceed as in Sec. 1.1.

1. The operator $Q_1(a,b)$ for any a and b, corresponds to Cartesian coordinates (t,x), since the operators that are really diagonalized are P_0 and P_1 (they correspond to a maximal Abelian subalgebra $\{P_0,P_1\}\in e(1,1)$).

2. The operator $Q_2=K^2$ also corresponds to subgroup type coordinates, namely pseudopolar coordinates

$$
\begin{aligned}
&t=r\cosh\alpha, \qquad x=r\sinh\alpha,\\
&0\le r<\infty, \qquad \infty<\alpha<\infty.
\end{aligned}
\tag{1.57}
$$

These coordinates only cover part of the pseudo-Euclidean plane, since we have $t^2-x^2=r^2\ge 0$. By interchanging t and x in (1.57) we can parametrize the part with $t^2-x^2=-r^2$.

The operators $Q_3,\dots Q_{11}$ can lead to separable coordinates via the algorithm of Eq. (1.17). Two problems can and do occur. The first is that the roots of Eq. (1.17) may coincide: $\rho_1=\rho_2$. Then we do not obtain separable coordiantes. This happens in precisely one case, namely that of the operator Q_5.

To other problem that may occur is that the eigenvalues $\rho(t,x)$ may be complex at least in a part of the (x,t) plane. This part of the plane will then not be covered by the corresponding coordinates (ρ_1,ρ_2).

The results of this analysis are presented in Table 2 and essentially agree with those of Kalnins [36].

1.4. The Systems of Coordinates on S_2. The Lie algebra of isometry group $O(3)$ is given by

$$
L_i=-\epsilon_{ikj}u_k\frac{\partial}{\partial u_j}, \quad [L_i,L_k]=\epsilon_{ikj}L_j, \quad i,k,j=1,2,3,
\tag{1.58}
$$

where u_i are the Cartesian coordinates in the ambient Euclidean space E_3. On the sphere S_2 we have $u_1^2+u_2^2+u_3^2=R^2$. The Casimir operator is

$$
C=R^2\Delta_{\mathrm{LB}}=L_1^2+L_2^2+L_3^2
\tag{1.59}
$$

Table 2. **Orthogonal coordinate systems on the two-dimensional pseudo-Euclidean plane** $E_{1,1}$

Coordinate system	Integrals of motion	Solution of Helmholtz equation
I. Cartesian $t,\ x$	$Q_C = P_0 P_1$	Product of exponentials
II. Pseudo-polar $t = r\cosh\tau_2,\ x = r\sinh\tau_2$ $r \geq 0,\ -\infty < \tau_2 < \infty$	$Q_S = K^2$	Product of Bessel function and exponential
III. Parabolic of type I $t = 1/2(u^2+v^2),\ x = uv$ $v \geq 0,\ -\infty < u < \infty$	$Q_P^{\mathrm{I}} = \{P_1, K\}$	Product of parabolic cylinder function for $t^2 - x^2 > 0$
IV. Parabolic of type II $t = uv, \quad x = 1/2(u^2+v^2)$ $v \geq 0,\ -\infty < u < \infty$	$Q_P^{\mathrm{II}} = \{P_0, K\}$	Product of parabolic cylinder function for $x^2 - t^2 > 0$
V. Parabolic of type III $t = 1/2(\eta-\zeta)^2 - (\eta+\zeta)$, $x = 1/2(\eta-\zeta)^2 + (\eta+\zeta)$ $-\infty < \eta, \zeta < \infty$	$Q_P^{\mathrm{III}} = \{P_0, K\} + \{P_1, K\}$ $+(P_0 - P_1)^2$	Products of two linear combinations of Airy functions for $x + t > 0$
VI. Hyperbolic of type I $t = 1/2\left(\cosh\dfrac{\eta-\zeta}{2} + \sinh\dfrac{\eta+\zeta}{2}\right)$, $x = 1/2\left(\cosh\dfrac{\eta-\zeta}{2} - \sinh\dfrac{\eta+\zeta}{2}\right)$ $-\infty < \eta, \zeta < \infty$	$Q_H^{\mathrm{I}} = K^2 - l^2 P_0 P_1$	Product of Mathieu equation solutions with argument displaced by $i\pi/2$
VII. Hyperbolic of type II $t = \left(\sinh(\eta-\zeta) + \mathrm{e}^{\eta+\zeta}\right)$, $x = \left(\sinh(\eta-\zeta) - \mathrm{e}^{\eta+\zeta}\right)$ $-\infty < \eta, \zeta < \infty$	$Q_H^{\mathrm{II}} = K^2 + (P_1 + P_2)^2$	Product of two solutions of Bessel's equation, one with real and one with imaginary arguments
VIII. Hyperbolic of type III $t = \left(\cosh(\eta-\zeta) + \mathrm{e}^{\eta+\zeta}\right)$, $x = \left(\cosh(\eta-\zeta) - \mathrm{e}^{\eta+\zeta}\right)$ $-\infty < \eta, \zeta < \infty$	$Q_H^{\mathrm{III}} = K^2 - (P_1 + P_2)^2$	Product of two solutions of Bessel's equation
IX. Elliptic of type I $t = D\sinh\eta\cosh\zeta$, $x = D\cosh\eta\sinh\zeta$ $-\infty < \eta, \zeta < \infty$	$Q_E^{\mathrm{I}} = K^2 + D^2 P_1^2$	Product of two solutions of the nonperiodic Mathieu equation
X. Elliptic of type II (i) $t = d\cosh\eta\cosh\zeta$, $x = d\sinh\eta\sinh\zeta$ $-\infty < \eta < \infty,\ \zeta \geq 0$ (ii) $t = d\cos\eta\cos\zeta$, $x = d\sin\eta\sin\zeta$ $0 < \eta < 2\pi, 0 \leq \zeta < \pi$	$Q_E^{\mathrm{II}} = K^2 - d^2 P_1^2$	Product of two solutions (i) of the nonperiodic Mathieu equation (ii) of the periodic Mathieu equation

and the Laplace–Beltrami Eq. (1.10) for S_2 has the form

$$\Delta_{\rm LB}\Psi = -\frac{\ell(\ell+1)}{R^2}\Psi, \quad \Psi_{lk}(\alpha,\beta) = \Xi_{lk}(\alpha)\Phi_{lk}(\beta), \tag{1.60}$$

where $\ell = 0, 1, 2, ...$ The second-order operator $\hat{Q}$ of Eq. (1.21) is given by

$$\hat{Q} = A_{ik}L_iL_k, \quad A_{ik} = A_{ki}. \tag{1.61}$$

The transformation matrix for $O(3)$ can be represented as

$$g = \\ = \begin{pmatrix} \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma & -\cos\alpha\cos\beta\sin\gamma - \sin\alpha\cos\gamma & \cos\alpha\sin\beta \\ \sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\gamma & -\sin\alpha\cos\beta\sin\gamma + \cos\alpha\cos\gamma & \sin\alpha\sin\beta \\ -\sin\beta\cos\gamma & \sin\beta\sin\gamma & \cos\beta \end{pmatrix}, \tag{1.62}$$

where (α,β,γ) are the Euler angles.

The matrix A_{ik} can be diagonalized to give

$$\hat{Q}(a_1,a_2,a_3) \equiv Q = a_1L_1^2 + a_2L_2^2 + a_3L_3^2. \tag{1.63}$$

For $a_1 = a_2 = a_3$ we have $Q \sim 0$. If two eigenvalues of A_{ik} are equal, e. g., $a_1 = a_2 \neq a_3$, or $a_1 \neq a_2 = a_3$, or $a_1 = a_3 \neq a_2$, we can transform the operator Q into the operators: $Q(0,0,1) = L_3^2$, $Q(1,0,0) = L_1^2$ or $Q(0,1,0) = L_2^2$, respectively. The corresponding separable coordinates on S_2 are the three types of spherical ones

$$\begin{aligned} u_1 &= R\sin\theta\cos\varphi = R\cos\theta' = R\sin\theta''\sin\varphi'', \\ u_2 &= R\sin\theta\sin\varphi = R\sin\theta'\cos\varphi' = R\cos\theta'', \\ u_3 &= R\cos\theta = R\sin\theta'\sin\varphi' = R\sin\theta''\cos\varphi'', \end{aligned} \tag{1.64}$$

where $\varphi \in [0,2\pi)$, $\theta \in [0,\pi]$. They correspond to the group reduction $O(3) \supset O(2)$ and $X = L_i^2$ is invariant under $O(2)$ and under reflections in all coordinate planes.

The $O(3)$ unitary irreducible representation matrix elements of (1.62) result in the well-known transformation formula for spherical functions $Y_{lm}(\theta,\varphi)$ [8,37]

$$Y_{l,m'}(\theta',\varphi') = \sum_{m=-l}^{l} D^l_{mm'}(\alpha,\beta\gamma)\,Y_{l,m}(\theta,\varphi), \tag{1.65}$$

where $D^l_{m_1,m_2}(\alpha,\beta,\gamma)$ are the Wigner D-functions

$$D^\ell_{m,m'}(\alpha,\beta,\gamma) = \mathrm{e}^{-im\alpha}\,d^\ell_{m,m'}(\beta)\,\mathrm{e}^{-im'\gamma}, \tag{1.66}$$

$$d^{\ell}_{m,m'}(\beta) = \frac{(-1)^{m-m'}}{(m-m')!}\sqrt{\frac{(\ell+m)!\,(\ell-m')!}{(\ell-m)!\,(\ell+m')!}}\left(\cos\frac{1}{2}\beta\right)^{2\ell-m+m'} \times$$
$$\times\left(\sin\frac{1}{2}\beta\right)^{m-m'} F\left[\begin{matrix} m-\ell,\ -m'-\ell \\ m-m'+1 \end{matrix}; -\tan^2\frac{1}{2}\beta\right] \quad (1.67)$$

and the spherical angles (θ, φ) and (θ', φ') are related by

$$\begin{aligned} \cos\theta' &= \cos\theta\cos\beta + \sin\theta\sin\beta\cos(\varphi-\alpha), \\ \cot(\varphi'+\gamma) &= \cot(\varphi-\alpha)\cos\beta - \frac{\cot\theta\sin\beta}{\sin(\varphi-\alpha)}. \end{aligned} \quad (1.68)$$

In particular, $Y_{lm}(\theta, \varphi)$ corresponding to the solution of Laplace–Beltrami equation in the systems of coordinates (1.64) are related by the formulas

$$Y_{l,m'}(\theta', \varphi') = \sum_{m=-l}^{l} D^{l}_{mm'}\left(0, \frac{\pi}{2}, \frac{\pi}{2}\right) Y_{l,m}(\theta, \varphi), \quad (1.69)$$

$$Y_{l,m''}(\theta'', \varphi'') = \sum_{m=-l}^{l} D^{l}_{mm''}\left(\frac{\pi}{2}, \frac{\pi}{2}, 0\right) Y_{l,m}(\theta, \varphi), \quad (1.70)$$

$$Y_{l,m''}(\theta'', \varphi'') = \sum_{m'=-l}^{l} D^{l}_{m'm''}\left(0, \frac{\pi}{2}, \frac{\pi}{2}\right) Y_{l,m'}(\theta', \varphi'). \quad (1.71)$$

When all three eigenvalues a_i are different, then the separable coordinates in Eq. (1.60) are elliptic ones [38–40]. These can be written in algebraic form, as

$$\begin{gathered} u_1^2 = R^2\frac{(\rho_1-a_1)(\rho_2-a_1)}{(a_2-a_1)(a_3-a_1)}, \quad u_2^2 = R^2\frac{(\rho_1-a_2)(\rho_2-a_2)}{(a_3-a_2)(a_1-a_2)}, \\ u_3^2 = R^2\frac{(\rho_1-a_3)(\rho_2-a_3)}{(a_1-a_3)(a_2-a_3)} \end{gathered} \quad (1.72)$$

with $a_1 \le \rho_1 \le a_2 \le \rho_2 \le a_3$.

In trigonometric form we put

$$\rho_1 = a_1 + (a_2-a_1)\cos^2\phi, \quad \rho_2 = a_3 - (a_3-a_2)\cos^2\theta, \quad (1.73)$$

and obtain

$$\begin{gathered} u_1 = R\sqrt{1-k'^2\cos^2\theta}\cos\phi, \quad u_2 = R\sin\theta\sin\phi, \\ u_3 = R\sqrt{1-k^2\cos^2\phi}\cos\theta, \quad 0 \le \phi < 2\pi, \quad 0 \le \theta \le \pi, \end{gathered} \quad (1.74)$$

Table 3. **Orthogonal coordinate systems on two-dimensional sphere S_2**

Coordinate systems	Integrals of motion	Solution of Helmholtz equation	Limiting systems on E_2
I. Spherical $u_1 = R\sin\theta\cos\varphi$, $u_2 = R\sin\theta\sin\varphi$, $u_3 = R\cos\theta$ $\varphi \in [0, 2\pi), \theta \in [0, \pi]$	$Q_S = L_3^2$	Product of associated Legendre polynomials and exponential	Polar Cartesian
II. Elliptic $u_1 = R\,\mathrm{sn}\,(\alpha, k)\,\mathrm{dn}\,(\beta, k')$, $u_2 = R\,\mathrm{cn}\,(\alpha, k)\,\mathrm{cn}\,(\beta, k')$, $u_3 = R\,\mathrm{dn}\,(\alpha, k)\,\mathrm{sn}\,(\beta, k')$ $\alpha \in [-K, K]$, $\beta \in [-2K', 2K']$	$Q_E = k'^2 L_3^2 - k^2 L_1^2$	Product of two Lamé polynomials	Elliptic Polar Cartesian Parabolic*

*After rotation.

where

$$k^2 = \frac{a_2 - a_1}{a_3 - a_1} = \sin^2 f, \qquad k'^2 = \frac{a_3 - a_2}{a_3 - a_1} = \cos^2 f, \qquad k^2 + k'^2 = 1. \tag{1.75}$$

The Jacobi elliptic version of elliptic coordinates is obtained by putting

$$\rho_1 = a_1 + (a_2 - a_1)\,\mathrm{sn}^2(\alpha, k), \quad \rho_2 = a_2 + (a_3 - a_2)\,\mathrm{cn}^2(\beta, k'). \tag{1.76}$$

We obtain

$$\begin{aligned} u_1 &= R\,\mathrm{sn}\,(\alpha, k)\,\mathrm{dn}\,(\beta, k'), \quad u_2 = R\,\mathrm{cn}\,(\alpha, k)\,\mathrm{cn}\,(\beta, k'), \\ u_3 &= R\,\mathrm{dn}\,(\alpha, k)\,\mathrm{sn}\,(\beta, k'), \quad -K \le \alpha \le K, \quad -2K' \le \beta \le 2K', \end{aligned} \tag{1.77}$$

where $\mathrm{sn}\,(\alpha, k)$, $\mathrm{cn}\,(\alpha, k')$ and $\mathrm{dn}\,(\beta, k)$ are the Jacobi elliptic functions with modulus k and k', and K and K' are the complete elliptic integrals [41].

The interfocal distance for the ellipses on the upper hemisphere is equal to $2fR$. The results are given in Table 3.

1.5. Systems of Coordinates on H_2. The isometry group for the hyperboloid H_2: $u_0^2 - u_1^2 - u_2^2 = R^2$, where u_i $(i = 0, 1, 2)$ are the Cartesian coordinates in the ambient space $E_{2,1}$, is $O(2,1)$. We choose a standard basis K_1, K_2, L_3 for the Lie algebra $o(2,1)$:

$$K_1 = -\left(u_0\partial_{u_2} + u_2\partial_{u_0}\right), \ K_2 = -\left(u_0\partial_{u_1} + u_1\partial_{u_0}\right), \ L_3 = -\left(u_1\partial_{u_2} - u_2\partial_{u_1}\right)$$

with commutation relations

$$[K_1, K_2] = -L_3, \quad [L_3, K_1] = K_2, \quad [K_2, L_3] = K_1. \tag{1.78}$$

The Casimir operator is

$$C = R^2\,\Delta_{\rm LB} = K_1^2 + K_2^2 - L_3^2 \tag{1.79}$$

and the Laplace–Beltrami equation (1.10) is given by

$$\Delta_{\rm LB}\Psi = \frac{l(l+1)}{R^2}\Psi, \quad \Psi_{l\lambda}(\zeta_1,\zeta_2) = \Xi_{l\lambda}(\zeta_1)\Phi_{l\lambda}(\zeta_2), \tag{1.80}$$

where ℓ for principal series of the unitary irreducible representations has the form

$$\ell = -\frac{1}{2} + i\rho, \quad 0 < \rho < \infty. \tag{1.81}$$

The second-order operator $\hat{Q}$ of Eq. (1.21):

$$\begin{aligned} Q = aK_1^2 + b(K_1K_2 + K_2K_1) + cK_2^2 + d(K_1L_3 + L_3K_1) +& \\ + e(K_2L_3 + L_3K_2) + fL_3^2& \end{aligned} \tag{1.82}$$

can be used to classify all coordinate systems on H_2. The classification of the operators Q can be reduced to a classification of the normal forms of the elements of the Jordan algebra jo(2,1) [35]. There are 9 inequivalent forms, in one-to-one correspondence with the 9 existing separable coordinate systems [15, 42, 43]. All the coordinate systems are orthogonal ones.

The normal forms of the operator Q and the corresponding coordinates are given in Table 4. Cases I, II, and III correspond to subgroup type coordinates. The corresponding subgroups are $O(2)$, $O(1,1)$ and $E(1)$, respectively. The $O(1,1)$ subgroup in the equidistant coordinates acts in the 01 plane. We could also have chosen the 02 plane (i. e., permuted u_1 and u_2).

The elliptic and hyperbolic coordinates of cases IV and V are given in algebraic form. Equivalently, they can be expressed, e. g., in terms of Jacobi elliptic functions. This makes it possible to express the coordinates in the ambient space directly, rather than their squares. Indeed, if we put

$$\varrho_1 = a_1 - (a_1 - a_3)\,{\rm dn}^2(\alpha, k), \qquad \varrho_2 = a_1 - (a_1 - a_2)\,{\rm sn}^2(\beta, k') \tag{1.83}$$

and

$$k^2 = \frac{a_2 - a_3}{a_1 - a_3}, \qquad k'^2 = \frac{a_1 - a_2}{a_1 - a_3}, \qquad k^2 + k'^2 = 1 \tag{1.84}$$

into the expressions

$$\begin{gathered} u_0^2 = R^2\frac{(\varrho_1 - a_3)(\varrho_2 - a_3)}{(a_1 - a_3)(a_2 - a_3)}, \quad u_1^2 = R^2\frac{(\varrho_1 - a_2)(\varrho_2 - a_2)}{(a_2 - a_3)(a_1 - a_2)}, \\ u_2^2 = R^2\frac{(\varrho_1 - a_1)(a_1 - \varrho_2)}{(a_1 - a_2)(a_1 - a_3)}, \end{gathered} \tag{1.85}$$

Table 4. **Orthogonal systems of coordinate on two-dimensional hyperboloid** H_2

Coordinate systems and integrals of motion	Coordinate	Limiting systems on E_2	Limiting systems on $E_{1,1}$
I. Pseudo-spherical $\tau > 0, \varphi \in [0, 2\pi)$ $Q_S = L_3^2$	$u_0 = R\cosh\tau$ $u_1 = R\sinh\tau\cos\varphi$ $u_2 = R\sinh\tau\sin\varphi$	Polar	Cartesian
II. Equidistant $\tau_{1,2} \in \mathbb{R}$ $Q_{EQ} = K_1^2$	$u_0 = R\cosh\tau_1\cosh\tau_2$ $u_1 = R\cosh\tau_1\sinh\tau_2$ $u_2 = R\sinh\tau_1$	Cartesian	Polar
III. Horocyclic $\tilde{y} > 0, \tilde{x} \in \mathbb{R}$ $Q_{HO} = (K_1 + L_3)^2$	$u_0 = R(\tilde{x}^2 + \tilde{y}^2 + 1)/2\tilde{y}$ $u_1 = R(\tilde{x}^2 + \tilde{y}^2 - 1)/2\tilde{y}$ $u_2 = R\tilde{x}/\tilde{y}$	Cartesian	Rectangular Cartesian coordinates rotated by $\pi/4$ (nonorthogonal)
IV. Elliptic $a_3 < a_2 < \rho_2 < a_1 < \rho_1$ $Q_E = L_3^2 + \sinh^2 f K_2^2$	$u_0^2 = R^2\dfrac{(\rho_1 - a_3)(\rho_2 - a_3)}{(a_1 - a_3)(a_2 - a_3)}$ $u_1^2 = R^2\dfrac{(\rho_1 - a_2)(\rho_2 - a_2)}{(a_1 - a_2)(a_2 - a_3)}$ $u_2^2 = R^2\dfrac{(\rho_1 - a_1)(a_1 - \rho_2)}{(a_1 - a_2)(a_1 - a_3)}$	Elliptic Parabolic Cartesian	Elliptic I, II, III Cartesian
V. Hyperbolic $\rho_2 < a_3 < a_2 < a_1 < \rho_1$ $Q_H = K_2^2 - \sin^2\alpha L_3^2$	$u_0^2 = R^2\dfrac{(\rho_1 - a_2)(a_2 - \rho_2)}{(a_1 - a_2)(a_2 - a_3)}$ $u_1^2 = R^2\dfrac{(\rho_1 - a_3)(a_3 - \rho_2)}{(a_1 - a_3)(a_2 - a_3)}$ $u_2^2 = R^2\dfrac{(\rho_1 - a_1)(a_1 - \rho_2)}{(a_1 - a_2)(a_1 - a_3)}$	Cartesian	Elliptic II Parabolic I
VI. Semihyperbolic $\mu_{1,2} > 0$ $Q_{SH} = -\{K_1, L_3\}$	$\dfrac{u_0^2 + u_1^2}{R^2} = (1 + \mu_1^2)(1 + \mu_2^2)$ $\dfrac{u_0^2 - u_1^2}{R^2} = (1 + \mu_1\mu_2)$ $u_2 = R\sqrt{\mu_1\mu_2}$	Parabolic Cartesian	Cartesian
VII. Elliptic-parabolic $a \in \mathbb{R}, \vartheta \in (-\pi/2, \pi/2)$ $Q_{EP} = (K_1 + L_3)^2 + K_2^2$	$u_0 = R\dfrac{\cosh^2 a + \cos^2\vartheta}{2\cosh a\cos\vartheta}$ $u_1 = R\dfrac{\sinh^2 a - \sin^2\vartheta}{2\cosh a\cos\vartheta}$ $u_2 = R\tan\vartheta\tanh a$	Parabolic	Hyperbolic II
VIII. Hyperbolic-parabolic $b > 0, \vartheta \in (0, \pi)$ $Q_{HP} = (K_1 + L_3)^2 - K_2^2$	$u_0 = R\dfrac{\cosh^2 b + \cos^2\vartheta}{2\sinh b\sin\vartheta}$ $u_1 = R\dfrac{\sinh^2 b - \sin^2\vartheta}{2\sinh b\sin\vartheta}$ $u_2 = R\cot\vartheta\coth b$	Cartesian	Hyperbolic III

End of Table 4

Coordinate systems and integrals of motion	Coordinate	Limiting systems on E_2	Limiting systems on $E_{1,1}$
IX. Semicircular-parabolic $\xi, \eta > 0$ $Q_{SCP} = \{K_1, K_2\} + \{K_2, L_3\}$	$u_0 = R\dfrac{(\xi^2+\eta^2)^2+4}{8\xi\eta}$ $u_1 = R\dfrac{(\xi^2+\eta^2)^2-4}{8\xi\eta}$ $u_2 = R\dfrac{\eta^2-\xi^2}{2\xi\eta}$	Cartesian	Does not correspond to a separable coordinate system

we obtain the elliptic coordinates in Jacobi form

$$\begin{aligned} u_0 &= R\,\mathrm{sn}\,(\alpha,k)\,\mathrm{dn}\,(\beta,k'), \quad u_1 = i\,R\,\mathrm{cn}\,(\alpha,k)\,\mathrm{cn}\,(\beta,k'), \\ u_2 &= i\,R\,\mathrm{dn}\,(\alpha,k)\,\mathrm{sn}\,(\beta,k'), \quad \alpha \in (iK', iK''+2K), \quad \beta \in [0,4K'). \end{aligned} \tag{1.86}$$

2. CONTRACTIONS OF THE LIE ALGEBRA AND CASIMIR OPERATOR

2.1. Contractions from $o(3)$ to $e(2)$. We shall use R^{-1} as the contraction parameter. To realize the contraction explicitly, let us introduce homogeneous or Beltrami coordinates on the sphere, putting

$$x_\mu = R\frac{u_\mu}{u_3} = \frac{u_\mu}{\sqrt{1-(u_1^2+u_2^2)/R^2}}, \quad \mu = 1,2. \tag{2.1}$$

Geometrically (x_1, x_2) correspond to a projection from the centre of the sphere to a tangent plane at the North pole. In this parametrization the metric tensor has the following form

$$g_{\mu\nu} = \frac{1}{1+r^2/R^2}\left[\delta_{\mu\nu} + \frac{x_\mu x_\nu}{r^2}\frac{1}{1+r^2/R^2}\right], \quad r^2 = x_\mu x_\mu. \tag{2.2}$$

The Laplace–Beltrami operator corresponds to

$$\begin{aligned} \Delta_{\mathrm{LB}} = \left(1+\frac{r^2}{R^2}\right)\left[\frac{\partial^2}{\partial x_\mu^2} + \frac{x_\mu}{R^2}\frac{\partial}{\partial x_\mu} + \frac{1}{R^2}\left(x_\mu\frac{\partial}{\partial x_\mu}\right)^2\right] = \\ = \left(\pi_1^2 + \pi_2^2 + \frac{L_3^2}{R^2}\right), \end{aligned} \tag{2.3}$$

where

$$\pi_\mu = \left(\frac{\partial}{\partial x_\mu} + \frac{x_\mu x_\nu}{R^2}\frac{\partial}{\partial x_\nu}\right), \quad L_3 = \left(x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1}\right). \tag{2.4}$$

Using the connection between operators π_μ and the generators of the $O(3)$ group

$$-\frac{L_1}{R} = \pi_2, \qquad \frac{L_2}{R} = \pi_1, \qquad L_3 = -(x_1\pi_2 - x_2\pi_1),$$

we obtain the following commutation relations

$$[L_3, \pi_1] = \pi_2, \qquad [L_3, \pi_2] = -\pi_1, \qquad [\pi_1, \pi_2] = \frac{L_3}{R^2}, \tag{2.5}$$

so that for $R \to \infty$ the $o(3)$ algebra contracts to the $e(2)$ one. Moreover the momenta π_μ contract to $P_\mu = \partial/\partial x_\mu$, $(\mu = 1, 2)$ and the $o(3)$ Laplace–Beltrami operator (2.3) contracts to the $e(2)$ one:

$$\Delta_{\text{LB}} = \pi_1^2 + \pi_2^2 + \frac{L_3^2}{R^2} \to \Delta = (P_1^2 + P_2^2). \tag{2.6}$$

2.2. Contractions from $o(2,1)$ to $e(2)$. As in Sec. 2.1, let us introduce the Beltrami coordinates on the hyperboloid H_2 putting

$$x_\mu = R\frac{u_\mu}{u_0} = R\frac{u_\mu}{\sqrt{R^2 + u_1^2 + u_2^2}}, \quad \mu = 1, 2. \tag{2.7}$$

The $O(2,1)$ generators can be expressed as:

$$\begin{aligned} -\frac{K_1}{R} &\equiv \tilde{\pi}_2 = \frac{\partial}{\partial x_2} - \frac{x_2}{R^2}\left(x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2}\right), \\ -\frac{K_2}{R} &\equiv \tilde{\pi}_1 = p_1 - \frac{x_1}{R^2}\left(x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2}\right), \\ L_3 &= x_1\pi_2 - x_2\pi_1. \end{aligned}$$

The commutation relations of the o(2,1) algebra (1.78) in terms of the new operators take the form

$$[\tilde{\pi}_1, \tilde{\pi}_2] = -\frac{L_3}{R^2}, \quad [L_3, \tilde{\pi}_1] = \tilde{\pi}_2, \quad [\tilde{\pi}_2, L_3] = \tilde{\pi}_1, \tag{2.8}$$

so, that for $R \to \infty$ the o(2,1) algebra contracts to e(2) and the momenta $\tilde{\pi}_\mu$ to $P_\mu = \partial/\partial x_\mu$. The o(2,1) Laplace–Beltrami operator (1.2) contracts to the $e(2)$ one:

$$\Delta_{\text{LB}} = \tilde{\pi}_1^2 + \tilde{\pi}_2^2 - \frac{L_3^2}{R^2} \to \Delta = (P_1^2 + P_2^2). \tag{2.9}$$

2.3. Contractions from $o(2,1)$ to $e(1,1)$. Let us introduce Beltrami coordinates on hyperboloid H_2

$$y_0 = R\frac{u_0}{u_2} = R\frac{u_0}{\sqrt{u_0^2+u_1^2-R^2}}, \quad y_1 = R\frac{u_1}{u_2} = R\frac{u_1}{\sqrt{u_0^2+u_1^2-R^2}}. \tag{2.10}$$

The $O(2,1)$ generators can be expressed as

$$\begin{aligned} -\frac{K_1}{R} &\equiv \tilde{\tilde{\pi}}_1 = \frac{\partial}{\partial y_0} - \frac{y_0}{R^2}\left(y_0\frac{\partial}{\partial y_0} + y_1\frac{\partial}{\partial y_1}\right), \\ -\frac{L_3}{R} &\equiv \tilde{\tilde{\pi}}_2 = \frac{\partial}{\partial y_1} + \frac{y_1}{R^2}\left(y_0\frac{\partial}{\partial y_0} + y_1\frac{\partial}{\partial y_1}\right), \\ -K_2 &\equiv K = y_0\tilde{\tilde{\pi}}_2 + y_1\tilde{\tilde{\pi}}_1. \end{aligned} \tag{2.11}$$

The commutators of the $o(2,1)$ algebra (1.78) in the new operators $(\tilde{\tilde{\pi}}_1, \tilde{\tilde{\pi}}_2, K)$ take the form

$$\left[\tilde{\tilde{\pi}}_1, \tilde{\tilde{\pi}}_2\right] = \frac{K}{R^2}, \quad \left[K, \tilde{\tilde{\pi}}_1\right] = -\tilde{\tilde{\pi}}_2, \quad \left[\tilde{\tilde{\pi}}_2, K\right] = \tilde{\tilde{\pi}}_1, \tag{2.12}$$

so, that for $R \to \infty$ the $o(2,1)$ algebra contracts to the $e(1,1)$ one. The $o(2,1)$ Laplace–Beltrami operator contracts to the $e(1,1)$ one:

$$\Delta_{\mathrm{LB}} = \tilde{\tilde{\pi}}_1^2 - \tilde{\tilde{\pi}}_2^2 + \frac{K^2}{R^2} \to \frac{\partial^2}{\partial y_0^2} - \frac{\partial^2}{\partial y_1^2}, \tag{2.13}$$

and Eq. (1.80) transforms for large $\ell \sim pR$ to the one-dimensional Klein–Gordan equation.

$$\frac{\partial^2\psi}{\partial y_0^2} - \frac{\partial^2\psi}{\partial y_1^2} + p^2\psi = 0. \tag{2.14}$$

3. CONTRACTION FOR SYSTEMS OF COORDINATES

3.1. Contractions and Coordinate Systems on S_2

1. Spherical coordinates on S_2 to polar on E_2. We consider the spherical coordinate (1.64) with the parametr $a_1 = a_2$ and put

$$\tan\theta = \frac{r}{R}.$$

In the contraction limit $R \to \infty$, $\theta \to 0$ we have

$$x_1 = R\frac{u_1}{u_3} \to x = r\cos\varphi, \qquad x_2 = R\frac{u_2}{u_3} \to y = r\sin\varphi$$

and

$$Q_S = L_3^2 \to L^2.$$

2. Spherical coordinate on S_2 to Cartesian on E_2. We choose the spherical coordinate (1.64) with $a_2 = a_3$. Putting

$$\cos\theta' \sim \frac{x}{R} \sim 0, \qquad \cos\varphi' \sim \frac{y}{R} \sim 0$$

and taking the limit $R \to \infty$ and $\theta' \to \frac{\pi}{2}$, $\varphi' \to \frac{\pi}{2}$, we obtain

$$\frac{1}{R^2} Q_S = \frac{L_1^2}{R^2} = \pi_1^2 \to P_1^2 \sim Q_C$$

and

$$x_1 = R\,\frac{\cot\theta'}{\sin\varphi'} \to x, \quad x_2 = R\,\cot\varphi' \to y.$$

It is easy to see that for the case $a_1 = a_3$ the corresponding spherical system of coordinates (1.64) contracts to Cartesian coordinates on E_2 for $R \to \infty$.

3. Elliptic coordinates on S_2 to elliptic coordinates on E_2. We take Q in its general form, equivalent to

$$Q_E = L_3^2 - \left(\frac{a_2 - a_1}{a_3 - a_2}\right) L_1^2. \tag{3.1}$$

We put

$$\frac{R^2}{a_3 - a_1} = \frac{D^2}{a_2 - a_1}, \tag{3.2}$$

and in the limit $R^2 \sim a_3 \to \infty$ obtain

$$Q_E = L_3^2 - \left(\frac{a_3 - a_1}{a_3 - a_2}\right) \frac{D^2}{R^2}\, L_1^2 \to L_3^2 - D^2 P_2^2 \sim I_E. \tag{3.3}$$

For the coordinates we put

$$\rho_1 = a_1 + (a_2 - a_1)\cos^2\eta, \qquad \rho_2 = a_1 + (a_2 - a_1)\cosh^2\xi, \tag{3.4}$$

and for $R^2 \sim a_3 \to \infty$, using Eq. (3.2), we obtain Eq. (3.41), i.e., elliptic coordinates on the plane E_2.

4. *Elliptic coordinates on S_2 to Cartesian coordinates on E_2.* We start from the coordinates (1.72) but change the ordering of the parameters a_i, which corresponds the interchange of coordinates $u_3 \leftrightarrow u_2$, i. e., put

$$a_1 \leq \rho_1 \leq a_3 \leq \rho_2 \leq a_2 \tag{3.5}$$

and choose $a_3 - a_1 = a_2 - a_3 \equiv a$. Then we have

$$Q_E = a(L_2^2 - L_1^2). \tag{3.6}$$

Introducing the new coordinates by

$$\frac{a_3 - \rho_1}{a} = \xi_1, \qquad \frac{\rho_2 - a_3}{a} = \xi_2, \tag{3.7}$$

we can rewrite the (1.72) in the form

$$u_1^2 = \frac{R^2}{2}(1 - \xi_1)(1 + \xi_2), \quad u_2^2 = \frac{R^2}{2}(1 + \xi_1)(1 - \xi_2), \quad u_2^2 = R^2 \xi_1 \xi_2. \tag{3.8}$$

Using Eq. (2.1) we have for Beltrami coordinates

$$x_1^2 = R^2 \frac{(1 - \xi_1)(1 + \xi_2)}{2\xi_1\xi_2}, \qquad x_2^2 = R^2 \frac{(1 + \xi_1)(1 - \xi_2)}{2\xi_1\xi_2}. \tag{3.9}$$

From equation (3.9) we obtain

$$\xi_{2,1} = \frac{R^2}{R^2 + x_1^2 + x_2^2} \left\{ \left[1 + \frac{x_1^2 + x_2^2}{R^2} + \frac{(x_1^2 - x_2^2)^2}{4R^4} \right]^{1/2} \mp \frac{x_1^2 - x_2^2}{2R^2} \right\}. \tag{3.10}$$

Taking now the limit $R \to \infty$ we have

$$\xi_1 \to 1 - \frac{x^2}{R^2}, \quad \xi_2 \to 1 - \frac{y^2}{R^2}, \tag{3.11}$$

and hence x_1 and x_2 of Eq. (3.9) go into Cartesian coordinates:

$$x_1 \to x, \qquad x_2 \to y. \tag{3.12}$$

For the integral of motion in the limit $R^2 \sim a \to \infty$ we have

$$\frac{1}{aR^2} Q_E = (\pi_1^2 - \pi_2^2) \to P_1^2 - P_2^2 = Q_C. \tag{3.13}$$

5. *Elliptic coordinates on S_2 to parabolic coordinates on E_2.* We take the operator (1.72) with $a_1 \leq \rho_1 \leq a_2 \leq \rho_2 \leq a_3$ and choose the parameter

$a_3 - a_2 = a_2 - a_1 \equiv a$. We must first «undo» the diagonalization (1.26) by a rotation through $\pi/4$. The operator (1.63) transforms into

$$\frac{1}{aR}Q_E = -\frac{1}{R}(L_1L_3 + L_3L_1) = (L_3\pi_2 + \pi_2L_3), \tag{3.14}$$

with the correct limit (1.39) for $R \to \infty$. The coordinates (1.77) on S_2 are rotated into

$$\begin{gathered} u_1 = \frac{R}{\sqrt{2}}(\operatorname{sn}\alpha\operatorname{dn}\beta + \operatorname{dn}\alpha\operatorname{sn}\beta), \quad u_2 = R\operatorname{cn}\alpha\operatorname{cn}\beta, \\ u_3 = \frac{R}{\sqrt{2}}(\operatorname{dn}\alpha\operatorname{sn}\beta - \operatorname{sn}\alpha\operatorname{dn}\beta), \end{gathered} \tag{3.15}$$

with modulus $k = k' = 1/\sqrt{2}$ for all Jacobi elliptic functions.

From Eq. (3.15) we obtain

$$\begin{gathered} \operatorname{sn}\alpha = \frac{1}{\sqrt{2}}\left[\left(1+\frac{u_1}{R}\right)^{1/2}\left(1-\frac{u_3}{R}\right)^{1/2} - \left(1-\frac{u_1}{R}\right)^{1/2}\left(1+\frac{u_3}{R}\right)^{1/2}\right], \\ \sqrt{2}\operatorname{dn}\beta = \frac{1}{\sqrt{2}}\left[\left(1+\frac{u_1}{R}\right)^{1/2}\left(1-\frac{u_3}{R}\right)^{1/2} + \left(1-\frac{u_1}{R}\right)^{1/2}\left(1+\frac{u_3}{R}\right)^{1/2}\right]. \end{gathered} \tag{3.16}$$

Equations (3.16) suggest the limiting procedure. Indeed we put

$$\operatorname{sn}\alpha = -1 + \frac{u^2}{2R}, \quad \sqrt{2}\operatorname{dn}\beta = 1 + \frac{v^2}{2R}. \tag{3.17}$$

In the limit $R \to \infty$ we obtain

$$x_1 \to x = \frac{u^2 - v^2}{2}, \qquad x_2 \to y = uv, \tag{3.18}$$

i. e., the parabolic coordinates (1.39).

3.2. Contractions of Coordinate Systems from H_2 to E_2

1. Pseudo-spherical coordinates on H_2 to polar coordinates on E_2. In the limit $R \to \infty$, $\tau \to 0$ putting $\tanh\tau \sim r/R$ we have:

$$Q_S = L_3^2 \to L_3^2,$$

and for Beltrami coordinates (2.7) we obtain:

$$x_1 = R\frac{u_1}{u_0} \to x = r\cos\varphi, \quad x_2 = R\frac{u_2}{u_0} \to y = r\sin\varphi.$$

2. Equidistant coordinates on H_2 to Cartesian on E_2. For Beltrami coordinates (2.7) we have:

$$x_1 = R\tanh\tau_2, \quad x_2 = R\tanh\tau_1/\cosh\tau_2. \tag{3.19}$$

Taking the limit $R\to\infty$, $\tau_1,\tau_2\to 0$ and putting $\sinh\tau_1 \sim y/R$, $\sinh\tau_2 \sim x/R$ in (3.19) we obtain $x_1\to x$, $x_2\to y$ and

$$\frac{Q_{EQ}}{R^2} = \pi_1^2 \to P_1^2 \sim Q_C.$$

3. Horocyclic coordinates on H_2 to Cartesian on E_2. For variables $\tilde{x},\tilde{y}$ we obtain:

$$\tilde{x} = \frac{u_2}{u_0-u_1}, \quad \tilde{y} = \frac{R}{u_0-u_1}.$$

In the limit $R\to\infty$ we get: $\tilde{x}\to y/R$, $\tilde{y}\to 1+x/R$ and Beltrami coordinates go into Cartesian ones

$$x_1 = R\,\frac{\tilde{x}^2+\tilde{y}^2-1}{\tilde{x}^2+\tilde{y}^2+1} \to x, \qquad x_2 = \frac{2\tilde{x}R}{\tilde{x}^2+\tilde{y}^2+1} \to y.$$

For integral of motion we have:

$$\frac{Q_{HO}}{R^2} = \pi_2^2 + \frac{L_3^2}{R^2} - \frac{1}{R}\{\pi_2, L_3\} \to P_2^2 \sim Q_C.$$

4. Elliptic coordinates on H_2 to elliptic coordinates on E_2. We put

$$\frac{R^2}{a_2-a_3} = \frac{D^2}{a_1-a_2} \tag{3.20}$$

and in the limit $R^2\sim(-a_3)\to\infty$ obtain:

$$Q_E = L_3^2 + \frac{D^2}{R^2}K_2^2 \to L^2 + D^2p_1^2 \sim Q_E,$$

where $2D$ is the focal distance. Writing the coordinates as

$$\rho_1 = a_1 + (a_1-a_2)\sinh^2\xi, \qquad \rho_2 = a_2 + (a_1-a_2)\cos^2\eta$$

and using Eq. (3.20) in the limit $R^2\sim(-a_3)\to\infty$ we get the ordinary elliptic coordinates on E_2 plane [10, 15].

5. Elliptic coordinates on H_2 to Cartesian on E_2. We make a special choice of the parameters a_i: $a_1-a_2 = a_2-a_3$ and determine new variables $\xi_{1,2}$ by the formula

$$\xi_{1,2} = \frac{\varrho_{1,2}-a_2}{a_1-a_2} = \frac{u_0^2+u_2^2}{2R^2} \pm \sqrt{\left(\frac{u_0^2+u_2^2}{2R^2}\right)^2 - \frac{u_1^2}{R^2}}. \tag{3.21}$$

Considering the limit $R \to \infty$ we obtain: $\xi_1 \sim 1 + 2y^2/R^2$, $\xi_2 \sim x^2/R^2$ and the Beltrami coordinate (2.7) takes the Cartesian form

$$x_1 = R\frac{u_1}{u_0} = R\sqrt{\frac{2\xi_1\xi_2}{(\xi_1+1)(\xi_2+1)}} \to x,$$
$$x_2 = R\frac{u_2}{u_0} = R\sqrt{\frac{(\xi_1-1)(1-\xi_2)}{(\xi_1+1)(\xi_2+1)}} \to y.$$

The operator Q_E goes to Cartesian one

$$\frac{Q_E}{R^2} = \frac{L_3^2}{R^2} + \pi_1^2 \to P_1^2 \sim Q_C.$$

6. Elliptic coordinates on H_2 to parabolic on E_2. We start from the rotated elliptic coordinates

$$\begin{pmatrix} u_0' \\ u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} \cosh f & \sinh f & 0 \\ \sinh f & \cosh f & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \end{pmatrix} = $$
$$= \begin{pmatrix} u_0 \cosh f + u_1 \sinh f \\ u_0 \sinh f + u_1 \cosh f \\ u_2 \end{pmatrix}, \quad (3.22)$$

where $\sinh^2 f = (a_1 - a_2)/(a_2 - a_3)$. We choose $a_2 - a_3 = a_1 - a_2 \equiv a$. Then for rotated elliptic coordiantes (3.22) we get

$$\begin{gathered} u_0 = \frac{R}{\sqrt{2}}(\operatorname{sn}\alpha \operatorname{dn}\beta + i\sqrt{2}\operatorname{cn}\alpha\operatorname{cn}\beta), \\ u_1 = \frac{R}{\sqrt{2}}(i\operatorname{cn}\alpha\operatorname{cn}\beta + \sqrt{2}\operatorname{sn}\alpha\operatorname{dn}\beta), \quad u_2 = iR\operatorname{dn}\alpha\operatorname{sn}\beta, \end{gathered} \quad (3.23)$$

with modulus $k = k' = 1/\sqrt{2}$ for all Jacobi elliptic function. The integral of motion transforms into

$$Q_{E'} = 3\,L_3^2 - \sqrt{2}\,(K_1 L_3 + L_3 K_1), \quad (3.24)$$

with the correct limit to (1.36). From Eq. (3.23) we obtain

$$\operatorname{cn}\alpha = -\frac{i}{2}\sqrt{\left(1 + \frac{u_1'}{R\sqrt{2}} - \frac{u_0'}{R}\right)^2 + \frac{u_2'^2}{2R^2}} + \frac{i}{2}\sqrt{\left(1 - \frac{u_1'}{R\sqrt{2}} + \frac{u_0'}{R}\right)^2 + \frac{u_2'^2}{2R^2}},$$

$$\operatorname{cn}\beta = \frac{1}{2}\sqrt{\left(1 + \frac{u_1'}{R\sqrt{2}} - \frac{u_0'}{R}\right)^2 + \frac{u_2'^2}{2R^2}} + \frac{1}{2}\sqrt{\left(1 - \frac{u_1'}{R\sqrt{2}} + \frac{u_0'}{R}\right)^2 + \frac{u_2'^2}{2R^2}},$$

and therefore for large R we have

$$-i\,\mathrm{cn}\,\alpha \simeq 1 - \frac{1}{2\sqrt{2}}\frac{u^2}{R}, \quad \mathrm{cn}\,\beta \simeq 1 + \frac{1}{2\sqrt{2}}\frac{v^2}{R}.$$

In the limit $R \to \infty$ we obtain

$$x_1 \to x = \frac{u^2 - v^2}{2}, \quad x_2 \to y = uv,$$

i. e., the parabolic coordinates (1.39).

3.3. Contractions of Coordinate Systems from H_2 to $E_{1,1}$

1. Equidistant coordinates on H_2 to pseudo-spherical ones on $E_{1,1}$ plane. For Beltrami coordinates (2.10) we have:

$$y_0 = R \coth\tau_1 \cosh\tau_2, \quad y_1 = R\coth\tau_1 \sinh\tau_2. \tag{3.25}$$

Taking the limit $R \to \infty$, $\tau_1 \to i\dfrac{\pi}{2} + \dfrac{r}{R}$ and putting

$$\coth\tau_1 = \tanh\frac{r}{R} \sim \frac{r}{R}, \tag{3.26}$$

we obtain

$$y_0 \to t = r\cosh\tau_2, \quad y_1 \to x = r\sinh\tau_2, \tag{3.27}$$

where $0 \le r < \infty$, $-\infty < \tau_2 < \infty$. For the integral of motion we get

$$Q_{EQ} = K_2^2 \to Q_S = L_3^2. \tag{3.28}$$

2. Pseudo-spherical coordinates on H_2 to Cartesian coordinates on $E_{1,1}$. For coordinates (2.10) we have

$$y_0 = R\frac{\coth\tau}{\cos\varphi}, \quad y_1 = R\cot\varphi. \tag{3.29}$$

Taking the limit $R \to \infty$, $\tau \to i\pi/2$, $\varphi \to \dfrac{\pi}{2}$ and putting

$$\coth\tau \sim \frac{t}{R}, \quad \cot\varphi \sim \frac{x}{R}, \tag{3.30}$$

we see that Beltrami coordinates go into Cartesian ones

$$y_0 \to t, \quad y_1 \to x. \tag{3.31}$$

For the integral of motion we obtain

$$\frac{Q_S}{R^2} = \frac{L_3^2}{R^2} \to P_2^2 \sim Q_C. \tag{3.32}$$

4. CONTRACTION OF BASIS FUNCTIONS ON S_2 AND H_2

4.1. Contraction of Spherical Basis and Interbasis Expansions

1. Spherical basis on S_2 to polar on E_2. We start from the standard spherical functions $Y_{lm}(\theta,\phi)$ as basis functions of irreducible representations of the group $O(3)$ (see, e. g., Ref. 37)

$$Y_{lm}(\theta,\phi) = (-1)^{(m+|m|)/2}\left[\frac{2l+1}{2}\frac{(l+|m|)!}{(l-|m|)!}\right]^{1/2}\frac{(\sin\theta)^{|m|}}{2^{|m|}|m|!}\times$$
$$\times {}_2F_1\left(-l+|m|, l+|m|+1; |m|+1; \sin^2\frac{\theta}{2}\right)\frac{\mathrm{e}^{im\phi}}{\sqrt{2\pi}}. \quad (4.1)$$

In the contraction limit $R\to\infty$ we put

$$\tan\theta \sim \theta \sim \frac{r}{R}, \quad l \sim kR. \quad (4.2)$$

Using the asymptotic formulas

$$\lim_{R\to\infty} {}_2F_1\left(-kR, kR; |m|+1; \frac{r^2}{4R^2}\right) = {}_0F_1\left(|m|+1; -\frac{k^2r^2}{4}\right) \quad (4.3)$$
$$\lim_{z\to\infty}\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} = z^{\alpha-\beta} \quad (4.4)$$

and formula

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \frac{1}{\Gamma(\nu+1)}{}_0F_1\left(\nu+1; -\frac{z^2}{4}\right), \quad (4.5)$$

we obtain

$$\lim_{\substack{R\to\infty\\ \theta\to 0}} \frac{1}{\sqrt{R}} Y_{lm}(\theta,\phi) = (-1)^{(m+|m|)/2}\sqrt{k}J_{|m|}(kr)\frac{\mathrm{e}^{im\phi}}{\sqrt{2\pi}}. \quad (4.6)$$

The result (4.6) is not new [37]. The point is that this asymptotic formula is obtained very naturally in the context of group contractions applied to the separation of variables.

2. Spherical basis on S_2 to Cartesian on E_2. We start from the coordinates (θ',ϕ') in Eq. (1.64), but drop the primes, and write the corresponding spherical functions as

$$Y_{lm}(\theta,\phi) = \frac{\sqrt{2l+1}}{2\pi} \mathrm{e}^{im\phi} (\sin\theta)^{|m|} \times$$
$$\times \begin{cases} (-1)^{\frac{l+m}{2}} \left[\frac{\Gamma(\frac{l+m+1}{2})\Gamma(\frac{l-m+1}{2})}{\Gamma((\frac{l+m+2}{2})\Gamma(\frac{l-m+2}{2})}\right]^{1/2} {}_2F_1\left(-\frac{l-m}{2}, \frac{l+m+1}{2}; \frac{1}{2}; \cos^2\theta\right) \\ (-1)^{\frac{l+m-1}{2}} \left[\frac{\Gamma(\frac{l+m+2}{2})\Gamma(\frac{l-m+2}{2})}{\Gamma((\frac{l+m+1}{2})\Gamma(\frac{l-m+1}{2})}\right]^{1/2} 2\cos\theta\, {}_2F_1\left(-\frac{l-m-1}{2}, \frac{l+m+2}{2}; \frac{3}{2}; \cos^2\theta\right) \end{cases} \tag{4.7}$$

for $l+m$ even and odd, respectively, we now put

$$l \sim kR, \qquad m \sim k_2 R, \qquad \theta \sim \frac{\pi}{2}, \qquad \phi \sim \frac{\pi}{2}, \tag{4.8}$$

and

$$\sin\theta \to 1, \quad \cos\theta \to \frac{x}{R}, \quad \cos\phi \to \frac{y}{R}. \tag{4.9}$$

The ${}_2F_1$ hypergeometric functions simplify to ${}_0F_1$ ones, the Γ functions also simplify and the final result is that under the contraction we have

$$\begin{aligned} \lim_{R\to\infty} (-1)^{-((l+m)/2)} Y_{lm}(\theta,\phi) &= \sqrt{\frac{k}{k_1}} \frac{\mathrm{e}^{ik_2 y}}{\pi} \begin{cases} {}_0F_1\left(\frac{1}{2}; \frac{-k_1^2 x^2}{4}\right) \\ -i(k_1 x) {}_0F_1\left(\frac{3}{2}; \frac{-{k_1}^2 x^2}{4}\right) \end{cases} \\ &= \sqrt{\frac{k}{k_1}} \frac{\mathrm{e}^{ik_2 y}}{\sqrt{\pi}} \begin{cases} \cos k_1 x \\ -i \sin k_1 x \end{cases} \end{aligned} \tag{4.10}$$

with ${k_1}^2 + {k_2}^2 = k^2$ and for $l+m$ even and odd, respectively. The parity properties of Y_{lm} under the exchange $\theta \to \pi - \theta$ have led to the appearance of $\cos k_1 x$ and $\sin k_1 x$ in Eq. (4.10), instead of the usual Cartesian coordinate solution $\exp i(k_1 x + k_2 y)$.

Finally note that the factor $\sqrt{k/k_1}$ in formula (4.10) is connected with the contraction of Kronecker symbols to delta function

$$\delta_{ll'} \to \frac{1}{R}\delta(k-k') = \frac{1}{R}\frac{k}{k'}\delta(k_1 - k_1').$$

3. Contraction in interbasis expansions. Let us now consider the contraction $R \to \infty$ for the interbasis expansion (1.69). The contraction of basis functions was presented in the formulas (4.6) and (4.10). In order to obtain the corresponding limit we need the asymptotic behavior of the «little» Wigner d function for a large R. It is easy to see that the expression of d function for the angle $\pi/2$ in terms of hypergeometric functions ${}_2F_1$ (see for example [37]) is not applicable for the contraction limit when $\ell \to \infty$ and $m \to \infty$ simultaneously. To make this contraction we use an integral representation for the function $d^l_{m_2,m_1}(\beta)$ [37]

$$d^l_{m_2,m_1}(\beta) = \frac{i^{m_2-m_1}}{2\pi}\left[\frac{(l+m_2)!(l-m_2)!}{(l+m_1)!(l-m_1)!}\right]^{1/2} \times \\ \times \int_0^{2\pi}\left(\mathrm{e}^{i\frac{\varphi}{2}}\cos\frac{\beta}{2} + i\,\mathrm{e}^{-i(\varphi/2)}\sin\frac{\beta}{2}\right)^{\ell-m_1} \times \\ \times \left(\mathrm{e}^{-i(\varphi/2)}\cos\frac{\beta}{2} + i\,\mathrm{e}^{i(\varphi/2)}\sin\frac{\beta}{2}\right)^{\ell+m_1}\mathrm{e}^{im_2\varphi}d\varphi, \quad (4.11)$$

which for the particular case of $\beta = \pi/2$ can be presented in the following form

$$d^l_{m_2,m_1}\left(\frac{\pi}{2}\right) = (-1)^{(l-m_1)/2}\,\frac{2^l}{\pi}\left[\frac{(l+m_2)!(l-m_2)!}{(l+m_1)!(l-m_1)!}\right]^{1/2} \times \\ \times \int_0^{\pi}(\sin\alpha)^{l-m_1}(\cos\alpha)^{l+m_1}\mathrm{e}^{2im_2\alpha}d\alpha. \quad (4.12)$$

Using now the formulas [44]

$$\cos(2n\alpha) = T_n(\cos 2\alpha), \qquad \sin(2n\alpha) = \sin 2\alpha U_{n-1}(\cos 2\alpha),$$

where $T_l(x)$ and $U_l(x)$ are Tchebyshev polynomials of the first and second kind. After integrating over α, we obtain a representation of the Wigner d function for angles $\pi/2$ in terms of the hypergeometrical function ${}_3F_2(1)$

$$d^l_{m_2,m_1}\left(\frac{\pi}{2}\right) = \frac{(-1)^{\frac{l-m_1}{2}}}{\sqrt{\pi}l!}\sqrt{(l+m_2)!(l-m_2)!}\times$$
$$\times\begin{cases}\left\{\dfrac{\Gamma\left(\frac{l+m_1+1}{2}\right)\Gamma\left(\frac{l-m_1+1}{2}\right)}{\Gamma\left(\frac{l+m_1}{2}+1\right)\Gamma\left(\frac{l-m_1}{2}+1\right)}\right\}^{1/2} {}_3F_2\left(\begin{matrix}-m_2, m_2, \frac{l+m_1+1}{2} \\ \frac{1}{2}, l+1\end{matrix}\middle|\, 1\right), & (l+m_1)-\text{even}, \\ \dfrac{2il}{(l+1)}\left\{\dfrac{\Gamma\left(\frac{l+m_1}{2}+1\right)\Gamma\left(\frac{l-m_1}{2}+1\right)}{\Gamma\left(\frac{l+m_1+1}{2}\right)\Gamma\left(\frac{l-m_1+1}{2}\right)}\right\}^{1/2} {}_3F_2\left(\begin{matrix}-m_2+1, m_2+1, \frac{l+m_1}{2}+1 \\ \frac{3}{2}, l+2\end{matrix}\middle|\, 1\right), & (l+m_1)-\text{odd}.\end{cases} \quad (4.13)$$

For large R we put

$$l \sim kR, \quad m_1 \sim k_1 R, \quad \theta_1 \sim \frac{r}{R}, \quad \theta_1' \sim \frac{y}{R}, \quad \theta_2' \sim \frac{x}{R}, \quad (4.14)$$

where $k^2 = k_1^2 + k_2^2$. Using the asimptotic formulas for ${}_3F_2(1)$ function (4.3) and Γ function (4.4), we get

$$\lim_{R\to\infty} (-1)^{-\frac{l-|m_1|}{2}} \sqrt{R}\, d^l_{m_2,m_1}\left(\frac{\pi}{2}\right) = \sqrt{\frac{2}{\pi k}} \times$$
$$\times \begin{cases} \left(\dfrac{k^2}{k_2}\right)^{1/4} {}_2F_1\left(-m_2, m_2; \dfrac{1}{2}; \dfrac{k+k_1}{2k}\right), \\ -im_2\left(\dfrac{k_2}{k^2}\right)^{1/4} {}_2F_1\left(-m_2+1, m_2+1; \dfrac{3}{2}; \dfrac{k+k_1}{2k}\right), \end{cases}$$
$$= (-1)^{(3m_2)/2}\sqrt{\frac{2}{\pi k_2}} \begin{cases} \cos m_2\varphi, \\ i\sin m_2\varphi, \end{cases} \tag{4.15}$$

with $\cos\varphi = k_1/k$ and for $(\ell + m_1)$ even or odd, respectively.

Multiplying now the interbasis expansion (1.69) by the factor $(-1)^{-\frac{l-|m_1|}{2}}$, and taking the contraction limit $R \to \infty$ we obtain ($\theta \equiv \theta_2$, $m \equiv m_2$)

$$\mathrm{e}^{ik_1x}\left\{\begin{array}{c} \cos k_2y \\ \sin k_2y \end{array}\right\} = \sum_{m=-\infty}^{\infty} (i)^{|m|}\left\{\begin{array}{c} \cos m\varphi \\ -\sin m\varphi \end{array}\right\} J_{|m|}(kr)\,\mathrm{e}^{im\theta}, \tag{4.16}$$

or in exponential form

$$\mathrm{e}^{ikr\cos(\theta-\varphi)} = \sum_{m=-\infty}^{\infty} (i)^m\, J_m(kr)\,\mathrm{e}^{im(\theta-\varphi)}. \tag{4.17}$$

The inverse expansion is

$$J_m(kr)\,\mathrm{e}^{im\theta} = \frac{(-i)^m}{2\pi}\int_0^{2\pi} \mathrm{e}^{im\varphi - ikr\cos(\theta-\varphi)}d\varphi. \tag{4.18}$$

For $\theta = 0$ the two last formulas are equivalent to the well-known formulas in the theory of Bessel functions [44], namely expansions of plane waves in terms of cylindrical ones and vice versa.

4.2. Solutions of the Lamé Equation. Let us consider Eq. (1.60) on the sphere S_2 and separate variables in the elliptic coordinates (1.72). We obtain two ordinary differential equation of the form

$$\frac{d^2\psi}{d\rho^2} + \frac{1}{2}\left\{\frac{1}{\rho-a_1} + \frac{1}{\rho-a_2} + \frac{1}{\rho-a_3}\right\}\frac{d\psi}{d\rho} +$$
$$+ \frac{1}{4}\left\{\frac{\lambda - l(l+1)\rho}{(\rho-a_1)(\rho-a_2)(\rho-a_3)}\right\}\psi = 0, \tag{4.19}$$

or equivalently

$$4\sqrt{P(\rho)}\,\frac{d}{d\rho}\sqrt{P(\rho)}\,\frac{d\psi}{d\rho}-\{l(l+1)\rho-\lambda\}\psi=0, \qquad (4.20)$$

where

$$P(\rho)=(\rho-a_1)(\rho-a_2)(\rho-a_3).$$

Equation (4.19) is the Lamé equation in algebraic form. It is a Fuchsian type equation with 4 regular singularities (at a_1, a_2, a_3, and ∞) [38–40, 45–47].

Its general solution can be represented by a series expansion about any one of the singular points a_k as

$$\psi(\rho)=(\rho-a_1)^{\alpha_1/2}(\rho-a_2)^{\alpha_2/2}(\rho-a_3)^{\alpha_3/2}\sum_{t=0}^{\infty}b_t^{(k)}(\rho-a_k)^t, \qquad (4.21)$$

where we have

$$\alpha_j(\alpha_j-1)=0, \quad j=1,2,3$$

and can choose k equal to 1, 2, or 3.

Substituting (4.21) into the Lamé equation (4.19) we obtain a three-term recursion relation for b_t^k

$$\beta_t^{(k)}b_{t+1}^{(k)}+[\gamma_t^{(k)}+\lambda-l(l+1)a_k]b_t^{(k)}+ \\ +(2t+\alpha-l-2)(2t+\alpha+l-1)b_{t-1}^{(k)}=0 \qquad (4.22)$$

with

$$\begin{gathered}\alpha=\alpha_1+\alpha_2+\alpha_3, \quad \alpha_{ik}=\alpha_i-\alpha_k, \quad b_{-1}=0, \\ \beta_t^{(k)}=4(a_i-a_k)(a_j-a_k)(t+1)(t+\alpha_k+1/2) \quad (i,j,k \text{ cyclic}), \\ \gamma_t^{(k)}=-(a_i-a_k)(2t+\alpha_k+\alpha_j)^2-(a_j-a_k)(2t+\alpha_k+\alpha_i)^2.\end{gathered} \qquad (4.23)$$

The expansion (4.21) represents a Lamé function. Since we are interested in representations of $O(3)$, the sum in $\psi(\rho)$ must be a polynomial of order N, i. e., we must have

$$b_N\neq 0, \quad b_{N+1}=b_{N+2}=\ldots=0 \qquad (4.24)$$

for some N. The condition for this is that we have

$$l=2N+\alpha, \qquad (4.25)$$

and we obtain a secular equation for the eigenvalues λ, i. e., the separation constant in elliptic coordinates, by requiring that the determinant of the homogeneous linear system (3.14) for $\{b_0, b_1, \dots, b_N\}$ should vanish. Since N and l must be integers, Eq. (4.25) implies that α and l must have the same parity.

Numerous further properties of the Lamé polynomials, in the context of representations of the group $O(3)$, in the $O(3) \supset D_2$ basis, were established, e. g., in Refs. 38–40.

Here let us just represent the basis functions as

$$\Psi_{l\lambda}^{pq}(\rho_1, \rho_2) = A_{l\lambda}^{pq} \psi_{l\lambda}^{pq}(\rho_1) \psi_{l\lambda}^{pq}(\rho_2), \tag{4.26}$$

where $A_{l\lambda}^{pq}$ is some normalization constant. The labels p, q take values ± 1 and identify representations of D_2. For each value of l the values of p, q, and λ label $(2l+1)$ different states. Since the given representations (p, q) of D_2 can figure more than once in the reduction of a representation of $O(3)$ corresponding to the given l, we are faced with a «missing label problem», resolved by the quantum number λ, i. e., the operator Q of Eq. (1.63).

The expansions that we shall use for the Lamé polynomials in (4.26) are as in Eq. (4.21), but the summation over t is from $t = 0$ to $t = N$.

4.3. Elliptic Basis on S_2 to Cartesian Basis on E_2. We choose elliptic coordinates on S_2 as in Eq. (1.72), but with $a_1 < a_3 < a_2$, as in Eq. (3.5). We write the basis functions as in Eq. (4.26) with

$$\begin{aligned} \psi_{l\lambda}(\rho_1) &= (\rho_1 - a_1)^{\alpha_1/2} (\rho_1 - a_2)^{\alpha_2/2} (\rho_1 - a_3)^{\alpha_3/2} \sum_{t=0}^{N} b_t^{(1)} (\rho_1 - a_1)^t, \\ \psi_{l\lambda}(\rho_2) &= (\rho_2 - a_1)^{\alpha_1/2} (\rho_2 - a_2)^{\alpha_2/2} (\rho_2 - a_3)^{\alpha_3/2} \sum_{t=0}^{N} b_t^{(2)} (\rho_2 - a_2)^t \end{aligned} \tag{4.27}$$

as in Eq. (4.21). The coefficients $b_t^{(j)}$ $(j = 1, 2)$ satisfy the recursion relation (4.22) and we have $N = (l - \alpha)/2$. We use the coordinates ξ_1 and ξ_2 introduced in Eq. (3.7) (for $a \equiv a_3 - a_1 = a_2 - a_3$). Equation (4.27) reduces to

$$\begin{aligned} \psi_{l\lambda}(\xi_1) &= (-1)^{(\alpha_2+\alpha_3)/2} a^{\alpha/2} (1-\xi_1)^{\alpha_1/2} (1+\xi_1)^{\alpha_2/2} \xi_1^{\alpha_3/2} \sum_{t=0}^{N} C_t^1 (1-\xi_1)^t, \\ \psi_{l\lambda}(\xi_2) &= (-1)^{\alpha_2/2} a^{\alpha/2} (1 - \xi_2)^{\alpha_2/2} (1 + \xi_2)^{\alpha_1/2} \xi_2^{\alpha_3/2} \sum_{t=0}^{N} C_t^2 (1 - \xi_2)^t \end{aligned} \tag{4.28}$$

with $C_t^{(1)} = a^t b_t$, $C_t^{(2)} = (-a)^t b_t$. The recursion relations (4.22) now imply

$$
\begin{aligned}
&8(t+1)\left(t+\alpha_1+\frac{1}{2}\right)C_{t+1}^{(1)} + \{\mu^{(1)} - 2(2t+\alpha_1+\alpha_3)^2 - \\
&-(2t+\alpha_1+\alpha_2)^2\}C_t^{(1)} + (2t+\alpha-l-2)(2t+\alpha+l-1)C_{t-1}^{(1)} = 0, \\
&-8(t+1)\left(t+\alpha_2+\frac{1}{2}\right)C_{t+1}^{(2)} + \{\mu^{(2)} + +2(2t+\alpha_2+\alpha_3)^2 + \\
&+(2t+\alpha_1+\alpha_2)^2\}C_t^{(2)} - (2t+\alpha-l-2)(2t+\alpha+l-1)C_{t-1}^{(2)} = 0,
\end{aligned}
\tag{4.29}
$$

where

$$
\mu^{(j)} = \frac{1}{a}[\lambda - a_j l(l+1)], \quad j = 1, 2. \tag{4.30}
$$

The contraction limit is taken using Eq. (3.7) to relate $\xi_{1,2}$ to the Cartesian coordinates on E_2. Taking $l \sim kR$ we find

$$
\mu^{(1)} \to 2R^2 k_1^2, \qquad \mu^{(2)} \to -2R^2 k_2^2, \qquad k = \sqrt{k_1^2 + k_2^2}. \tag{4.31}
$$

For $R \to \infty$ the recursion relations (4.29) simplify to two-term ones that can be solved to obtain

$$
C_t^{(j)} = \frac{R^{2t}}{(\alpha_j + 1/2)_t}\left(\frac{-k_j^2}{4}\right)^t \frac{1}{t!} \tag{4.32}
$$

with

$$
\left(\alpha_j + \frac{1}{2}\right)_t = \left(\alpha_j + \frac{1}{2}\right)\left(\alpha_j + \frac{3}{2}\right)\ldots\left(\alpha_j - \frac{3}{2} + t\right),
$$
$$
t \geq 1, \quad \left(\alpha_j + \frac{1}{2}\right)_0 = 1.
$$

Substituting (4.3) into (4.28) we obtain

$$
\begin{aligned}
\psi_{l\lambda}(\xi_1) &= (-1)^{(\alpha_2+\alpha_3)/2}\frac{a^{\alpha/2}}{R^{\alpha_1}}x^{\alpha_1}{}_0F_1\left(\alpha_1 + \frac{1}{2}; -\frac{k_1^2 x^2}{4}\right), \\
\psi_{l\lambda}(\xi_2) &= (-1)^{\alpha_2/2}\frac{a^{\alpha/2}}{R^{\alpha_2}}y^{\alpha_2}{}_0F_1\left(\alpha_2 + \frac{1}{2}; -\frac{k_2^2 y^2}{4}\right).
\end{aligned}
\tag{4.33}
$$

Using now the formula (4.10) we find the contraction limit:

$$A_{l\lambda}^{pq}(R)\psi_{l\lambda}(\xi_1,\xi_2) \to A_{l\lambda}^{pq}(R)\psi_{k_1}(x)\psi_{k_2}(y) = A_{l\lambda}^{pq}(-1)^{\alpha_3/2}a^{\alpha/2} \times$$
$$\times \begin{cases} \cos k_1 x \cos k_2 y, & \alpha_1 = 0, \alpha_2 = 0 \\ -\dfrac{1}{k_2 R}\cos k_1 x \sin k_2 y, & \alpha_1 = 0, \alpha_2 = 1 \\ -\dfrac{1}{k_1 R}\sin k_1 x \cos k_2 y, & \alpha_1 = 1, \alpha_2 = 0 \\ -\dfrac{1}{k_1 k_2 R^2}\sin k_1 x \sin k_2 y, & \alpha_1 = 1, \alpha_2 = 1. \end{cases} \tag{4.34}$$

4.4. Elliptic Basis on S_2 to Elliptic Basis on E_2. Let us start from the elliptic coordinates (1.72) with $a_1 \le \rho_1 \le a_2 \le \rho_2 \le a_3$. We take the limit $R \to \infty$, $a_3 \to \infty$ with $\sqrt{a_3}/R$, a_1 and a_2 finite. We introduce a constant D as in Eq. (3.2). Elliptic coordinates on the plane E_2 are introduced via Eq. (3.4), so that the Cartesian coordinates (x, y) are expressed in terms of the elliptic ones (ξ, η) as in Eq. (1.41). Let us first take the limit in the separated equations (4.19). Going over to the variables (ξ, η) from (ρ_1, ρ_2) we obtain for $R \to \infty$:

$$\frac{d^2\psi_1}{d\eta^2} + \left\{\mu - \frac{k^2D^2}{2}\left(\frac{a_2+a_1}{a_2-a_1}\right) - \frac{k^2D^2}{2}\cos 2\eta\right\}\psi_1 = 0, \tag{4.35}$$

$$\frac{d^2\psi_2}{d\xi^2} + \left\{\mu - \frac{k^2D^2}{2}\left(\frac{a_2+a_1}{a_2-a_1}\right) - \frac{k^2D^2}{2}\cosh 2\xi\right\}\psi_2 = 0 \tag{4.36}$$

with

$$\mu = \frac{\lambda}{a_3}, \qquad l \sim kR.$$

In (4.35) we recognize the standard form of the Mathieu equation, whereas Eq. (4.36) is a modified Mathieu equation [41]. Thus, in the contraction limit, Lamé functions will go over into Mathieu ones. Moreover, periodic solutions of the Lamé equation go over into periodic solutions of Eq. (4.35).

The contraction limit can also be taken directly in the Lamé polynomials, using the expansion (4.21) (cut off at $t = N$). The result that we obtain is

$$\lim_{R\to\infty}\frac{\Psi_{l\lambda}(\rho_1)}{R^{\alpha_3}} = (a_2-a_1)^{\alpha/2}\frac{(-1)^{(\alpha_2+\alpha_3)/2}}{D^{\alpha_3}} \times$$
$$\times (\cos\eta)^{\alpha_1}(\sin\eta)^{\alpha_2}\sum_{t=0}^{\infty} C_t(\cos\eta)^{2t}, \tag{4.37}$$

$$\lim_{R\to\infty}\frac{\Psi_{l\lambda}(\rho_2)}{R^{\alpha_3}} = (a_2-a_1)^{\alpha/2}\frac{(-1)^{\alpha_3/2}}{D^{\alpha_3}}\times$$
$$\times(\cosh\xi)^{\alpha_1}(\sinh\xi)^{\alpha_2}\sum_{t=0}^{\infty}C_t(\cosh\xi)^{2t}, \quad (4.38)$$

where the expansion coefficients C_t satisfy recursion relations obtained from Eq. (4.22), namely

$$4(t+1)(t+1/2+\alpha_1)C_t+\{\mu-(2t+\alpha_1+\alpha_2)^2\}C_t-k^2D^2C_t=0. \quad (4.39)$$

4.5. Elliptic Basis on S_2 to Parabolic Basis on E_2. Let us consider the contraction limit for the Lamé equations (4.19). To do this we use equations (1.76) with $a_3-a_2=a_2-a_1=a$, i.e., $k=k'=1/\sqrt{2}$, together with Eq. (3.17), to obtain

$$\rho_1\sim a_1+a\left(-1+\frac{u^2}{2R}\right), \quad \rho_2\sim a_1+a\left(1+\frac{v^2}{R}\right). \quad (4.40)$$

The equation (4.19) for $\rho=\rho_1$ and $\rho=\rho_2$ in the limit $R\to\infty$, with $l^2\sim k^2R^2$ and $\lambda-a_2l(l+1)=\mu Ra$, yields the two equations

$$\frac{d^2\psi_1}{du^2}+(k^2u^2+\mu)\psi_1=0, \qquad \frac{d^2\psi_2}{dv^2}+(k^2v^2-\mu)\psi_2=0, \quad (4.41)$$

respectively.

Thus the Lamé equations in the contraction limit go over into the equations (4.41) for parabolic cylinder functions [44]. The same is of course true for solutions. The expansion (4.21) is not suitable for the contraction limit. In view of Eq. (3.17) we need expansions in terms of the variables $(1+\operatorname{sn}\alpha)$ and $(1-\sqrt{2}\operatorname{dn}\beta)$. This is not hard to do, following for instance methods used in Ref. 48 to relate the wave functions of a two-dimensional hydrogen atom, calculated in different coordinate systems. The formulas are cumbersome, so we shall not present them here.

4.6. Contractions of Basis Functions from H_2 to E_2 and $E_{1,1}$

1. *Pseudo-spherical basis on H_2 to polar basis on E_2*. The pseudo-spherical eigenfunctions $\Psi_{\rho m}(\tau,\varphi)$ normalized to the Dirac delta-function, have the form:

$$\Psi_{\rho m}(\tau,\varphi)=\sqrt{\frac{\rho\sinh\pi\rho}{2\pi^2R}}\left|\Gamma\left(\frac{1}{2}+i\rho+|m|\right)\right|\times$$
$$\times P^{|m|}_{i\rho-1/2}(\cosh\tau)\exp{(im\varphi)}, \quad (4.42)$$

where $m = 0, \pm 1, \pm 2, \ldots$ In the contraction limit $R \to \infty$ we put: $\tanh\tau \sim \tau \sim r/R$, $\rho \sim kR$. Rewriting the Legendre function in terms of hypergeometric function as [49]

$$P^{|m|}_{i\rho-1/2}(\cosh\tau) = \frac{\Gamma(1/2 + i\rho + |m|)}{\Gamma(1/2 + i\rho - |m|)} \frac{1}{|m|! 2^{|m|}} \, {}_2F_1 \times \\ \times \left(\frac{1}{2} + |m| + i\rho, \frac{1}{2} + |m| - i\rho; 1 + |m|; -\sinh^2 \frac{\tau}{2} \right).$$

Then using the asymptotic formula for hypergeometrical function ${}_2F_1$ and Γ function

$$\lim_{|y|\to\infty} \mid \Gamma(x + iy) \mid \exp\left(\frac{\pi}{2}|y|\right) |y|^{1/2 - x} = \sqrt{2\pi}, \tag{4.43}$$

we obtain in the contraction limit $R \to \infty$:

$$\lim_{R\to\infty} \Psi_{\rho m}(\tau, \varphi) = \sqrt{k} J_{|m|}(kr) \frac{\mathrm{e}^{im\varphi}}{\sqrt{2\pi}}.$$

2. *Pseudo-spherical basis on H_2 to Cartesian basis on $E_{1,1}$.* Taking the Legendre function in Eq. (4.42) in terms of two hypergeometric functions [49]

$$P^{m}_{i\rho-1/2}(\cosh\tau) = \frac{\sqrt{\pi} 2^m (\sinh\tau)^{-m}}{\Gamma\left(\frac{3}{4} - \frac{m + i\rho}{2}\right) \Gamma\left(\frac{3}{4} - \frac{m - i\rho}{2}\right)} \times \\ \times \left\{ 2\cosh\tau \frac{\Gamma\left(\frac{3}{4} - \frac{m + i\rho}{2}\right) \Gamma\left(\frac{3}{4} - \frac{m - i\rho}{2}\right)}{\Gamma\left(\frac{1}{4} - \frac{m + i\rho}{2}\right) \Gamma\left(\frac{1}{4} - \frac{m - i\rho}{2}\right)} \times \right. \\ \times {}_2F_1\left(\frac{3}{4} - \frac{m + i\rho}{2}, \frac{3}{4} - \frac{m - i\rho}{2}; \frac{3}{2}; \cosh^2\tau\right) + \\ \left. + {}_2F_1\left(\frac{1}{4} - \frac{m + i\rho}{2}, \frac{1}{4} - \frac{m - i\rho}{2}; \frac{1}{2}; \cosh^2\tau\right) \right..$$

Putting for large R

$$\rho \sim kR, \; m \sim k_1 R, \quad \coth\tau \sim \frac{t}{R}, \quad \cot\varphi \sim \frac{x}{R}, \quad k^2 + k_1^2 = k_0^2.$$

Using two asymptotic formulas

$$\lim_{R\to\infty} {}_2F_1\left(\frac{1}{4}-\frac{m+i\rho}{2}, \frac{1}{4}-\frac{m-i\rho}{2}; \frac{1}{2}; \cosh^2\tau\right) = \\ = {}_0F_1\left(\frac{1}{2}; -\frac{k_0^2 t^2}{4}\right) = \cos(k_0 t),$$

$$\lim_{R\to\infty} {}_2F_1\left(\frac{3}{4}-\frac{m+i\rho}{2}, \frac{3}{4}-\frac{m-i\rho}{2}; \frac{3}{2}; \cosh^2\tau\right) = \\ = {}_0F_1\left(\frac{3}{2}; -\frac{k_0^2 t^2}{4}\right) = \frac{\sin(k_0 t)}{k_0 t}$$

and formula (4.4) we finally obtain

$$\lim_{R\to\infty} \sqrt{R}|\Gamma(i\rho)|\Psi_{\rho m}(\tau,\varphi) = \sqrt{\frac{2}{k_0}}\, \mathrm{e}^{ik_0 t - ik_1 x}. \tag{4.44}$$

4.7. Contractions for Equidistant Basis on H_2

1. *Equidistant basis on H_2 to Cartesian basis on E_2.* In the equidistant system the normalized eigenfunctions $\Psi_{\rho\lambda}(\tau_1,\tau_2)$ have the form:

$$\Psi_{\rho\lambda}(\tau_1,\tau_2) = \sqrt{\frac{\rho\sinh\pi\rho}{\cosh^2\pi\lambda+\sinh^2\pi\rho}}(\cosh\tau_1)^{-1/2} P^{i\rho}_{i\lambda-1/2}(-\tanh\tau_1)\, \mathrm{e}^{i\lambda\tau_2}.$$

To perform the contraction we write the Legendre function in terms of hypergeometric function [49]

$$P^{i\rho}_{i\lambda-1/2}(-\tanh\tau_1) = \frac{\sqrt{\pi}2^{i\rho}(\cosh\tau_1)^{-i\rho}}{\Gamma\left(\frac{3}{4}-a\right)\Gamma\left(\frac{3}{4}-b\right)} \times \\ \times \left\{ {}_2F_1\left(\frac{1}{4}+a, \frac{1}{4}+b; \frac{1}{2}; \tanh^2\tau_1\right) + 2\tanh\tau_1 \frac{\Gamma\left(\frac{3}{4}-a\right)\Gamma\left(\frac{3}{4}-b\right)}{\Gamma\left(\frac{1}{4}-a\right)\Gamma\left(\frac{1}{4}-b\right)} \times \right. \\ \left. \times\, {}_2F_1\left(\frac{3}{4}-a, \frac{3}{4}-b; \frac{3}{2}; \tanh^2\tau_1\right) \right\},$$

where $a = i(\rho-\lambda)/2$; $b = i(\rho+\lambda)/2$. For large R we put $\rho \sim kR$, $\lambda \sim k_1 R$; $\tau_2 \sim x/R$, $\tau_1 \sim y/R$, where x, y are the Cartesian coordinates. Then using the asymptotic formulas:

$$\lim_{R\to\infty} {}_2F_1\left(\frac{1}{4}+a, \frac{1}{4}+b; \frac{1}{2}; \tanh^2\tau_1\right) = {}_0F_1\left(\frac{1}{2}; -\frac{y^2k_2^2}{4}\right) = \cos k_2 y,$$
$$\lim_{R\to\infty} {}_2F_1\left(\frac{3}{4}-a, \frac{3}{4}-b; \frac{3}{2}; \tanh^2\tau_1\right) = {}_0F_1\left(\frac{3}{2}; -\frac{y^2k_2^2}{4}\right) = \frac{1}{k_2 y}\sin k_2 y,$$

where $k_1^2+k_2^2=k^2$, we finally get

$$\lim_{R\to\infty} \Psi_{\rho\lambda}(\tau_1,\tau_2) = \sqrt{\frac{k}{\pi k_2}}\exp\left(ik_1x+ik_2y\right).$$

2. *Contraction from equidistant basis on H_2 to polar on $E_{1,1}$.* Writting the Legendre function in terms of hypergeometric functions [49]

$$P^{i\rho}_{i\lambda-1/2}(\tanh\tau_1) = \frac{1}{\sqrt{2\pi}}(\sinh\tau_1)^{i\rho}\left\{2^{-i\lambda}(\coth\tau_1)^{i\lambda+1/2}\frac{\Gamma(-i\lambda)}{\Gamma(1/2-i(\rho+\lambda))}\times\right.$$
$$\times\, {}_2F_1\left(\frac{1}{4}-\frac{i(\rho-\lambda)}{2}, \frac{3}{4}-\frac{i(\rho-\lambda)}{2}; 1+i\lambda; \coth^2\tau_1\right) + 2^{i\lambda}(\coth\tau_1)^{-i\lambda+1/2}\times$$
$$\left.\times\,\frac{\Gamma(i\lambda)}{\Gamma(1/2-i(\rho-\lambda)}\,{}_2F_1\left(\frac{1}{4}-\frac{i(\rho+\lambda)}{2}, \frac{3}{4}-\frac{i(\rho+\lambda)}{2}; 1-i\lambda;\, \coth^2\tau_1\right)\right\}.$$

Putting for large R: $\rho\sim kR$ and $\cosh\tau_1\sim r/R$, and using the asymptotic formulas for hypergeometric functions

$$\lim_{R\to\infty} {}_2F_1\left(\frac{1}{4}-\frac{i(\rho-\lambda)}{2}, \frac{3}{4}-\frac{i(\rho-\lambda)}{2};\, 1+i\lambda;\, \coth^2\tau_1\right) =$$
$$= \Gamma(1+i\lambda)\left(\frac{kr}{2}\right)^{-i\lambda} J_{i\lambda}(kr),$$
$$\lim_{R\to\infty} {}_2F_1\left(\frac{1}{4}-\frac{i(\rho+\lambda)}{2}, \frac{3}{4}-\frac{i(\rho+\lambda)}{2};\, 1-i\lambda;\, \coth^2\tau_2\right) =$$
$$= \Gamma(1-i\lambda)\left(\frac{kr}{2}\right)^{+i\lambda} J_{-i\lambda}(kr),$$

we obtain

$$\lim_{R\to\infty}\frac{1}{\sqrt{R}}\Psi_{\rho\lambda}(\alpha,\tau_2) = \sqrt{\frac{k}{2}}H^{(1)}_{i\lambda}(kr)\mathrm{e}^{i\lambda(\tau_2+i(\pi/2))},$$

where $H^{(1)}_{\nu}(z)$ is the first kind of Hankel function.

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REFERENCES

1. *Inönü E., Wigner E. P.* On the Contractions of Groups and Their Representations // Proc. Nat. Acad. Sci. (US). 1953. V. 39. P. 510.
2. *Kirillov A. A., Neretin Yu. A.* The Variety A_n of n-Dimensional Lie Algebra Structures // Amer. Math. Soc. Transl. Ser. 2. 1987. V. 137. P. 21–31.
3. *Saletan E. J.* Contraction of Lie Groups // J. Math. Phys. 1961. V. 2. P. 1–21.
4. *Gilmore R.* Lie Groups, Lie Algebras and Some of Their Applications. N. Y.: Wiley, 1974.
5. *Moody R. V., Patera J.* Discrete and Continuous Graded Contractions of Representations of Lie Algebra // J. Phys. A: Math. Gen. 1991. V. 24. P. 2227–2258.
6. *de Montigny M., Patera J.* Discrete and Continuous Graded Contractions of Lie Algebras and Superalgebras // Ibid. P. 525–549.
7. *Couture M. et al.* Graded Contractions of $sl(3, C)$ // J. Math. Phys. 1991. V. 32. P. 2310–2318.
8. *Vilenkin N. Ya.* Special Functions and the Theory of Group Representation // Amer. Math. Soc. Providence, R. I., 1968.
9. *Talman J. D.* Special Functions: A Group Theoretic Approach. N. Y.: Benjamin, 1968.
10. *Miller W., Jr.* Lie Theory and Special Functions. N. Y.: Academic Press, 1968.
11. *Exton H.* q-Hypergeometric Functions and Applications. Chichester: Ellis Horwood, 1983.
12. *Gasper G., Rahman M.* Basic Hypergeometric Series. Cambridge University Press, 1990.
13. *Floreanini R., Vinet L.* // J. Math. Phys. 1992. V. 33. P. 1358.
14. *Winternitz P., Fris I.* Invariant Expansions of Relativistic Amplitudes and the Subgroups of the Proper Lorentz Group // Sov. J. Nucl. Phys. 1965. V. 1. P. 636.
15. *Winternitz P., Lukac L., Smorodinsky Ya. A.* Quantum Numbers in the Little Groups of the Poincaré Group // Sov. J. Nucl. Phys. 1968. V. 7. P. 139.
16. *Mardoyan L. G. et al.* Hidden Symmetry, Separation of Variables and Interbasis Expansions in the Two-Dimensional Hydrogen Atom // J. Phys. A: Math. Gen. 1985. V. 18. P. 455–466.
17. *Miller W., Jr.* Symmetry and Separation of Variables. Addison Wesley Publishing Company, 1987.
18. *Kalnins E. G.* Separation of Variables for Riemannian Spaces of Constant Curvature. Longman, 1986.
19. *Miller W., Jr., Patera J., Winternitz P.* Subgroups of Lie Groups and the Separation of Variables // J. Math. Phys. 1981. V. 22. P. 251.

20. *Izmest'ev A. A. et al.* Contraction of Lie Algebras and Separation of Variables // J. Phys. A: Math. Gen. 1996. V. 29. P. 5940–5962.

21. *Izmest'ev A. A. et al.* Contraction of Lie Algebras and Separation of Variables. Two-Dimensional Hyperboloid // Intern. J. of Mod. Phys. A. 1997. V. 12(1). P. 53–61.

22. *Izmest'ev A. A. et al.* Lie Algebra Contractions for Overlap Functions. JINR Preprint E2-99-21. Dubna, 1999.

23. *Izmest'ev A. A., Pogosyan G. S., Sissakian A. N.* Contraction of Lie Algebras and Separation of Variables. From Two-Dimensional Hyperboloid to Two-Dimensional Minkowsky Space // Proc. «Quantum Theory and Symmetries» (QTS1) / Ed. H.-D. Doebner et al. Singapore, 2000. P. 583–591.

24. *Izmest'ev A. A., Pogosyan G. S.* Contraction of Lie Algebras and Separation of Variables on Three-Dimensional Sphere // Proc. «Physical Applications and Mathematical Aspects of Geometry, Groups, and Algebras» (GROUP21) / Ed.: H.-D. Doebner, W. Scherer, P. Nattermann. Singapore, 1997. V. 1. P. 137.

25. *Kalnins E. G., Miller W., Jr., Pogosyan G. S.* Contractions of Lie Algebras: Applications to Special Functions and Separation of Variables // J. Phys. A: Math. Gen. 1999. V. 32. P. 4709–4732.

26. *Izmest'ev A. A. et al.* Contraction of Lie Algebras and Separation of Variables. N-dimensional Sphere // J. Math. Phys. 1999. V. 40. P. 1549–1573.

27. *Pogosyan G. S. et al.* Graf's Addition Theorem Obtained from $SO(3)$ Contraction // Theor. Math. Phys. 2001. V. 129(2). P. 1501–1503 (in Russian: Teor. Matem. Fiz. 2001. V. 129(2). P. 227–229).

28. *Izmest'ev A. A. et al.* Contractions of Lie Algebras and the Separation of Variables. Interbases Expansions // J. Phys. A: Math. Gen. 2001. V. 34. P. 521–554.

29. *Izmest'ev A. A. et al.* Contractions and Interbases Expansions on N-Sphere // Proc. of Conf. «Quantum Theory and Symmetry» (QTS-2) Krakow, Poland, July 16–21, 2001.

30. *Calzada J. A. et al.* Contraction of Superintegrable Hamiltonian Systems // J. Math. Phys. 2000. V. 41. P. 317–336.

31. *Heredero H. R. et al.* Lie Algebra Contractions and Symmetries of the Toda Hierarchy // J. Phys. A: Math. Gen. 2000. V. 33. P. 5025–5040.

32. Lecons sur la théorie générale des surface / Ed. by G. Darboux. N. Y.: Chelsea, 1972. V. 1–4.

33. *Koenigs G.* Sur les géodésiques a intégrales quadratiques // Ibid. V. 4. P. 368–404.

34. *Liouville J.* Memoire sur L'integration des equations differentielles du mouvement d'un nombre quelconque de points materiels // J. Math. Pures Appl. 1849. V. 14. P. 257–299.

35. *Djokovic D. Z. et al.* Normal Forms of Elements of Classical Real and Complex Lie and Jordan Algebras // J. Math. Phys. 1983. V. 24. P. 1062–1072.

36. *Kalnins E. G.* On the Separation of Variables for the Laplace Equation $\Delta\Psi + K^2\Psi$ in Two- and Three-Dimensional Minkowsky Space // SIAM J. Math. Anal. 1975. V. 6(2).

37. *Varshalovich D. A., Moskalev A. N., Khersonskii V. K.* Quantum Theory of Angular Momentum. L.: Nauka, 1975.

38. *Patera J., Winternitz P.* A New Basis for the Representations of the Rotation Group. Lamé and Heun Polynomials // J. Math. Phys. 1973. V. 14. P. 1130.

39. *Lukac I., Smorodinskii Ya. A.* Wave Functions for the Asymmetric Top // Sov. Phys. JETP. 1970. V. 30. P. 728.

40. *Lukac I.* A Complete Set of the Quantum-Mechanical Observables on a Two-Dimensional Sphere // Theor. Math. Phys. 1973. V. 14. P. 271–281.

41. *Bateman H., Erdélyi A.* Higher Transcendental Functions. N. Y.: McGraw-Hill, 1955. V. 3.

42. *Olevskii M. N.* Triorthogonal Systems in Spaces of Constant Curvature in which the Equation $\Delta_2 u + \lambda u = 0$ Allows the Complete Separation of Variables // Math. Sb. 1950. V. 27. P. 379 (in Russian).

43. *Grosche C., Pogosyan G. S., Sissakian A. N.* Path Integral Approach to Superintegrable Potentials. Two-Dimensional Hyperboloid // Phys. Part. Nucl. 1996. V. 27(3). P. 244–278; FECAY. V. 27. P. 593–674.

44. *Bateman H., Erdélyi A.* Higher Transcendental Functions. N. Y.: McGraw-Hill, 1953. V. 2.

45. *Whittaker E. T., Watson G. N.* A Course of Modern Analysis. Cambridge University Press, 1927. V. 2.

46. *Hobson E. W.* The Theory of Spherical and Ellipsoidal Harmonics. N. Y., 1965.

47. *Arscott F. M.* Periodic Differential Equations. N. Y.: Macmillan, 1964.

48. *Mardoyan L. G. et al.* Two-Dimensional Hydrogen Atom. I. Elliptic Basis // Teor. Matem. Fiz. 1984. V. 61. P. 99–117.

49. *Bateman H., Erdélyi A.* Higher Transcendental Functions. N. Y.: McGraw-Hill, 1953. V. 1.