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The review is devoted to a relativistic formulation of the first Dirac quantization of QED (1927) and its generalization to the non-Abelian theories (Yang–Mills and QCD) with the topological degeneration of initial data. Using the Dirac variables we give a systematic description of relativistic nonlocal bound states in QED with a choice of the time axis of quantization along the eigenvectors of their total momentum operator. We show that the Dirac variables of the non-Abelian fields are topologically degenerated, and there is a pure gauge Higgs effect in the sector of the zero winding number that leads to a nonperturbative physical vacuum in the form of the Wu–Yang monopole. Phases of the topological degeneration in the new perturbation theory are determined by an equation of the Gribov ambiguity of the constraint-shell gauge defined as an integral of the Gauss equation with zero initial data. The constraint-shell non-Abelian dynamics includes zero mode of the Gauss-law differential operator, and a rising potential of the instantaneous interaction, that rearranges the perturbation series and changes the asymptotic freedom formula. The Dirac variables with the topological degeneration of initial data describe color confinement in the form of quark-hadron duality as a consequence of summing over the Gribov copies. A solution of $U(1)$ problem is given by mixing the zero mode with η_0 meson. We discuss reasons why all these physical effects disappear for arbitrary gauges of physical sources in the standard Faddeev–Popov integral.

Обзор посвящен релятивистской формулировке первого дираковского квантования и его обобщению на неабелевы теории Янга–Миллса и КХД с топологическим вырождением начальных данных. Мы используем релятивистски-ковариантное определение дираковских калибровочно-инвариантных переменных, заданных на поверхности связей (Полубаринов, 1965), для систематического описания нелокальных связанных состояний в КЭД, выбирая ось времени вдоль собственных векторов оператора полного импульса физических состояний, чтобы удовлетворить условию Маркова–Юкавы (т. е. условию неприводимости нелокальных представлений группы Пуанкаре). Показано, что прямое обобщение дираковских переменных на неабелеву теорию ведет к их топологическому вырождению в форме грибовских копий калибровки, которая является интегралом уравнения Гаусса с нулевыми начальными данными. Уравнение Грибова однозначно определяет фазы топологических преобразований и их «носитель» в виде ифракрасно-регуляризованного монополя Ву–Янга. Динамика неабелевых теорий на поверхности связей включает нулевую моду уравнения связи и растущий потенциал одновременного взаимодействия токов. Проблема грибовских нулей детерминанта Фаддеева–Попова решается путем построения адекватного интеграла Фейнмана, содержащего интегрирование по нулевым модам. Релятивистская формулировка КХД на поверхности связей с топологическим вырождением начальных данных описывает конституентные массы кварков и глюонов, модификацию формулы асимптотической свободы, спонтанное нарушение киральной симметрии, конфайнмент цветных состояний в форме кварк-адронной дуальности (как следствие усреднения по топологическому вырождению) и дополнительную массу девятого псевдоскалярного мезона (как следствие смешивания этого мезона с нулевой модой). Мы обсуждаем причины исчезновения всех этих эффектов при переходе к интегралу Фаддеева–Попова с произвольной калибровкой.

INTRODUCTION

The first papers by Dirac [1], Heisenberg, Pauli [2], and Fermi [3] on the quantization of electrodynamics ran into difficulties of the determination of physical variables. The interpretation of all gauge components as independent variables contradicts the quantum principles, whereas exclusion of nonphysical variables contradicts the relativistic principles.

The consistent quantum description of gauge constrained systems was considered by Dirac, Schwinger, Feynman, Faddeev, Gribov, and other physicists [1, 4–7] as one of the most fundamental problems of theoretical physics in the 20th century.

The first quantization of electrodynamics belongs to Dirac [1] who disregarded the relativistic principles and excluded nonphysical components by the reduction of the initial action to the solution of the Gauss law constraint, i. e., the equation for the time-like component of the gauge vector field. The Gauss law connects initial data of the time-like component with the data of all other fields. But this *constraint-shell* method had a set of defects, including nonlocality, explicit noncovariance as the dependence on the external time axis of quantization, and complexity. Feynman–Schwinger–Tomonaga formulation of QED admitted a simpler method based on the extended dynamics where all components were considered on equal footing with fixing a relativistic gauge.

At the beginning of the 1960s, Feynman found that the naive generalization of his method of construction of QED did not work for the non-Abelian theory. The unitary S matrix in the non-Abelian theory was obtained in the form of the Faddeev–Popov (FP) path integral [8] by the brilliant application of the theory of connections in vector bundle. There is an opinion that the FP path integral is the highest level of quantum description of gauge relativistic constrained systems. In any case, just this FP integral was the basis to prove renormalizability of the unified theory of electroweak interactions in papers by 't Hooft and Veltman awarded the Nobel prize in 1999.

Nevertheless, in the context of the first Dirac quantization and its Hamiltonian generalizations [9, 10], the intuitive status of the FP integral was so evident that two years after the paper [8], Faddeev gave the foundation of the FP integral by the construction of the unitary S matrix [6] for an «equivalent non-Abelian unconstrained system» derived by resolving constraints in terms of the radiation variables of the Hamiltonian description.

Faddeev showed that on the one hand, the constraint-shell dynamics is compatible with the simplest quantization by the standard Feynman path integral, on the other hand, this Feynman integral is equivalent to the FP integral in an arbitrary gauge. This equivalence was proved by the change of variables in the Feynman path integral that removed the time-like vector of the canonical quantization into the phase factors of physical source terms. These phase factors disappear for S-matrix elements on the mass shell of elementary particles. In other words, Faddeev proved the equivalence of the *constraint-shell* approach with quantization of gauge theories by the gauge-fixing method only for scattering amplitudes [6] where all color particle-like excitations of the fields are on their mass-shell. But the scattering amplitudes for color particles are nonobservable in QCD. The observables are hadrons as colorless bound states where elementary particles are off the mass-shell. Just for this case, the Faddeev theorem of equivalence of different gauges becomes problematic even for QED in the sector of instantaneous bound states, as the FP integral in a relativistic gauge loses all propagators with analytic properties that lead to instantaneous bound states identified with observable atoms.

The Faddeev equivalence theorem [6] was proved before the revelation of nontrivial topological properties [11–13] and the Gribov ambiguities [7]. These new facts were studied mainly on the level of the FP integral. The topological degeneration of classical vacuum was incorporated into the FP integral in the form of the instanton solutions [11].

It seems more natural, first, to study the topological degeneration of all non-Abelian physical states at the more fundamental level of the *constraint-shell dynamics* compatible with the Feynman path integral and, then, to obtain the corresponding non-Abelian FP path integral analyzing possible Gribov copies.

The first quantization of electrodynamics was fulfilled just at the level of the *constraint-shell dynamics* by Dirac [1]. The Dirac reduction is based on the honest resolution of the Gauss law constraint and introduction of the Dirac nonlocal gauge-invariant physical variables [1, 14, 15], instead of gauge-fixing. The Dirac method is the way to distinguish the unique (radiation) gauge of physical sources.

The present review is devoted to the generalization of the Dirac variables [1, 14, 15] to the non-Abelian theories with the topological degeneration of the initial data in QCD in the class of functions of nontrivial topological transformations [16–20].

We present here a set of arguments in favor of that the constructed *equivalent unconstrained system* contains the most interesting physical effects of hadronization and confinement in QCD that can be hidden in explicit solutions of constraints and equations of motion [18, 19, 21, 22]. We show also that the FP integral in an arbitrary relativistic gauge loses all these effects. The relativistic covariant properties of the Dirac variables [23] allow us to use the Markov–Yukawa [24] prescription of construction of the multilocal irreducible representations of the Poincare group and to formulate the generalized S-matrix formalism where the time axis of the Hamiltonian description is proportional to eigenvectors of the total momenta operator of any physical state [25–31].

1. DIRAC VARIABLES IN QED

1.1. Gauge-Fixing Method of 1967. We formulate the statement of the problem using QED. It is given by the action

$$W[A,\psi,\bar{\psi}] = \int dx \left[-\frac{1}{4}(F_{\mu\nu})^2 + \bar{\psi}(i\nabla\!\!\!/(A) - m^0)\psi \right], \qquad (1)$$

$$\nabla_\mu(A) = \partial_\mu - ieA_\mu, \quad \nabla\!\!\!/ = \nabla_\mu \gamma^\mu, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \qquad (2)$$

This action contains gauge fields more than independent degrees of freedom. First of all it is invariant with respect to gauge transformations

$$A_\mu^\Lambda = A_\mu + \partial_\mu \Lambda, \quad \psi^\Lambda = \exp[ie\Lambda]\psi. \qquad (3)$$

One supposes that this invariance allows us to remove one field degree of freedom with the help of arbitrary gauges

$$F(A_\mu) = 0, \quad F(A_\mu^u) = M_F u \neq 0, \qquad (4)$$

where the second equation means that the gauge fixes field unambiguously.

The standard quantization of the gauge field system in the gauge $F(A_\mu) = 0$ is based on the FP path integral [8]

$$Z^{\rm FP}[s^F, \bar{s}^F, J^F] = \\ = \int \prod_\mu DA_\mu^F D\psi^F D\bar{\psi}^F \Delta_{\rm FP}^F \delta(F(A^F)) \exp\left(iW[A^F, \psi^F\bar{\psi}^F] + iS^F\right), \qquad (5)$$

where $\Delta_{\rm FP}^F = \det M_F$ is FP determinant and

$$S^F = \int d^4x \left(\bar{s}^F \psi^F + \bar{\psi}^F s^F + A_\mu^F J^\mu \right) \qquad (6)$$

are the sources. The foundation of the intuitive FP integral by the canonical quantization was made in Faddeev's paper [6].

After the Faddeev papers of 1967–1969, there dominates very popular opinion that all physical results do not depend on a gauge $F(A_\mu) = 0$. A gauge is defined for reasons of simplicity and convenience, including the condition $\det M_F \neq 0$; the opposite case $\det M_F = 0$ is called the Gribov ambiguity [7]. All applications of quantum gauges theory after 1967–1969, including investigation of the Gribov ambiguity [32], topological degeneration of initial data [11], hadronization [33], parton models [34], etc., were fulfilled at the level of the FP integral (5).

While, in Faddeev's paper [6] there was established the range of validity of the FP integral and its gauge independence. Faddeev managed to prove the FP integral by the canonical quantization only in the sector of scattering amplitudes for elementary particles on their mass-shell. This proof becomes doubtful for bound states where elementary particles are off mass-shell, the latter is crucial for QCD where only bound states are observable.

The present review is devoted to consideration of the problems of bound states, Gribov copies, and topological degenerations of initial data at the more fundamental level of the Dirac canonical quantization of 1927 [1].

1.2. Dirac Reduction and Dirac Variables. The situation with canonical quantization of gauge theories before the FP revolution of 1967 was presented in the review by I. V. Polubarinov [14]. Igor Vasil'evich intended to send his manuscript to «Usp. Fiz. Nauk». It is a pity that this nice review was not published. After 1967, Polubarinov tried to include the FP integral in his review, but he was not satisfied by the level of the physical foundation of the FP scheme of quantization in the context of the first (constraint-shell) formulation of QED that belongs to Dirac [1]. We reproduce here this formulation.

The Dirac quantization was based on a definite frame of reference distinguished by the time axis $\eta_\mu = (1,0,0,0)$, which allows us to establish boundary conditions and initial data. In special relativity, Einstein identified a frame of reference with a set of the watches, rulers, and other physical instruments for measurement of physical quantities (that include in our case a spectrum of the field excitations). The classical equations are split on the Gauss law constraint

$$\frac{\delta W}{\delta A_0} = 0 \Rightarrow \Delta A_0 = \partial_i \partial_0 A_i + j_0 \quad (\Delta = \partial_i \partial_i,\ j_\mu = e\bar{\psi}\gamma_\mu \psi), \tag{7}$$

and equations of motion

$$\frac{\delta W}{\delta A_k} = 0 \Rightarrow \partial_0^2 A_k - \partial_k \partial_0 A_0 - (\delta_{ki}\Delta - \partial_k \partial_i) A_i = j_k, \tag{8}$$

$$\frac{\delta W}{\delta \psi} = 0 \Rightarrow \bar{\psi}(i\nabla\!\!\!/(A) + m^0) = 0, \quad \frac{\delta W}{\delta \bar{\psi}} = 0 \Rightarrow (i\nabla\!\!\!/(A) - m^0)\psi = 0. \tag{9}$$

The problem of canonical quantization meets with the nondynamic status of the time component $A_0 = A_\mu \eta_\mu$. The nondynamic status of A_0 is not compatible with quantization of this component as the fixation of A_0 (by the Gauss law), and its zero momentum $E_0 = \partial \mathcal{L}/\partial(\partial_0 A_0) = 0$ contradicts the commutation relation and uncertainty principle. To keep quantum principle, Dirac excluded the time component using the Gauss law constraint (7): an explicit solution of this Gauss law

$$A_0(t,x) = a_0[A] + \frac{1}{\Delta} j_0(t,x), \tag{10}$$

where

$$a_0[A] = \frac{1}{\Delta}\partial_i\partial_0 A_i(t,x), \tag{11}$$

connects initial data of $A_0(t_0,x)$ with a set of the initial data of the longitudinal component $\partial_i\partial_0 A_i(t,y)$, and current $j_0(t,y)$ in the whole space, here

$$\frac{1}{\Delta}f(x) = -\frac{1}{4\pi}\int d^3y \frac{f(y)}{|x-y|} \tag{12}$$

is the Coulomb kernel of the nonlocal distribution. One can substitute the solution (10) into equation for spatial component (8)

$$\frac{\delta W}{\delta A_i}\Bigg|_{\delta W/\delta A_0=0} \Rightarrow \left[\delta_{ik} - \partial_i\frac{1}{\Delta}\partial_k\right](\partial_0^2 - \Delta)A_k = j_i - \partial_i\frac{1}{\Delta}\partial_0 j_0. \tag{13}$$

We can see that the constraint-shell equations (13) contain only two transverse physical variables as the gauge-invariant functionals

$$A_i^*(t,\mathbf{x}) = \left[\delta_{ik} - \partial_i\frac{1}{\Delta}\partial_k\right]A_k. \tag{14}$$

Dirac generalized these gauge-invariant variables onto all fields by gauge transformations

$$\begin{gathered}\sum_{a=1,2} e_k^a A_a^D = A_k^D[A] = v[A]\left(A_k + i\frac{1}{e}\partial_k\right)(v[A])^{-1},\\ \psi^D[A,\psi] = v[A]\psi,\end{gathered} \tag{15}$$

where the gauge factor [1] is defined by

$$v[A] = \exp\left\{ie\int_{t_0}^{t} dt' a_0(t')\right\}. \tag{16}$$

Using the gauge transformations (3)

$$a_0^\Lambda = a_0 + \partial_0\Lambda \Rightarrow v[A^\Lambda] = \exp[ie\Lambda(t_0,\mathbf{x})]v[A]\,\exp[ie\Lambda(t,\mathbf{x})], \tag{17}$$

we can find that initial data of the gauge-invariant Dirac variables (15) are degenerated with respect to stationary gauge transformations

$$\begin{gathered}A_i^D[A^\Lambda] = A_i^D[A] + \partial_i\Lambda(t_0,\mathbf{x}),\\ \psi^D[A^\Lambda,\psi^\Lambda] = \exp[ie\Lambda(t_0,\mathbf{x})]\psi^D[A,\psi].\end{gathered} \tag{18}$$

The Dirac variables (15) as the functionals of initial fields satisfy the Gauss law constraint

$$\partial_0 \left(\partial_i A_i^D(t, \mathbf{x})\right) \equiv 0. \tag{19}$$

Thus, explicit resolving of the Gauss law allows us to remove two degrees of freedom and to reduce the gauge group into the subgroup of the stationary gauge transformations (18).

We can fix a stationary phase $\Lambda(t_0, \mathbf{x}) = \Phi_0(\mathbf{x})$ by an additional constraint in the form of the time integral of the Gauss law constraint (19) with zero initial data

$$\partial_i A_i^D = 0. \tag{20}$$

We call this equation the *constraint-shell «gauge»*. This «gauge» restricts initial data to a phase distinguished by the equations $\Delta\Phi_0(\mathbf{x}) = 0$. Nontrivial solutions of this equation we call the degeneration of initial data and the Gribov copies of the constraint-shell «gauge». The degeneration of initial data is determined by topological properties of the manifold of stationary gauge transformations in the class of functions with a finite density of energy. In the case of the three-dimensional QED there is only a trivial solution $\Phi_0(\mathbf{x}) = 0$. In this case

$$\psi^D = \psi^*, \quad A^D = A_i^*.$$

In the one-dimensional QED and non-Abelian theory the degeneration of initial data is the evidence of a zero mode of the Gauss law constraint that describes the topological excitation of gauge field with the Coleman spectrum of the electric tension [17, 19, 35, 36]. We consider this zero mode in the Section devoted to Yang–Mills theory.

Dirac constructed an *equivalent unconstrained system*, the equations of which reproduce the equations of the initial theory (1).

$$\begin{aligned} W^* &= W|_{\delta W/\delta A_0 = 0} = \\ &= \int d^4x \frac{1}{2}\left[\sum_{a=1,2} (\partial_\mu A_a^* \partial^\mu A_a^*) + \frac{1}{2} j_0^* \frac{1}{\Delta} j_0^* - j_i^* A_i^* + \bar\psi^*(i\partial\!\!\!/ - m)\psi^*\right]. \end{aligned} \tag{21}$$

To derive this *equivalent unconstrained system* that contains only physical variables, Dirac [1, 14] proposed to change the order of constraining and varying. He substituted the solution of the Gauss law constraint (10) into the initial action (1) in the rest frame $\eta_\mu = (1, 0, 0, 0)$ and introduced the Dirac variables (15).

To combine the nonlocal physical variables A_i^* and variational principle formulated for local fields, we can introduce three independent variables. In this case, the Dirac action should be added by the Lagrange multiplier [14]

$$W_{\text{eff}} = W^* + \int d^4x \lambda_L(x) \partial_i A_i^*. \tag{22}$$

The local equations of the *equivalent unconstrained system* (22)

$$\frac{\delta W_{\text{eff}}}{\delta A_i^*} = 0, \quad \frac{\delta W_{\text{eff}}}{\delta \lambda_L} = 0, \quad \frac{\delta W^*}{\delta \psi^*} = 0, \quad \frac{\delta W^*}{\delta \bar{\psi}^*} = 0 \tag{23}$$

completely coincide with the equations of the initial constrained action (7)–(9). Equations (23) reproduce the constraint (20) and lead to the equation for the Lagrange multiplier

$$\Delta \lambda_L(t, \mathbf{x}) = 0. \tag{24}$$

The latter coincides with the equation for the stationary phase. In three-dimensional QED both λ and Φ_0 are equal to zero in the class of functions where the physical fields are defined.

In three-dimensional QED there are only three subtle differences of the *equivalent unconstrained system* (22) from the initial gauge theory (1). First of them is the origin of the current conservation law. In the initial constrained system (1), the current conservation law $\partial_0 j_0 = \partial_i j_i$ follows from the equations for the gauge fields; whereas the similar law $\partial_0 j_0^* = \partial_i j_i^*$ in the *equivalent unconstrained system* (21) follows only from the classical equations for the fermion fields. This difference becomes essential in quantum theory. In the second case, we cannot use the current conservation law, if the quantum fermions are off mass-shell, in particular, in an atom. What we observe in an atom? The bare fermions, or *dressed* ones (15)? Dirac supposed [1] that we can observe only *gauge invariant* quantities of the type of the *dressed* fields. Really, we can convince that *dressed* fields (15) as nonlocal functionals from initial gauge fields are invariant with respect to the time-dependent gauge transformations of these initial fields (3).

The gauge invariance with respect to the time-dependent gauge transformations is the second difference of the nonlocal Dirac variables (15) from the initial fields of the constrained system (1) with usual transformational properties with respect to the gauge and Lorentz transformations.

The gauge constraint $\partial_i A_i = 0$, in the gauge-fixing method, is associated with the relativistic noncovariance. Whereas, the observable nonlocal variables (15) depend on the time axis by the *relativistic covariant* manner. Polubarinov's review [14] was mainly devoted to the relativistic covariant formulation of the Dirac quantization [1].

The gauge-fixing method and its terminology «the Coulomb gauge» do not reflect these three properties of the Dirac observables in the *constraint-shell*

QED (21): the off-mass-shell nonconservation of the current, gauge invariance (3), and relativistic covariance.

In fact, the term *gauge* (20) means a *choice of nonlocal variables*, or more exactly, a *gauge of physical sources* associated with these variables.

This Dirac construction of a relativistic covariant *equivalent unconstrained system* can be generalized also on massive vector theories [37]. The generalization of the first Dirac quantization on the non-Abelian theory, including QCD [16–18, 20, 22], is the topic of the present review.

1.3. Relativistic Covariance. Relativistic transformations of the Dirac variables are discussed in detail in the Polubarinov review [14] (see also [38–40]). If we make usual relativistic transformations of the initial fields A_i, A_0, ψ with the parameter ϵ_i

$$\begin{aligned}\delta_L^0 A_k &= \epsilon_i(x'_i\partial_{0'} - x'_0\partial_{i'})A_k(x') + \epsilon_k A_0, \\ \delta_L^0 \psi &= \epsilon_i(x'_i\partial_{0'} - x'_0\partial_{i'})\psi(x') + \frac{1}{4}\epsilon_k[\gamma_i, \gamma_j]\psi(x'),\end{aligned} \tag{25}$$

then the physical variables (15) suffer the Heisenberg–Pauli transformations [2]

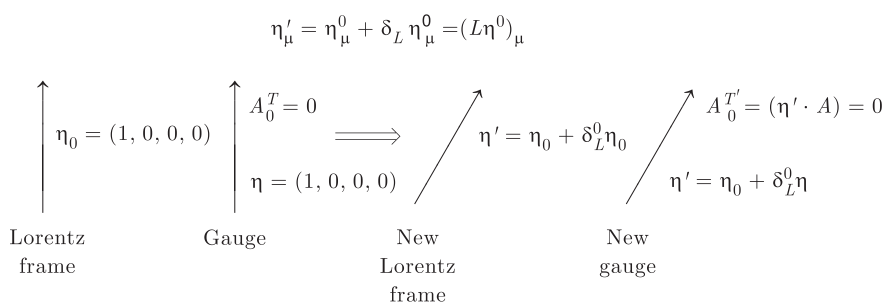
$$A_k^*[A_i + \delta_L^0 A] - A_k^*[A] = \delta_L^0 A_k^* + \partial_k \Lambda, \tag{26}$$

$$\psi^*[A + \delta_L^0 A, \psi + \delta_L^0 \psi] - \psi^*[A, \psi] = \delta_L^0 \psi^* + ie\Lambda(x')\psi^*, \tag{27}$$

were

$$\Lambda[A^*, j_0^*] = \epsilon_k \frac{1}{\partial^2}\left(\partial_0 A_k^* + \partial_k \frac{1}{\Delta} j_0^*\right). \tag{28}$$

These transformations were interpreted by Heisenberg and Pauli [2] (with reference to the unpublished note by von Neumann) as the transition from the Coulomb gauge with respect to the time axis in the rest frame $\eta_\mu^0 = (1, 0, 0, 0)$ to the Coulomb gauge with respect to the time axis in the moving frame (see the Figure).



These transformations correspond to the «change of variables»

$$\psi^*(\eta), A^*(\eta) \to \psi^*(\eta'), A^*(\eta'), \tag{29}$$

so that they became transverse with respect to the new time axis η' (or, from the point of view of the «gauge-fixing» method of reduction, the transformations (25), (26)) correspond to the «change of gauge».

In result we got the relativistic covariant separation of the interaction on the Coulomb potential (instantaneous with respect to the time axis η_μ) and on the retardation.

The Coulomb interaction has the covariant form

$$W_C = \int d^4x d^4y \frac{1}{2} j^*_\eta(x) V_C(z^\perp) j^*_\eta(y) \delta(\eta \cdot z). \tag{30}$$

Here

$$j^*_\eta = e\bar{\psi}^* \not{\eta} \psi^*, \quad z^\perp_\mu = z_\mu - \eta_\mu(z \cdot \eta), \quad z_\mu = (x-y)_\mu, \tag{31}$$

$$V_C(r) = -\frac{1}{4\pi r}, \quad r = |\mathbf{z}|. \tag{32}$$

Finite Lorentz transformations from the time axis $\eta^{(1)}$ to the time axis $\eta^{(2)}$ were constructed in paper [14] using the gauge transformations

$$ieA^{(2)} = U_{(2,1)}[ieA^{(1)} + \partial]U^{-1}_{(2,1)}, \quad \psi^{(2)} = U_{(2,1)}\psi^{(1)}, \tag{33}$$

where $U_{2,1} = v_{(2)}v^{-1}_{(1)}$, and $v_{(2)}, v_{(1)}$ are the Dirac gauge factors (16) for the time axes $\eta^{(2)}$ and $\eta^{(1)}$, respectively.

1.4. Quantization and Feynman Path Integral. The initial action (1) is not compatible with quantum principles. The Dirac formulation of the *equivalent unconstrained system* keeps the quantum principles by the value of excluding the nonphysical components. We quantize the *equivalent unconstrained system* with gauge-invariant physical variables (15). The corresponding commutation relations

$$i\big[\partial_0 A^*_i(\mathbf{x},t),\ A^*_j(\mathbf{y},t)\big] = \left(\delta_{ij} - \partial_i \frac{1}{\Delta}\partial_j\right)\delta^3(\mathbf{x}-\mathbf{y}),$$
$$\big\{\hat{\psi}^{*+}(\mathbf{x},t), \hat{\psi}^*(\mathbf{y},t)\big\} = \delta^3(\mathbf{x}-\mathbf{y})$$

lead to the generating functional for Green's function of the obtained unconstrained system in the form of the Feynman path integral

$$Z^*_\eta[s, \bar{s}^*, J^*] \ = \ \int \prod_j DA^*_j D\psi^* D\bar{\psi}^* \mathrm{e}^{iW^*[A^*,\psi^*,\bar{\psi}^*] + iS^*}, \tag{34}$$

with external source terms

$$S^* = \int d^4x \left(\bar{s}^*\psi^* + \bar{\psi}^* s^* + J_i^* A_i^*\right). \tag{35}$$

By the construction of the *unconstrained system* this generating functional is *gauge-invariant* and *relativistic covariant*. Relativistic transformation properties of the quantum fields should repeat the ones of the Dirac variables (15) as nonlocal functionals of the initial fields. As was shown in papers [4,14,23,38–40], quantum theory with the gauge-invariant Belinfante energy-momentum tensor on the constraint

$$\begin{gathered} T_{\mu\nu} = F_{\mu\lambda}F_\nu^\lambda + \bar{\psi}\gamma_\mu[i\partial_\nu + eA_\nu]\psi - g_{\mu\nu}L + \frac{i}{4}\partial_\lambda[\bar{\psi}\Gamma^\lambda_{\mu\nu}\psi], \\ \Gamma^\lambda_{\mu\nu} = \frac{1}{2}[\gamma^\lambda\gamma_\mu]\gamma_\nu - g_{\mu\nu}\gamma^\lambda - \delta^\lambda_\nu\gamma_\mu \end{gathered} \tag{36}$$

completely reproduced the symmetry properties of the «classical» theory (25) (26), (28)

$$i\epsilon_k[M_{0k}, \psi^*] = \delta^0_L\psi^* + ie\Lambda[A^*, j_0^*]\psi^*, \quad M_{0k} = \int d^3x[x_kT_{00} - tT_{0k}]. \tag{37}$$

This Lorentz transformation of the operator quantization means the change of the time axis on the level of the Feynman path integral

$$Z^*_{L\eta}[s^*, \bar{s}^*, J^*] = Z^*_\eta[Ls^*, L\bar{s}^*, LJ^*]. \tag{38}$$

This scheme of quantization depends on a choice of the time axis. If one chooses a definite frame of reference with the initial time axis, any Lorentz transformation turns this time axis by the relativistic covariant manner. In this meaning, the *constraint dynamics* is *relativistic covariant*. Another problem is to find conditions when measurable physical quantities and results of theoretical calculations do not depend on the time axis (identified with a physical device). This independence exists only for scattering amplitudes of particles onto their mass-shell [6]. In this case, one can say about the *relativistic invariance* of the scattering amplitudes of the local degrees of freedom. But, it is well known that the Green functions (in particular, one-particle Green function) and instantaneous bound states depend on the time axis. In general case, measurable quantities in electrodynamics depend on the time axis and other parameters of a physical device including its size and energy resolution [17].

If a nonlocal process depends on the time axis, one should establish a principle of the choice of this time axis.

We shall use the generalization of the Markov–Yukawa principle [24] supposed in [18, 25, 41, 42]: in concrete calculations the time axis is chosen to be parallel to a total momentum of any state.

In particular, this choice and the nonlocal relativistic transformations (37) remove all infrared divergences from the one-particle Green function in the radiation variables [18, 41, 42]

$$i(2\pi)^4\delta^4(p-q)G(p-q) = \int d^4d^4y \exp{(ipx - iqy)}\langle 0|T\bar{\psi}^*(x)\psi^*(y)|0\rangle,$$

$$G(p) = G_0(p) + G_0(p)\Sigma(p)G_0(p) + O(\alpha^4), \quad G_0(p) = [\not{p} - m]^{-1},$$

$$\Sigma(p) = \frac{\alpha}{8\pi^3 i}\int \frac{d^4q}{q^2+i\epsilon}\left[\left(\delta_{ij} - q_i\frac{1}{\mathbf{q}^2}q_j\right)\gamma_i G_0(p+q)\gamma_j + \right.$$
$$\left. + \gamma_0 G_0(p+q)\gamma_0\frac{1}{\mathbf{q}^2}\right] = \frac{\alpha}{4\pi}\Pi(p), \tag{39}$$

where $\Pi(p)$ is

$$m(3D+4) - D(\not{p}-m) + \frac{1}{2}(\not{p}-m)^2\left[\frac{(\not{p}+m)}{p^2}\left(\ln\frac{m^2-p^2}{m^2}\right)\times\right.$$
$$\left.\times\left(1+\frac{\not{p}(\not{p}-m)}{2p^2}\right) - \frac{\not{p}}{2p^2}\right],$$

and D is an ultra-violet divergence. The transition to another Lorentz frame $p'_\mu = (p'_0, \mathbf{p}' \neq 0)$ is accompanied by the additional diagrams $\langle 0|T\bar{\psi}^*(x)\delta_L\psi^*(y)|0\rangle$ induced by the transformation (37). As a result, in another frame we get the same relativistic covariant expression depending on the new momentum p' (see details in [41, 42]).

1.5. Gauge Equivalence Theorem and FP Integral. Thus, the constraint-shell generational functional (34) is relativistic-covariant (38) and gauge-invariant by the construction. The main difference of this functional from the intuitive FP integral (5) is the information contained in the solution of the constraint, i. e., the electrostatic phenomena of the instantaneous interaction, including the Coulomb-like bound states.

The Faddeev–Popov integral for the generating functional of Green functions in the gauge $(F(A) = 0$ can be obtained from the Feynman integral [6] by two steps: a) a change of variables, and b) a change of sources.

a) *The change of variables* is fulfilled by the Dirac factors (15), (16)

$$A^*_k[A^F] = v[A^F]\left(A^F_k + i\frac{1}{e}\partial_k\right)(v[A^F])^{-1}, \tag{40}$$

$$\psi^*[A^F] = v[A^F]\psi,$$
$$v[A^F] = \exp\left\{ie\frac{1}{\Delta}\partial^j A^F_j\right\}. \tag{41}$$

This change introduces additional degrees of freedom and the FP determinant $\Delta^F_{\rm FP}$ of the transition to new variables of integration. These degrees are removed by additional constraints $F(A) = 0$. Finally, the *constraint-shell* functional Z^* (34) takes the equivalent form of the FP path integral

$$Z^*[s^*, \bar{s}^*, J^*] = \int \prod_{\mu} DA^F_\mu D\psi^F D\bar{\psi}^F \Delta^F_{\rm FP} \delta(F(A^F))\, {\rm e}^{iW[A^F, \psi^F \bar{\psi}^F] + iS^*}, \quad (42)$$

all electrostatic monopole physical phenomena that depend on the time axis are concentrated in the Dirac gauge factor $v(A)$ that accompanies the physical sources $\bar{s}^*, s^*, J^*$,

$$S^* = \int d^4x \left((v[A^F])^{-1} \bar{s}^* \psi^F + \bar{\psi}^F (v[A^F])^{-1} s^* + J^*_i A^*_i [A^F] \right). \quad (43)$$

Thus, this change of variables corresponds to the rearrangement of the Feynman diagrams. In particular, after the change of variables, the sum of the Coulomb kernel and transverse photon propagator

$$\mathcal{K}^R(J) = J_0^{(1)} \frac{1}{\mathbf{q}^2} J_0^{(2)} + J_i^{(1)} \left(\delta_{ij} - q_i \frac{1}{\mathbf{q}^2} q_j \right) \frac{1}{q_0^2 - \mathbf{q}^2} J_j^{(2)}$$

converts into the identically equivalent sum of the Feynman gauge propagator $\mathcal{K}^F$ and the longitudinal term $\mathcal{K}^L$:

$$\mathcal{K}^R(J) \equiv \mathcal{K}^F(J) + \mathcal{K}^L(J), \quad (44)$$

where

$$\mathcal{K}^F(J) = -\left[J_0^{(1)} J_0^{(2)} - J_i^{(1)} J_i^{(2)} \right] \frac{1}{q_0^2 - \mathbf{q}^2},$$

$$\mathcal{K}^L(J) = \left[(q_0 J_0^{(1)})(q_0 J_0^{(2)}) - (q_i J_i^{(1)})(q_j J_j^{(2)}) \right] \frac{1}{\mathbf{q}^2 (q_0^2 - \mathbf{q}^2)}.$$

b) The next step is *the change of sources*

$$S^* \Rightarrow S^F = \int d^4x \left(\bar{s}^F \psi^F + \bar{\psi}^F s^F + A^F_\mu J^\mu \right). \quad (45)$$

In the result, we get the original FP integral (5)

$$Z^{\rm FP}[s^F, \bar{s}^F, J^F] = \\ = \int \prod_{\mu} DA^F_\mu D\psi^F D\bar{\psi}^F \Delta^F_{\rm FP} \delta(F(A^F))\, {\rm e}^{iW[A^F, \psi^F \bar{\psi}^F] + iS^F} \quad (46)$$

without the Dirac factors. Changing the sources we lose the Dirac factor together with the class of the **spurious diagrams** that remembered the electrostatic phenomena and instantaneous bound states. One of these spurious diagrams is the longitudinal term $\mathcal{K}^L$ in equation (44). This longitudinal term disappears only on the mass-shell (because of the current conservation law $J_0^{(1,2)}q_0 = J_i^{(1,2)}q_i$). As we have seen before, the Dirac nonlocal gauge-invariant observable currents do not satisfy the conservation law for particles off the mass-shell, in particular, in bound states.

Really, the FP perturbation theory in the relativistic gauge (46) contains only photon propagators with the light-cone singularities forming the Wick–Cutkosky bound states with the spectrum differing from the observed one which corresponds to the instantaneous Coulomb interaction [43]. The Wick–Cutkosky bound states have the problem of a tachyon and the probability interpretation. These problems were solved by the quasipotential method [44] that introduces instantaneous bound states, and the corresponding time axis as the Markov–Yukawa prescription [24].

Thus, in QED, the fundamental constraint-shell functional (34) coincides with the FP integral (42) in the sector of scattering amplitudes for elementary particles on their mass-shell [6] that is not actual for solution of the problem of hadronization in QCD. For description of the bound state sector in gauge theories, including QCD, we have three alternatives: 1) the FP integral (42), 2) its quasipotential approach to bound states at the level of the FP integral [44], and 3) the constraint-shell functional (34) added by the Markov–Yukawa prescription of the choice of the time axis.

After the Faddeev papers of 1967–1969 solutions of all problems of gauge theories, including the description of bound states, the Gribov ambiguity [7], topological degeneration [11, 12], were considered only at the level of the FP integral (42).

The task of the present paper is to consider these problems at the most fundamental level of the Hamiltonian approach to quantization of gauge theories. It is based on a definite frame of reference, that includes a choice of the time axis, initial data, boundary conditions, normalization of wave functions, bound states, etc. It is useful to recall the following words by Max Born about quantum theory ([45, p. 108]): «The clue is the point ..., that quantum mechanics does not describe a situation in an objective external world, but a definite experimental arrangement for observing a section of the external word. Without this idea even the formulation of a dynamical problem in quantum theory is impossible. But if it is acceptable, the fundamental indeterminacy in the physical predictions becomes natural as no experimental arrangement can ever be absolute precise». Following to Max Born one can say that the Hamiltonian description of any quantum system is determined by «a definite experimental arrangement for observing a section of the external word». If it is acceptable, the dependence of the quantum description of nonlocal processes on a frame of reference becomes natural as any experimental arrangement is included in this frame. The Hamiltonian method determines the energy spectrum of physical states. Therefore, a direction of the total momentum of any physical state is distinguished. In the case of bound states, the time axis is chosen along the total momentum of any bound state in the context of the Markov–Yukawa solution [24, 25, 46] of the problem of the relativistic covariance in QED.

2. QED OF BOUND STATES: SPECTRUM AND S MATRIX

2.1. Markov–Yukawa Prescription. One of the first definitions of the physical bound states in QED belongs to Lord Eddington: «A proton yesterday and electron today do not make an atom» [46]. It is clear that we can observe experimentally two particles as a bound state $\mathcal{M}(x,y)$ at one and the same time

$$\mathcal{M}(x,y) = \mathrm{e}^{iMX_0}\psi(z_i)\delta(z_0), \tag{47}$$

where X_μ and z_μ are the total and relative coordinates

$$X_\mu = \frac{(x+y)_\mu}{2}, \quad z_\mu = (x-y)_\mu. \tag{48}$$

This principle of the simultaneity has more deep mathematical meaning [25, 47] as the constraint of irreducible nonlocal representations of the Poincare group for arbitrary bilocal field $\mathcal{M}(x,y) = \mathcal{M}(z|X)$

$$z_\mu \frac{\partial}{\partial X_\mu}\mathcal{M}(z|X) = 0, \quad \mathcal{M}(z|X) \equiv \mathcal{M}(x,y). \tag{49}$$

This constraint is not connected with the dynamics of interaction and realized the Eddington simultaneity.

The general solution of the irreducibility constraint (49) can be written in the form of the expansion of the bilocal field $\mathcal{M}(z|X)$ with respect to «in» and «out» plane waves

$$\mathcal{M}(z|X) = \sum_A \int \frac{d^3\mathcal{P}_A}{\sqrt{(2\pi)^3\omega_A}} \times \\ \times \left[\mathrm{e}^{i\mathcal{P}_A X}\Phi_{\mathcal{P}_A}(z_A^\perp)a^{(+)}_{\mathcal{P}_A} + \mathrm{e}^{-i\mathcal{P}_A X}\bar{\Phi}_{\mathcal{P}_A}(z_A^\perp)a^{(-)}_{\mathcal{P}_A}\right]\delta\left(\frac{\mathcal{P}_A z}{\sqrt{\mathcal{P}^2}}\right), \tag{50}$$

where $a^{(\pm)}_{\mathcal{P}_A}$ are coefficients of the expansion. $\Phi_{\mathcal{P}_A}(z^{\perp}_A)$, $\bar{\Phi}_{\mathcal{P}_A}(z^{\perp}_A)$ are the normalized amplitudes in the space of relative coordinates orthogonal to the total momentum of an atom $\mathcal{P}_A$

$$(z^{\perp}_A)_\mu = z_\mu - \mathcal{P}_{A\mu}\left(\frac{\mathcal{P}_A \cdot z}{\mathcal{P}^2}\right). \tag{51}$$

It is clear, that at the point of the existence of the bound state with the definite total momentum $\mathcal{P}_{A\mu}$ any instantaneous interaction (30) with the time axis η_μ parallel to this momentum $\eta_\mu \sim \mathcal{P}_{A\mu}$

$$\eta_\mu \mathcal{M}(z|X) \sim \mathcal{P}_{A\mu}\mathcal{M}(z|X) = \frac{1}{i}\frac{\partial}{\partial X_\mu}\mathcal{M}(X|z) \tag{52}$$

is much greater than any «retardation» interaction [48]. It is just our principle of the choice of the time axis of the Dirac quantization of a gauge theory. A time axis is chosen to be parallel to the total momentum of a considered state. In particular, for bound states this choice means that the coordinate of the potential coincides with the space of the relative coordinates of the bound state wave function in accordance with the Markov–Yukawa prescription [24] and the Eddington concept of simultaneity [46]. In this case, we get the relativistic covariant dispersion law and invariant mass spectrum. The relativistic generalization of the Coulomb potential is not only the change of the form of the potential, but also the change of a direction of its motion in four-dimensional space to lie along the total momentum of the bound state. The relativistic covariant unitary perturbation theory in terms of such relativistic instantaneous bound states has been constructed in [25]. In this perturbation theory, each instantaneous bound state in QED has a proper equivalent unconstrained system of the Dirac quantization. The manifold of frames corresponds to the manifold of «equivalent unconstrained systems». In this case, the bilocal fields (60) automatically belong to the irreducible representation of the Poincare group [47].

By analogy, we introduce for the N-local field the total and relative coordinates

$$X_\mu = \frac{1}{N}\sum_{i=1}^{N} x_{i\mu}, \quad z^{(i)}_\mu = x_{i\mu} - X_\mu \tag{53}$$

which are connected by the identity

$$\sum_{i=1}^{N} z^{(i)}_\mu = 0.$$

Then, the generalization of the Markov–Yukawa condition takes the form

$$z^{(i)}_\mu \frac{\partial}{\partial X_\mu}\Phi(z^{(1)}_\mu, z^{(2)}_\mu, \ldots, z^{(N)}_\mu) = 0 \quad (i = 1, 2, \ldots, N). \tag{54}$$

Let $\mathcal{P}_\mu$ be the eigenvalue of the operator for the total 4-momentum, and η_μ be the unit vector in the direction $\mathcal{P}(\eta_\mu \sim \mathcal{P}_\mu)$. Owing to the condition (54) the N-local function

$$\underline{\Phi}(p_\mu^{\perp(1)}, p_\mu^{\perp(2)}, \ldots, p_\mu^{\perp(N)}|\mathcal{P}),$$

being the Fourier transform of $\Phi(z_\mu^{(1)}, X_\mu)$ with respect to all coordinates, depends only on the transverse relative momenta

$$p_\mu^{(i)\perp} = p_\mu^{(i)} - \eta_\mu(p^{(i)} \cdot \eta), \quad \sum_{i=1}^{N} p_\mu^{(i)\perp} = 0. \tag{55}$$

See also the generalization of the Markov–Yukawa condition for three-local [49] and N-local [50] cases.

2.2. Effective Lagrangian of Bilocal Fields. The constraint-shell QED allows us to construct the «bound state» relativistic covariant perturbation theory with respect to «retardation» [25, 48]. Our solution of the problem of relativistic invariance of the nonlocal objects is the choice of the time axis as a vector operator with eigenvalues proportional to total momenta of bound states (52) [25]. In this case, the relativistic covariant unitary S matrix can be defined as the Feynman path integral

$$Z^*_{\hat{\eta}}[s, \bar{s}^*, J^*] = \langle * | \int D\psi^* D\bar{\psi}^* \, \mathrm{e}^{iW^*_C[\psi^*, \bar{\psi}^*] + iS^*} | * \rangle, \tag{56}$$

where

$$\langle * | F | * \rangle = \int \prod_j DA^*_j \, \mathrm{e}^{iW^*_0[A^*]} F \tag{57}$$

is the averaging over transverse photons

$$W_C[\psi, \bar{\psi}] = \int d^4x \Big[\bar{\psi}(x)(i\partial\!\!\!/ - ie A\!\!\!/^* - m^0)\psi(x) + \\ + \frac{1}{2} \int d^4y (\psi(y)\bar{\psi}(x)) \mathcal{K}^{(\eta)}(z^\perp \mid X)(\psi(x)\bar{\psi}(y)) \Big]. \tag{58}$$

Here $\partial\!\!\!/ = \partial^\mu \gamma_\mu$, $\mathcal{K}^{(\eta)}$ is the kernel

$$\mathcal{K}^{(\eta)}(z^\perp \mid X) = \eta\!\!\!/ V(z^\perp)\delta(z \cdot \eta)\eta\!\!\!/, \\ (\eta\!\!\!/ = \eta^\mu \gamma_\mu = \gamma \cdot \eta, \quad z^\perp_\mu = z_\mu - \eta_\mu(z \cdot \eta)), \tag{59}$$

and z and X are the relative and total coordinates defined in equation (48), and $V(z^{\perp})$ is the potential depending on the transverse component of the relative coordinate with respect to the time axis η

$$\eta_\mu \sim i\frac{\partial}{\partial X_\mu}. \tag{60}$$

In the context of this perturbation theory the bound state total momentum operators (23) can form the «new quantum numbers» of the type the Isgur–Wise ones [51, 52].

It seems that the most straightforward way for constructing a theory of bound states is the redefinition of action (58) in terms of bilocal fields by means of the Legendre transformation [53, 54]

$$\begin{aligned}\frac{1}{2}\int d^4xd^4y(\psi(y)\bar{\psi}(x))\mathcal{K}(x,y)(\psi(x)\bar{\psi}(y)) = \\ = -\frac{1}{2}\int d^4xd^4y\mathcal{M}(x,y)\mathcal{K}^{-1}(x,y)\mathcal{M}(x,y) + \\ + \int d^4xd^4y(\psi(x)\bar{\psi}(y))\mathcal{M}(x,y),\end{aligned} \tag{61}$$

where $\mathcal{K}^{-1}$ is the inverse of the kernel (59). Following Ref. 54, we introduce the short-hand notation

$$\int d^4xd^4y\psi(y)\bar{\psi}(x)(i\partial\!\!\!/ - ie A\!\!\!/^* - m^0)\delta^{(4)}(x-y) = (\psi\bar{\psi}, -G_A^{-1}), \tag{62}$$

$$\int d^4xd^4y(\psi(x)\bar{\psi}(y))\mathcal{M}(x,y) = (\psi\bar{\psi}, \mathcal{M}). \tag{63}$$

After quantization (or integration) over N_c fermion fields and normal ordering, the functional (56) takes the form

$$Z^*_{\hat{\eta}}[s, \bar{s}^*, J^*] = \langle 165 * | \int \prod D\mathcal{M} e^{iW_{\text{eff}}[\mathcal{M}] + iS_{\text{eff}}[\mathcal{M}]} | * \rangle, \tag{64}$$

where

$$W_{\text{eff}}[\mathcal{M}] = (\psi\bar{\psi}, (-G_A^{-1} + \mathcal{M})) - \frac{1}{2}(\mathcal{M}, \mathcal{K}^{-1}\mathcal{M}) \tag{65}$$

is the effective action, and

$$S_{\text{eff}}[\mathcal{M}] = (s^*\bar{s^*}, (G_A^{-1} - \mathcal{M})^{-1}) \tag{66}$$

is the source term. The effective action can be decomposed in the form

$$W_{\rm eff}[\mathcal{M}] = -\frac{1}{2}N_c(\mathcal{M}, \mathcal{K}^{-1}\mathcal{M}) + iN_c\sum_{n=1}^{\infty}\frac{1}{n}\Phi^n. \tag{67}$$

Here $\Phi \equiv G_A\mathcal{M}$, Φ^2, Φ^3, etc., mean the following expressions

$$\Phi(x,y) \equiv G_A\mathcal{M} = \int d^4z G_A(x,z)\mathcal{M}(z,y),$$
$$\Phi^2 = \int d^4x d^4y \Phi(x,y)\Phi(y,x), \tag{68}$$
$$\Phi^3 = \int d^4x d^4y d^4z \Phi(x,y)\Phi(y,z)\Phi(z,x), \text{ etc.}$$

As a result of such quantization, only the contributions with inner fermionic lines (but no scattering and dissociation channel contribution) are included in the effective action since we are interested only in the bound states.

The requirement for the choice of the time axis (60) in bilocal dynamics is equivalent to Markov–Yukawa condition [24] (69)

$$z_\mu i\frac{\partial\mathcal{M}(z|X)}{\partial X_\mu} = 0, \quad \mathcal{M}(z|X) \equiv \mathcal{M}(x,y), \tag{69}$$

where $z_\mu = (x-y)_\mu$ and $X_\mu = (1/2)(x+y)_\mu$ are relative and total coordinates.

2.3. Quantization of Bilocal Fields. The first step to the quantization of the effective action is the determination of its minimum

$$N_c^{-1}\frac{\delta W_{\rm eff}(\mathcal{M})}{\delta\mathcal{M}} = -\mathcal{K}^{-1}\mathcal{M} + i\sum_{n=1}^{\infty}G_A(\mathcal{M}G_A)^n \equiv$$
$$\equiv -\mathcal{K}^{-1}\mathcal{M} + \frac{i}{G_A^{-1} - \mathcal{M}} = 0. \tag{70}$$

We denote the corresponding classical solution for the bilocal field by $\Sigma(x-y)$. It depends only on the difference $x-y$ because of translation invariance of vacuum solutions.

The next step is the expansion of the effective action around the point of minimum $\mathcal{M} = \Sigma + \mathcal{M}'$,

$$W_{\rm eff}(\Sigma + \mathcal{M}') = W_{\rm eff}^{(2)} + W_{\rm int},$$
$$W_{\rm eff}^{(2)}(\mathcal{M}') = W_Q(\Sigma) + N_c\left[-\frac{1}{2}\mathcal{M}'\mathcal{K}^{-1}\mathcal{M}' + \frac{i}{2}(G_\Sigma\mathcal{M}')^2\right], \tag{71}$$
$$W_{\rm int} = \sum_{n=3}^{\infty}W^{(n)} = iN_c\sum_{n=3}^{\infty}\frac{1}{n}(G_\Sigma\mathcal{M}')^n, \quad (G_\Sigma = (G_A^{-1} - \Sigma)^{-1}),$$

and the representation of the small fluctuations $\mathcal{M}'$ as a sum over the complete set of orthonormalized solutions Γ, of the classical equation

$$\left.\frac{\delta^2 W_{\rm eff}(\Sigma+\mathcal{M}')}{\delta\mathcal{M}^2}\right|_{\mathcal{M}'=0}\Gamma=0 \tag{72}$$

with a set of quantum numbers (H) including masses $M_H=\sqrt{\mathcal{P}_\mu^2}$ and energies $\omega_H=\sqrt{\boldsymbol{\mathcal{P}}^2+M_H^2}$

$$\mathcal{M}'(z|X)=\sum_H\int\frac{d^3\boldsymbol{\mathcal{P}}}{(2\pi)^{3/2}\sqrt{2\omega_H}}\int\frac{d^4q}{(2\pi)^4}\times$$
$$\times\{\mathrm{e}^{i\boldsymbol{\mathcal{P}}\mathbf{X}}\Gamma_H(q^\perp|\mathcal{P})a_H^+(\mathcal{P})+\mathrm{e}^{-i\boldsymbol{\mathcal{P}}\mathbf{X}}\bar{\Gamma}_H(q^\perp|-\mathcal{P})a_H^-(\mathcal{P})\}. \tag{73}$$

The bound state creation and annihilation operators obey the commutation relations

$$\left[a_{H'}^-(\boldsymbol{\mathcal{P}}'),a_H^+(\boldsymbol{\mathcal{P}})\right]=\delta_{H'H}\delta^3(\boldsymbol{\mathcal{P}}'-\mathcal{P}), \tag{74}$$
$$\left[a_H^\pm(\mathcal{P}),a_{H'}^\pm(\mathcal{P}')\right]=0. \tag{75}$$

The corresponding Green function takes the form

$$\mathcal{G}(q^\perp,p^\perp|\mathcal{P})=$$
$$=\sum_H\left\{\frac{\Gamma_H(q^\perp|\mathcal{P})\bar{\Gamma}_H(p^\perp|-\mathcal{P})}{(\mathcal{P}_0-\omega_H-i\varepsilon)2\omega_H}-\frac{\Gamma_H(p^\perp|\mathcal{P})}{(\mathcal{P}_0-\omega_H-i\varepsilon)2\omega_H}\right\}. \tag{76}$$

To normalize vertex functions, we can use the «free» part of effective action (71) for the quantum bilocal meson $\mathcal{M}$ with the commutation relations (74). The substitution of the off-shell $\sqrt{\mathcal{P}^2}\neq M_H$ decomposition (73) into the «free» part of effective action defines the reverse Green function of the bilocal field $\mathcal{G}(\mathcal{P}_0)$

$$W_{\rm eff}^{(0)}[\mathcal{M}]=2\pi\delta(\mathcal{P}_0-\mathcal{P}'_0)\sum_H\int\frac{d\boldsymbol{\mathcal{P}}}{\sqrt{2\omega}}a_H^+(\boldsymbol{\mathcal{P}})a_H^-(-\boldsymbol{\mathcal{P}})\mathcal{G}_H^{-1}(\mathcal{P}_0), \tag{77}$$

where $\mathcal{G}_H^{-1}(\mathcal{P}_0)$ is the reverse Green function which can be represented as the difference of two terms

$$\mathcal{P}_H^{-1}(\mathcal{P}_0)=I(\sqrt{\mathcal{P}^2})-I(M_H^{ab}(\omega)), \tag{78}$$

where $M_H^{ab}(\omega)$ is the eigenvalue of the equation for small fluctuations (72) and

$$I(\sqrt{\mathcal{P}^2}) = N_c \int \frac{d^3q^\perp}{(2\pi)^3} \left\{ \frac{i}{2\pi} \int dq_0 \times \right.$$
$$\left. \times \ \mathrm{tr} \left[G_{\Sigma_b} \left(q - \frac{\mathcal{P}}{2} \right) \bar{\Gamma}^H_{ba}(q^\perp | -\mathcal{P}) G_{\Sigma_a} \left(q + \frac{\mathcal{P}}{2} \right) \Gamma^H_{ab}(q^\perp | \mathcal{P}) \right] \right\}. \quad (79)$$

According to quantum field theory, the normalization condition is defined by formula

$$2\omega = \left. \frac{\partial \mathcal{G}^{-1}(\mathcal{P}_0)}{\partial \mathcal{P}_0} \right|_{\mathcal{P}_0 = \omega(\mathcal{P}_1)} = \left. \frac{dM(\mathcal{P}_0)}{d\mathcal{P}_0} \frac{dI(M)}{dM} \right|_{\mathcal{P}_0 = \omega}. \quad (80)$$

Finally, we get that solutions of equation (72) satisfy the normalization condition [55]

$$iN_c \frac{d}{d\mathcal{P}_0} \int \frac{d^4q}{(2\pi)^4} \, \mathrm{tr} \left[\underline{G}_\Sigma \left(q - \frac{\mathcal{P}}{2} \right) \bar{\Gamma}_H(q^\perp | -\mathcal{P}) \times \right.$$
$$\left. \times \underline{G}_\Sigma \left(q + \frac{\mathcal{P}}{2} \right) \Gamma_H(q^\perp | \mathcal{P}) \right] = 2\omega_H, \quad (81)$$

and

$$\underline{G}_\Sigma(q) = \frac{1}{\not{q} - \underline{\Sigma}(q^\perp)}, \quad \underline{\Sigma}(q) = \int d^4x \Sigma(x) \, \mathrm{e}^{iqx} \quad (82)$$

is the fermion Green function.

2.4. Schwinger–Dyson Equation: the Fermion Spectrum. The equation of stationarity (70) can be rewritten form of Schwinger–Dyson (SD) equation

$$\Sigma(x-y) = m^0 \delta^{(4)}(x-y) + i\mathcal{K}(x,y) G_\Sigma(x-y). \quad (83)$$

It describes the spectrum of Dirac particles in bound states. In the momentum space with

$$\underline{\Sigma}(k) = \int d^4x \Sigma(x) \, \mathrm{e}^{ikx}$$

for the Coulomb type kernel we obtain the following equation for the mass operator ($\underline{\Sigma}$)

$$\underline{\Sigma}(k) = m^0 + i \int \frac{d^4q}{(2\pi)^4} \underline{V}(k^\perp - q^\perp) \not{\eta} \underline{G}_\Sigma(q) \not{\eta}, \quad (84)$$

where $G_\Sigma(q) = (\not{q} - \underline{\Sigma}(q))^{-1}$, $\underline{V}(k^\perp)$ means the Fourier transform of the potential; $k^\perp_\mu = k_\mu - \eta_\mu(k\cdot\eta)$ is the transverse with respect to η_μ relative momentum. The quantity $\underline{\Sigma}$ depends only on the transverse momentum

$$\underline{\Sigma}(k) = \underline{\Sigma}(k^\perp),$$

because of the instantaneous form of the potential $\underline{V}(k^\perp)$ in any frame. The fermion spectrum can be obtained by solving the Schwinger–Dyson equation (83). We may integrate it over the longitudinal momentum $q_0 = (q\cdot\eta)$ using the representation

$$\underline{\Sigma}_a(q) = \not{q}^\perp + E_a(q^\perp)S_a^{-2}(q^\perp) \tag{85}$$

for the self-energy with

$$S_a^{-2}(q^\perp) = \exp\{-\hat{\not{q}}^\perp 2\upsilon_a(q^\perp)\}, \quad \hat{q}^\perp_\mu = q^\perp_\mu/|q^\perp|, \tag{86}$$

where S_a is the Foldy–Wouthuysen type transformation matrix with the parameter υ_a.

Then, one has

$$\underline{G}_{\Sigma_a} = [q_0\not{\eta} - E_a(q^\perp)S_a^{-2}(q^\perp)]^{-1} =$$

$$= \left[\frac{\Lambda^{(\eta)}_{(+)a}(q^\perp)}{q_0 - E_a(q^\perp) + i\epsilon} + \frac{\Lambda^{(\eta)}_{(-)a}(q^\perp)}{q_0 + E_a(q^\perp) + i\epsilon}\right]\not{\eta}, \tag{87}$$

where

$$\Lambda^{(\eta)}_{(\pm)a}(q^\perp) = S_a(q^\perp)\Lambda^{(\eta)}_{(\pm)}(0)S_a^{-1}(q^\perp), \quad \Lambda^{(\eta)}_{(\pm)}(0) = (1\pm\not{\eta})/2 \tag{88}$$

are the operators separating the states with positive ($+E_a$) and negative ($-E_a$) energies. As a result, we obtain the following equations for the one-particle energy E and the angle υ:

$$E_a(k^\perp)\cos 2\upsilon(k^\perp) = m_a^0 + \frac{1}{2}\int \frac{d^3q^\perp}{(2\pi)^3}\underline{V}(k^\perp - q^\perp)\cos 2\upsilon(q^\perp), \tag{89}$$

$$E_a(k^\perp)\sin 2\upsilon(k^\perp) = |k^\perp| + \frac{1}{2}\int \frac{d^3q^\perp}{(2\pi)^3}\underline{V}(k^\perp - q^\perp)|k^\perp\cdot q^\perp|\sin 2\upsilon(q^\perp). \tag{90}$$

2.5. Bethe–Salpeter Equation: Bound-State Spectrum. Equations for the spectrum of the bound states (72) can be rewritten in the form of the Bethe–Salpeter (BS) one [56]

$$\Gamma = i\mathcal{K}(x,y)\int d^4z_1 d^4z_2 G_\Sigma(x - z_1)\Gamma(z_1,z_2)G_\Sigma(z_2 - y). \tag{91}$$

This equation in the momentum space

$$\underline{\Gamma}(q|\mathcal{P}) = \int d^4x d^4y \, \mathrm{e}^{i(x+y)/2} \, \mathrm{e}^{i(x-y)q} \Gamma(x,y)$$

for the Coulomb type kernel takes the form

$$\underline{\Gamma}(k,\mathcal{P}) = i \int \frac{d^4q}{(2\pi)^4} \underline{V}(k^\perp - q^\perp) \not\eta \left[\underline{G}_\Sigma \left(q + \frac{\mathcal{P}}{2} \right) \Gamma(q|\mathcal{P}) \underline{G}_\Sigma \left(q - \frac{\mathcal{P}}{2} \right) \right] \not\eta, \quad (92)$$

where $\underline{V}(k^\perp)$ means the Fourier transform of the potential; $k_\mu^\perp = k_\mu - \eta_\mu (k \cdot \eta)$ is the relative momentum transversal with respect to η_μ; $\mathcal{P}_\mu$ is the total momentum.

The quantity $\underline{\Gamma}$ depends only on the transversal momentum

$$\underline{\Gamma}(k|\mathcal{P}) = \underline{\Gamma}(k^\perp|\mathcal{P}),$$

because of the instantaneous form of the potential $\underline{V}(k^\perp)$ in any frame.

We consider the Bethe–Salpeter equation (91) after integration over the longitudinal momentum q_0. The vertex function takes the form

$$\Gamma_{ab}(k^\perp|\mathcal{P}) = \int \frac{d^3q^\perp}{(2\pi)^3} \underline{V}(k^\perp - q^\perp) \not\eta \psi_{ab}(q^\perp) \not\eta, \quad (93)$$

where the bound state wave function ψ_{ab} is given by

$$\psi_{ab}(q^\perp) = \not\eta \left[\frac{\bar\Lambda_{(+)a}(q^\perp) \Gamma_{ab}(q^\perp|\mathcal{P}) \Lambda_{(-)b}(q^\perp)}{E_T - \sqrt{\mathcal{P}^2} + i\epsilon} + \right.$$
$$\left. + \frac{\bar\Lambda_{(-)a}(q^\perp) \Gamma_{ab}(q^\perp|\mathcal{P}) \Lambda_{(+)b}(q^\perp)}{E_T + \sqrt{\mathcal{P}^2} - i\epsilon} \right] \not\eta, \quad (94)$$

$E_T = E_a + E_b$ means the sum of one particle energies of the two particles (a) and (b) defined by (89), (90), and the notation (88)

$$\bar\Lambda_{(\pm)}(q^\perp) = S^{-1}(q^\perp) \Lambda_{(\pm)}(0) S(q^\perp) = \Lambda_{(\pm)}(-q^\perp) \quad (95)$$

has been introduced.

Acting with the operators (88) and (95) on equation (93), one gets the equations for the wave function ψ in an arbitrary moving reference frame

$$(E_T(k^\perp) \mp \sqrt{\mathcal{P}^2}) \Lambda^{(\eta)}_{(\pm)a}(k^\perp) \psi_{ab}(k^\perp) \Lambda^{(\eta)}_{(\mp)b}(-k^\perp) =$$
$$= \Lambda^{(\eta)}_{(\pm)a}(k^\perp) \left[\int \frac{d^3q^\perp}{(2\pi)^3} \underline{V}(k^\perp - q^\perp) \psi_{ab}(q^\perp) \right] \Lambda^{(\eta)}_{(\mp)b}(-k^\perp). \quad (96)$$

All these equations (93) and (96) have been derived without any assumption about the smallness of the relative momentum $|k^{\perp}|$ and for an arbitrary total momentum

$$\mathcal{P}_{\mu} = \left(\sqrt{M_A^2 + \boldsymbol{\mathcal{P}}^2}, \quad \boldsymbol{\mathcal{P}} \neq 0\right).$$

We expand the function Ψ on the projection operators

$$\Psi = \Psi_+ + \Psi_-, \quad \Psi_{\pm} = \Lambda_{\pm}^{(\eta)} \Psi \Lambda_{\mp}^{(\eta)}. \tag{97}$$

According to Eq. (94), Ψ satisfies the identities

$$\Lambda_{+}^{(\eta)} \Psi \Lambda_{+}^{(\eta)} = \Lambda_{-}^{(\eta)} \Psi \Lambda_{-}^{(\eta)} \equiv 0, \tag{98}$$

which permit the determination of an unambiguous expansion of Ψ in terms of the Lorentz structures:

$$\Psi_{\pm(a,b)} = S_{(a)}^{-1} \left\{ \gamma_5 L_{\pm(a,b)}(q^{\perp}) + (\gamma_{\mu} - \eta_{\mu} \not{\eta}) N^{\mu}_{\pm(a,b)} \right\} \Lambda_{\mp}^{(\eta)}(0) S_{(b)}^{-1}, \tag{99}$$

where $L_{\pm} = L_1 \pm L_2$, $N_{\pm} = N_1 \pm N_2$. In the rest frame $\eta_{\mu} = (1,0,0,0)$ we get

$$N^{\mu} = (0, N^i), \quad N^i(q) = \sum_{a=1,2} N_{\alpha}(q) e^i_{\alpha}(q) + \Sigma(q) \hat{q}^i.$$

The wave functions L, N^{α}, Σ satisfy the equations

1. *Pseudoscalar particles*

$$\begin{aligned} M\overset{0}{L}_2 &= E\overset{0}{L}_1 + \int \frac{d\mathbf{q}}{(2\pi)^3} V(\mathbf{p}-\mathbf{q}) (\mathtt{c}_p^- \mathtt{c}_q^- + \xi \mathtt{s}_p^- \mathtt{s}_q^-) \overset{0}{L}_1, \\ M\overset{0}{L}_1 &= E\overset{0}{L}_2 + \int \frac{d\mathbf{q}}{(2\pi)^3} V(\mathbf{p}-\mathbf{q}) (\mathtt{c}_p^+ \mathtt{c}_q^+ + \xi \mathtt{s}_p^+ \mathtt{s}_q^+) \overset{0}{L}_2, \end{aligned} \tag{100}$$

$$\xi = \hat{p}_i \cdot \hat{q}_i.$$

2. *Vector particles*

$$\begin{aligned} M\overset{0}{N}{}_2^{\alpha} = E\overset{0}{N}{}_1^{\alpha} + &\int \frac{d\mathbf{q}}{(2\pi)^3} V(\mathbf{p}-\mathbf{q}) \{ (\mathtt{c}_p^- \mathtt{c}_q^- \underline{\delta}^{\alpha\beta} + \\ &+ \mathtt{s}_p^- \mathtt{s}_q^- (\underline{\delta}^{\alpha\beta} \xi - \eta^{\alpha} \underline{\eta}^{\beta})) \overset{0}{N}{}_1^{\beta} + (\eta^{\alpha} \mathtt{c}_p^- \mathtt{c}_q^+) \overset{0}{\Sigma}_1 \}, \\ M\overset{0}{N}{}_1^{\alpha} = E\overset{0}{N}{}_2^{\alpha} + &\int \frac{d\mathbf{q}}{(2\pi)^3} V(\mathbf{p}-\mathbf{q}) \{ (\mathtt{c}_p^+ \mathtt{c}_q^+ \underline{\delta}^{\alpha\beta} + \\ &+ \mathtt{s}_p^+ \mathtt{s}_q^+ (\underline{\delta}^{\alpha\beta} \xi - \eta^{\alpha} \underline{\eta}^{\beta})) \overset{0}{N}{}_2^{\beta} + (\eta^{\alpha} \mathtt{c}_p^+ \mathtt{c}_q^-) \overset{0}{\Sigma}_2 \}, \end{aligned} \tag{101}$$

$$\eta^{\alpha} = \hat{q}_i \hat{e}^{\alpha}_i(p), \quad \underline{\eta}^{\alpha} = \hat{p}_i \hat{e}^{\alpha}_i(q), \quad \underline{\delta}^{\alpha\beta} = \hat{e}^{\alpha}_i(q) \hat{e}^{\beta}_i(p).$$

3. *Scalar particles*

$$M\overset{0}{\Sigma}_2 = E\overset{0}{\Sigma}_1 + \int \frac{d\mathbf{q}}{(2\pi)^3} V(\mathbf{p}-\mathbf{q})\{(\xi \mathtt{c}_p^+ \mathtt{c}_q^+ + \mathtt{s}_p^+ \mathtt{s}_q^+)\overset{0}{\Sigma}_1 + (\underline{\eta}^\beta \mathtt{c}_p^- \mathtt{c}_q^+)\overset{0}{N}{}_1^\beta\},$$
$$(102)$$
$$M\overset{0}{\Sigma}_1 = E\overset{0}{\Sigma}_2 + \int \frac{d\mathbf{q}}{(2\pi)^3} V(\mathbf{p}-\mathbf{q})\{(\xi \mathtt{c}_p^- \mathtt{c}_q^- + \mathtt{s}_p^- \mathtt{s}_q^-)\overset{0}{\Sigma}_2 + (\underline{\eta}^\beta \mathtt{c}_p^+ \mathtt{c}_q^-)\overset{0}{N}{}_2^\beta\}.$$

Here, in all equations,

$$\mathtt{c}^\pm(p) = \cos[v_a(p) \pm v_b(p)], \quad \mathtt{s}^\pm(p) = \sin[v_a(p) \pm v_b(p)] \tag{103}$$

v_a, v_b are the Foldy–Wouthuysen matrices of particles (a, b); and $E = E^a + E^b$ is the sum of one-particle energies.

The normalization of these solutions is uniquely determined by equation (81)

$$\frac{2N_c}{M_L} \sum \frac{d^3q}{(2\pi)^3} \left\{ L_1(q) L_2^+(q) + L_2(q) L_1^+(q) \right\} = 1, \tag{104}$$

$$\frac{2N_c}{M_N} \sum \frac{d^3q}{(2\pi)^3} \left\{ N_1^\mu(q) N_2^{\mu+}(q) + N_2^\mu(q) N_1^{\mu+}(q) \right\} = 1, \tag{105}$$

$$\frac{2N_c}{M_\Sigma} \sum \frac{d^3q}{(2\pi)^3} \left\{ \Sigma_1(q) \Sigma_2^+(q) + \Sigma_2(q) \Sigma_1^+(q) \right\} = 1. \tag{106}$$

The description of instantaneous relativistic bound states in a hot and dense medium can be found in [57, 58].

2.6. Schrödinger Equation. If the atom is at rest ($\mathcal{P}_\mu = (M_A, 0, 0, 0)$) equation (96) coincides with the Salpeter equation [56]. If one assumes that the current mass m^0 is much larger than the relative momentum $|q^\perp|$, then the coupled equations (93) and (96) turn into the Schrödinger equation. In the rest frame ($\mathcal{P}_0 = M_A$) equations (89) and (90) for a large mass ($m^0/|q^\perp| \to \infty$) describe a nonrelativistic particle

$$E_a(\mathbf{k}) = \sqrt{(m_a^0)^2 + \mathbf{k}^2} \simeq m_a^0 + \frac{1}{2}\frac{\mathbf{k}^2}{m_a^0},$$

$$\tan 2v = \frac{k}{m^0} \to 0, \quad S(\mathbf{k}) \simeq 1, \quad \Lambda_{(\pm)} \simeq \frac{1 \pm \gamma_0}{2}.$$

Then, in equation (96) only the state with positive energy remains

$$\psi \simeq \psi_{(+)} = \Lambda_{(+)}\gamma_5 \sqrt{4\mu}\psi_{\rm Sch}, \quad \Lambda_{(-)}\psi\Lambda_{(+)} \simeq 0,$$

where $\mu = m_a m_b/(m_a + m_b)$. And finally the Schrödinger equation results in

$$\left[\frac{1}{2\mu}\mathbf{k}^{-2} + (m_a^0 + m_b^0 - M_A)\right]\psi_{\rm Sch}(\mathbf{k}) = \int \frac{d\mathbf{q}}{(2\pi)^3}\underline{V}(\mathbf{k}-\mathbf{q})\psi_{\rm Sch}(\mathbf{q}), \quad (107)$$

with the normalization $\int d^3q|\psi_{\rm Sch}|^2/(2\pi)^3 = 1$.

For an arbitrary total momentum $\mathcal{P}_\mu$, equation (107) takes the form

$$\left[-\frac{1}{2\mu}(k_\nu^\perp)^{-2} + (m_a^0 + m_b^0 - \sqrt{\mathcal{P}^2})\right]\psi_{\rm Sch}(k^\perp) = \\ = \int \frac{d^3q^\perp}{(2\pi)^3}\underline{V}(k^\perp - q^\perp)\psi_{\rm Sch}(q^\perp), \quad (108)$$

and describes a relativistic atom with nonrelativistic relative momentum $|k^\perp| \ll m_{a,b}^0$. In the framework of such a derivation of the Schrödinger equation, it is sufficient to define the total coordinate as $X = (x+y)/2$, independently of the magnitude of the masses of the two particles forming an atom.

In particular, the Coulomb interaction leads to a positronium at rest with the bilocal wave function

$$\Phi_P^{\alpha\beta}(t|\mathbf{z}) = \eta_P(t)\left(\frac{1+\gamma_0}{2}\gamma_5\right)^{\alpha\beta}\underline{\psi}_{\rm Sch}(\mathbf{z})\sqrt{\frac{m_e}{2}}, \\ \underline{\psi}_{\rm Sch}(\mathbf{z}) = \int \frac{d^3p}{(2\pi)^3}\,\mathrm{e}^{(i\mathbf{pz})}\psi_{\rm Sch}(\mathbf{p}), \quad (109)$$

where $\underline{\psi}_{\rm Sch}(\mathbf{z})$ is the Schrödinger normalizable wave function of the relative motion

$$\left(-\frac{1}{m_e}\frac{d^2}{d\mathbf{z}^2} - \frac{\alpha}{|\mathbf{z}|}\right)\underline{\psi}_{\rm Sch}(\mathbf{z}) = \epsilon\underline{\psi}_{\rm Sch}(\mathbf{z}), \quad \left(\int d^3z \|\ \underline{\psi}_{\rm Sch}(z)\ \|^2 = 1\right), \quad (110)$$

$M_P = (2m_e - \epsilon)$ is the mass of a positronium; $(1+\gamma_0)/2$ is the projection operator on the state with positive energies of an electron and positron. We have chosen the total-motion variable $\eta_P(t)$ in (109) so that the effective action for the total motion of the positronium with anomaly term have the form similar to the η_0-meson ones [59]

$$W_{\rm eff} = \int dt\left\{\frac{1}{2}\left(\dot\eta_P^2 - M_P^2\eta_P{}^2\right)V_{(3)} + C_P\eta_P\dot X[A]\right\}, \quad (111)$$

where

$$C_P = \frac{\sqrt{2}}{m_e}8\pi^2\left(\frac{\underline{\psi}_{\rm Sch}(0)}{m_e^{3/2}}\right), \quad (112)$$

and

$$\frac{d}{dt}X[A] = \int d^3x F^*_{\mu\nu}F^{\mu\nu}, \quad X[A] = \frac{e^2}{16\pi}\int d^3x(\epsilon_{ijk}A_i\partial_j A_k) \tag{113}$$

is the «winding number» functional (i. e., anomalous term) that describes the two γ decay of a positronium.

2.7. Spontaneous Chiral Symmetry Breaking. The solution of the set of SD and Salpeter equations (89), (90), (96) was considered in the numerous papers [60–62] (see also review [58]) for different potentials. One of the main results of these paper was the pure quantum effect of spontaneous chiral symmetry breaking for non-Coulomb potentials. The instantaneous interaction, in this case, leads to rearrangement of the perturbation series and strongly changes the spectrum of elementary excitations and bound states in contrast to the naive perturbation theory.

To demonstrate this effect and estimate possibility of the considered relativistic equations, we consider the opposite case of massless particles, $m_a^0 = m_b^0 \to 0$ for an arbitrary potential. Suppose that in this case equations (89), (90)

$$2E_a(k^\perp)\cos 2\upsilon(k^\perp) = \int \frac{d^3q^\perp}{(2\pi)^3}\underline{V}(k^\perp - q^\perp)\cos 2\upsilon(q^\perp), \tag{114}$$

$$2E_a(k^\perp)\sin 2\upsilon(k^\perp) = |k^\perp| + \int \frac{d^3q^\perp}{(2\pi)^3}\underline{V}(k^\perp - q^\perp)|\hat{k}^\perp \cdot \hat{q}^\perp|\sin 2\upsilon(q^\perp) \tag{115}$$

have a nontrivial solution $\upsilon(k^\perp) \neq 0$. This solution describes the spontaneous breakdown of chiral symmetry [25, 31, 58, 60–62].

It can easily be seen that equations (114) and (115) are identical with (96) for the bound state wave function with zero eigenvalue, $\mathcal{P}_\mu^2 = 0$ and

$$\begin{gathered}\Lambda_{(+)}\psi\bar{\Lambda}_{(-)} = \Lambda_{(-)}\psi\bar{\Lambda}_{(+)} \equiv \psi, \\ 2E_a(k^\perp)\psi(k^\perp) = \int \frac{d^3q^\perp}{(2\pi)^3}\underline{V}(k^\perp - q^\perp)\psi(q^\perp).\end{gathered} \tag{116}$$

Therefore,

$$\psi = \cos 2\upsilon(k^\perp)/F, \tag{117}$$

where F is a proportionality constant determined by the normalization (104) [25]

$$F = \frac{4N_c}{M_H}\int \frac{d^3q}{(2\pi)^3}L_2\cos(2\upsilon(q)). \tag{118}$$

In this way, the coupled equations (89), (90), and (96) contain the Goldstone mode that accompanies spontaneous breakdown of chiral symmetry. Thus, in the framework of instantaneous action we get the proof of the Goldstone theorem in the bilocal variant.

Just this example represents a model for the construction of a low-energy theory of light mesons, in which the pion is considered in two different ways, as a quark–antiquark bound state and as a Goldstone particle. So, it turns out that our relativistic instantaneous model for bound states can, in the lowest order of radiative corrections, also describe mesons. It was shown that the spontaneous symmetry breaking is absent for the pure Coulomb potential in QED [61]. The spontaneous symmetry breaking and the Goldstone meson in QCD are realized for the potential of hadronization. What is an origin of this potential in QCD?

Recall that in QED the Coulomb potential is the consequence of the resolving the Gauss law constraint. This Coulomb potential is lost in any relativistic invariant gauge of sources in FP representation of the path integral. The defect of the pure relativistic artificial gauges is the problem of a tachion and other nonphysical states in the bound state spectrum [55]. The incorporation of the simultaneity for these gauges by the quasipotential approach [44] could not describe both the spontaneous chiral symmetry breaking with the Goldstone meson and the S-matrix elements of interactions of some instantaneous bound states (papers on this topic are absent in literature).

2.8. The Relativistic Equation for Multiparticle Systems. To derive the relativistic covariant equations for many-particle systems, we use the operator approach [30, 60, 61] with the Hamiltonian given by

$$\mathcal{H} = \int d\mathbf{x}\bar{\psi}(i\partial_i\gamma_i + m^0)\psi + \\ + \frac{1}{2}\int d\mathbf{x}d\mathbf{y}(\psi_i^+(\mathbf{x})\psi_j(\mathbf{x}))V(\mathbf{x}-\mathbf{y})(\psi_k^+(\mathbf{y})\psi_l(\mathbf{y})). \quad (119)$$

The first step for constructing the physical states consists in the definition of the one-quasi-particle creation (a^+, b^+) and annihilation (a, b) operators with the help of the Bogoliubov fermion expansion [63]

$$\psi_\alpha(\mathbf{x}) = \sum_s \int \frac{d\mathbf{q}}{(2\pi)^{3/2}}\, \mathrm{e}^{i\mathbf{q}\mathbf{x}}[a_s(\mathbf{q})\mu_\alpha(\mathbf{q},s) + b_s^+(-\mathbf{q})\nu_\alpha(-\mathbf{q},s)]. \quad (120)$$

Here $\mu_\alpha(\mathbf{q},s)$ and $\nu_\alpha(-\mathbf{q},s)$ are the coefficients determined from the Schrödinger equation for the one-particle energy

$$\langle a_s(\mathbf{q})|\hat{H}|a_s^+(\mathbf{q}')\rangle = E(\mathbf{q})\langle 0|a_s(\mathbf{q})a_s^+(\mathbf{q}')|0\rangle. \quad (121)$$

They can be represented via the Foldy–Wouthuysen matrix (86) as

$$\mu_\alpha(\mathbf{q},s) = S(\mathbf{q})_{\alpha\beta}\mu_\beta(0,s), \quad \nu_\alpha(-\mathbf{q},s) = S(-\mathbf{q})_{\alpha\beta}\nu_\beta(0,s)$$

with

$$S_{\alpha\alpha'}(\mathbf{q})\left[\sum_s \mu_{\alpha'}(0,s)\mu^+_{\beta'}(0,s)\right]S^{-1}_{\beta'\beta}(\mathbf{q}) = \\ = \left(S\frac{1+\gamma_0}{2}S^{-1}\right)_{\alpha\beta} \equiv (\Lambda^0_+(\mathbf{q}))_{\alpha\beta},$$

$$S_{\alpha\alpha'}(-\mathbf{q})\left[\sum_s \nu_{\alpha'}(0,s)\nu^+_{\beta'}(0,s)\right]S^{-1}_{\beta'\beta}(-\mathbf{q}) = \\ = \left(S\frac{1-\gamma_0}{2}S^{-1}\right)_{\alpha\beta} \equiv (\Lambda^0_-(-\mathbf{q}))_{\alpha\beta}.$$

Λ^0_+ and Λ^0_- are projection operators on states with positive, resp., negative energy. Then, equation (121) takes the form of the Schwinger–Dyson equation (89), (90) which can compactly be written as

$$E(p)S^{-2}(\mathbf{p}) = m^0 + p_i\gamma_i + \frac{1}{2}\int \frac{d\mathbf{q}}{(2\pi)^3}\underline{V}(\mathbf{p}-\mathbf{q})S^{-2}(\mathbf{q}). \tag{122}$$

After inserting (120) into (119) the Hamiltonian can be given in the following manner:

$$\mathcal{H} = E_0 + H_1 + :H_4:,\quad E_0 = \langle 0|\mathcal{H}|0\rangle,\quad H_1 = \sum_{(1)} E(\mathbf{p}_1)(a^+_1 a_1 + b^+_1 b_1),$$
$$:H_4: = \frac{2}{3}\sum_{1,2,3,4,}\delta^{(4)}(p_1 - p_2 + p_3 - p_4)\underline{V}(\mathbf{p}_1 - \mathbf{p}_3)\times \\ \times\{a^+_1 b^+_{\hat 2} a^+_3 b^+_{\hat 4}\mu^*_1\nu^*_{\hat 2}\mu^*_3\nu^*_{\hat 4} + a^+_1 b^+_{\hat 2} b_{\hat 3} a_1 \mu^*_1\nu_{\hat 2}\nu^*_{\hat 3}\mu_4 + \\ + b_{\hat 1} a_2 a^+_3 b^+_{\hat 4}\nu^*_{\hat 1}\mu_2\mu^*_3\nu_{\hat 4} + b_{\hat 1} a_2 b_{\hat 3} a_4 \nu^*_{\hat 1}\mu_2\nu^*_{\hat 3}\mu_4 + \ldots\} + \ldots \tag{123}$$

The following abbreviations have been used in (123):

$$\sum_I = \sum_{s_I}\int\frac{d\mathbf{p}_I}{(2\pi)^{3/2}},\quad \{I\} = \{p_I, s_I\},\quad \{\hat I\} = \{-p_I, -s_I\},\quad I = 1,2,3,4.$$

For diagonalizing the Hamiltonian (123) with respect to pair correlations $(a^+_1 b^+_{\hat 2})$, $(b_{\hat 3} a_4)$ one defines a new vacuum as the coherent state

$$|0\rangle\rangle_\alpha = \exp\left\{\sum_{1,2,3,4}\alpha(1,\hat 2,\hat 3,4)[(a^{+i_1}_1 b^{+i_1}_{\hat 2})(b^{+j_1}_{\hat 3} a^{+j_1}_4)]\right\}|0\rangle \tag{124}$$

and the creation operator for the bound state (of pair correlation)

$$B^{+}(n)=\sum_{1,2}\delta(\mathbf{p}_1-\mathbf{p}_2)[X_{+}(1,\hat{2})a^{+i}(1)b^{+i}(\hat{2})-X_{-}(\hat{1},2)b^{j}(\hat{1})a^{j}(1)]. \quad (125)$$

The coefficients X_+ and X_- are determined from the Schrödinger equation for the two-particle energy M_B,

$${}_{\alpha}\langle\langle 0|B(n)(H_1+H_4)B^{+}(n)|0\rangle\rangle_{\alpha}=M_B\ {}_{\alpha}\langle\langle 0|B(n)B^{+}(n)|0\rangle\rangle_{\alpha}, \quad (126)$$

and the parameter α in (126) is given with the help of the definition of the annihilation operator $B(n)$ for the pair correlation

$$B(n)|0\rangle\rangle_{\alpha}=0. \quad (127)$$

Equation (126) coincides with equation (96) in the rest frame (the Salpeter equation) for the meson spectrum

$$\begin{aligned}(E_1(\mathbf{p})+E_2(\mathbf{p})\mp M_B)\psi_{\pm\pm}(\mathbf{p})=&\\ =\Lambda_{\pm}(\mathbf{p})[\hat{I}_{\mathbf{pq}}\times(\psi_{++}(\mathbf{q})+\psi_{--}(\mathbf{q}))]\Lambda_{\pm}(-\mathbf{p})&\end{aligned} \quad (128)$$

up to the notation

$$\begin{aligned}\psi=\psi_{++}+\psi_{--},\quad \psi_{\pm\pm}&=\Lambda_{\pm}\psi\Lambda_{\mp},\\ \psi_{++}(\mathbf{p})_{\alpha\beta}&=\sum_{s_1,s_2}X_{+}(\mathbf{p},\mathbf{p},s_1,s_2)\mu^{+}_{\alpha}(\mathbf{p},s_1)\nu_{\beta}(\mathbf{p},s_2),\\ \psi_{--}(\mathbf{p})_{\alpha\beta}&=\sum_{s_1,s_2}X_{-}(\mathbf{p},\mathbf{p},s_1,s_2)\nu^{+}_{\alpha}(\mathbf{p},s_1)\mu_{\beta}(\mathbf{p},s_2),\\ \hat{I}_{\mathbf{pq}}\times\psi(q)&=\int\frac{d^3q}{(2\pi)^3}V(\mathbf{p}-\mathbf{q})\psi(\mathbf{q}).\end{aligned} \quad (129)$$

The one-particle energies $E_1(\mathbf{p})$, $E_2(\mathbf{p})$ in (129) are defined via the Schwinger–Dyson equation (122).

Notice that equations of the type (122), (129) are well known from the nonrelativistic many-body theory (Landau's theory of fermi liquids [64], Random Phase Approximation [65]) and play an essential role in the description of elementary excitation in atomic nuclei [66]. Their relativistic analogies describing the Goldstone pion and the constituent masses of the light quarks are equations (89), (90), (96).

Thus the Green function method discussed in Secs. 2.2.–2.5, and the operator approach lead to one and the same equations and complement each other. The first allows one to make easily the relativistic generalization and to construct the effective bound state interaction Lagrangian, whereas the second yields an adequate interpretation of quantum states and enables one to describe more complicated system (in QCD, baryons and other many-quark states [30]).

Let us construct by means of the quasiparticle operator method the relativistic equation for a three-particle system. In the «coherent» vacuum (125) the creation operator of a three-particle system consists not only of creation operators for particles (a^+) but also of annihilation operators for antiparticles (b) with the same quantum numbers

$$\underline{B}^+ = \sum_{1,2,3} \delta(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3)[X_{+++}(1,2,3)a^{i(+)}(1)a^{j(+)}(2)a^{l(+)}(3) + \\ + X_{--+}(1,2,3)b^{i(+)}(1)b^{i(+)}(2)b^{i(+)}(3) + \text{interchange of } (1,2,3)]\epsilon^{ijk}. \quad (130)$$

The «baryon» functions are as follows:

$$\psi_{+++}(1,2,3)_{\alpha\beta\gamma} = \sum_{s_1 s_2 s_3} \mu_\alpha^+(1)\mu_\beta^+(2)\mu_\gamma^+(3)X_{+++}(1,2,3),$$

$$\psi_{--+}(1,2,3)_{\alpha\beta\gamma} = \sum_{s_1 s_2 s_3} \nu_\alpha^+(1)\nu_\beta^+(2)\nu_\gamma^+(3)X_{--+}(1,2,3),$$

etc. Then, the eigenvalue equation for the Hamiltonian operator

$${}_\alpha\langle\langle 0|\underline{B}H\underline{B}^+|0\rangle\rangle_\alpha \quad (131)$$

is equivalent to the following system for the «baryons» wave functions ψ_{+++}, $\psi_{--+}, \psi_{-+-}, \psi_{+--}$.

$$\left[\begin{pmatrix}+\\+\\+\\-\end{pmatrix} E(1) \begin{pmatrix}+\\+\\-\\+\end{pmatrix} E(2) \begin{pmatrix}+\\-\\+\\+\end{pmatrix} E(3) \begin{pmatrix}-\\+\\+\\+\end{pmatrix} M_B\right] \psi_{\begin{pmatrix}+++\\--+\\-+-\\+--\end{pmatrix}}(1,2,3) =$$

$$= \frac{2}{3}\Lambda_{\begin{pmatrix}+\\-\\-\\+\end{pmatrix}}(1)\Lambda_{\begin{pmatrix}+\\-\\+\\-\end{pmatrix}}(2)\Lambda_{\begin{pmatrix}+\\+\\-\\-\end{pmatrix}}(3) \times$$

$$\times \left\{ \hat{I}_{\underline{1},\underline{2}} \left[\psi_{\begin{pmatrix}+++\\--+\\-+-\\+--\end{pmatrix}}(\underline{1},\underline{2},3) + \psi_{\begin{pmatrix}--+\\+++\\+--\\-+-\end{pmatrix}}(\underline{1},\underline{2},3) \right] + \right.$$

$$+\hat{I}_{\underline{2},\underline{3}}\left[\psi_{\begin{pmatrix}+&+&+\\-&-&+\\-&+&-\\+&-&-\end{pmatrix}}(1,\underline{2},\underline{3})+\psi_{\begin{pmatrix}+&-&-\\-&+&-\\-&-&+\\+&+&+\end{pmatrix}}(1,\underline{2},\underline{3})\right]+$$

$$+\hat{I}_{\underline{1},\underline{3}}\left[\psi_{\begin{pmatrix}+&+&+\\-&-&+\\-&+&-\\+&-&-\end{pmatrix}}(\underline{1},2,\underline{3})+\psi_{\begin{pmatrix}-&+&-\\+&-&-\\+&+&+\\-&-&+\end{pmatrix}}(\underline{1},2,\underline{3})\right]\Bigg]\Bigg\}, \qquad (132)$$

where

$$I_{\underline{1},\underline{2}}\psi(\underline{1},\underline{2},3)=\int\frac{d\mathbf{q}}{(2\pi)^3}\underline{V}(\mathbf{q})\psi(\mathbf{p}_1-\mathbf{q},\mathbf{p}_2+\mathbf{q},\mathbf{p}_3),$$
$$\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3=0. \qquad (133)$$

Equation (133) is the analogue of the Salpeter equation (128) for a bound state consisting of three particles. In the same notations Eq. (129) has the form

$$\left[\begin{pmatrix}+\\+\end{pmatrix}E_1(1)\begin{pmatrix}+\\+\end{pmatrix}E_2(\hat{2})\begin{pmatrix}-\\+\end{pmatrix}M_B\right]\bar{\psi}_{\pm\pm}(1,\hat{2})=$$
$$=\frac{4}{3}\Lambda_{\pm}(1)\{\hat{I}_{\underline{1},\underline{\hat{2}}}[\psi_{++}(\underline{1},\underline{\hat{2}})+\psi_{--}(\underline{1},\underline{\hat{2}})]\}\Lambda_{\pm}(\hat{2})$$

with the condition $p_1=p_2=p$ and with the taking into account the identities

$$\int d\mathbf{q}\underline{V}(\mathbf{p}-\mathbf{q})\psi(\mathbf{q})=\int d\mathbf{q}\underline{V}(\mathbf{q})\psi(\mathbf{q}+\mathbf{p})=$$
$$=\int d\mathbf{q}\underline{V}(\mathbf{q})\psi(\mathbf{p}_1+\mathbf{q},-\mathbf{p}_2-\mathbf{q})|_{p_1=p_2=p},\quad \psi(p_1,-p_2)=\psi(1,\hat{2}).$$

The nonrelativistic reduction [56] from the Salpeter Eq. (96) to the Schrödinger equation,

$$E_a(\mathbf{p})\simeq\sqrt{m_a^2+\mathbf{p}^2}\simeq m_a+\frac{1}{2}\frac{\mathbf{p}^2}{m_a},$$
$$S_a(\mathbf{p})\simeq 1,\quad \psi_{+++}\equiv\psi\gg\psi_{\begin{pmatrix}+&-&-\\-&+&-\\-&-&+\end{pmatrix}},$$

leads in our case to the well-known nonrelativistic equation for the wave function of three-particle bound states

$$\left[\frac{\mathbf{p}_1^2}{2m_1}+\frac{\mathbf{p}_2^2}{2m_2}+\frac{\mathbf{p}_3^2}{2m_3}-(M_{\underline{B}}-m_1-m_2-m_3)\right]\psi(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)=$$
$$=\frac{2}{3}[\hat{I}_{\underline{1},\underline{2}}\psi(\underline{\mathbf{p}}_1,\underline{\mathbf{p}}_2,\mathbf{p}_3)+\hat{I}_{\underline{2},\underline{3}}\psi(\mathbf{p}_1,\underline{\mathbf{p}}_2,\underline{\mathbf{p}}_3)+\hat{I}_{\underline{1},\underline{3}}\psi(\underline{\mathbf{p}}_1,\mathbf{p}_2,\underline{\mathbf{p}}_3)]. \quad (134)$$

Here, the condition (134), which means the choice of the rest frame $\mathcal{P}_\mu = (M_{\underline{B}}, 0, 0, 0)$, has to be fulfilled.

Notice that the Jacobi coordinates, which allow one to write the Hamiltonian in the term of two relative momenta, have sense only in the nonrelativistic limit.

To describe the three-particle bound system in an arbitrary reference frame it is sufficient to substitute in (132) all relative momenta $\mathbf{p}_i$ by the transversal ones, $p_\mu^{\perp(i)}$, and the projection operators $\Lambda_\pm(\mathbf{p})$ by the operators

$$\Lambda_\pm(\mathbf{p}^\perp)=S(\mathbf{p}^\perp)\frac{M_{\underline{B}}\pm \not{P}}{2M_{\underline{B}}}S(\mathbf{p}^\perp)^{-1}.$$

In the same way one can generalize the equation (132) and its relativization for an arbitrary N-particle state.

The method for constructing relativistic wave functions of many-quark system explained above unambiguously enables one to build from the nonrelativistic bound state wave function

$$\chi_{\alpha_1,\alpha_2,\ldots,\alpha_N}\,\mathrm{e}^{iMX_0}\Phi_{\alpha_1,\alpha_2,\ldots,\alpha_N}(\mathbf{p}^1,\mathbf{p}^2,\ldots,\mathbf{p}^N),\quad \sum_i \mathbf{p}(i)=0$$

relativistic wave functions for the same bound states with the total momentum $\mathcal{P}_\mu = (\omega = \sqrt{\mathbf{P}^2+M^2}, \mathbf{P})$,

$$\chi_{\alpha_1,\alpha_2,\ldots,\alpha_N}\,\mathrm{e}^{i\mathcal{P}X}\Lambda_{+\alpha_1\alpha_1'}(p^{(1)\perp})\Lambda_{+\alpha_2\alpha_2'}(p^{(2)\perp})\cdots\Lambda_{+\alpha_N\alpha_N'}(p^{(N)\perp})\times$$
$$\times\Phi_{\alpha_1',\alpha_2',\ldots,\alpha_N'}(p^{(1)\perp},p^{(2)\perp},\ldots,p^{(N)\perp}),\quad \sum_i p_\mu^{(i)\perp}=0.$$

Here $\chi_{\alpha_1,\alpha_2,\ldots,\alpha_N}$ is the matrix selecting one or another representation of the Lorentz group with a definite spin. (A representation of the Poincare group that preserves the one-time dependence of wave functions, see in Ref. 67).

2.9. Relativistic Covariant Unitary S Matrix for Bound States. The achievement of the relativistic covariant constraint-shell quantization of gauge theories is the description of both the spectrum of bound states and their S-matrix elements.

It is convenient to write the matrix elements for the action (65), (71) in terms of the field operator

$$\Phi'(x,y) = \int d^4x_1 G_\Sigma(x-x_1)\mathcal{M}'(x_1,y) = \Phi'(z|X).$$

Using the decomposition over the bound state quantum numbers (H)

$$\Phi'(z|X) = \sum_H \int \frac{d^3\boldsymbol{\mathcal{P}}}{(2\pi)^{3/2}\sqrt{2\omega_H}} \int \frac{d^4q}{(2\pi)^4} \times \\ \times \{\mathrm{e}^{i\boldsymbol{\mathcal{P}}\mathbf{X}}\Phi_H(q^\perp|\mathcal{P})a_H^+(\mathcal{P}) + \mathrm{e}^{-i\boldsymbol{\mathcal{P}}\mathbf{X}}\bar{\Phi}_H(q^\perp|-\mathcal{P})a_H^-(\mathcal{P})\}, \quad (135)$$

where

$$\Phi_{H(ab)}(q^\perp|\mathcal{P}) = G_{\Sigma a}(q+\mathcal{P}/2)\Gamma_{H(ab)}(q^\perp|\mathcal{P}), \quad (136)$$

we can write the matrix elements for the interaction $W^{(n)}$ (71) between the vacuum and the n-bound state

$$\langle H_1\mathcal{P}_1, ..., H_n\mathcal{P}_n|iW^{(n)}|0\rangle = \\ = -i(2\pi)^4\delta^4\left(\sum_{i=1}^n \mathcal{P}_i\right)\prod_{j=1}^n\left[\frac{1}{(2\pi)^3 2\omega_j}\right]^{1/2} M^{(n)}(\mathcal{P}_1, ..., \mathcal{P}_n), \quad (137)$$

$$M^{(n)} = \int \frac{id^4q}{(2\pi)^4 n}\sum_{\{i_k\}}\Phi_{H_{i_1}}^{a_1,a_2}(q|\mathcal{P}_{i_1})\Phi_{H_{i_2}}^{a_2,a_3}\left(q - \frac{\mathcal{P}_{i_1}+\mathcal{P}_{i_2}}{2}|\mathcal{P}_{i_2}\right) \times \\ \times\Phi_{H_{i_3}}^{a_3,a_4}\left(q - \frac{2\mathcal{P}_{i_2}+\mathcal{P}_{i_1}+\mathcal{P}_{i_3}}{2}|\mathcal{P}_{i_3}\right)\cdots \\ \cdots\Phi_{H_{i_n}}^{a_n,a_1}\left(q - \frac{2(\mathcal{P}_{i_2}+\ldots+\mathcal{P}_{i_{n-1}})+\mathcal{P}_{i_1}+\mathcal{P}_{i_n}}{2}|\mathcal{P}_{i_n}\right),$$

($\{i_k\}$ denotes permutations over i_k).

Expressions (73), (76), (135), (137) represent Feynman rules for the construction of a quantum field theory with the action (71) in terms of bilocal fields.

It was shown [27] that the separable approximation of the constraint-shell gauge theory of bound states leads to the well-known Nambu–Jona-Lasinio model [33, 68] and the phenomenological chiral Lagrangians [69, 70] used for the description of the low-energy meson physics. Thus, the constraint-shell gauge theory of bound states is sufficient for describing the spectrum and interaction of hadrons as extended objects (without introducing the ideology of bags and string).

In the context of the constraint-shell gauge theory, to solve the problem of hadronization in QCD, one needs to answer the questions:

i) What is the origin of the potential of hadronization in the non-Abelian theory?

ii) How to combine the Schrödinger equation for heavy quarkonia (that is derived by the residuum of poles of the quark Green functions) with the quark confinement [71]?

iii) What is the origin of the additional mass of the ninth pseudoscalar meson [72]?

3. DIRAC VARIABLES IN YANG–MILLS THEORY WITH THE TOPOLOGICAL DEGENERATION OF PHYSICAL STATES

3.1. Constraint-Shell Radiation Variables in Perturbation Theory. We consider the Yang–Mills theory with the local $SU(2)$ group in four-dimensional Minkowskian space-time

$$W[A_\mu] = -\frac{1}{4}\int d^4x F^a_{\mu\nu}F^{\mu\nu}_a = \frac{1}{2}\int d^4x \left(F^{a\,2}_{0i} - B^{a\,2}_i\right), \tag{138}$$

where the standard definitions of non-Abelian electric tension F^a_{0i}

$$F_{0i} = \partial_0 A^a_i - D(A)^{ab}_i A^b_0, \quad D^{ab}_i = \left(\delta^{ab}\partial_i \ + g\epsilon^{acb}A^c_i\right)$$

and magnetic one B^a_i

$$B^a_i = \epsilon_{ijk}\left(\partial_j A^a_k + \frac{g}{2}\,\epsilon^{abc}\,A^b_j\,A^c_k\right)$$

are used. The action (138) is invariant with respect to gauge transformations $u(t, \mathbf{x})$

$$\hat{A}^u_i := u(t,\mathbf{x})\left(\hat{A}_i + \partial_i\right)u^{-1}(t,\mathbf{x}), \quad \psi^u := u(t,\mathbf{x})\psi, \tag{139}$$

where $\hat{A}_\mu = g(\tau^a/2i)A^a_\mu$.

Solutions of the non-Abelian equations

$$\frac{\delta W}{\delta A^a_0} = 0 \Longrightarrow \left[D^2(A)\right]^{ac} A^c_0 = D^{ac}_i(A)\partial_0 A^c_i, \tag{140}$$

$$\frac{\delta W}{\delta A^a_i} = 0 \Longrightarrow \left[\delta_{ij}D^2_k(A) - D_j(A)D_i(A)\right]^{ac} A^c_j = = D^{ac}_0(A)\left[\partial_0 A^c_i - D(A)^{cb}_i A^b_0\right] \tag{141}$$

are determined by boundary conditions and initial data.

The first Gauss equation (140) is the constraint. It connects initial data of A_0^a with the ones of the spatial components A_i^a. To remove nonphysical variables, we can honestly solve this constraint in the form of the naive perturbation series

$$A_0^c = a_0^c[A_i] = \frac{1}{\Delta}\partial_0\partial_i A_i^c + \ldots \tag{142}$$

The resolving of the constraint and the substitution of this solution into the equations of motion distinguishes the gauge-invariant nonlocal (radiation) variables. After the substitution of this solution into (141), the lowest order of the equation (141) in the coupling constant contains only transverse fields

$$(\partial_0^2 - \Delta)A_k^{cT} + \ldots = 0, \quad A_i^{cT} = \left[\delta_{ik} - \partial_i \frac{1}{\Delta}\partial_k\right] A_k^c + \ldots \tag{143}$$

This perturbation theory is well known as the radiation [4] (or Coulomb [6]) «gauge» with the generational functional of the Green functions in the form of the Feynman integral

$$Z_F[l^{(0)}, J^{aT}] = \int\int \prod_{c=1}^{c=3} [d^2A^{cT} d^2E^{cT}] \times \\ \times \exp\left\{ iW^T_{l^{(0)}}[A^T, E^T] - i\int d^4x [J_k^{cT} A_k^{cT}] \right\} \tag{144}$$

with the constraint-shell action (138)

$$W^T_{l^{(0)}}[A^T, E^T] = W^I \Big|_{\delta W^I/\delta A_0 = 0} \tag{145}$$

given in the first-order formalism

$$W^I = \int dt \int d^3x \left\{ F_{0i}^c E_i^c - \frac{1}{2}[E_i^c E_i^c + B_i^c B_i^c] \right\}. \tag{146}$$

The constraint

$$\frac{\delta W^I}{\delta A_0} = 0 \Rightarrow D_i^{cd}(A) E_i^d = 0 \tag{147}$$

is solved in terms of the radiation variables

$$E_i^c = E_i^{Tc} + \partial_i \sigma, \quad \partial_i E_i^{Tc} = 0, \tag{148}$$

where the functions σ^a take the form [6]

$$\sigma^a[A^T, E^T] = \left(\frac{1}{D_i(A)\partial_i}\right)^{ac} \epsilon^{cbd} A_k^{Tb} E_k^{Td}. \tag{149}$$

The operator quantization of the Yang–Mills theory in terms of the radiation variables belongs to Schwinger [4] who proved the relativistic covariance of the radiation variables (143). This means that the radiation fields are transformed as the nonlocal functional

$$\begin{gathered} \hat{A}_k^T[A] = v^T[A](\hat{A}_k + \partial_k)(v^T[A])^{-1}, \\ \hat{A}_k^T = \mathrm{e}\frac{A_k^{Ta}\tau^a}{2i}, \end{gathered} \tag{150}$$

where the matrix $v^T[A]$ is found from the condition of transversality $\partial_k \hat{A}_k^T = 0$. At the level of the Feynman integral, as we have seen in QED, the relativistic covariance means the relativistic transformation of sources [41].

The definition (150) can be treated as the transition to the new variables that allows us to rewrite the Feynman integral in the form of the FP integral [6,73,74]

$$\begin{aligned} Z_F[l^{(0)}, J^{aT}] = \int\int \prod_{c=1}^{c=3} [d^4A^c]\delta(\partial_i A_i^c) \operatorname{Det}[D_i(A)\partial_i] \times \\ \times \exp\left\{iW[A] - i\int d^4x(J_k^{cT} A_k^{cT}[A])\right\}. \end{aligned} \tag{151}$$

It was proved [6, 73, 74] that on the mass-shell of the radiation fields the scattering amplitudes do not depend on the factor $v^T[A]$. It remained only to reply the question: Why does nobody observe these scattering amplitudes of mass-shell non-Abelian radiation fields? There are some possible answers to this question: the infrared unstability of the naive perturbation theory [75, 76], the Gribov ambiguity, or zero of the FP determinant [7], the topological degeneration of the physical states [17, 19, 71]. In any case, in the non-Abelian theory there are not observable physical processes for which the gauge-equivalence theorem is valid.

3.2. Topological Degeneration of Initial Data. One can find a lot of solutions of equations of classical electrodynamics. The nature chooses two types of functions: the monopole that determines nonlocal electrostatic phenomena (including instantaneous bound states), and multipoles that determine the spatial components of gauge fields with a nonzero magnetic tension.

Spatial components of the non-Abelian fields considered above as radiation variables (143) in the naive perturbation theory are also defined as multipoles. In the non-Abelian theory, there is the reason to count that *the spatial components* *of the non-Abelian fields belong to the monopole class of functions* like the time component of the Abelian fields.

This fact was revealed by the authors of instantons [11]. Instantons satisfy the duality equation in the Euclidean space, so that the instanton action coincides with the Chern–Simons functional (Pontryagin index)

$$\nu[A] = \frac{g^2}{16\pi^2} \int\limits_{t_{\text{in}}}^{t_{\text{out}}} dt \int d^3x F^a_{\mu\nu} \widetilde{F}^{a\mu\nu} = \\ = X[A_{\text{out}}] - X[A_{\text{in}}] = n(t_{\text{out}}) - n(t_{\text{in}}), \quad (152)$$

where

$$X[A] = -\frac{1}{8\pi^2} \int\limits_V d^3x \epsilon^{ijk} \operatorname{Tr}\left[\hat{A}_i \partial_j \hat{A}_k - \frac{2}{3} \hat{A}_i \hat{A}_j \hat{A}_k\right], \\ A_{\text{in,out}} = A(t_{\text{in,out}}, x) \quad (153)$$

is the topological winding number functional of the gauge fields and n is a value of this functional for a classical vacuum

$$\hat{A}_i = L^n_i = v^{(n)}(\mathbf{x}) \partial_i v^{(n)}(\mathbf{x})^{-1}. \quad (154)$$

The manifold of all classical vacua (154) in the non-Abelian theory represents the group of three-dimensional paths lying on the three-dimensional space of the $SU_c(2)$ manifold with the homotopy group $\pi_{(3)}(SU_c(2)) = Z$. The whole group of stationary matrices is split into topological classes marked by integer numbers (the degree of the map) defined by the expression

$$\mathcal{N}[n] = -\frac{1}{24\pi^2} \int d^3x \epsilon^{ijk} \operatorname{Tr}[L^n_i L^n_j L^n_k] = n \quad (155)$$

which counts how many times a three-dimensional path $v(\mathbf{x})$ turns around the $SU(2)$ manifold when the coordinate x_i runs over the space where it is defined.

In 1976, Gribov V. N. suggested to treat instantons as Euclidean solutions interpolating between classical vacua with different degrees of map.

The degree of a map (155) can be considered as the condition for normalization that determines a class of functions where the classical vacua L^n_i (154) are given. In particular, to obtain Eq. (155) we should choose a classical vacuum in the form

$$v^{(n)}(\mathbf{x}) = \exp(n\hat{\Phi}_0(\mathbf{x})), \quad \hat{\Phi}_0 = -i\pi \frac{\tau^a x^a}{r} f_0(r) \quad (r = |\mathbf{x}|), \quad (156)$$

where the function $f_0(r)$ satisfies the boundary conditions

$$f_0(0) = 0, \quad f_0(\infty) = 1. \tag{157}$$

The normalization (155) points out that the vacuum values L_i^n of spatial components A_i belong to monopole-type class of functions. To show that these classical values are not sufficient to describe a physical vacuum in the non-Abelian theory, we consider a quantum instanton, i. e., the corresponding zero vacuum solution of the Schrödinger equation

$$\hat{H}\Psi_0[A] = 0 \quad \left(\hat{H} = \int d^3x[\hat{E}^2 + B^2],\ \hat{E} = \frac{\delta}{i\delta A}\right). \tag{158}$$

It can be constructed using the winding number functional (153) and its derivative

$$\frac{\delta}{\delta A_i^c} X[A] = \frac{g^2}{8\pi^2} B_i^c(A). \tag{159}$$

The vacuum wave functional in terms of the winding number (153) takes the form of a plane wave [16]

$$\Psi_0[A] = \exp\left(iP_N X[A]\right) \tag{160}$$

for nonphysical values of the topological momentum $P_N = \pm i8\pi^2/g^2$ [16, 20]. We would like to drew attention of a reader to that in QED this type of the wave functional belongs to nonphysical part of a spectrum like the wave function of an oscillator $(\hat{p}^2 + q^2)\psi_0 = 0$. The value of this nonphysical plane wave functional for classical vacuum (154) coincides with quasiclassical instanton wave function

$$\begin{aligned} \exp\left(iW[A_{\text{instanton}}]\right) = \Psi_0[A = L_{\text{out}}] \times \Psi_0^*[A = L_{\text{in}}] = \\ = \exp\left(-\frac{8\pi^2}{g^2}[n_{\text{out}} - n_{\text{in}}]\right). \end{aligned} \tag{161}$$

This exact relation between a classical instanton and its quantum version (158) points out that classical instantons are also nonphysical solutions, they are permanently tunneling in Euclidean space-time between classical vacua with the zero energy that does not belong to physical spectrum*.

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*The author is grateful to V. N. Gribov for the discussion of the problem of instantons during a visit to Budapest, May 1996.

3.3. Physical Vacuum and the Gauge Higgs Effect. The next step is the assertion [77] about the topological degeneration of initial data of not only classical vacuum but all physical fields with respect to stationary gauge transformations

$$\hat{A}_i^{(n)}(t_0,\mathbf{x}) = v^{(n)}(\mathbf{x})\hat{A}_i^{(0)}(t_0,\mathbf{x}){v^{(n)}(\mathbf{x})}^{-1} + L_i^n, \qquad L_i^n = v^{(n)}(\mathbf{x})\partial_i {v^{(n)}(\mathbf{x})}^{-1}. \tag{162}$$

The stationary transformations $v^n(\mathbf{x})$ with $n=0$ are called the small ones; and those with $n \neq 0$, the large ones [77].

The group of transformations (162) means that *the spatial components of the non-Abelian fields with a nonzero magnetic tension* $B(A) \neq 0$ *belong to the monopole class of functions* like the time component of the Abelian fields. In this case, non-Abelian fields with a nonzero magnetic tension contain a nonperturbative monopole-type term, and spatial components can be decomposed in a form of a sum of the monopole $\Phi_i^{(0)}(\mathbf{x})$ and multipoles $\bar{A}_i$

$$\hat{A}_i^{(0)}(t_0,\mathbf{x}) = \hat{\Phi}_i^{(0)}(\mathbf{x}) + \hat{\bar{A}}_i^{(0)}(t_0,\mathbf{x}). \tag{163}$$

The multipole is considered as a weak perturbative part with asymptotics at the spatial infinity

$$\bar{A}_i(t_0,\mathbf{x})|_{\rm asymp} = O\left(\frac{1}{r^{1+l}}\right) \quad (l>1). \tag{164}$$

Nielsen and Olesen [75], and Matinyan and Savidy [76] introduced a vacuum magnetic tension, using the fact that all asymptotically free theories are unstable, and the perturbation vacuum is not the lowest stable state.

The extension of the topological classification of classical vacua to all initial data of spatial components helps us to choose a vacuum monopole with a zero value of the winding number functional (153)

$$X[A = \Phi_i^{c(0)}] = 0, \quad \left.\frac{\delta X[A]}{\delta A_i^c}\right|_{A=\Phi^{(0)}} \neq 0. \tag{165}$$

The zero value of the winding number, transversality, and spherical symmetry fix a class of initial data for spatial components

$$\hat{\Phi}_i = -i\frac{\tau^a}{2}\epsilon_{iak}\frac{x^k}{r^2}f(r). \tag{166}$$

They contain only one function $f(r)$. The classical equation for this function takes the form

$$D_k^{ab}(\Phi_i)F_{kj}^b(\Phi_i) = 0 \Rightarrow \frac{d^2 f}{dr^2} + \frac{f(f^2-1)}{r^2} = 0. \tag{167}$$

We can see three solutions of this equation

$$f_1^{\mathrm{PT}} = 0, \quad f_1^{\mathrm{WY}} = \pm 1 \quad (r \neq 0). \tag{168}$$

The first solution corresponds to the naive unstable perturbation theory with the asymptotic freedom formula.

Two nontrivial solutions are well known. They are the Wu–Yang monopoles applied for the construction of physical variables in the current literature [78]. As it was shown in paper [22] the Wu–Yang monopole leads to rising potentials of the instantaneous interaction of the quasiparticle current. This interaction rearranges the perturbation series, leads to the gluon constituent mass, and removes the asymptotic freedom formula [29, 31] as the origin of unstability.

The Wu–Yang monopole is a solution of classical equations everywhere besides the origin of coordinates $r = 0$. The corresponding magnetic field is

$$B_i^a(\Phi_k) = \frac{x^a x^i}{gr^4}. \tag{169}$$

Following Wu and Yang [21], we consider the whole finite space volume, excluding an ϵ-region around the singular point. To remove a singularity at the origin of coordinates and regularize its energy, the Wu–Yang monopole is considered as the limit of zero size $\epsilon \to 0$ for the Bogomol'nyi–Prasad–Sommerfeld (BPS) monopole [79]

$$f_1^{\mathrm{WY}} \Rightarrow f_1^{\mathrm{BPS}} = \left[1 - \frac{r}{\epsilon \sinh(r/\epsilon)}\right] \tag{170}$$

with the finite energy

$$\int d^3x [B_i^a(\Phi_k)]^2 \equiv V\langle B^2 \rangle = \frac{4\pi}{g^2 \epsilon} \equiv \frac{1}{\alpha_s \epsilon}. \tag{171}$$

In this case, the BPS regularization of the Wu–Yang monopole is the analogue of the infrared regularization in QED by the introduction of the photon mass that also violates the initial equations of motion. The size of the BPS monopole is chosen so that the parameter ϵ disappears in the infinite volume limit

$$\epsilon = \frac{1}{\alpha_s \langle B^2 \rangle V}, \tag{172}$$

and the vacuum energy-density of the monopole solution $\langle B^2 \rangle$ is removed by a finite counter-term in the Lagrangian

$$\bar{\mathcal{L}} = \mathcal{L} - \frac{\langle B^2 \rangle}{2}.$$

This vacuum magnetic tension is the crucial difference of the topological degeneration of fields in Minkowski space from the topological degeneration of classical vacua of the instantons in the Euclidean one.

The problem is to formulate the Dirac quantization of weak perturbations of the non-Abelian fields in the presence of the nonperturbative monopole taking into account the topological degeneration of all initial data.

3.4. Dirac Method and Gribov Copies. Instead of artificial equations (4), (5) of the gauge-fixing method [8]

$$F(A_\mu) = 0, \quad F(A_\mu^u) = M_F u \neq 0 \Rightarrow$$
$$\Rightarrow Z^{\rm FP} = \int \prod_\mu DA_\mu \det\, M_F \delta(F(A))\, {\rm e}^{iW} \qquad (173)$$

we repeat the Dirac *constraint-shell* formulation resolving the constraint (140)

$$\frac{\delta W}{\delta A_0^a} = 0 \Rightarrow \left[D^2(A)\right]^{ac} A_0^c = D_i^{ac}(A)\partial_0 A_i^c \qquad (174)$$

with nonzero initial data

$$\partial_0 A_i^c = 0 \Rightarrow A_i^c(t,\mathbf{x}) = \Phi_i^{c(0)}(\mathbf{x}). \qquad (175)$$

The vacuum magnetostatic field $\Phi_i^{c(0)}$ has a zero value of the winding number (153) $X[\Phi_i^{c(0)}] = 0$ and satisfies the classical equations everywhere besides the small region near the origin of coordinates of the size of

$$\epsilon \sim \frac{1}{\int d^3x B^2(\Phi)} \equiv \frac{1}{\langle B^2 \rangle V}, \qquad (176)$$

that disappears in the infinite volume limit.

The second step is the consideration of the perturbation theory (163) where the constraint (174) takes the form

$$\left[D^2(\Phi^{(0)})\right]^{ac} A_0^{c(0)} = \partial_0[D_i^{ac}(\Phi^{(0)}) A_i^{c(0)}]. \qquad (177)$$

Dirac proposed [1] that the time component A_0 (the quantization of which contradicts to quantum principles) can be removed by gauge transformation, so that the constraint (177) takes the form

$$\partial_0[D_i^{ac}(\Phi^{(0)}) A_i^{c(0)}] = 0. \qquad (178)$$

We defined the *constraint-shell gauge*

$$[D_i^{ac}(\Phi^{(0)}) A_i^{c(0)}] = 0 \qquad (179)$$

as the zero initial data of this constraint (178).

The topological degeneration of initial data means that not only classical vacua but also all fields $A_i^{(0)} = \Phi_i^{(0)} + \bar{A}_i^{(0)}$ in the gauge (179) are degenerated

$$\hat{A}_i^{(n)} = v^{(n)}(\mathbf{x})(\hat{A}_i^{(0)} + \partial_i)v^{(n)}(\mathbf{x})^{-1}, \quad v^{(n)}(\mathbf{x}) = \exp[n\Phi_0(\mathbf{x})]. \tag{180}$$

The winding number functional (153) after the transformation (162) takes the form

$$X[A_i^{(n)}] = X[A_i^{(0)}] + \mathcal{N}(n) + \frac{1}{8\pi^2}\int d^3x \epsilon^{ijk}\,\mathrm{Tr}\,[\partial_i(\hat{A}_j^{(0)} L_k^n)], \tag{181}$$

where $\mathcal{N}(n) = n$ is given by Eq. (155).

The constraint-shell gauge (179) keeps its form in each topological class

$$D_i^{ab}(\Phi_k^{(n)})\bar{A}_i^{(n)b} = 0, \tag{182}$$

if the phase $\Phi_0(\mathbf{x})$ satisfies the equation of the Gribov ambiguity

$$[D_i^2(\Phi_k^{(0)})]^{ab}\Phi_0^b = 0. \tag{183}$$

In this case, the topological degeneration means the existence of the manifold of the Gribov copies of the constraint-shell gauge (179). One can show [22] that the Gribov equation (183) together with

$$X[\Phi^{(n)}] = n \tag{184}$$

are compatible with the unique solution of classical equations. It is just the Wu–Yang monopole considered before. The nontrivial solution of the equation for the Gribov phase (183) in this case is well known:

$$\hat{\Phi}_0 = -i\pi\frac{\tau^a x^a}{r} f_0^{\mathrm{BPS}}(r), \quad f_0^{\mathrm{BPS}}(r) = \left[\frac{1}{\tanh(r/\epsilon)} - \frac{\epsilon}{r}\right], \tag{185}$$

it is the Bogomol'nyi–Prasad–Sommerfeld (BPS) monopole [79].

Thus, instead of the topological degenerated classical vacuum for the instanton calculation (that is in the physically unattainable region), we have the topological degenerated Wu–Yang monopole (180)

$$\hat{\Phi}_i^{(n)} := v^{(n)}(\mathbf{x})[\hat{\Phi}_i^{(0)} + \partial_i]v^{(n)}(\mathbf{x})^{-1}, \quad v^{(n)}(\mathbf{x}) = \exp[n\Phi_0(\mathbf{x})], \tag{186}$$

and the topological degenerated multipoles

$$\hat{\bar{A}}_i^{(n)} := v^{(n)}(\mathbf{x})\hat{\bar{A}}_i^{(0)} v^{(n)}(\mathbf{x})^{-1}. \tag{187}$$

The Gribov copies are evidence of a zero mode in the left-hand side of both the constraints (177) and (174).

$$\left[D_i^2\,(\Phi^{(0)})\right]^{ac} A_0^c = 0. \tag{188}$$

A nontrivial solution of this equation

$$A_0^c(t,\mathbf{x}) = \dot{N}(t)\Phi_0^c(\mathbf{x}) \tag{189}$$

can be removed from the local equations of motion by the gauge transformation (a la Dirac of 1927) to convert the fields into the Dirac variables

$$\hat{A}_i^{(N)} = \exp\,[N(t)\hat{\Phi}_0(\mathbf{x})][\hat{A}_i^{(0)} + \partial_i]\,\exp\,[-N(t)\hat{\Phi}_0(\mathbf{x})]. \tag{190}$$

But this solution (189) cannot be removed from the constraint-shell action $W^* = \int dt\dot{N}^2 I/2 + \ldots$ and from the winding number $X[A^{(N)}] = N + X[A^{(0)}]$. Finally we obtained the Feynman path integral

$$Z_F = \int DN \prod_{i,c}[DE_i^{c(0)} DA_i^{c(0)}]\,\mathrm{e}^{iW^*} \tag{191}$$

that does not coincide with any artificial gauge (173).

We consider the derivation of this integral (191) in detail further.

3.5. Topological Dynamics. The repetition of the Dirac definition of the observable variables in QED allowed us to determine the vacuum fields and the phase of their topological degeneration in the form of the Gribov copies of the constraint-shell gauge.

The degeneration of initial data is the evidence of the zero mode of the Gauss law constraint. In the lowest order of the considered perturbation theory, this constraint (188) has the solution (189) with the nontrivial vacuum electric field

$$F_{i0}^b = \dot{N}(t)D_i^{bc}(\Phi_k^{(0)})\Phi_0^c(\mathbf{x}). \tag{192}$$

We call the new variable $N(t)$ the winding number variable, as the vacuum Chern–Simons functional is equal to the difference of the *in* and *out* values of this variable

$$\nu[A_0, \Phi^{(0)}] = \frac{g^2}{16\pi^2}\int\limits_{t_{\rm in}}^{t_{\rm out}} dt \int d^3x F_{\mu\nu}^a \widetilde{F}^{a\mu\nu} =$$
$$= \frac{\alpha_s}{2\pi}\int d^3x F_{i0}^b B_i^b(\Phi^{(0)})[N(t_{\rm out}) - N(t_{\rm in})] = N(t_{\rm out}) - N(t_{\rm in}). \tag{193}$$

The winding number functional admits its generalization to noninteger degrees of a map [22]

$$X[\Phi^{(N)}] = N, \quad N \neq n, \tag{194}$$

where

$$\left(\hat{\Phi}_i^{(N)} = \mathrm{e}^{N\hat{\Phi}_0}[\hat{\Phi}_i^{(0)} + \partial_i]\,\mathrm{e}^{-N\hat{\Phi}_0}\right).$$

Thus, we can identify the global variable $N(t)$ with the winding number degree of freedom in the Minkowskian space described by the action

$$W_N = \int d^4x \frac{1}{2}(F_{0i}^c)^2 = \int dt \frac{\dot{N}^2 I}{2}, \tag{195}$$

where the functional

$$I = \int\limits_V d^3x (D_i^{ac}(\Phi_k)\Phi_0^c)^2 = \frac{4\pi^2}{\alpha_s^2}\frac{1}{V\langle B^2\rangle} \tag{196}$$

does not contribute in local equations of motion. The topological degeneration of all fields converts into the degeneration of only one global topological variable $N(t)$ with respect to a shift of this variable on integers: $(N \Rightarrow N+n,\ n = \pm 1, \pm 2, \dots)$. Thus, the topological variable is a free rotator with the instanton-type wave function (160) in the Minkowskian space-time

$$\Psi_N = \exp\{iP_N N\}, \quad P_N = \dot{N}I = 2\pi k + \theta, \tag{197}$$

where k is a number of the Brilloin zone, and θ is the θ angle. In contrast with the instanton wave function (160), the spectrum of the topological momentum is real and belongs to physical values. Finally, equations (195) and (197) determine the finite spectrum of the global electric tension (192)

$$F_{i0}^b = \dot{N}[D_i(\Phi^{(0)})A_0]^b = \alpha_s\left(\frac{\theta}{2\pi} + k\right)B_i^b(\Phi^{(0)}). \tag{198}$$

It is the analogue of the Coleman spectrum of the electric tension in the $\mathrm{QED}_{(1+1)}$ [35]. The application of the Dirac quantization to the 1-dimensional electrodynamics $\mathrm{QED}_{(1+1)}$ in paper [36] demonstrates the universality of the Dirac variables and their adequacy to the description of topological dynamics with a nontrivial homotopy group.

3.6. Zero Mode of Gauss Law and Dirac Variables. The constraint-shell theory is obtained by the explicit resolution of the Gauss law constraint

$$\frac{\delta W}{\delta A_0^a} = 0 \Rightarrow \left[D^2(A)\right]^{ac} A_0^c = D_i^{ac}(A)\partial_0 A_i^c \tag{199}$$

and next in dealing with the initial action on surface of these solutions

$$W^* = W[A_\mu]\big|_{\delta W/\delta A_0^a=0}. \tag{200}$$

The result of similar solution in QED was the electrostatics and the Coulomb-like atoms. In the non-Abelian case, the topological degeneration in the form of the Gribov copies means that a general solution of the Gauss law constraint (199) contains the zero mode $\mathcal{Z}$. A general solution of the inhomogeneous equation (199) is a sum of the zero-mode solution $\mathcal{Z}^a$ of the homogeneous equation

$$(D^2(A))^{ab}\mathcal{Z}^b = 0, \tag{201}$$

and a particular solution $\tilde{A}_0^a$ of the inhomogeneous one

$$A_0^a = \mathcal{Z}^a + \tilde{A}_0^a. \tag{202}$$

The zero-mode $\mathcal{Z}^a$ at the spatial infinity has been represented in the form of a sum of the product of a new topological variable $\dot{N}(t)$ and the Gribov phase $\Phi_0(\mathbf{x})$ and weak multipole corrections

$$\hat{\mathcal{Z}}(t,\mathbf{x})|_{\text{asymp}} = \dot{N}(t)\hat{\Phi}_0(\mathbf{x}) + O\left(\frac{1}{r^{1+l}}\right), \quad (l > 0). \tag{203}$$

In this case, the single one-parametric variable $N(t)$ reproduces the topological degeneration of all field variables, if the Dirac variables are defined by the gauge transformations

$$\begin{gathered} 0 = U_{\mathcal{Z}}(\hat{\mathcal{Z}} + \partial_0)U_{\mathcal{Z}}^{-1}, \\ \hat{A}_i^* = U_{\mathcal{Z}}(\hat{A}_i^{(0)} + \partial_i)U_{\mathcal{Z}}^{-1}, \quad A_i^{(0)} = \Phi_i^{(0)} + \bar{A}_i^{(0)} \end{gathered} \tag{204}$$

where the spatial asymptotic of $U_{\mathcal{Z}}$ is

$$U_{\mathcal{Z}} = T\exp\left.\left[\int^t dt'\hat{\mathcal{Z}}(t',\mathbf{x})\right]\right|_{\text{asymp}} = \exp\left[N(t)\hat{\Phi}_0(\mathbf{x})\right]. \tag{205}$$

The topological degeneration of all fields converts into the degeneration of only one global topological variable $N(t)$ with respect to a shift of this variable on integers: $(N \Rightarrow N+n,\ n = \pm 1, \pm 2, \ldots)$.

3.7. Constraining with the Zero Mode. Let us formulate an equivalent unconstrained system for the YM theory in the monopole class of functions in the presence of the zero mode $\mathcal{Z}^b$ of the Gauss law constraint

$$A_0^a = \mathcal{Z}^a + \tilde{A}_0^a, \quad F_{0k}^a = -D_k^{ab}(A)\mathcal{Z}^b + \tilde{F}_{0k}^a \quad ((D^2(A))^{ab}\mathcal{Z}^b = 0). \tag{206}$$

To obtain the constraint-shell action

$$W_{\mathrm{YM}}(\text{constraint}) = \mathcal{W}_{\mathrm{YM}}[\mathcal{Z}] + \tilde{W}_{\mathrm{YM}}[\tilde{F}], \tag{207}$$

we use the evident decomposition

$$F^2 = (-D\mathcal{Z} + \tilde{F})^2 = (D\mathcal{Z})^2 - 2\tilde{F}D\mathcal{Z} + (\tilde{F})^2 = \\ = \partial(\mathcal{Z}(D\mathcal{Z})) - 2\partial(\mathcal{Z}\tilde{F}) + (\tilde{F})^2 \tag{208}$$

and the Gauss Eqs. $D\tilde{F} = 0$ and $D^2\mathcal{Z} = 0$ which show that the zero mode part $\mathcal{W}_{\mathrm{YM}}$ of the constraint-shell action (207) is the sum of two surface integrals

$$\mathcal{W}_{\mathrm{YM}}[\mathcal{Z}] = \\ = \int dt \int d^3x \left[\frac{1}{2}\partial_i(\mathcal{Z}^a D_i^{ab}(A)\mathcal{Z}^b) - \partial_i(\tilde{F}_{0i}^a \mathcal{Z}^a)\right] = \mathcal{W}^0 + \mathcal{W}', \tag{209}$$

where the first one $\mathcal{W}^0$ is the kinetic term and the second one $\mathcal{W}'$ describes the coupling of the zero-mode to the local excitations. These surface terms are determined by the asymptotics of the fields $(\mathcal{Z}^a, A_i^a)$ at spatial infinity (203), (164) which we denoted by $(\dot{N}(t)\Phi_0^a(\mathbf{x}), \Phi_i^a(\mathbf{x}))$. The fluctuations $\tilde{F}_{0i}^a$ belong to the class of multipoles. Since the surface integral over monopole-multipole couplings vanishes, the fluctuation part of the second term obviously drops out. The substitution of the solution with the asymptotic (203) into the first surface term of Eq. (209) leads to the zero-mode action (195).

The action for the equivalent unconstrained system of the local excitations,

$$\tilde{W}_{\mathrm{YM}}[\tilde{F}] = \\ = \int d^4x \left\{E_k^a \dot{A}_k^{a(0)} - \frac{1}{2}\left\{E_k^2 + B_k^2(A^{(0)}) + [D_k^{ab}(\Phi^{(0)})\tilde{\sigma}^b]^2\right\}\right\}, \tag{210}$$

is obtained in terms of variables with zero degree of map

$$\hat{\tilde{F}}_{0k} = U_{\mathcal{Z}}\hat{F}_{0k}^{(0)}U_{\mathcal{Z}}^{-1}, \quad \hat{A}_i = U_{\mathcal{Z}}(\hat{A}_i^{(0)} + \partial_i)U_{\mathcal{Z}}^{-1}, \\ \hat{A}_i^{(0)}(t,\mathbf{x}) = \hat{\Phi}_i^{(0)}(\mathbf{x}) + \hat{\tilde{A}}_i^{(0)}(t,\mathbf{x}) \tag{211}$$

by decomposing the electrical components of the field strength tensor $F_{0i}^{(0)}$ into transverse E_i^a and longitudinal $F_{0i}^{a\,L} = -D_i^{ab}(\Phi^{(0)})\tilde{\sigma}^b$ parts, so that

$$F_{0i}^{a(0)} = E_i^a - D_i^{ab}(\Phi^{(0)})\tilde{\sigma}^b, \quad D_k^{ab}(\Phi^{(0)})E_k^b = 0. \tag{212}$$

Here the function $\tilde{\sigma}^b$ is determined from the Gauss equation

$$\left((D^2(\Phi^{(0)}))^{ab} + g\epsilon^{adc}\tilde{A}_i^{d(0)}D_i^{cb}(\Phi^{(0)})\right)\tilde{\sigma}^b = -g\epsilon^{abc}\tilde{A}_i^{a(0)}E_i^c. \tag{213}$$

Due to gauge-invariance, the dependence of the action for local ecxitations on the zero mode disappears, and we got the ordinary generalization of the Coulomb gauge [4, 6] in the presence of the Wu–Yang monopole.

3.8. Feynman Path Integral. The Feynman path integral over the independent variables includes the integration over the topological variable $N(t)$

$$Z_F[J] = \int \prod_t dN(t)\tilde{Z}[J^U], \tag{214}$$

where

$$\tilde{Z}[J^U] = \int \prod_{t,x} \left\{ \prod_{a=1}^{3} \frac{[d^2A_a^{(0)}d^2E_a^{(0)}]}{2\pi} \right\} \times \\ \times \exp i\left\{\mathcal{W}_{\mathrm{YM}}(\mathcal{Z}) + \tilde{W}_{\mathrm{YM}}(A_a^{(0)}) + S[J^U]\right\}. \tag{215}$$

As we have seen above, the functionals $\tilde{W}, S$ are given in terms of the variables which contain the nonperturbative phase factors $U = U_{\mathcal{Z}}$ (205) of the topological degeneration of initial data. These factors disappear in the action $\tilde{W}$, but not in the source

$$S[J^U] = \int d^4x J_i^a \bar{A}_i^a, \quad \hat{\bar{A}}_i = U(\hat{A}_a^{(0)})U^{-1} \tag{216}$$

what reflects the fact of the topological degeneration of the physical fields.

The constraint-shell formulation distinguishes a bare «gluon» as a weak deviation of the monopole with the index ($n = 0$), and an observable (physical) «gluon» averaged over the topological degeneration (i. e., Gribov's copies) [17]

$$\bar{A}^{\mathrm{Phys}} = \lim_{L\to\infty} \frac{1}{2L} \sum_{n=-L}^{n=+L} \bar{A}^{(n)}(\mathbf{x}) \sim \delta_{r,0}, \tag{217}$$

whereas in QED the constraint-shell field is a transversal photon.

3.9. Rising Potential Induced by Monopole. We can calculate the instantaneous Green function

$$(D^2(\Phi^{(0)}))^{ab}(\mathbf{x})G^{bc}(\mathbf{x},\mathbf{y}) = \delta^{ac}\delta^3(x-y). \tag{218}$$

In the presence of the Wu–Yang monopole we have

$$(D^2)^{ab}(\mathbf{x}) = \delta^{ab}\Delta - \frac{n^a n^b + \delta^{ab}}{r^2} + 2\left(\frac{n_a}{r}\partial_b - \frac{n_b}{r}\partial_a\right),$$

and $n_a(x) = x_a/r$; $r = |\mathbf{x}|$. Let us decompose G^{ab} into a complete set of orthogonal vectors in color space

$$G^{ab}(\mathbf{x},\mathbf{y}) = \left[n^a(x)n^b(y)V_0(z) + \sum_{\alpha=1,2} e^a_\alpha(x)e^b_\alpha(y)V_1(z)\right], \quad (z = |\mathbf{x}-\mathbf{y}|).$$

Substituting the latter into the first equation, we get

$$\frac{d^2}{dz^2}V_n + \frac{2}{z}\frac{d}{dz}V_n - \frac{n}{z^2}V_n = 0, \quad n = 0,\ 1.$$

The general solution for the last equation is

$$V_n(|\mathbf{x}-\mathbf{y}|) = d_n|\mathbf{x}-\mathbf{y}|^{l_1^n} + c_n|\mathbf{x}-\mathbf{y}|^{l_2^n}, \quad n = 0,\ 1, \tag{219}$$

where d_n, c_n are constants, and l_1^n, l_2^n can be found as roots of the equation $(l^n)^2 + l^n = n$, i. e.,

$$l_1^n = -\frac{1+\sqrt{1+4n}}{2}, \quad l_2^n = \frac{-1+\sqrt{1+4n}}{2}. \tag{220}$$

It is easy to see that for $n = 0$ we get the Coulomb-type potential $d_0 = -1/4\pi$,

$$l_1^0 = -\frac{1+\sqrt{1}}{2} = -1, \quad l_2^0 = \frac{-1+\sqrt{1}}{2} = 0, \tag{221}$$

and for $n = 1$ the «golden section» potential with

$$l_1^1 = -\frac{1+\sqrt{5}}{2} \approx -1.618, \quad l_2^1 = \frac{-1+\sqrt{5}}{2} \approx 0.618. \tag{222}$$

The last potential (in the contrast with the Coulomb-type one) can lead to rearrangement of the naive perturbation series of the type of the spontaneous chiral symmetry breaking. This potential can be considered as the origin of the «hadronization» of quarks and gluons in QCD [31].

3.10. FP Path Integral. Thus, we can say that the Dirac variables with the topological degeneration of initial states in a non-Abelian theory determine the physical origin of hadronization and confinement as nonlocal monopole effects. The Dirac variables distinguish a unque gauge. In QED, it is the Coulomb gauge; whereas, in YM theory, it is covariant generalization of the Coulomb gauge in the presence of a monopole.

If we pass to another gauges of physical sources on the level of the FP integral in relativistic gauges, all the monopole effects of the degeneration and rising potential can be lost (as the Coulomb potential is lost in QED in relativistic invariant gauges). Recall that to prove the equivalence of the Feynman integral to the Faddeev–Popov integral in an arbitrary gauge, we change variables and concentrate all monopole effects in the phase factors before the physical sources. The change of the sources removes all these effects.

The change of the sources was possible in the Abelian theory only for the scattering amplitudes [6] when all particle-like excitations of the fields are on their mass-shell. However, for the cases of nonlocal bound states and other phenomena where these fields are off their mass-shell the Faddeev theorem of equivalence of different «gauges» is not valid.

For the non-Abelian theory all states become nonlocal, and the range of validity of the equivalence theorem is equal to zero.

3.11. A Free Rotator: Topological Confinement. We have seen in Section 3, that the topology can be the origin of color confinement as complete destructive interference of the phase factors of the topological degeneration of initial data.

The mechanical analogue of the topological degeneration of initial data is the free rotator $N(t)$ with the action of free particle

$$W(N_{\rm out}, N_{\rm in}|t_1) = \int\limits_0^{t_1} dt \frac{\dot{N}^2}{2} I, \quad p = \dot{N} I, \quad H_0 = \frac{p^2}{2I} \tag{223}$$

given on a ring where the points $N(t)+n$ (n is integer) are physically equivalent. Instead of an initial data $N(t=0) = N_{\rm in}$ in the mechanics in the space with the trivial topology, the observer of the rotator has the manifold of initial data $N^{(n)}(t=0) = N_{\rm in} + n$; $n = 0, \pm 1, \pm 2, \ldots$

An observer does not know where is the rotator. It can be at points $N_{\rm in}, N_{\rm in} \pm 1, N_{\rm in} \pm 2, N_{\rm in} \pm 3, \ldots$ Therefore, he should average a wave function

$$\Psi(N) = {\rm e}^{ipN}$$

over all values of the topological degeneration with the θ-angle measure $\exp{(i\theta n)}$.

In the result, we obtain the wave function

$$\Psi(N)_{\rm observ} = \lim_{L\to\infty} \frac{1}{2L} \sum_{n=-L}^{n=+L} {\rm e}^{i\theta n} \Psi(N+n) = \exp\{i(2\pi k + \theta)N\}, \qquad (224)$$

where k is integer. In the opposite case $p \neq 2\pi k + \theta$, the corresponding wave function (i. e., the probability amplitude) disappears $\Psi(N)_{\rm observ} = 0$ due to the complete destructive interference.

The consequence of this topological degeneration is that a part of values of the momentum spectrum becomes unobservable in the comparison with the trivial topology.

This fact can be treated as confinement of those values which do not coincide with the discreet ones

$$p_k = 2\pi k + \theta, \quad 0 \le \theta \le \pi. \qquad (225)$$

The observable spectrum follows also from the constraint of the equivalence of the point N and $N+1$

$$\Psi(N) = {\rm e}^{i\theta} \Psi(N+1), \quad \Psi(N) = {\rm e}^{ipN}. \qquad (226)$$

In the result we obtain the spectral decomposition of the Green function of the free rotator (223) (as the probability amplitude of transition from the point $N_{\rm in}$ to $N_{\rm out}$) over the observable values of spectrum (225)

$$G(N_{\rm out}, N_{\rm in}|t_1) \equiv \langle N_{\rm out}| \exp(-i\hat{H}t_1)|N_{\rm in}\rangle = \\ = \frac{1}{2\pi} \sum_{k=-\infty}^{k=+\infty} \exp\left[-i\frac{p_k^2}{2I}t_1 + ip_k(N_{\rm out} - N_{\rm in})\right]. \qquad (227)$$

Using the connection with the Jacobian theta-functions [81]

$$\Theta_3(Z|\tau) = \sum_{k=-\infty}^{k=+\infty} \exp[i\pi k^2\tau + 2ikZ] = (-i\tau)^{-1/2} \exp\left[\frac{Z^2}{i\pi\tau}\right] \Theta_3\left(\frac{Z}{\tau}\middle| -\frac{1}{\tau}\right),$$

we can represent expression (227) in the form of the sum over all paths

$$G(N_{\rm out}, N_{\rm in}|t_1) = \sqrt{\frac{I}{(i4\pi t_1)}} \times \\ \times \sum_{n=-\infty}^{n=+\infty} \exp[i\theta n] \exp[+iW(N_{\rm out}, N_{\rm in} + n|t_1)], \qquad (228)$$

where

$$W(N_{\rm out} + n, N_{\rm in}|t_1) = \frac{(N_{\rm out} + n - N_{\rm in})^2 I}{2t_1}$$

is the rotator action (223).

3.12. Confinement as Destructive Interference. The similar topological confinement as a complete destructive interference of phase factor of the topological degeneration (i. e., a pure quantum effect) can be in the «classical non-Abelian field theory». Recall that at the time of the first paper of Dirac [1] (1927), the so-called «classical relativistic field theory» was revealed in the papers of Schrödinger, Fock, Klein, Weyl [82, 83] as a type of *relativistic quantum mechanics*, i. e., the result of the primary quantization. The phase of the gauge transformations was introduced by Weyl [83] as a pure quantum quantity.

The free rotator shows us that the topological degeneracy can be removed, if all Green functions are averaged over values of the topological variable and all possible angles of orientation of the monopole unit vector ($\mathbf{n} = \mathbf{x}/r$) (164) in the group space (instead of the instanton averaging over interpolations between different vacua).

Averaging over all parameters of the degenerations can lead to a complete destructive interference of all color amplitudes [17, 19, 71]. In this case, only colorless («hadron») states have to form a complete set of physical states. Using the example of a free rotator, we have seen that the disappearance of a part of physical states due to the topological degeneration (confinement) does not violate the composition law for Green functions

$$G_{ij}(t_1,t_3) = \sum_h G_{ih}(t_1,t_2)G_{hj}(t_2,t_3), \tag{229}$$

defined as the amplitude of the probability to find a system with the Hamiltonian H in a state j at the time t_3, if at the time t_1 this system was in a state i, where (i,j) belongs to a complete set of all states $\{h\}$:

$$G_{ij}(t_1,t_3) = \langle i| \exp\left(-i\int\limits_{t_1}^{t_3} dtH\right)|j\rangle.$$

The particular case of this composition law (229) is the unitarity of S matrix

$$SS^+ = I \Rightarrow \sum_h \langle i|S|h\rangle\langle h|S^+|j\rangle = \langle i|j\rangle$$

known as the law of probability conservation for S-matrix elements ($S = I + iT$)

$$\sum_h \langle i|T|h\rangle\langle h|T^*|j\rangle = 2\,\mathrm{Im}\,\langle i|T|j\rangle. \tag{230}$$

The left side of this law is the analogue of the spectral series of the free rotator (227). The destructive interference keeps only colorless «hadron» states. Whereas, the right side of this law far from resonances can be presented in the form of the perturbation series over the Feynman diagrams that follow from the Hamiltonian. Due to gauge invariance $H[A^{(n)},q^{(n)}] = H[A^{(0)},q^{(0)}]$, this Hamiltonian does not depend on the Gribov phase factors and it contains the perturbation series in terms of only the zero-map fields (i. e., in terms of constituent color particles) that can be identified with the Feynman partons. The Feynman path integral as the generation functional of this perturbation series is the analogue of the sum over all path of the free rotator (228).

Therefore, confinement in the spirit of the complete destructive interference of color amplitudes [16, 17, 71] and the law of probability conservation for S-matrix elements (230) leads to Feynman quark-hadron duality, that is foundation of all the parton model [84] and QCD application [34]. The quark-parton duality gives the method of direct experimental measurement of the quark and gluon quantum numbers from the deep inelastic scattering cross section [84]. For example, according to Particle Data Group, the ratio of the sum of the probabilities of τ-decay hadron modes to the probability of τ-decay muon mode is

$$\frac{\sum\limits_h w_{\tau\to h}}{w_{\tau\to\mu}} = 3.3 \pm 0.3.$$

This is the left-hand side of Eq. (230) normalized to the value of the lepton mode probability of τ decay. On the right-hand side of Eq. (230), we have the ratio of the imaginary part of the sum of quark-gluon diagrams (in terms of constituent fields free from the Gribov phase factors) to the one of the lepton diagram. In the lowest order of QCD perturbation on the right-hand side, we get the number of colors N_c and, therefore,

$$3.3 \pm 0.3 = N_c.$$

Thus in the constraint-shell QCD we can understand not only «why we do not see quarks», but also «why we can measure their quantum numbers». This mechanism of confinement due to the quantum interference of phase factors (revealed by the explicit resolving the Gauss law constraint [17, 19, 71]) disappears after the change of «physical» sources $A^*J^* \Rightarrow AJ$ that is called the transition to another gauge in the gauge-fixing method.

3.13. $U(1)$**-Problem.** We have seen that the bilocal linearization of the four fermion interaction leads to an effective bilocal field η_M action [33, 54] in both QED (111) and QCD.

This meson action includes Abelian anomalies in the pseudoscalar isosinglet channel [59, 89–91] (positronium $\eta_M = \eta_P$, in QED; and $\eta_M = \eta_0$ meson, in QCD).

We have chosen the total-motion variable $\eta_M(t)$ so that the effective action for the total motion of the pseudoscalar bound state with anomaly term has the universal form for all gauge theories ($\mathrm{QED}_{(3+1)}$, $\mathrm{QED}_{(1+1)}$, $\mathrm{QCD}_{(3+1)}$) in terms of physical variables similar to (111)

$$W_{\text{eff}} = \int dt \left\{ \frac{1}{2} \left(\dot{\eta}_M^2 - M_P^2 \eta_M{}^2 \right) V + C_M \eta_P \dot{X}[A^{(N)}] \right\}, \tag{231}$$

where in 3-dimensional $\mathrm{QED}_{(3+1)}$ the constant C_M is given by Eq. (112),

$$C_M = C_P = \frac{\sqrt{2}}{m_e} 8\pi^2 \left(\frac{\psi_{\text{Sch}}(0)}{m_e^{3/2}} \right);$$

in 1-dimensional $\mathrm{QED}_{(1+1)}$ [36], $C_M = 2\sqrt{\pi}$; and in 3-dimensional $\mathrm{QCD}_{(3+1)}$,

$$C_M = C_\eta = \frac{N_f}{F_\pi} \sqrt{\frac{2}{\pi}}, \quad (N_f = 3).$$

$X[A]$ is the «winding number» functional. In $\mathrm{QED}_{(3+1)}$ the «winding number» functional defined by Eq. (113) describes two γ decays of a positronium. In $\mathrm{QED}_{(1+1)}$ and $\mathrm{QCD}_{(3+1)}$ these winding number functionals

$$\begin{gathered} \dot{X}_{\text{QED}}(A^{(N)}) = \frac{e}{4\pi} \int\limits_{-V/2}^{V/2} dx F_{\mu\nu} \epsilon^{\mu\nu} = \dot{N}(t) \Rightarrow F_{01} = \frac{2\pi \dot{N}}{eV}, \\ \dot{X}_{\text{QCD}}[A^{(N)}] = \frac{g^2}{16\pi^2} \int d^3x G^a_{\mu\nu}{}^* G^a_{\mu\nu} = \dot{N}(t) + \dot{X}[A^{(0)}] \end{gathered} \tag{232}$$

contain the independent topological variable. Recall that in QCD with the Wu–Yang monopole, we obtained the normalizable zero mode

$$G^a_{0i} = \dot{N} D^{ab}_i(\Phi) \Phi^b_0 = \dot{N} B^a_i(\Phi) \frac{2\pi}{\alpha_s V \langle B^2 \rangle},$$

so that

$$\frac{g^2}{8\pi^2} \int d^3 D^{ab}_i(\Phi) \Phi^b_0 B^a_i(\Phi) = 1.$$

In $\mathrm{QED}_{(1+1)}$ and $\mathrm{QCD}_{(3+1)}$ the effective action should be added by the topological dynamics of the zero mode with the actions

$$W_{\text{QED}} = \frac{1}{2} \int dt \int\limits_{-V/2}^{V/2} dx F_{01}^2 = \int dt \frac{\dot{N}^2 I_{\text{QED}}}{2},$$

$$W_{\text{QCD}} = \frac{1}{2} \int dt \int\limits_V d^3x G_{0i}^2 = \int dt \frac{\dot{N}^2 I_{\text{QCD}}}{2},$$

where

$$I_{\rm QED} = \left(\frac{2\pi}{e}\right)^2 \frac{1}{V},$$

$$I_{\rm QCD} = \left(\frac{2\pi}{\alpha_s}\right)^2 \frac{1}{V\langle B^2\rangle}.$$

It is easy to show that the diagonalization of the total Lagrangian of the type of

$$L = \left[\frac{\dot{N}^2 I}{2} + C_M \eta_M \dot{N}\right] = \left[\frac{(\dot{N} + C_M\eta_M/I)^2 I}{2} - \frac{C_M^2}{2IV}\eta_M^2 V\right]$$

leads to additional mass of the pseudoscalar bound state in both $\mathrm{QED}_{(1+1)}$ and $\mathrm{QCD}_{(3+1)}$

$$\triangle M^2 = \frac{C_M^2}{IV}.$$

In $\mathrm{QED}_{(1+1)}$ this formula describes the well-known Schwinger mass

$$\triangle M^2 = \frac{C_M^2}{I_{\rm QED}V} = \frac{e^2}{\pi},$$

whereas in $\mathrm{QCD}_{(3+1)}$ we obtain the additional mass of η_0 meson

$$L_{\rm eff} = \frac{1}{2}[\dot{\eta}_0^2 - \eta_0^2(t)(m_0^2 + \triangle m_\eta^2)]V, \tag{233}$$

$$\triangle m_\eta{}^2 = \frac{C_\eta^2}{I_{\rm QCD}V} = \frac{N_f^2}{F_\pi^2}\frac{\alpha_s^2\langle B^2\rangle}{2\pi^2}. \tag{234}$$

This result allows us to estimate the value of the vacuum chromomagnetic field in QCD

$$\langle B^2\rangle = \frac{2\pi^3 F_\pi^2 \triangle m_\eta{}^2}{N_f^2\alpha_s^2} = \frac{0.06\ \mathrm{GeV}^4}{\alpha_s^2}.$$

After calculation we can remove infrared regularization $V \to \infty$.

Thus, the Dirac constraint-shell formulation of gauge theories [1] allows us to describe on equal footing the set of well-known results on anomalous interactions of pseudoscalar bound states in gauge theories including the anomalous decay of a positronium in $\mathrm{QED}_{(D=1+3)}$, the Schwinger mass in $\mathrm{QED}_{(D=1+1)}$, and $U(1)$-problem in $\mathrm{QCD}_{(D=1+3)}$. These results include also the zero probability of two gluon decays of a pseudoscalar meson due to confinement as the destructive interference of the Gribov copies for gauge theories with the homotopy group $\pi_{(D-1)}(G) = Z$.

CONCLUSION

Why does the non-Abelian gauge theory lead to the spontaneous chiral symmetry breaking, quark-hadron duality, rising potential, and color confinement?

The formulation of the theory of strong interactions is completed if the theory can explain and unify all working phenomenological schemes (chiral Lagrangians with the Goldstone mesons, the parton model based of quark-parton duality, the Schrödinger equation with rising potentials for heavy quarkonia, etc.) from the first principles (dynamic, quantum, and symmetric).

The pure water of the theory of strong interactions can clean the muddy swamp of phenomenology if there are constructive explanations of the paradoxes:

i) How to combine the observation of nonlocal gauge-invariant variables with the variational principles formulated for local fields?

ii) How to combine the dependence of the Hamiltonian approach to quantization on the time axis with the relativistic covariance of S-matrix elements?

iii) How to reconcile the presence of poles of color particle Green functions (that needs to ground the parton model and to obtain the Shrödinger equation) with nonobservability of physical states corresponding to these poles?

The first two questions belong also to the consistent scheme of the description of the nonlocal bound-state sector (spectrum and interactions) in QED. Nevertheless, these questions become essentially actual in QCD where observables are only colorless bound states. In this case, the theorem of equivalence of different gauges [6, 73, 74] on the mass-shell of elementary particles is not adequate to the physical situation of the description of observable nonlocal objects where local elementary particles are off mass-shell. For nonlocal bound states, even in QED, the dependence on the time-axis and gauge exists. All peculiarities of bound states (including time initial data, spatial boundary conditions, normalization of wave functions, time evolution) reflect the choice of their rest frame of reference distinguished by the axis of time chosen to lie along the total momentum of any bound state in order to obtain the relativistic-covariant dispersion law and invariant mass spectrum.

Thus, the answers to the first two questions (i, ii) are the *equivalent unconstrained system* obtained by resolving the Gauss law constraint and the Markov–Yukawa prescription of the choice of the time axis parallel to the eigenvector of the bound-state total momentum operator. In this case, we can combine the relativistic and quantum principles of the dynamic description of gauge fields.

The last question iii) belongs only to QCD. The answer is the topological degeneration of initial data in the non-Abelian theory that leads to the pure gauge Higgs effect of appearance of a physical vacuum in the form of the Wu–Yang monopole. This vacuum determines the effects of hadronization and confinement in QCD omitted by the conventional gauge-fixing method.

We brought out that the physical vacuum leads to rising potentials, chiral Lagrangians, and an additional mass of the η_0 meson (as a consequence of the rearrangement of the perturbation series). The averaging over the Gribov copies of the topological degeneration leads to the color confinement and quark-hadron duality as a consequence of the destructive interference of the Gribov phase factors. This confinement is one of the consequences of the Dirac definition of measurable variables that keeps quantum principles. This confinement (omitted by all other methods of quantization) appears as pure quantum effect that gives the theoretical basis of the quark-hadron duality as the experimental method of measurement of the quark-gluon quantum numbers.

Here we can recall the words by J. C. Maxwell in the Introduction of his *A Treatise on Electricity and Magnetism* (Oxfrord, 1873): «The most important aspect of any phenomenon from mathematical point of view is that of a measurable quantity. I shall therefore consider electrical phenomena chiefly with a view to their measurement, describing the methods of measurement, and defining the standards on which they depend».

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