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GAUGE-INVARIANT FIELD VARIABLES AND THE ROLE OF THE LORENTZ GAUGE CONDITIONS

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We study two classes of gauge-invariant fields introduced by Fock and Dirac. It is shown that such field variables satisfy the Lorentz gauge condition as a secondary constraint according to the Dirac classification. For the field in the axial gauge considered as a primary constraint a simple inversion formula is derived, which expresses that field through the strength tensor and is analogous to the formula that occurs in the Fock gauge.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Калибровочно-инвариантные полевые переменные и роль условий Лоренца

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Изучаются два класса калибровочно-инвариантных полей, введенных в рассмотрение Фоком и Дираком. Показано, что такие полевые переменные удовлетворяют калибровочному условию Лоренца в качестве вторичной связи, согласно классификации Дирака. Для поля, на которое наложено в качестве первичной связи условие аксиальной калибровки, найдена простая формула обращения, выражающая его через тензор напряженности, аналогичная формуле, которая имеет место в калибровке Фока.

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There are two ways of introducing of fields that with the property to be invariant under the local gauge transformations

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \lambda(x); \quad \psi(x) \to e^{ig\lambda(x)} \psi(x). \qquad (1)$$

The first class of gauge-invariant fields is defined with the help of a specified gauge transformation under the primary electromagnetic field $A_\mu(x)$ with the gauge parameter $\lambda(x)$ taken in the form (that was introduced for the first time by V.A.Fock [1]):

$$\lambda(x) = \Lambda(x \mid \xi) = -\int_{\xi}^{x} dz^{\mu} A_{\mu}(z), \tag{2}$$

so that

$$A_{\mu}(x) \to B_{\mu}(x \mid \xi) = A_{\mu}(x) + \partial_{\mu}\Lambda(x \mid \xi),$$
$$\psi(x) \to \Psi(x \mid \xi) = e^{ig\Lambda(x \mid \xi)}\psi(x). \tag{3}$$

Here ξ is an arbitrary fixed point from the Minkowski space. Field variables (3) were studied in papers[2-6]. For the contour of integration in (2), according to[1], a piece of a straight line is taken: $z^{\mu} = \xi^{\mu} + t(x-\xi)^{\mu}$, $0 \leq t \leq 1$. The field variables $B_{\mu}(x|\xi)$ and $\Psi(x|\xi)$ that are introduced by formulae (2) and (3) are gauge invariant under the local gauge transformations (1) performed for the primary fields $A_{\mu}(x)$ and $\psi(x)$. The field $B_{\mu}(x|\xi)$, that is gauge invariant under the gauge transformations (1) for the field $A_{\mu}(x)$, satisfies the Fock gauge condition*[1]

$$(x - \xi)^{\mu} B_{\mu}(x \mid \xi) = 0. \tag{4}$$

(That is why we shall call these fields $B_{\mu}(x|\xi)$, $\Psi(x|\xi)$, and $\Psi(x|\xi)$ as fields of the Fock class). The field $B_{\mu}(x|\xi)$ is none other than the field $A_{\mu}(x)$, taken in the Fock gauge: $B_{\mu}(x|\xi) = A^{F}_{\mu}(x|\xi)$. It satisfies the well-known inversion formula[1]

$$B_{\mu}(x \mid \xi) = -\int_{0}^{1} dt\, t\, (x - \xi)^{\nu} F_{\mu\nu}(\xi + t(x - \xi)), \tag{5}$$

that expresses the field through the strength tensor $F_{\mu\nu}(x) = \partial_{\nu}A_{\mu}(x) - \partial_{\mu}A_{\nu}(x)$ being a gauge-invariant quantity.

The second class of gauge-invariant field variables was introduced by Dirac[7] and is defined as follows (for details see ref.[9])

$$B_{\mu}(x \mid f) = A_{\mu}(x) - \int dy \frac{\partial}{\partial x_{\mu}} f^{\nu}(x - y) A_{\nu}(y), \tag{6}$$

$$\Psi(x \mid f) = e^{-ig\int dy\, f^{\nu}(x-y) A_{\nu}(y)} \psi(x), \tag{7}$$

where $f^{\nu}(x-y)$ is a real function that obeys the condition $\partial^{\mu} f_{\mu}(z) = \delta(z)$. The formula (6) can be written in the mo-

*This gauge was rediscovered later by a number of authors[2-5].

mentum space as follows:

$$B_\mu(p|f) = A_\mu(p) - ip_\mu f^\nu(p) A_\nu(p), \tag{8}$$

with the conditions

$$ip_\nu f^\nu(p) = 1; \quad f^*_\nu(p) = f_\nu(-p). \tag{9}$$

It is easy to check with the help of (9) that the gauge-invariant field (8) can be expressed analogously to (5) through the strength tensor:

$$B_\mu(p|f) = f^\nu(p) F_{\nu\mu}(p). \tag{10}$$

It is clear from formula (1) that on the Maxwell equations

$$p^2 A_\mu(p) - p_\mu p^\nu A_\nu(p) = (\equiv \frac{1}{i} p^\nu F_{\nu\mu}(p)) = 0$$

the field $B_\mu(p|f)$ satisfies the Lorentz gauge condition

$$p^\mu B_\mu(p|f) = 0. \tag{11}$$

Dirac has defined the so-called second class of constraints as the constraints that are obtained with the help of Lagrange equations of motion (in our case with the help of Maxwell equations) and the constraints of the first class (that is equivalent in our case to the use of restriction conditions (9) on the functions $f_\nu(p)$). From this definition we conclude that in the class of gauge-invariant Dirac fields defined by formulae (6)-(9) the Lorentz gauge condition (11) appears as the constraint of the second class. It is evident that the field (8) satisfies the Maxwell equation.

As has been shown in [9], the solution of relations (9) that meets the requirement of symmetric description of particles and antiparticles has the form

$$f^\nu(p) = -i \,\mathrm{P.V.} \frac{n^\nu}{(np)}, \tag{12}$$

where n_ν is an arbitrary 4-vector. With the help of (8) it is easy to find that the gauge-invariant field $B_\mu(p|f)$ coincides with the primary field $A_\mu(p)$ in the gauge

$$f^\nu(p) A_\nu(p) = 0, \tag{13}$$

that with the choice of $f_\nu(p)$ in the form of (12) transforms into the condition of axial gauge $n^\mu A_\mu = 0$ for the field $A_\mu(p)$. From here and from (10) it follows that in the axial gauge for the field $A_\mu^{ax}(p)$, as a particular case of formula (10), there exists the expression for $A_\mu^{ax}(p)$ through a gauge-invariant strength tensor $F_{\nu\mu}(p)$

$$A_\mu^{ax}(p) = iF_{\nu\mu}(p)\frac{n^\nu}{(np)} \tag{14}$$

(the principal value for the denominator in (14) is supposed). Let us mention that previously the inversion formula was established only for the field taken in the Fock gauge (see formula (5)). The inversion formula (14) that we have found for the field $A_\mu(p)$ in the axial gauge and formula (10) for the whole Dirac class of gauge invariant fields were never met in the literature so we consider their derivation as a new result.

As a particular case of formula (11) for the electromagnetic field that obeys the condition of axial gauge as the constraint of the first class we obtain the Lorentz gauge condition

$$\partial^\mu A_\mu^{ax}(x) = 0, \tag{15}$$

that appears as a constraint of the second class.

Let us return to the Fock field $B_\mu(x|\xi)$ (2). Using (1) with $z^\mu = \xi^\mu + t(x-\xi)^\mu$ we get

$$\Box_x \Lambda(x|\xi) = -\int_0^1 dt[(x-\xi)^\mu A_\mu(\xi + t(x-\xi)) - 2\frac{\partial}{\partial x_\nu}A_\nu(\xi+t(x-\xi))]. \tag{16}$$

Making use of the Maxwell equation $\Box A_\mu + \partial_\mu \partial^\nu A_\nu = 0$ we find

$$\Box_x \Lambda(x|\xi) = -\int_0^1 dt[t^2\frac{d}{dt}(\frac{\partial}{\partial z_\mu}A(z)) + 2t\frac{\partial}{\partial z_\mu}A(z)],$$
$$z^\mu = \xi^\mu + t(x-\xi)^\mu. \tag{17}$$

Then integrating (17) by parts we finally obtain

$$\Box_x \Lambda(x|\xi) = -\partial^\mu A_\mu(x). \tag{18}$$

Taking 4-divergence of (2) and allowing for (18) we get for the gauge-invariant Fock field $B_\mu(x|\xi)$ the Lorentz condition*

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*As has been mentioned before, the field $B_\mu(x|\xi)$ coincides with the field $A_\mu(x)$ in the Fock gauge, for which the validity of the Lorentz condition, as we have got informed, has been established independently by V.N.Kapshay.

$$\partial^{\mu} B_{\mu}(x | \xi) = 0, \tag{19}$$

which is also obtained with the use of the equations of motion (Maxwell equations) and with the gauge parameter $\lambda(x)$ taken in the form (2). Thus, the condition (19) also appears to be the secondary constraint.

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