УДК 539.182

ON THE HARMONIC OSCILLATOR REALISATION OF q-OSCILLATORS

D.Gangopadhyay, A.P.Isaev

The general version of the bosonic harmonic oscillator realisation of bosonic q-oscillators is given. It is shown that the currently known realisation is a special case of our general solution.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

О реализации q-осцилляторов гармоническими осцилляторами

Д.Гангопадья, А.П.Исаев

Получено общее представление для бозонных q-осцилляторов в терминах обычных бозонных осцилляторов. Показано, что известное до сих пор представление получается как частный случай из нашего общего решения.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Recently, there has been much interest in quantum Lie algebras which first appeared in the investigations of the quantum inverse scattering problem while studying the Yang — Baxter equations [1]. These quantum algebras can be considered as a "deformation" of the Lie algebra with the numerical deformation parameter s or $q = e^s$, such that the usual Lie algebra is reproduced in the limit $s \to 0$, i.e. $q \to 1$. It has been shown that this structure essentially connects with quasi-triangular Hopf algebras and its generalisation to all simple Lie algebras has been given [2]. There also exists the quantum generalisation of the Jordan-Schwinger mapping for $su(2)_q$ algebra [3]. Moreover a q-oscillator realisation of many other quantum algebras has also been obtained [4,5]. In ref.4 a harmonic oscillator representation of the q-oscillators was also given. The motive of this paper is to show that the harmonic oscillator realisation of the q-oscillators admits a more general solution than the one currently in vogue [4].

The basic equations characterising the q-deformed bosonic oscillator system are

$$aa^+ - q a^+ a = q^{-N}, \quad N^+ = N, \tag{1}$$

$$[N, a] = -a, \quad Na = a(N-1), \tag{2}$$

$$[N, a^+] = a^+, \quad Na^+ = a^+(N+1), \tag{3}$$

where a, a^+ are annihilation and creation operators and N is the number operator.

Consider the case when q is complex. Then (1) implies

$$aa^+ - q^* a^+ a = (q^*)^{-N}. \tag{4}$$

So from (1) and (2) we get

$$a^+ a = \frac{q^{-N} - (q^*)^{-N}}{q^* - q}. \tag{5}$$

Multiplying (5) by a and then commuting a to the right in the right-hand side term we obtain

$$aa^+ = \frac{q^{-N-1} - (q^*)^{-N-1}}{q^* - q}. \tag{6}$$

Substituting (5) and (6) in (1) then gives

$$q^{-N}(q^* - q^{-1}) = (q^*)^{-N}(q - (q^*)^{-1}). \tag{7}$$

Now taking $q = |q| e^{i\alpha}$, $q^* = |q| e^{-i\alpha}$ and putting these in (7) we have

$$e^{-i\alpha(N+1)}\left(|q| - \frac{1}{|q|}\right) = e^{i\alpha(N+1)}\left(|q| - \frac{1}{|q|}\right). \tag{8}$$

Equation (8) has two solutions

$$|q| = \frac{1}{|q|} \quad \text{i.e.} \quad |q| = 1 \tag{9a}$$

and

$$e^{-2i\alpha(N+1)} = 1 \qquad \text{i.e.} \quad \alpha = \frac{\pi}{N+1} m \tag{9b}$$

with m being some integer. The second solution is not appropriate for us as we consider q as a number and not as an operator. Let us take the first solution (9a) viz. $q = e^{i\alpha}$. Then eqs. (5) and (6) can be rewritten as

$$a^+a = [N], \quad aa^+ = [N+1], \tag{10}$$

where $[x] = (q^x - q^{-x})/(q - q^{-1})$. It is straightforward to verify that (10) is indeed a solution of (1) even if q is real.

Let us address ourselves to determining the representation of the operators a and a^+ in terms of ordinary oscillators $\hat{a}$, $\hat{a}^+$ described by

$$[\hat{a}, \hat{a}^+] = 1, \quad \hat{N} = \hat{a}^+\hat{a} = \hat{a}\hat{a}^+ - 1,$$
$$[\hat{N}, \hat{a}] = -\hat{a}, \quad [\hat{N}, \hat{a}^+] = \hat{a}^+. \tag{11}$$

where $\hat{N}$ is the usual number operator. We now find the solutions for a, a^+ and N satisfying equations (1), (2) and (3) together with

$$[\hat{N}, N] = 0, \; [\hat{N}, a] = -a, \; [\hat{N}, a^+] = a^+. \tag{11b}$$

From (11b) one immediately has

$$N = \Phi(q, \hat{N}), \quad a = \hat{a} f(q, \hat{N}) \tag{12a}$$

with Φ and f some arbitrary functions at this moment. Reality of f and (12a) then give

$$a^+ = f(q, \hat{N}) \hat{a}^+. \tag{12b}$$

Substituting (12) in (1) yields

$$f^2(q, \hat{N}+1)(\hat{N}+1) - q f^2(q, \hat{N}) \hat{N} = q^{-\Phi(q,\hat{N})} = q^{-N}. \tag{13}$$

With $q = e^s$ this means

$$\Phi(q, \hat{N}) = -\frac{1}{s} \ln[f^2(q, \hat{N}+1)(\hat{N}+1) - q f^2(q, \hat{N}) \hat{N}]. \tag{14}$$

Now from (2) and (3) we have

$$q^{-N} a = a q^{-N+1}, \tag{15a}$$

$$q^{-N} a^+ = a^+ q^{-N-1}. \tag{15b}$$

Putting equation (13) in (15a) results in the functional equation

$$F(q, \hat{N}) \left(\frac{1}{q} + q\right) - F(q, \hat{N}-1) - F(q, \hat{N}+1) = 0, \tag{16}$$

where $F(q, \hat{N}) = f^2(q, \hat{N})\hat{N}$. The same equation is also obtainable from (15b).

In order to solve eq.(16) for $F(q, \hat{N})$ note that

$$F(q, \hat{N}) \to \hat{N} \tag{17}$$

for $s \to 0$ or $q \to 1$. This is simply because $f(q, \hat{N}) \to 1 (a \to \hat{a})$ when $q \to 1$.

Hence, we have the following systems of equations:

$$(q + \frac{1}{q}) F(q, N) - F(q, N-1) - F(q, N+1) = 0, \tag{18a}$$

$$F(1, N) = N, \quad \Phi(1, \hat{N}) = \hat{N}, \tag{18b}$$

$$F(q, \hat{N}+1) - qF(q, \hat{N}) = q^{-\Phi(q, \hat{N})} = q^{-N}. \tag{18c}$$

The last of these equations is essentially equation (13). From (18c) we have

$$F(q, 1) = qF(q, 0) + q^{-\Phi(q, 0)}. \tag{19}$$

From (18a) and (19) we get

$$F(q, 2) = q^2 F(q, 0) + (q + q^{-1}) q^{-\Phi(q, 0)}.$$

A little algebra then leads to the general term

$$F(q, N) = q^N F(q, 0) + [N] q^{-\Phi(q, 0)}. \tag{20}$$

It is readily verified that (20) satisfies (18a). Hence (20) is the solution of (18a) for arbitrary $F(q, 0)$ and $\Phi(q, 0)$. Moreover, note that if $F \equiv F(q, N)$ is a solution of (18a), then $F \equiv \tilde{F}(q, -N)$ is also a solution.

It is by now obvious that we may write the general solution as

$$F(q, N) = \frac{q^N \Phi_1(q) - q^{-N} \Phi_2(q)}{q - q^{-1}}, \tag{21}$$

where $\Phi_{1,2}$ are arbitrary functions with the restriction that $\Phi_{1,2}(1) = 1$. Then, using $F(q, \hat{N}) = f^2(q, \hat{N})\hat{N}$ we arrive at

$$f(q, N) = \sqrt{\frac{q^{\hat{N}}\Phi_1 - q^{-\hat{N}}\Phi_2}{\hat{N}(q - q^{-1})}}$$

so that

$$a = \hat{a}\sqrt{\frac{(q^{\hat{N}}\Phi_1 - q^{-\hat{N}}\Phi_2)}{\hat{N}(q - q^{-1})}}, \quad a^+ = \sqrt{\frac{(q^{\hat{N}}\Phi_1 - q^{-\hat{N}}\Phi_2)}{\hat{N}(q - q^{-1})}}\,\hat{a}^+$$

$$N = \hat{N} - \frac{1}{S}\ln\Phi_2. \tag{22}$$

That the solutions (22) satisfy all the fundamental relations may be easily established. Choosing $\Phi_1 = \Phi_2 = 1$ gives the presently known realisation /4/.

Thus we prove that taking into account the additional conditions (11b), the representation (22) is the most general.

A similar analysis for fermionic q-oscillators leads to the known result $b = \hat{b}$, $b^+ = \hat{b}^+$, and $M = \hat{M}$ after imposing the requirement $M = M^2$ for the number operator.

It is our pleasure to thank A.T.Filippov and J.Lukierski for enlightening discussions.

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Received on August 27, 1990.