

## CONDITIONAL DENSITY MATRIX: SYSTEMS AND SUBSYSTEMS IN QUANTUM MECHANICS

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A new quantum mechanical notion — Conditional Density Matrix — proposed by the authors [5,6] is discussed and is applied to describe some physical processes. This notion is a natural generalization of von Neumann density matrix for such processes as divisions of quantum systems into subsystems and reunifications of subsystems into new joint systems. Conditional Density Matrix assigns a quantum state to a subsystem of a composite system on condition that another part of the composite system is in some pure state.

### INTRODUCTION

A problem of a correct quantum mechanical description of divisions of quantum systems into subsystems and reunifications of subsystems into new joint systems attracts a great interest due to the present development of quantum communication.

Although the theory of such processes finds room in the general scheme of quantum mechanics proposed by von Neumann in 1927 [1], even now these processes are often described in a fictitious manner. For example, the authors of classical photon teleportation experiment [2] write:

*The entangled state contains no information on the individual particles; it only indicates that two particles will be in the opposite states. The important property of an entangled pair is that as soon as a measurement on one particle projects it, say, onto  $|\leftrightarrow\rangle$  the state of the other one is determined to be  $|\updownarrow\rangle$ , and vice versa. How could a measurement on one of the particles instantaneously influence the state of the other particle, which can be arbitrary far away? Einstein, among many other distinguished physicists, could simply not accept this «spooky action at a distance». But this property of entangled states has been demonstrated by numerous experiments.*

### 1. THE GENERAL SCHEME OF QUANTUM MECHANICS

It was W. Heisenberg who in 1925 formulated a kinematic postulate of quantum mechanics [3]. He proposed that there exists a connection between matrices and physical variables:

$$\text{variable } \mathcal{F} \iff \text{matrix } (\hat{F})_{mn}.$$

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In the modern language the kinematic postulate looks like:

Each dynamical variable  $\mathcal{F}$  of a system  $\mathcal{S}$  corresponds to a linear operator  $\hat{F}$  in Hilbert space  $\mathcal{H}$ :

$$\text{dynamical variable } \mathcal{F} \iff \text{linear operator } \hat{F}.$$

The dynamics is given by the famous Heisenberg's equations formulated in terms of commutators,

$$\frac{d\hat{F}}{dt} = \frac{i}{\hbar} [\hat{H}, \hat{F}].$$

To compare predictions of the theory with experimental data, it was necessary to understand how one can determine the values of dynamical variables in the given state. W. Heisenberg gave a partial answer to this problem:

If matrix that corresponds to the dynamical variable is diagonal, then its diagonal elements define possible values for the dynamical variable, i. e., its spectrum,

$$(\hat{F})_{mn} = f_m \delta_{mn} \iff \{f_m\} \text{ is spectrum } \mathcal{F}.$$

The general solution of the problem was given by von Neumann in 1927. He proposed the following procedure for calculation of average values of physical variables:

$$\langle \mathcal{F} \rangle = \text{Tr} (\hat{F} \hat{\rho}).$$

Here, operator  $\hat{\rho}$  satisfies three conditions:

- 1)  $\hat{\rho}^\dagger = \hat{\rho}$ ,
- 2)  $\text{Tr} \hat{\rho} = 1$ ,
- 3)  $\forall \psi \in \mathcal{H} \langle \psi | \hat{\rho} \psi \rangle \geq 0$ .

By the formula for average values, von Neumann found out the correspondence between linear operators  $\hat{\rho}$  and states of quantum systems:

$$\text{state of a system } \rho \iff \text{linear operator } \hat{\rho}.$$

In this way, the formula for average values becomes quantum mechanical definition of the notion «a state of a system». The operator  $\hat{\rho}$  is called Density Matrix.

From the relation

$$(\langle \mathcal{F} \rangle)^* = \text{Tr} (\hat{F}^\dagger \hat{\rho})$$

one can conclude that Hermitian-conjugate operators correspond to complex-conjugate variables, and Hermitian operators correspond to real variables,

$$\begin{aligned} \mathcal{F} \leftrightarrow \hat{F} &\iff \mathcal{F}^* \leftrightarrow \hat{F}^\dagger, \\ \mathcal{F} = \mathcal{F}^* &\iff \hat{F} = \hat{F}^\dagger. \end{aligned}$$

The real variables are called *observables*.

From the properties of density matrix and the definition of positively definite operators

$$\hat{F}^+ = \hat{F}, \quad \forall \psi \in \mathcal{H}, \langle \psi | \hat{F} \psi \rangle \geq 0,$$

it follows that the average value of nonnegative variable is nonnegative. Moreover, the average value of nonnegative variable is equal to zero if and only if this variable equals zero. Now it is easy to give the following definition: *variable  $\mathcal{F}$  has a definite value in the state  $\rho$  if and only if its dispersion in the state  $\rho$  is equal to zero.*

In accordance to general definition of the dispersion of an arbitrary variable

$$\mathcal{D}(A) = \langle A^2 \rangle - (\langle A \rangle)^2,$$

the expression for dispersion of a quantum variable  $\mathcal{F}$  in the state  $\rho$  has the form

$$\mathcal{D}_\rho(\mathcal{F}) = \text{Tr}(\hat{Q}^2 \hat{\rho}),$$

where  $\hat{Q}$  is an operator:

$$\hat{Q} = \hat{F} - \langle \mathcal{F} \rangle \hat{E}.$$

If  $\mathcal{F}$  is observable then  $\hat{Q}^2$  is a positive-definite variable. It follows that the dispersion of  $\mathcal{F}$  is nonnegative. And all this makes clear the above-given definition.

Since density matrix is a positive definite operator and its trace equals 1, we see that its spectrum is pure discrete, and it can be written in the form

$$\hat{\rho} = \sum_n p_n \hat{P}_n,$$

where  $\hat{P}_n$  is a complete set of self-conjugate projective operators:

$$\hat{P}_n^+ = \hat{P}_n, \quad \hat{P}_m \hat{P}_n = \delta_{mn} \hat{P}_m, \quad \sum_n \hat{P}_n = \hat{E}.$$

Numbers  $\{p_n\}$  satisfy the condition

$$p_n^* = p_n, \quad 0 \leq p_n, \quad \sum_n p_n \text{Tr} \hat{P}_n = 1.$$

It follows that  $\hat{\rho}$  acts according to the formula

$$\hat{\rho} \Psi = \sum_n p_n \sum_{\alpha \in \Delta_n} \phi_{n\alpha} \langle \phi_{n\alpha} | \Psi \rangle.$$

The vectors  $\phi_{n\alpha}$  form an orthonormal basis in the space  $\mathcal{H}$ . Sets  $\Delta_n = \{1, \dots, k_n\}$  are defined by degeneration multiplicities  $k_n$  of eigenvalues  $p_n$ .

Now the dispersion of the observable  $\mathcal{F}$  in the state  $\rho$  is given by the equation

$$\mathcal{D}_\rho(\mathcal{F}) = \sum_n p_n \sum_{\alpha \in \Delta_n} \|\hat{Q} \phi_{n\alpha}\|^2.$$

All terms in this sum are nonnegative. Hence, if the dispersion is equal to zero, then

$$\text{if } p_n \neq 0, \text{ then } \hat{Q}\phi_{n\alpha} = 0.$$

Using the definition of the operator  $\hat{Q}$ , we obtain

$$\text{if } p_n \neq 0, \text{ then } \hat{F}\phi_{n\alpha} = \phi_{n\alpha}\langle F \rangle.$$

In other words, *if an observable  $\mathcal{F}$  has a definite value in the given state  $\rho$ , then this value is equal to one of the eigenvalues of the operator  $\hat{F}$ .*

In this case, we have

$$\begin{aligned} \hat{\rho}\hat{F}\phi_{n\alpha} &= \phi_{n\alpha}p_n\langle \mathcal{F} \rangle, \\ \hat{F}\hat{\rho}\phi_{n\alpha} &= \phi_{n\alpha}\langle \mathcal{F} \rangle p_n, \end{aligned}$$

which proves the commutativity of operators  $\hat{F}$  and  $\hat{\rho}$ .

It is well known that, if  $\hat{A}$  and  $\hat{B}$  are commutative self-conjugate operators, then there exists self-conjugate operator  $\hat{T}$  with nondegenerate spectrum such that  $\hat{A}$  and  $\hat{B}$  are functions of  $\hat{T}$ :

$$\begin{aligned} \hat{T}\Psi &= \sum_{n\alpha} \phi_{n\alpha} t_{n\alpha} \langle \phi_{n\alpha} | \Psi \rangle, \\ t_{n\alpha}^* &= t_{n\alpha}, \quad t_{n\alpha} \neq t_{n'\alpha'}, \quad \text{if } (n, \alpha) \neq (n', \alpha'). \\ \hat{F}\Psi &= \sum_{n\alpha} \phi_{n\alpha} f_1(t_{n\alpha}) \langle \phi_{n\alpha} | \Psi \rangle, \\ \hat{\rho}\Psi &= \sum_{n\alpha} \phi_{n\alpha} f_2(t_{n\alpha}) \langle \phi_{n\alpha} | \Psi \rangle. \end{aligned}$$

Suppose that  $\hat{F}$  is an operator with nondegenerate spectrum; then *if the observable  $\mathcal{F}$  with nondegenerate spectrum has a definite value in the state  $\rho$ , then it is possible to represent the density matrix of this state as a function of the operator  $\hat{F}$ .*

The operator  $\hat{F}$  can be written in the form

$$\begin{aligned} \hat{F} &= \sum_n f_n \hat{P}_n, \\ \hat{P}_n^+ &= \hat{P}_n, \quad \hat{P}_m \hat{P}_n = \delta_{mn} \hat{P}_m, \quad \text{tr}(\hat{P}_n) = 1, \quad \sum_n \hat{P}_n = \hat{E}. \end{aligned}$$

The numbers  $\{f_n\}$  satisfy the conditions

$$f_n^* = f_n, \quad f_n \neq f_{n'}, \quad \text{if } n \neq n'.$$

We obviously have

$$\hat{F} = \sum_n f_n \hat{P}_n.$$

From

$$\begin{aligned}\langle F \rangle &= \sum_n p_n f_n = f_N, \\ \langle F^2 \rangle &= \sum_n p_n f_n^2 = f_N^2,\end{aligned}$$

we get

$$p_n = \delta_{nN}.$$

In this case, density matrix is a projective operator satisfying the condition

$$\hat{\rho}^2 = \hat{\rho}.$$

It acts as

$$\hat{\rho}\Psi = \Psi_N \langle \Psi_N | \Psi \rangle,$$

where  $|\Psi\rangle$  is a vector in Hilbert space.

The average value of an arbitrary variable in this state is equal to

$$\langle \mathcal{A} \rangle = \langle \Psi_N | \hat{A} \Psi_N \rangle.$$

It is so-called *pure* state. If the state is not pure, it is known as *mixed*.

Suppose that every vector in  $\mathcal{H}$  is a square integrable function  $\Psi(x)$ , where  $x$  is a set of continuous and discrete variables. Scalar product is defined by the formula

$$\langle \Psi | \Phi \rangle = \int dx \Psi^*(x) \Phi(x).$$

For simplicity, we assume that every operator  $\hat{F}$  in  $\mathcal{H}$  acts as follows:

$$(\hat{F}\Psi)(x) = \int F(x, x') dx' \Psi(x').$$

That is, for any operator  $\hat{F}$  there is an integral kernel  $F(x, x')$  associated with this operator:

$$\hat{F} \iff F(x, x').$$

Certainly, we may use  $\delta$  function if necessary.

Now the average value of the variable  $\mathcal{F}$  in the state  $\rho$  is given by equation

$$\langle \mathcal{F} \rangle_\rho = \int F(x, x') dx' \rho(x', x) dx.$$

Here, the kernel  $\rho(x, x')$  satisfies the conditions

$$\begin{aligned}\rho^*(x, x') &= \rho(x', x), \\ \int \rho(x, x) dx &= 1, \\ \forall \Psi \in \mathcal{H} \int \Psi(x) dx \rho(x, x') dx' \Psi(x') &\geq 0.\end{aligned}$$

## 2. COMPOSITE SYSTEM AND REDUCED DENSITY MATRIX

Suppose the variables  $x$  are divided into two parts:  $x = \{y, z\}$ . Suppose also that the space  $\mathcal{H}$  is a direct product of two spaces  $\mathcal{H}_1, \mathcal{H}_2$ :

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2.$$

So, there is a basis in the space that can be written in the form

$$\phi_{an}(y, z) = f_a(y)v_n(z).$$

The kernel of operator  $\hat{F}$  in this basis looks like

$$\hat{F} \iff F(y, z; y', z').$$

In quantum mechanics it means that the system  $S$  is a unification of two subsystems  $S_1$  and  $S_2$ :

$$S = S_1 \cup S_2.$$

The Hilbert space  $\mathcal{H}$  corresponds to the system  $S$  and the spaces  $\mathcal{H}_1$  and  $\mathcal{H}_2$  correspond to the subsystems  $S_1$  and  $S_2$ .

Now suppose that a physical variable  $\mathcal{F}_1$  depends on variables  $y$  only. The operator that corresponds to  $\mathcal{F}_1$  has a kernel

$$F_1(y, z; y', z') = F_1(y, y')\delta(z - z').$$

The average value of  $F_1$  in the state  $\rho$  is equal to

$$\langle F_1 \rangle_\rho = \int F(y, y') dy' \rho_1(y', y) dy,$$

where the kernel  $\rho_1$  is defined by the formula

$$\rho_1(y, y') = \int \rho(y, z; y', z) dz.$$

The operator  $\hat{\rho}_1$  satisfies all the properties of Density Matrix in  $S_1$ . Indeed, we have

$$\begin{aligned} \rho_1^*(y, y') &= \rho_1(y', y), \\ \int \rho_1(y, y) dy &= 1, \\ \forall \Psi_1 \in \mathcal{H}_\infty \int \Psi_1(y) dy \rho_1(y, y') dy' \Psi_1(y') &\geq 0. \end{aligned}$$

The operator

$$\hat{\rho}_1 = \text{Tr}_2 \hat{\rho}_{1+2}$$

is called Reduced Density Matrix. Thus, the state of the subsystem  $S_1$  is defined by reduced density matrix.

The reduced density matrix for the subsystem  $S_2$  is defined analogously:

$$\hat{\rho}_2 = \text{Tr}_1 \hat{\rho}_{1+2}.$$

Quantum states  $\rho_1$  and  $\rho_2$  of subsystems are defined uniquely by the state  $\rho_{1+2}$  of the composite system.

Suppose the system  $S$  is in a pure state, then a quantum state of the subsystem  $S_1$  is defined by the kernel

$$\rho_1(y, y') = \int \Psi(y, z) dz \Psi^*(y', z).$$

If the function  $\Psi(y, z)$  is the product

$$\Psi(y, z) = f(y)w(z), \quad \int |w(z)|^2 dz = 1,$$

then subsystem  $S_1$  is a pure state, too:

$$\rho_1(y, y') = f(y)f^*(y'), \quad \int |f(y)|^2 dy = 1.$$

As was proved by von Neumann, it is the only case when purity of composite system is inherited by its subsystems.

Let us consider an example of a system in a pure state having subsystems in mixed states. Let the wave function of composite system be

$$\Psi(y, z) = \frac{1}{\sqrt{2}}(f(y)w(z) \pm f(z)w(y)),$$

where  $\langle f|w \rangle = 0$  and  $\langle f|f \rangle = \langle w|w \rangle = 1$ . The density matrix of the subsystem  $S_1$  has the kernel

$$\rho_1(y, y') = \frac{1}{2}(f(y)f^*(y') + w(y)w^*(y')).$$

The kernel of the operator  $\rho_1^2$  has the form

$$\rho_1^2(y, y') = \frac{1}{4}(f(y)f^*(y') + w(y)w^*(y')).$$

Therefore, the subsystem  $S_1$  is in the mixed state. Moreover, its density matrix is proportional to unity operator. The previous property resolves the perplexities connected with Einstein–Podolsky–Rosen paradox.

### 3. EPR PARADOX

Anyway, it was Schrödinger who introduced a term «EPR paradox». The authors of EPR themselves always considered their article as a demonstration of inconsistency of present quantum mechanics rather than a particular curiosity.

The main conclusion of the paper «Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?» [4] published in 1935 (eight years after von Neumann's book) is the statement:

*...We proved that (1) the quantum-mechanical description of reality given by wave functions is not complete or (2) when the operators corresponding to two physical quantities do not commute, the two quantities cannot have simultaneous reality. Starting then with the assumption that the wave function does give a complete description of the physical reality, we arrived at the conclusion that two physical quantities, with noncommuting operators, can have simultaneous reality. Thus the negation of (1) leads to negation of only other alternative (2). We can thus focus to conclude that the quantum-mechanical description of physical reality given by wave function is not complete.*

After von Neumann's works this statement appears obvious. However, in order to clarify this point of view completely, we must understand what is «the physical reality» in EPR. In EPR paper the physical reality is defined as:

*If, without in any way disturbing a system, we can predict with certainty (i. e., with probability equal to unity) the value of physical quantity, then there exists an element of physical reality corresponding to this physical quantity.*

Such a definition of physical reality is a step back as compared to von Neumann's definition. By EPR definition, the state is actual only when at least one observable has an exact value. This point of view is incomplete and leads to inconsistency.

When a subsystem is separated, «the loss of observables» results directly from the definition of density matrix for the subsystem. «The occurrence» of observables in the chosen subsystem when the quantities are measured in another «subsidiary» subsystem can be naturally explained in the terms of conditional density matrix.

#### 4. CONDITIONAL DENSITY MATRIX

The average value of a variable with the kernel

$$F^c(x, x') = F_1(y, y')u(z)u^*(z'), \quad \int |u(z)|^2 dz = 1$$

is equal to

$$\langle F^c \rangle_\rho = p \int F_1(y, y') dy' \rho^c(y', y) dy,$$

where

$$\begin{aligned} \rho^c(y, y') &= \frac{1}{p} \int u^*(z) dz \rho(y, z; y', z') u(z') dz', \\ p &= \int u^*(z) dz \rho(y, z; y, z') u(z') dz' dy. \end{aligned}$$

Since we can represent  $p$  in the form

$$\begin{aligned} p &= \int P(z, z') dz' \rho_2(z'; z) dz, \\ P(z, z') &= u(z)u^*(z'), \end{aligned}$$

we see that  $p$  is an average value of a variable  $P$  of the subsystem  $S_2$ . Operator  $\hat{P}$  is a projector ( $\hat{P}^2 = \hat{P}$ ). Therefore, it is possible to consider the value  $p$  as a probability.

It is easy to demonstrate that the operator  $\hat{\rho}^c$  satisfies all the properties of density matrix. So the kernel  $\rho^c(y, y')$  defines some state of the subsystem  $S_1$ . What is this state?

According to the decomposition of  $\delta$  function

$$\delta(z - z') = \sum_n \phi_n(z) \phi_n^*(z'),$$

$\{\phi_n(z)\}$  being a basis in the space  $H_2$ , the reduced density matrix is represented in the form of the sum

$$\rho_1(y, y') = \sum p_n \rho_n^c(y, y').$$

Here,

$$\rho_n^c(y, y') = \frac{1}{p_n} \int \phi_n^*(z) dz \rho(y, z; y', z') \phi_n(z') dz',$$

and

$$\begin{aligned} p_n &= \int \phi_n^*(z) dz \rho(y, z; y, z') \phi_n(z') dz' dy, \\ &= \int \hat{P}_n(z, z') dz' \rho_2(z', z) dz. \end{aligned}$$

The numbers  $p_n$  satisfy the conditions

$$p_n^* = p_n, \quad p_n \geq 0, \quad \sum_n p_n = 1$$

and are connected with a probability distribution.

The basis  $\{\phi_n\}$  in the space  $H_2$  corresponds to some observable  $\hat{G}_2$  of the subsystem  $S_2$  with discrete nondegenerate spectrum. It is determined by the kernel

$$G_2(z, z') = \sum_n g_n \phi_n(z) \phi_n^*(z'), \quad g_n = g_n^*, \quad g_n \neq g_{n1} \quad \text{if} \quad n \neq n1.$$

The average value of  $G_2$  in the state  $\rho_2$  is equal to

$$\int dy \rho_2(z, z') dz' G(z', z) = \sum_n g_n \int dy \rho_2(z, z') dz' \phi_n(z') \phi_n^*(z') = \sum_n p_n g_n.$$

Thus, number  $p_n$  defines the chance that the observable  $\hat{G}_2$  has the value  $g_n$  in the state  $\rho_2$ . Obviously, the kernel  $\rho_n^c(y, y')$  in this case defines the state of system  $S_1$  on condition that the value of variable  $G_2$  is equal to  $g_n$ . Hence, it is natural to call operator  $\hat{\rho}_n^c$  Conditional Density Matrix (CDM) [5, 6]:

$$\hat{\rho}_{c1|2} = \frac{\text{Tr}_2(\hat{P}_2 \hat{\rho})}{\text{Tr}(\hat{P}_2 \hat{\rho})}.$$

It is (*conditional*) density matrix for the subsystem  $S_1$  on condition that the subsystem  $S_2$  is selected in a pure state  $\hat{\rho}_2 = \hat{P}_2$ . It is the most important case for quantum communication. Conditional density matrix satisfies all the properties of density matrix.

Conditional density matrix helps to clarify a sense of operations in some finest experiments.

## 5. EXAMPLES: SYSTEM AND SUBSYSTEMS

**5.1. Parapositronium.** As an example, we consider parapositronium, i. e., the system consisting of an electron and a positron. The total spin of the system is equal to zero. In this case the nonrelativistic approximation is valid, and the state vector of the system is represented in the form of the product

$$\Psi(\mathbf{r}_e, \sigma_e; \mathbf{r}_p, \sigma_p) = \Phi(\mathbf{r}_e, \mathbf{r}_p)\chi(\sigma_e, \sigma_p).$$

The spin wave function is equal to

$$\chi(\sigma_e, \sigma_p) = \frac{1}{\sqrt{2}}(\chi_{\mathbf{n}}(\sigma_e)\chi_{-\mathbf{n}}(\sigma_p) - \chi_{\mathbf{n}}(\sigma_p)\chi_{-\mathbf{n}}(\sigma_e)).$$

Here,  $\chi_{\mathbf{n}}(\sigma)$  and  $\chi_{(-\mathbf{n})}(\sigma)$  are the eigenvectors of the operator that projects spin onto the vector  $\mathbf{n}$ :

$$(\boldsymbol{\sigma}\mathbf{n}) \chi_{\pm\mathbf{n}}(\sigma) = \pm\chi_{\pm\mathbf{n}}(\sigma).$$

The spin density matrix of the system is determined by the operator with the kernel

$$\rho(\sigma; \sigma') = \chi(\sigma_e, \sigma_p)\chi^*(\sigma'_e, \sigma'_p).$$

The spin density matrix of the electron is

$$\begin{aligned} \rho_e(\sigma, \sigma') &= \sum_{\xi} \chi(\sigma, \xi)\chi^*(\sigma', \xi) = \\ &= \frac{1}{2}(\chi_{\mathbf{n}}(\sigma)\chi_{(-\mathbf{n})}(\sigma') + \chi_{\mathbf{n}}(\sigma)\chi_{(-\mathbf{n})}(\sigma')) = \frac{1}{2}\delta(\sigma - \sigma'). \end{aligned}$$

In this state the electron is completely unpolarized.

If an electron passes through polarization filter, then the pass probability is independent of the filter orientation. The same fact is valid for the positron if its spin state is measured independently of the electron.

Now let us consider quite a different experiment. Namely, the positron passes through the polarization filter, and the electron polarization is simultaneously measured. The operator that projects the positron spin onto the vector  $\mathbf{m}$  (determined by the filter) is given by the kernel

$$P(\sigma, \sigma') = \chi_{\mathbf{m}}(\sigma)\chi_{\mathbf{m}}^*(\sigma').$$

Now the conditional density matrix of the electron is equal to

$$\rho_{e/p}(\sigma, \sigma') = \frac{\sum_{(\sigma, \sigma')} \chi_{\mathbf{m}}(\sigma)\chi_{\mathbf{m}}^*(\sigma')\chi(\sigma_e, \sigma')\chi^*(\sigma'_e, \sigma)}{\sum_{(\xi, \sigma, \sigma')} \chi_{\mathbf{m}}(\sigma)\chi_{\mathbf{m}}^*(\sigma')\chi(\xi, \sigma')\chi^*(\xi, \sigma)}.$$

The result of the summation is

$$\rho_{e/p}(\sigma, \sigma') = \chi_{(-\mathbf{m})}(\sigma) \chi_{(-\mathbf{m})}^*(\sigma').$$

Thus, if the polarization of the positron is selected with the help of polarizer in the state with well-defined spin, then the electron appears to be polarized in the opposite direction. Of course, this result is in an agreement with the fact that total spin of composite system is equal to zero. Nevertheless, this natural result can be obtained if positron and electron spins are measured simultaneously. In the opposite case, the more simple experiment shows that the direction of electron and positron spins are absolutely indefinite.

A. Einstein said, «*Raffiniert ist der Herr Gott, aber boschaft ist Er nicht.*»

**5.2. Quantum Photon Teleportation.** In the Innsbruck experiment [2] on a photon state teleportation, the initial state of the system is the result of the unification of the pair of photons 1 and 2 being in the antisymmetric state  $\chi(\sigma_1, \sigma_2)$  with summary angular momentum equal to zero and the photon 3 being in the state  $\chi_{\mathbf{m}}(\sigma_3)$  (that is, being polarized along the vector  $\mathbf{m}$ ). The joint system state is given by the density matrix

$$\rho(\sigma, \sigma') = \Psi(\sigma) \Psi^*(\sigma'),$$

where the wave function of the joint system is the product

$$\Psi(\sigma) = \chi(\sigma_1, \sigma_2) \chi_{\mathbf{m}}(\sigma_3).$$

Considering then the photon 2 only (without fixing the states of the photons 1 and 3), we find the photon 2 to be completely unpolarized with the density matrix

$$\rho(\sigma_2, \sigma'_2) = \text{Tr}_{(1,3)} \rho(\sigma_1, \sigma_2, \sigma_3; \sigma_1, \sigma'_2, \sigma_3) = \frac{1}{2} \delta(\sigma_2 - \sigma'_2).$$

However, if the photon 2 is registered when the state of the photons 1 and 3 has been determined to be  $\chi(\sigma_1, \sigma_3)$ , then the state of the photon 2 is given by the conditional density matrix

$$\rho_{2/\{1,3\}} = \frac{\text{Tr}_{(1,3)} (P_{1,3} \rho_{1,2,3})}{\text{Tr} (P_{1,3} \rho_{1,2,3})}.$$

Here,  $P_{1,3}$  is the projection operator

$$P_{1,3} = \chi(\sigma_1, \sigma_3) \chi^*(\sigma_1, \sigma_3).$$

To evaluate the conditional density matrix, it is convenient to first find the vectors

$$\phi(\sigma_1) = \sum_3 \chi_{\mathbf{m}}^*(\sigma_3) \chi(\sigma_1, \sigma_3)$$

and

$$\theta(\sigma_2) = \sum_1 \phi^*(\sigma_1) \chi(\sigma_1, \sigma_2).$$

The vector  $\theta$  equals to

$$\theta(\sigma_2) = -\frac{1}{2}\chi_{\mathbf{m}}(\sigma_2)$$

and the conditional density matrix of the photon 2 appears to be equal to

$$\rho_{2/\{1,3\}} = \chi_{\mathbf{m}}(\sigma_2)\chi_{\mathbf{m}}^*(\sigma_2').$$

Thus, if the subsystem consisting of the photons 1 and 3 is forced to be in the antisymmetric state  $\chi(\sigma_1, \sigma_3)$  (with total angular momentum equal to zero), then the photon 2 appears to be polarized along the vector  $\mathbf{m}$ .

**5.3. Entanglement Swapping.** In the recent experiment [7] in installation two pairs of correlated photons emerged simultaneously. The state of the system is described by the wave function

$$\Psi(\sigma) = \Psi(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = \chi(\sigma_1, \sigma_2)\chi(\sigma_3, \sigma_4).$$

The photons 2 and 3 are selected into antisymmetric state  $\chi(\sigma_2, \sigma_3)$ .

What is the state of pair of photons 1 and 4?

Conditional density matrix of the pair (1 and 4) is

$$\hat{\rho}_{14/23} = \frac{\text{Tr}_{23}(\hat{P}_{23}\hat{\rho}_{1234})}{\text{Tr}(\hat{P}_{23}\hat{\rho}_{1234})},$$

where operator that selects pair (2 and 3) is defined by

$$P_{23}(\sigma, \sigma') = \chi(\sigma_2, \sigma_3)\chi^*(\sigma_2', \sigma_3')$$

and density matrix of four-photon system is determined by kernel

$$\rho_{1234}(\sigma, \sigma') = \Psi(\sigma_1, \sigma_2, \sigma_3, \sigma_4)\Psi^*(\sigma_1', \sigma_2', \sigma_3', \sigma_4').$$

Direct calculation shows that the pair of the photons (1 and 4) has to be in pure state with the wave function

$$\Phi(\sigma_1, \sigma_4) = \chi(\sigma_1, \sigma_4).$$

The experiment confirms this prediction.

**5.4. Pairs of Polarized Photons.** Now consider a modification of the Innsbruck experiment. Let there be two pairs of photons (1, 2) and (3, 4). Suppose that each pair is in the pure antisymmetric state  $\chi$ . The spin part of the density matrix of the total system is given by the equation

$$\rho(\sigma, \sigma') = \Psi(\sigma)\Psi^*(\sigma'),$$

where

$$\Psi(\sigma) = \chi(\sigma_1, \sigma_2)\chi(\sigma_3, \sigma_4).$$

If the photons 2 and 4 pass through polarizers, they are polarized along  $\chi_{\mathbf{m}}(\sigma_2)$  and  $\chi_{\mathbf{s}}(\sigma_4)$ , then the wave function of the system is transformed into

$$\Phi(\sigma) = \chi_{\mathbf{n}}(\sigma_1)\chi_{\mathbf{m}}(\sigma_2)\chi_{\mathbf{r}}(\sigma_3)\chi_{\mathbf{s}}(\sigma_4).$$

Here,  $\mathbf{n}$ ,  $\mathbf{m}$  and  $\mathbf{r}$ ,  $\mathbf{s}$  are pairs of mutually orthogonal vectors.

Now the conditional density matrix of the pair of photons 1 and 3 is

$$\rho_{(1,3)/(2,4)}(\sigma, \sigma') = \Theta(\sigma_1, \sigma_3) \Theta^*(\sigma'_1, \sigma'_3).$$

The wave function of the pair is the product of wave functions of each photon with definite polarization

$$\Theta(\sigma_1, \sigma_3) = \chi_n(\sigma_1) \chi_r(\sigma_3).$$

We note that initial correlation properties of the system appear only when the photons pass through polarizers. Although the wave function of the system seems to be a wave function of independent particles, the initial correlation exhibits in correlations of polarizations for each pair. Pairs of polarized photons appear to be very useful in quantum communication.

**5.5. Quantum Realization of Vernam Communication Scheme.** Let us recall the main idea of Vernam communication scheme [8]. In this scheme, Alice encrypts her message (a string of bits denoted by the binary number  $m_1$ ) using a randomly generated key  $k$ . She simply adds each bit of the message with the corresponding bit of the key to obtain the scrambled text ( $s = m_1 \oplus k$ , where  $\oplus$  denotes the binary addition modulo 2 without carry). It is then sent to Bob, who decrypts the message by subtracting the key ( $s \ominus k = m_1 \oplus k \ominus k = m_1$ ). Because the bits of the scrambled text are as random as those of the key, they do not contain any information. This cryptosystem is, thus, proved secure in sense of information theory. Actually, today this is the only probably secure cryptosystem!

Although perfectly secure, the problem with this security is that it is essential for Alice and Bob to possess a common secret key, which must be at least as long as the message itself. They can only use the key for a single encryption. If they used the key more than once, Eve could record all of the scrambled messages and start to build up a picture of the plain texts and, thus, also of the key. (If Eve recorded two different messages encrypted with the same key, she could add the scrambled text to obtain the sum of the plain texts:  $s_1 \oplus s_2 = m_1 \oplus k \oplus m_2 \oplus k = m_1 \oplus m_2 \oplus k \oplus k = m_1 \oplus m_2$ , where we used the fact that  $\oplus$  is commutative.) Furthermore, the key has to be transmitted by some trusted means, such as a courier, or through a personal meeting between Alice and Bob. This procedure may be complex and expensive, and even may lead to a loophole in the system.

With the help of pairs of polarized photons we can overcome the shortcomings of the classical realization of Vernam scheme. Suppose Alice sends Bob pairs of polarized photons obtained according to the rules described in the previous section. Note that the concrete photons' polarizations are set up in Alice's laboratory and Eve does not know them. If the polarization of the photon 1 is set up by a random binary number  $p_i$  and the polarization of the photon 3 is set up by a number  $m_i \oplus p_i$ , then each photon (when considered separately) does not carry any information. However, Bob after obtaining these photons can add corresponding binary numbers and get the number  $m_i$  containing the information ( $m_i \oplus p_i \oplus p_i = m_i$ ).

In this scheme, a secret code is created during the process of sending and is transferred to Bob together with the information. It makes the usage of the scheme completely secure.

## CONCLUSION

Provided that the subsystem  $S_2$  of composite quantum system  $S = S_1 + S_2$  is selected (or will be selected) in a pure state  $\hat{P}_n$ , the quantum state of subsystem  $S_1$  is conditional density

matrix  $\hat{\rho}_{1c/2n}$ . Reduced density matrix  $\hat{\rho}_1$  is connected with conditional density matrices by an expansion:

$$\hat{\rho}_1 = \sum p_n \hat{\rho}_{1n/2n};$$

here

$$\sum \hat{P}_n = \hat{E}, \quad \sum p_n = 1.$$

The coefficients  $p_n$  are probabilities to find subsystem  $S_2$  in pure states  $\hat{P}_n$ .

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