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EFFECT OF THE RELATIVISTIC SPIN ROTATION ON TWO-PARTICLE SPIN COMPOSITION

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The effect of the relativistic spin rotation on two-particle spin states, conditioned by the setting of the spins of the particles in their rest frames and by the noncommutativity of the Lorentz transformations along noncollinear directions, is discussed. Particularly, the transition from the c.m.s. of two spin-1/2 particles to the laboratory is considered. When the vectors of the c.m.s. particle velocities are not collinear with the velocity vector of the c.m.s., the angles of the relativistic spin rotation for the two particles are different. As a result, the relative fractions of the singlet and triplet states in the relativistic system of two spin-1/2 particles with a nonzero vector of relative momentum depend on the concrete frame in which the two-particle system is analyzed.

Обсуждается влияние релятивистского вращения спина на двухчастичные спиновые состояния с учетом фиксации спинов в системах покоя частиц и некоммутативности преобразований Лоренца вдоль неколлинеарных направлений. В частности, рассмотрен переход из с. ц. м. двух частиц со спином 1/2 в лабораторную. Когда скорости частиц в с. ц. м. не коллинеарны скорости с. ц. м., углы релятивистского спинового вращения для двух частиц различны. В результате относительные доли синглетного и триплетного состояний в релятивистской системе двух частиц со спином 1/2 и ненулевым относительным импульсом зависят от конкретной системы, в которой анализируется двухчастичная система.

1. Earlier the spin correlations in two-particle quantum systems were analyzed in detail as a tool allowing one to measure the space-time characteristics of particle production [1–5], to study the two-particle interaction and the production dynamics (see [3,4] and references therein) and to verify the consequences of the quantum-mechanical coherence with the help of Bell-type inequalities [4].

The spin state of the system of two particles in an arbitrary frame is described by the two-particle density matrix, the elements of which, $\rho^{(1,2)}_{m_1 m_1';\, m_2 m_2'}$, are given in the representation of the spin projections of the first and second particle in the corresponding rest frames onto the common coordinate axis z (see, e.g., [3,5])****. However, one should take into account the relativistic spin rotation

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****The setting of the particle spins in their respective rest frames is based on the properties

conditioned by the additional rotation of the spatial axes at the successive Lorentz transformations along noncollinear directions [7–9]*. As a result, the *concrete* description of a particle spin state depends on the frame from which the transition to the particle rest frame is performed. Particularly, the total spin composition of the two-particle state with a nonzero vector of relative momentum is generally frame-dependent due to different relativistic rotation angles of the two spins at the transition to the frame moving in the direction which is not collinear with the velocity vectors of both particles.

Usually, it is convenient to consider the spin correlations in the centre-of-mass system (c.m.s.) of the particle pair. This is natural at the addition of the two-particle total spin and the relative orbital angular momentum into the conserved total angular momentum. In some cases, however, it may be useful to make transition to the laboratory, e.g., in the case when the particle scatterings are used as their spin analyzers [1]**. Denoting M_l and $\mathbf{p}_l = \pm\mathbf{k}$ the masses and c.m.s. momenta of the two particles, $l = 1, 2$, their respective c.m.s. velocities in the units of the velocity of light ($c = 1$) are $\mathbf{v}_l = \pm\mathbf{k}/\sqrt{\mathbf{k}^2 + M_l^2}$. Here and below the $\pm$ signs correspond to the first ($l = 1$) and second ($l = 2$) particle, respectively. We denote the corresponding laboratory velocities as $\widetilde{\mathbf{v}}_l$ and the laboratory velocity of the particle pair as $\mathbf{V}$. At the Lorentz transformation from the c.m.s. of the particle pair to the laboratory frame with parallel respective spatial axes, the spins of the first and the second particle (in their respective rest frames) rotate in opposite directions around the axis which is parallel to the vector $[\mathbf{kV}]$***. The rotation angles ω_l are given by the Stapp formula [7] (see also [8, 9]):

$$\sin \omega_l = \pm\gamma\gamma_l V v_l \sin \theta \frac{1 + \gamma + \gamma_l + \widetilde{\gamma}_l}{(1 + \gamma)(1 + \gamma_l)(1 + \widetilde{\gamma}_l)}, \tag{1}$$

where the positive sign corresponds to the direction of the nearest rotation from the vector $\mathbf{k}$ to the vector $\mathbf{V}$; θ is the angle between the vectors $\mathbf{k}$ and $\mathbf{V}$ ($0 \leqslant \theta \leqslant \pi$),

of the inhomogeneous Lorentz group and avoids the problem of the noncommutativity of the spin operators with the free Hamiltonian [6]. This circumstance was not understood in Ref. 2, where the unnecessary condition of nonrelativistic particle velocities was required.

*The relativistic rotation of the spatial axes leads to the nontransitivity of the parallelism in the theory of relativity (see [10] and references therein): generally, the parallel axes of the frames K_1 and K, K_2 and K do not imply the parallel axes of the frames K_1 and K_2. The axes of all the three frames could be mutually parallel if only their velocities were collinear (for example, if K_1 and K_2 were the rest frames of the two particles and K — their c.m.s.).

**In principle, this transition is not necessary since one can transform the four vectors defining the polarization analyzers first to the pair c.m.s. and then to the respective particle rest frames.

***The relativistic spin rotation is the purely kinematical effect: the angles of the space rotation coincide with the angles between the vectors of the resulting velocities at the relativistic addition of velocities $\mathbf{v}_l$ and $\mathbf{V}$ in the direct and reverse orders [11].

$v_l = |\mathbf{v}_l|$, $\widetilde{v}_l = |\widetilde{\mathbf{v}}_l|$, $V = |\mathbf{V}|$ and $\gamma_l = \left(1 - v_l^2\right)^{-1/2}$, $\widetilde{\gamma}_l = \left(1 - \widetilde{v}_l^2\right)^{-1/2}$, $\gamma = \left(1 - V^2\right)^{-1/2}$ are the Lorentz factors;

$$\widetilde{\gamma}_l = \gamma_l \gamma (1 \pm v_l V \cos \theta). \tag{2}$$

In the case of equal-mass particles the relations $\mathbf{v}_1 = -\mathbf{v}_2$, $\gamma_1 = \gamma_2$ hold (but $\widetilde{\gamma}_1 \neq \widetilde{\gamma}_2$ when $V \cos \theta \neq 0$).

Using the equality

$$(1 + \gamma + \gamma_l + \widetilde{\gamma}_l)^2 = 2(1 + \gamma)(1 + \gamma_l)(1 + \widetilde{\gamma}_l) - (\gamma^2 - 1)(\gamma_l^2 - 1) \sin^2 \theta,$$

one can write the analogous expressions for the cosines of the spin rotation angles:

$$\cos \omega_l = 1 - \frac{(\gamma - 1)(\gamma_l - 1)}{(1 + \widetilde{\gamma}_l)} \sin^2 \theta. \tag{1a}$$

In the case of the collinearity of the velocity vectors $\mathbf{v}_l$ and $\mathbf{V}$, when $\theta = 0$ or $\theta = \pi$, both the rotation angles are equal to zero.

At nonrelativistic velocities v_l in the c.m.s. of the particle pair ($\gamma_l \approx 1$, $\widetilde{\gamma}_l \approx \gamma$), the angles ω_l of the spin rotation are small and scale with v_l:

$$\omega_l \approx \pm \frac{\gamma}{\gamma + 1} v_l V \sin \theta. \tag{3}$$

In the ultrarelativistic limit, when $\gamma_l \to \infty$, $\widetilde{\gamma}_l / \gamma_l \to \gamma (1 \pm V \cos \theta)$, one has

$$\sin \omega_l \approx \pm V \sin \theta \frac{1 + \gamma(1 \pm V \cos \theta)}{(1 + \gamma)(1 \pm V \cos \theta)}, \quad \cos \omega_l \approx 1 - \frac{\gamma - 1}{\gamma(1 \pm V \cos \theta)} \sin^2 \theta. \tag{4}$$

Relations (4) are valid exactly for massless particles (photons, neutrinos). In this case the rotation angles coincide with the aberration angles (the angles between the vectors $\mathbf{v}_l$ and $\widetilde{\mathbf{v}}_l$); then the helicity (the spin projection of the particle onto the direction of its momentum) is the relativistic invariant [9].

Taking into account the relativistic spin rotation at the transition from the two-particle c.m.s. to the laboratory, the two-particle spin density matrix is transformed as follows:

$$\hat{\rho}'^{(1,2)} = \hat{D}^{(1)}(\omega_1) \otimes \hat{D}^{(2)}(\omega_2) \hat{\rho}^{(1,2)} \hat{D}^{(1)+}(\omega_1) \otimes \hat{D}^{(2)+}(\omega_2), \tag{5}$$

where

$$\hat{D}^{(l)}(\omega_l) = \exp\left(i\omega_l \hat{\mathbf{j}}_l \mathbf{n}\right) \tag{6}$$

are the matrices of the space rotations generated by the vector spin operators $\hat{\mathbf{j}}_l$, $\mathbf{n}$ is the unit vector parallel to the direction of the vector $[\mathbf{kV}]$.

2. In the case of two spin-1/2 particles, the two-particle spin density matrix has the structure [1, 3, 4]:

$$\hat{\rho}^{(1,2)} = \frac{1}{4}\left[\hat{I}^{(1)} \otimes \hat{I}^{(2)} + (\hat{\boldsymbol{\sigma}}^{(1)}\mathbf{P}_1) \otimes \hat{I}^{(2)} + \hat{I}^{(1)} \otimes (\hat{\boldsymbol{\sigma}}^{(2)}\mathbf{P}_2) + \right.$$

$$\left. + \sum_{i=1}^{3}\sum_{k=1}^{3}T_{ik}\hat{\sigma}_i^{(1)} \otimes \hat{\sigma}_k^{(2)}\right]. \quad (7)$$

Here $\hat{I}$ is the two-row unit matrix; $\hat{\boldsymbol{\sigma}}$ is the Pauli vector operator; $\mathbf{P}_l = \langle\hat{\boldsymbol{\sigma}}^{(l)}\rangle$ are the polarization vectors; $T_{ik} = \langle\hat{\sigma}_i^{(1)} \otimes \hat{\sigma}_k^{(2)}\rangle$ are the components of the correlation tensor, $\{1, 2, 3\} \equiv \{x, y, z\}$. The left and right indexes of the correlation tensor correspond to the rest frames of the first ($l = 1$) and second ($l = 2$) particle, respectively. The corresponding probability to select the particles with the polarizations $\boldsymbol{\zeta}^{(l)}$ can be obtained by the substitution of the matrices $\hat{\sigma}_i^{(l)}$ in expression (7) with the corresponding projections $\zeta_i^{(l)}$. Particularly, when analyzing the polarization states with the help of particle decays, the vector analyzing power $\boldsymbol{\zeta}^{(l)} = \alpha_l\mathbf{n}_l$, where α_l is the decay asymmetry corresponding to the decay analyzer unit vector $\mathbf{n}_l$. As a result [3, 4], the correlation between the decay analyzers is determined by the product of the decay asymmetries and the trace of the spin correlation tensor

$$T = T_{xx} + T_{yy} + T_{zz}.$$

For example, the angular correlation $\mathbf{n}_1\mathbf{n}_2 = \cos\theta_{12}$ between the directions of the three-momenta of the decay protons in the respective rest frames of two Λ-hyperons decaying into the channel $\Lambda \to p + \pi^-$ with the P-odd asymmetry $\alpha = 0.642$ is described by the normalized probability density

$$W(\cos\theta_{12}) = \frac{1}{2}\left(1 + \alpha^2\frac{T}{3}\cos\theta_{12}\right). \quad (8)$$

Clearly, the structure of both Eq. (7) and the corresponding angular distribution of the spin analyzers (e.g., Eq. (8)) does not depend on the system from which the transitions to the particle rest frames are performed. The system dependence manifests only through the relativistic rotations in the successive Lorentz transformations along noncolinear directions. The matrices of the space rotations due to the transition from the c.m.s. of two free spin-1/2 particles to the laboratory are the following:

$$\hat{D}^{(l)}(\omega_l) = \cos\frac{\omega_l}{2} + i\hat{\boldsymbol{\sigma}}^{(l)}\mathbf{n}\sin\frac{\omega_l}{2}. \quad (9)$$

Selecting the z-axis parallel to the direction of the vector $\mathbf{n} = [\mathbf{kV}]/|[\mathbf{kV}]|$, and the axes x and y in the plane perpendicular to this vector, the polarization vectors

and the spin correlation tensor transform at the transition to the laboratory in accordance with the (active) rotations around the z-axis by the angles ω_1 and ω_2 for the first and second particle, respectively:

$$P'_{lx} = P_{lx} \cos \omega_l - P_{ly} \sin \omega_l; \quad P'_{ly} = P_{ly} \cos \omega_l + P_{lx} \sin \omega_l; \quad P'_{lz} = P_{lz};$$

$$T'_{xx} = (T_{xx} \cos \omega_1 - T_{yx} \sin \omega_1) \cos \omega_2 - (T_{xy} \cos \omega_1 - T_{yy} \sin \omega_1) \sin \omega_2;$$

$$T'_{yy} = (T_{yy} \cos \omega_1 + T_{xy} \sin \omega_1) \cos \omega_2 + (T_{yx} \cos \omega_1 + T_{xx} \sin \omega_1) \sin \omega_2;$$

$$T'_{zz} = T_{zz};$$

(10)

$$T'_{xy} = (T_{xy} \cos \omega_1 - T_{yy} \sin \omega_1) \cos \omega_2 + (T_{xx} \cos \omega_1 - T_{yx} \sin \omega_1) \sin \omega_2;$$

$$T'_{yx} = (T_{yx} \cos \omega_1 + T_{xx} \sin \omega_1) \cos \omega_2 - (T_{yy} \cos \omega_1 + T_{xy} \sin \omega_1) \sin \omega_2;$$

$$T'_{xz} = T_{xz} \cos \omega_1 - T_{yz} \sin \omega_1; \quad T'_{zx} = T_{zx} \cos \omega_2 - T_{zy} \sin \omega_2;$$

$$T'_{yz} = T_{yz} \cos \omega_1 + T_{xz} \sin \omega_1; \quad T'_{zy} = T_{zy} \cos \omega_2 + T_{zx} \sin \omega_2.$$

Particularly, the trace of the spin correlation tensor transforms at the transition to the laboratory as:

$$T' = (T_{xx} + T_{yy}) \cos (\omega_1 - \omega_2) + (T_{xy} - T_{yx}) \sin (\omega_1 - \omega_2) + T_{zz} \qquad (11)$$

or, in the case of a symmetric tensor, as:

$$T' = T - 2 (T_{xx} + T_{yy}) \sin^2 \frac{\omega_1 - \omega_2}{2}. \qquad (12)$$

So, the c.m.s. trace T in Eq. (8) is substituted by the laboratory one T' calculated using Eqs. (11) or (12) together with Eqs. (1) and (2) for the spin rotation angles.

3. It was shown [3, 4] (see also [12, 13]) that the trace of the correlation tensor of a system of two spin-1/2 particles is the following linear combination of the relative fractions of singlet (the total spin $S = 0$) and triplet ($S = 1$) states:

$$T = \langle \hat{\boldsymbol{\sigma}}^{(1)} \otimes \hat{\boldsymbol{\sigma}}^{(2)} \rangle = \rho_t - 3\rho_s, \qquad \rho_t + \rho_s = 1. \qquad (13)$$

When we have the pure singlet state of the particle pair in its c.m.s. ($\rho_s = 1$, $\rho_t = 0, T_{ik} = -\delta_{ik}$, $T = -3$), the transformation to the laboratory gives

$$T' = -3 + 4 \sin^2 \frac{\omega_1 - \omega_2}{2}. \qquad (14)$$

It follows from Eqs. (13) and (14) that at the transition to the laboratory the relative fraction of the singlet state decreases in favor of a triplet state:

$$\rho'_s = \cos^2 \frac{\omega_1 - \omega_2}{2}, \qquad \rho'_t = \sin^2 \frac{\omega_1 - \omega_2}{2}. \qquad (15)$$

Thus the square of the total spin of two free particles with a nonzero vector of relative velocity is not a relativistic invariant (see [8]). Introducing the two-particle singlet state:

$$|\psi\rangle_{00} = \frac{1}{\sqrt{2}} \left(|+1/2\rangle_z^{(1)} \, |-1/2\rangle_z^{(2)} - |-1/2\rangle_z^{(1)} \, |+1/2\rangle_z^{(2)} \right) \qquad (16)$$

and the triplet state with the zero projection onto the rotation axis z:

$$|\psi\rangle_{10} = \frac{1}{\sqrt{2}} \left(|+1/2\rangle_z^{(1)} \, |-1/2\rangle_z^{(2)} + |-1/2\rangle_z^{(1)} \, |+1/2\rangle_z^{(2)} \right), \qquad (17)$$

the result in Eq. (15) also follows directly from the matrices of space rotations in Eq. (9), the singlet state in the two-particle c.m.s. is transformed into the following superposition of the singlet and triplet states in the laboratory:

$$|\psi_s'\rangle = \cos\frac{\omega_1 - \omega_2}{2} |\psi\rangle_{00} + i \sin\frac{\omega_1 - \omega_2}{2} |\psi\rangle_{10}. \qquad (18)$$

Similarly, the transformation of the pure triplet state $|\psi\rangle_{10}$ in the two-particle c.m.s. ($\rho_s = 0$, $\rho_t = 1$, $T_{zz} = -1$, $T_{xx} = T_{yy} = 1$, $T = 1$) to the laboratory gives

$$T' = 1 - 4\sin^2\frac{\omega_1 - \omega_2}{2}, \qquad (19)$$

the corresponding fractions being

$$\rho_s' = \sin^2\frac{\omega_1 - \omega_2}{2}, \qquad \rho_t' = \cos^2\frac{\omega_1 - \omega_2}{2}, \qquad (20)$$

in accordance with the transformation:

$$|\psi_t'\rangle = \cos\frac{\omega_1 - \omega_2}{2} |\psi\rangle_{10} + i \sin\frac{\omega_1 - \omega_2}{2} |\psi\rangle_{00}. \qquad (21)$$

In the case of the unpolarized triplet in the two-particle c.m.s. ($\rho_s = 0$, $\rho_t = 1$, $T_{ik} = \delta_{ik}/3$, $T = 1$ [1,4]), we have

$$T' = 1 - \frac{4}{3}\sin^2\frac{\omega_1 - \omega_2}{2}, \qquad (22)$$

$$\rho_s' = \frac{1}{3}\sin^2\frac{\omega_1 - \omega_2}{2}, \qquad \rho_t' = 1 - \frac{1}{3}\sin^2\frac{\omega_1 - \omega_2}{2}. \qquad (23)$$

Using Eqs. (1) and (2), it is easy to show that in the case of two spin-1/2 particles with the same masses ($\gamma_2 = \gamma_1$, $v_2 = v_1$) the measure of the spin mixing can be written in the form:

$$K \equiv \sin^2\frac{\omega_1 - \omega_2}{2} = (v_1 V)^2 \sin^2\theta \left[\left(\frac{1}{\gamma} + \frac{1}{\gamma_1} \right)^2 + (v_1 V)^2 \sin^2\theta \right]^{-1}. \qquad (24)$$

The maximum of the mixing factor K corresponds to the angle $\theta = \pi/2$. In the ultrarelativistic limit, when $\gamma_1 \gg 1$, $\gamma \gg 1$ and $\sin\theta \gg \max(1/\gamma, 1/\gamma_1)$, the factor K approaches unity. Then the singlet state in the two-particle c.m.s. becomes in the laboratory the triplet state with the zero projection of the total spin onto the spin rotation axis z and, *vice versa*.

Thus, the effect of the relativistic spin rotation leads to the dependence of the total spin composition (the singlet and triplet fractions in particular) on the concrete frame in which the system of two-particles, moving with different velocity vectors, is analyzed. The physical origin of this dependence is the violation of the parallelism of the spatial axes of the particle rest frames, except for the case when the Lorentz transformations to these frames are done along the directions collinear with the relative velocity (e.g., from the c.m.s. of the two particles).

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