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In this paper the Feynman path integral technique is applied to two-dimensional spaces of nonconstant curvature: these spaces are called Darboux spaces $D_{\rm I}$–$D_{\rm IV}$. We start each consideration in terms of the metric and then analyze the quantum theory in the separable coordinate systems. The path integral in each case is formulated and then solved in the majority of cases, the exceptions being quartic oscillators where no closed solution is known. The required ingredients are the path integral solutions for the linear potential, the harmonic oscillator, the radial harmonic oscillator, the modified Pöschl–Teller potential, and for spheroidal wave functions, respectively. The basic path integral solutions, which appear here in a complicated way, have been developed in recent work and are known. The final solutions are represented in terms of the corresponding Green functions and the expansions into the wave functions, respectively. We also sketch some limiting cases of the Darboux spaces, where spaces of constant negative and zero curvature emerge.

В настоящей работе фейнмановская техника континуального интегрирования применена к двумерным пространствам произвольной кривизны: эти пространства названы пространствами Дарбу $D_{\rm I}$–$D_{\rm IV}$. Рассмотрение каждого случая начато в терминах метрики, затем проанализирована квантовая теория в разделяющихся системах координат. В каждом случае формулируется континуальный интеграл, а затем находится решение для большинства случаев. Исключение сделано для ангармонического осциллятора четвертого порядка, когда замкнутого решения не существует. Необходимые ингредиенты являются решениями континуального интеграла для линейного потенциала, гармонического осциллятора, радиального гармонического осциллятора, модифицированного потенциала Пешля–Теллера и сфероидальных волновых функций соответственно. Основные решения континуального интегрирования, которые появляются здесь в сложной форме, были разработаны в недавних работах. Конечные решения представлены в терминах соответствующих функций Грина и разложений по волновым функциям. Бегло представлены предельные случаи пространств Дарбу, где возникают пространства постоянной отрицательной или нулевой кривизны.

1. INTRODUCTION

1.1. General Overview and Recent Work. In recent years there have been an enormous success in developing path integral techniques and in solving Feynman path integrals. After its invention by Feynman [10], the solution of the harmonic oscillator has been for a long time the only accessible path integral solution. Several textbooks made the path integral more popular, e.g., Feynman and Hibbs [12] and Schulman [60]. However, as a tool in quantum mechanics the path integral remained in a dormant state, whereas its main applications were in the field theory, e.g., [11, 14, 35, 50]. Only later on, the technique of radial path integrals, i.e., the Cartesian path integral formulated in spherical coordinates, was developed in the 1960's by Edwards and Gulyaev [9] and Peak and Inomata [59].

Matters shifted with the first calculation of the path integral for the Hydrogen atom [7, 8] by Duru and Kleinert. In the textbook [46] a lot of applications and a summary of the development of this technique can be found, including many references on the subject. The principal development in their approach was a technique called space-time transformation, sometimes also called «Duru–Kleinert transformation», which does not only perform a coordinate transformation in the path integral but also transforms the time-slicings $\epsilon = T/N$ into new time-slicings δ. This combined coordinate and time-transformation usually ends up in manipulating the action in the path integral in such a way that a given problem is transformed into one of the basic path integrals. These basic path integrals are, roughly speaking, the harmonic oscillator, the radial harmonic oscillator [59], the (modified) Pöschl–Teller potential path integrals [1, 6, 13], and the spheroidal path integral [24, 26]. In our book «Handbook of Feynman Path Integrals» [34] we have given a thorough overview of all the techniques how to handle and manipulate path integrals, and, most important, an up-to-date list of several hundred solvable path integrals and the corresponding references.

The separation of a particular quantum mechanical potential problem into more than one coordinate system has the consequence that there are additional integrals of motion and that the spectrum is degenerate. The Noether theorem [57] connects the particular symmetries of a Lagrangian, i.e., the invariances with respect to the dynamical symmetries, with conservation laws in classical mechanics and with observables in quantum mechanics, respectively. In the case of the isotropic harmonic oscillator one has in addition to the conservation of energy and the conservation of the angular momentum, the conservation of the quadrupole moment; in the case of the Coulomb problem one has in addition to the conservation of energy and the angular momentum, the conservation of the Pauli–Runge–Lenz vector. In total, the additional conserved quantities in these two examples add up to five functionally independent integrals of motion in classical mechanics, respectively observables in quantum mechanics.

A topic which appeared in the formulation of the radial path integral and for the (modified) Pöschl–Teller potential [1, 6, 13, 47] was the path integration over group spaces. This included the formulation and evaluation of the path integral in spaces of constant curvature, the curvature being positive (spheres), negative (hyperboloids), or zero (Euclidean and pseudo-Euclidean space). This opened a new rich field of investigations. It required separation of variables in the Schrödinger, respectively the Laplace–Beltrami equation $\Delta_{\rm LB} - E$, in various coordinate systems in these spaces. The case of two- and three-dimensional Euclidean space can be found, e.g., in [55] and in the textbooks [54, 56]: In two-dimensional flat space there are four coordinate systems and in three-dimensional space there are eleven coordinate systems which separate the Helmholtz, respectively the Schrödinger equation*.

A thorough study of separation of variables of the Laplace–Beltrami equation in spaces of nonzero constant curvature was first performed by Olevskiĭ [58], who studied the two- and three-dimensional cases of constant curvature. In particular, he found that on the two-dimensional hyperboloid there are nine coordinate systems; and on the three-dimensional hyberboloid, 34 coordinate systems which allow separation of variables in the Laplace–Beltrami equation. However, only in several cases, which exhibit symmetry properties, closed solutions in terms of higher transcendental functions are known. In many coordinate systems very little is known about the corresponding solutions of the eigenvalue equation in terms of special functions and no closed solution is known. These coordinate systems are usually parameter-dependent, e.g., ellipsoidal or paraboloid systems. Usually, only in the nonparameter-dependent coordinate systems, like spherical and parabolic coordinates, a well-developed theory of higher transcendental functions is known. In these cases, the relevant eigenfunctions can be expressed in terms of hypergeometric and degenerate hypergeometric functions and we find Legendre polynomials (in cases with a discrete spectrum) and functions (in cases with a continuous spectrum), the Bessel and Whittaker functions (in cases with a continuous spectrum) and numerous polynomial solutions (in cases with a discrete spectrum), and so on.

The only exception of parametric coordinate systems, where a developed theory of the corresponding higher transcendental functions exists, are the elliptic and spheroidal coordinate systems in a flat space [53] and on spheres [26]. The important point is that this theory allows one to expand the exponentiated invariant distance in terms of elliptic and spheroidal wave functions. These functions are one-parameter-generalized functions of the well-known spherical harmonics and Bessel functions. It also allows one to formulate the «spheroidal path integral» which can be added to the list of basic path integrals.

Based on [58], the theory of [53], and the thorough study of coordinate systems in spaces of constant curvature [37], we were able to find the solution of the path integral formulations in these spaces, expressed in its separating coordinate systems [23, 24]. This study included the two- and three-dimensional Euclidean and pseudo-Euclidean spaces, the two- and three-dimensional spheres, the two- and three-dimensional hyperboloids, imaginary Lobachevsky space, respectively the single-sheeted hyperbolic [22], $SU(u,v)$-path integration [18], hyperbolic spaces of rank one [19], and Hermitian hyperbolic spaces [19, 25].

*As pointed out in [2] there are 17 types of coordinate systems which R-separate the Laplace equation, i.e., $\Delta_{\rm LB}\Phi = 0$. However, these additional systems are very complicated except the case of the toroidal coordinate system.

Let us finally note that due to Kalnins, Miller, Winternitz, and coworkers there is in a series of papers an extensive study on separation of variables of the Schrödinger, respectively the Helmholtz equation in spaces of constant curvature, called «Lie Theory and Separation of Variables». We just would like to refer to [2, 3, 5, 40, 54] and the textbooks [24, 37] with their extensive reference list on this subject.

1.2. Introducing Darboux Spaces. An extension of the study of path integration on spaces of constant curvature is the investigation of path integral formulations in spaces of nonconstant curvature. Kalnins et al. [38, 39] denoted four types of two-dimensional spaces of nonconstant curvature, labeled by $D_{\rm I}$–$D_{\rm IV}$, which are called Darboux spaces [48]. In terms of the infinitesimal distance they are described by:

$$\text{(I)} \qquad ds^2 = (x+y)dxdy, \tag{1.1}$$

$$\text{(II)} \qquad ds^2 = \left(\frac{a}{(x-y)^2} + b\right)dxdy, \tag{1.2}$$

$$\text{(III)} \qquad ds^2 = \left(a\,\mathrm{e}^{-(x+y)/2} + b\,\mathrm{e}^{-x-y}\right)dxdy, \tag{1.3}$$

$$\text{(IV)} \qquad ds^2 = -\frac{a\left(\mathrm{e}^{(x-y)/2} + \mathrm{e}^{(y-x)/2}\right) + b}{\left(\mathrm{e}^{(x-y)/2} - \mathrm{e}^{(y-x)/2}\right)^2}dxdy, \tag{1.4}$$

a and b are additional (real) parameters. Kalnins et al. [38, 39] studied not only the solution of the free motion, but also emphasized on the superintegrable systems in theses spaces. Superintegrable means that in two dimensions at least three constants of motion must exist, which is by construction already fulfilled for the free motion. They found appropriate coordinate systems, and we will consider all of them. In the majority of the cases we will be able to find a solution, however in some cases this will be impossible due to the quartic anharmonicity of the problems in question.

The Gaussian curvature in a space with metric $ds^2 = g(u,v)(du^2 + dv^2)$ is given by ($g = \det g(u,v)$)

$$K = -\frac{1}{2g}\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right)\ln g. \tag{1.5}$$

Equation (1.5) will be used to discuss shortly the curvature properties of the Darboux spaces, including their limiting cases of constant curvature.

In the following sections we discuss each of the four Darboux spaces, we set up the Lagrangian, the Hamiltonian, the quantum operator, and formulate and solve (if this is possible) the corresponding path integral. We also discuss some of the limiting cases of the Darboux spaces, i.e., where we obtain a space of constant (zero or negative) curvature. In particular, for $D_{\rm II}$ we consider the limiting case $b = 0$ a little bit more explicitly: It gives the two-dimensional hyperboloid (with constant negative curvature). The other limiting case on $D_{\rm II}$ ($a = 0$, i.e., $\mathbb{R}^2$) is only sketched. In the case of $D_{\rm I}$ there is no limiting case, because we have no free parameter in the metric to choose from. In the two remaining Darboux spaces, $D_{\rm III}$ and $D_{\rm IV}$, the limiting case of $D_{\rm III}$ is not very difficult since it is the zero-curvature case $\mathbb{R}^2$ which emerges. In $D_{\rm IV}$ we sketch the matter with some notes.

In order to make the paper self-contained we provide in the Appendices some material about the basic path integral techniques and solutions. We set up the path integral formulation for general coordinates, including our lattice definition of the time-sliced path integral and shortly describe several transformation techniques, including coordinate transformation and the space-time transformation. Also, we summarize some important path integral solutions, like the (radial) harmonic oscillator, the linear potential, and the modified Pöschl–Teller potential, including the corresponding Green functions. These solutions including their generalization to related potentials are indispensable tools in the path integral investigation of Darboux spaces.

2. DARBOUX SPACE $D_{\rm I}$

We start with the consideration of the Darboux Space $D_{\rm I}$ and consider the following coordinate systems:

$$\begin{aligned}
&((u,v)\text{ coordinates:}) && x = u + iv, \quad y = u - iv\ (u \geqslant a), && (2.1)\\
&(\text{Rotated } (r,q) \text{ coordinates:}) && u = r\cos\vartheta + q\sin\vartheta, && (2.2)\\
& && v = -r\sin\vartheta + q\cos\vartheta \quad (\vartheta\in[0,\pi]), && (2.3)\\
&(\text{Displaced parabolic:}) && u = \frac{1}{2}(\xi^2-\eta^2) + a,\ v = \xi\eta && \\
& && (\xi\in\mathbb{R}, \eta > 0, a > 0). && (2.4)
\end{aligned}$$

The infinitesimal distance, i.e., the metric is given by

$$\begin{aligned}
ds^2 &= (x+y)dxdy, && (2.5)\\
((u,v)\text{ coordinates:}) &= 2u(du^2+dv^2), && (2.6)\\
(\text{Rotated } (r,q)\text{ coordinates:}) &= 2(r\cos\vartheta + q\sin\vartheta)(dr^2+dq^2), && (2.7)\\
(\text{Displaced parabolic:}) &= (\xi^2-\eta^2+2a)(\xi^2+\eta^2)(d\xi^2+d\eta^2). && (2.8)
\end{aligned}$$

We find, e.g., in the (u, v) system for the Gaussian curvature

$$K = \frac{1}{u^4}. \tag{2.9}$$

There is no further parameter in the metric, therefore this space is of nonconstant curvature throughout for all $u > a$ with a being some real constant $a > 0^*$.

2.1. The Path Integral in (u, v) Coordinates on $D_{\rm I}$. In order to set up the path integral formulation we follow our canonical procedure as presented in [34]. The Lagrangian and Hamiltonian are given by, respectively:

$$\mathcal{L}(u, \dot{u}, v, \dot{v}) = mu(\dot{u}^2 + \dot{v}^2), \quad \mathcal{H}(u, p_u, v, p_v) = \frac{1}{4mu}(p_u^2 + p_v^2), \tag{2.10}$$

and we must require $u > a$ for some $a > 0$, and $v \in [0, 2\pi]$ can be considered as a cyclic variable [39]. The canonical momenta are

$$p_u = \frac{\hbar}{i}\left(\frac{\partial}{\partial u} + \frac{1}{2u}\right), \qquad p_v = \frac{\hbar}{i}\frac{\partial}{\partial v}, \tag{2.11}$$

and for the quantum Hamiltonian we find

$$H = -\frac{\hbar^2}{2m}\frac{1}{2u}\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) = \frac{1}{2m}\frac{1}{\sqrt{2u}}(p_u^2 + p_v^2)\frac{1}{\sqrt{2u}}. \tag{2.12}$$

We formulate the path integral (first ignoring the half-space constraint):

$$\begin{aligned} K(u'', u', v'', v'; T) &= \lim_{N\to\infty}\left(\frac{m}{2\pi i\epsilon\hbar}\right)^N \prod_{j=1}^{N-1}\int 2u_j du_j dv_j \times \\ &\times \exp\left[\frac{im}{\hbar}\sum_{j=1}^N \widehat{u_j}(\Delta^2 u_j + \Delta^2 v_j)\right] = \qquad (2.13) \\ &= \int\limits_{u(t')=u'}^{u(t'')=u''} \mathcal{D}u(t) \int\limits_{v(t')=v'}^{v(t'')=v''} \mathcal{D}v(t) 2u \times \\ &\times \exp\left[\frac{im}{\hbar}\int_0^T u(\dot{u}^2 + \dot{v}^2)dt\right] \qquad (2.14) \end{aligned}$$

($\widehat{u_j} = \sqrt{u_j u_{j-1}}$). I have displayed the path integral in our lattice definition, which will be used throughout this paper. Due to this lattice definition of the path integral, we have no additional $\hbar^2$ potential because the dimension of the space

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*In [39] the condition $a \geqslant 1/2$ is imposed in order to embed $D_{\rm I}$ into a three-dimensional space with coordinates X, Y, Z such that $dX^2 + dY^2 + dZ^2 = 2u(du^2 + dv^2)$. For $dX^2 + dY^2 + dZ^2$ we have $v \in [0, 2\pi)$. For $dX^2 + dY^2 - dZ^2$ we have $v \in \mathbb{R}$.

of nonconstant curvature equals 2, c.f. (A.7). In this path integral we perform a time transformation according to $\Delta t_{(j)} = 2\widehat{u_j}\Delta s_{(j)}$, i.e., with time-transformation function $f(u) = 2u = \sqrt{g}$ (g is the determinant of the metric tensor), and we obtain:

$$K(u'', u', v'', v'; T) = \int_{-\infty}^{\infty} \frac{dE}{2\pi\hbar} \, \mathrm{e}^{-iET/\hbar} \int_{0}^{\infty} ds'' K(u'', u', v'', v'; s'') \tag{2.15}$$

with $K(s'')$ given by:

$$\begin{aligned} &K(u'', u', v'', v'; s'') = \\ &= \int\limits_{u(0)=u'}^{u(s'')=u''} \mathcal{D}u(s) \int\limits_{v(0)=v'}^{v(s'')=v''} \mathcal{D}v(s) \exp\left\{ \frac{i}{\hbar} \int_0^{s''} \left[\frac{m}{2}(\dot{u}^2 + \dot{v}^2) + 2uE \right] ds \right\} = \\ &\qquad = \sum_{l=-\infty}^{\infty} \frac{\mathrm{e}^{il(v''-v')}}{2\pi} \exp\left(-\frac{i}{\hbar} \frac{l^2\hbar^2}{2m} s'' \right) \int\limits_{u(0)=u'}^{u(s'')=u''} \mathcal{D}u(s) \times \\ &\qquad\qquad \times \exp\left[\frac{i}{\hbar} \int_0^{s''} \left(\frac{m}{2}\dot{u}^2 + 2uE \right) ds \right]. \end{aligned} \tag{2.16}$$

The feature that the time-transformation function equals $f = \sqrt{g}$ is a general feature of the Darboux-space path integration. I have separated the v-dependent part of the path integral in circular waves. The remaining path integral in the variable u is a path integral for the linear potential. Let us denote the path integrals in the variables u and v by $K_u(s'')$ and $K_v(s'')$, respectively. We obtain for the product of the two kernels with corresponding energy-dependent Green functions $G_u(E; \mathcal{E}_u)$ and $G_v(E; \mathcal{E}_v)$:

$$\begin{aligned} K(u'', u', v'', v'; T) &= \int_{-\infty}^{\infty} \frac{dE}{2\pi\hbar} \, \mathrm{e}^{-iET/\hbar} \int_0^{\infty} ds'' K_v(v'', v'; s'') K_u(u'', u'; s'') = \\ &= \int_{-\infty}^{\infty} \frac{dE}{2\pi\hbar} \, \mathrm{e}^{-iET/\hbar} \int_0^{\infty} ds'' \frac{1}{2\pi i} \int d\mathcal{E}_v \, \mathrm{e}^{-i\mathcal{E}_v s''/\hbar} G_v(E; v'', v'; \mathcal{E}_v) \frac{1}{2\pi i} \times \\ &\qquad \times \int d\mathcal{E}_u \, \mathrm{e}^{-i\mathcal{E}_u s''/\hbar} G_u(E; u'', u'; \mathcal{E}_u) = \\ &= \int_{-\infty}^{\infty} \frac{dE}{2\pi\hbar} \, \mathrm{e}^{-iET/\hbar} \frac{\hbar}{2\pi i} \int d\mathcal{E} G_v(E; v'', v'; -\mathcal{E}) G_u(E; u'', u'; \mathcal{E}). \end{aligned} \tag{2.17}$$

Alternatively, in terms of the Green function we get

$$G(u'',u',v'',v';E)=\frac{\hbar}{2\pi i}\int d\mathcal{E}G_v(E;v'',v';\mathcal{E})G_u(E;u'',u';-\mathcal{E}). \tag{2.18}$$

Inserting now the explicit form of the v-dependent kernel we obtain:

$$K(u'',u',v'',v';T)=\int\limits_{-\infty}^{\infty}\frac{dE}{2\pi\hbar}\,\mathrm{e}^{-iET/\hbar}\sum_{l=-\infty}^{\infty}\frac{\mathrm{e}^{il(v''-v')}}{2\pi}G_u\bigg(E;u'',u';-\frac{l^2\hbar^2}{2m}\bigg). \tag{2.19}$$

For the complete solution we must know the kernel $G_u(u'',u';\mathcal{E})$ which we obtain in the following way. The Green function for the linear potential $V(x)=kx$ is given by [34]

$$\begin{aligned}G^{(k)}(x'',x';\mathcal{E})&=\frac{4m}{3\hbar}\bigg[\bigg(x'-\frac{\mathcal{E}}{k}\bigg)\bigg(x''-\frac{\mathcal{E}}{k}\bigg)\bigg]^{1/2}\times\\&\times I_{1/3}\bigg[\frac{\sqrt{8mk}}{3\hbar}\bigg(x_<-\frac{\mathcal{E}}{k}\bigg)^{3/2}\bigg]K_{1/3}\bigg[\frac{\sqrt{8mk}}{3\hbar}\bigg(x_>-\frac{\mathcal{E}}{k}\bigg)^{3/2}\bigg]; \end{aligned}\tag{2.20}$$

I_ν and K_ν are modified Bessel functions [15], and $x_<$ and $x_>$ denote the smaller and larger of x' and x'', respectively. We have to identify $\mathcal{E}=-L^2\hbar^2/2m$, $k=-2E$, and $x=u$. In addition, we have to recall that the motion in u takes place only in the half-space $u>a$. In order to construct the Green function in the half-space $x>a$ we have to put Dirichlet boundary conditions at $x=a$ [20,21]. Therefore the Green function for the linear potential in the half-space $x>a$ is given by

$$G_{(x=a)}(u'',u';\mathcal{E})=G^{(k)}(u'',u';\mathcal{E})-\frac{G^{(k)}(u'',a;\mathcal{E})G^{(k)}(a,u';\mathcal{E})}{G^{(k)}(a,a;\mathcal{E})}. \tag{2.21}$$

Therefore we obtain finally:

$$\begin{aligned}G(u'',u',v'',v';E)&=\sum_{l=-\infty}^{\infty}\frac{\mathrm{e}^{il(v''-v')}}{2\pi}\frac{4m}{3\hbar}\bigg[\bigg(u'-\frac{l^2\hbar^2}{4mE}\bigg)\bigg(u''-\frac{l^2\hbar^2}{4mE}\bigg)\bigg]^{1/2}\times\\&\times\Bigg[\tilde{I}_{1/3}\bigg(u_<-\frac{l^2\hbar^2}{4mE}\bigg)\tilde{K}_{1/3}\bigg(u_>-\frac{l^2\hbar^2}{4mE}\bigg)-\\&-\frac{\tilde{I}_{1/3}\Big(a-\frac{l^2\hbar^2}{4mE}\Big)}{\tilde{K}_{1/3}\Big(a-\frac{l^2\hbar^2}{4mE}\Big)}\tilde{K}_{1/3}\bigg(u'-\frac{l^2\hbar^2}{4mE}\bigg)\tilde{K}_{1/3}\bigg(u''-\frac{l^2\hbar^2}{4mE}\bigg)\Bigg]. \end{aligned}\tag{2.22}$$

And $\tilde{I}_\nu(z)$ denotes

$$\tilde{I}_\nu(z) = I_\nu\left(\frac{4\sqrt{-mE}}{3\hbar}z^{3/2}\right),$$

with $\tilde{K}_\nu(z)$ similarly. The Green function (2.22) is rather complicated due to the boundary condition at $u = a$, and we will not evaluate the free-particle wave functions. Let us only consider the following point. According to Langguth and Inomata [49] the modified Bessel function $I_\nu(z)$ has the asymptotic expansion

$$I_\nu(z) \simeq \frac{1}{\sqrt{2\pi}}\mathrm{e}^{z-\nu^2/z} \qquad (\text{for } |z| \to \infty), \tag{2.23}$$

provided $\Re(z) > 0$. However, if $-3\pi/2 < \arg(z) < \pi/2$, one has

$$I_\nu(z) \simeq \frac{1}{\sqrt{2\pi}}\mathrm{e}^{z-\nu^2/z} + \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-z+\nu^2/z+i\pi(\nu+1/2)} \qquad (\text{for } |z| \to \infty). \tag{2.24}$$

(A similar consideration is valid for K_ν.) In our Green function (2.22) we now see that due to the $-$-sign in the square-root expression in the argument of the modified Bessel functions, its argument becomes purely imaginary, from which follows that for large u we get «plane» waves $\propto \mathrm{e}^{\pm i\kappa u^{3/2}}$ with some wave number κ, including in- and out-coming waves. This is the well-known feature of «plane waves» in the free particle motion, in the present case modified by the curvature of the space.

Let us note another particularity of the quantum motion in D_{I}. According to [39] the coordinates (u, v) can only be uniquely determined if we embed the space D_{I} into a three-dimensional Euclidean space with definite or indefinite metric, respectively. In the present calculation we have chosen the definite case. However, if we chose a metric according to $d^2X + d^2Y - d^2T = 2u(d^2u + d^2v)$, it is found that the variable v can vary in its range over the entire real line, i.e., $v \in \mathbb{R}$. In the separation in the path integral (2.16) the only difference would be that the summation over the discrete quantum number l is replaced by an integration over the continuous quantum number k, say, including the replacement $l \to k$ in all the following formulas from (2.16) on.

The problem of the exact range of the variables (u, v) we will encounter several times, and for this reason we will leave this range unspecified. Implicitly we assume that when terms $\propto u^{-2}$ appear, then u is in the range $u > 0$, i.e, a radial variable. In the other coordinate systems like parabolic, spherical, elliptic, etc., the usual range of variables is assumed. If however, a coordinate is treated within the range of $\mathbb{R}$, but is in fact restricted to be positive or larger than a definite number, then (2.21) must be applied to find the proper quantum solution.

2.2. The Path Integral in Rotated (r, q) Coordinates on D_{I}. In order to set up the path integral formulation we follow again our canonical procedure. The

Lagrangian and Hamiltonian are given by, respectively:

$$\mathcal{L}(r,\dot r,q,\dot q)=m(r\cos\vartheta+q\sin\vartheta)(\dot r^2+\dot q^2), \tag{2.25}$$

$$\mathcal{H}(r,p_r,q,p_q)=\frac{1}{4m(r\cos\vartheta+q\sin\vartheta)}(p_r^2+p_q^2). \tag{2.26}$$

The canonical momenta are

$$p_r = \frac{\hbar}{i}\left(\frac{\partial}{\partial r}+\frac{\cos\vartheta}{2(r\cos\vartheta+q\sin\vartheta)}\right), \tag{2.27}$$

$$p_q = \frac{\hbar}{i}\left(\frac{\partial}{\partial q}+\frac{\sin\vartheta}{2(r\cos\vartheta+q\sin\vartheta)}\right). \tag{2.28}$$

The quantum Hamiltonian has the form

$$H = -\frac{\hbar^2}{2m}\frac{1}{2(r\cos\vartheta+q\sin\vartheta)}\left(\frac{\partial^2}{\partial r^2}+\frac{\partial^2}{\partial q^2}\right)= \tag{2.29}$$

$$= \frac{1}{2m}\frac{1}{\sqrt{2(r\cos\vartheta+q\sin\vartheta)}}(p_r^2+p_q^2)\frac{1}{\sqrt{2(r\cos\vartheta+q\sin\vartheta)}}. \tag{2.30}$$

The path integral is formulated in the usual form*

$$K(r'',r',q'',q';T)=\int\limits_{r(t')=r'}^{r(t'')=r''}\mathcal{D}r(t)\int\limits_{q(t')=q'}^{q(t'')=q''}\mathcal{D}q(t)2(r\cos\vartheta+q\sin\vartheta)\times$$
$$\times\exp\left[\frac{im}{\hbar}\int\limits_0^T(r\cos\vartheta+q\sin\vartheta)(\dot r^2+\dot q^2)dt\right]=$$
$$=\int\frac{dE}{2\pi\hbar}\,\mathrm{e}^{-iET/\hbar}\int\limits_0^\infty ds''K(r'',r',q'',q';s''), \tag{2.31}$$

$$K(r'',r',q'',q';s'')=\int\limits_{r(0)=r'}^{r(s'')=r''}\mathcal{D}r(s)\exp\left[\frac{i}{\hbar}\int\limits_0^{s''}\left(\frac{m}{2}\dot r^2+2Er\cos\vartheta\right)ds\right]\times$$
$$\times\int\limits_{q(0)=q'}^{q(s'')=q''}\mathcal{D}q(s)\exp\left[\frac{i}{\hbar}\int\limits_0^{s''}\left(\frac{m}{2}\dot q^2+2Eq\sin\vartheta\right)ds\right]. \tag{2.32}$$

*We will assume $r,q\in\mathbb{R}$, otherwise (2.21) must be applied.

I have time-transformed the path integral with the time-transformation function $f(r,q) = 2(r\cos\vartheta + q\sin\vartheta) = \sqrt{g}$. Both path integrals are path integrals in r and q, respectively, for the linear potential. In order to find the Green function, we set this function in the variable r as $G_r(\mathcal{E})$ and in the variable q as $G_q(\mathcal{E})$, respectively. We get similarly as before:

$$K(r'',r',q'',q';T) = \int_{-\infty}^{\infty} \frac{dE}{2\pi\hbar}\,\mathrm{e}^{-iET/\hbar}\frac{\hbar}{2\pi i}\int d\mathcal{E} G_r(r'',r';-\mathcal{E})G_q(q'',q';\mathcal{E}). \tag{2.33}$$

Together with the solution of the linear potential, this gives the solution

$$\begin{aligned} K(r'',r',q'',q';T) = \int dE\,\mathrm{e}^{-iET/\hbar}\int d\mathcal{E}\biggl(\frac{m}{\hbar^2}\sqrt{\frac{2}{E\cos\vartheta}}\biggr)^{2/3}\times \\ \times\,\mathrm{Ai}\Biggl[-\biggl(r''-\frac{\mathcal{E}}{2E\cos\vartheta}\biggr)\biggl(\frac{4mE\cos\vartheta}{\hbar^2}\biggr)^{1/3}\Biggr]\times \\ \times\,\mathrm{Ai}\Biggl[-\biggl(r'-\frac{\mathcal{E}}{2E\cos\vartheta}\biggr)\biggl(\frac{4mE\cos\vartheta}{\hbar^2}\biggr)^{1/3}\Biggr]\times \\ \times\biggl(\frac{m}{\hbar^2}\sqrt{\frac{2}{E\sin\vartheta}}\biggr)^{2/3}\mathrm{Ai}\Biggl[-\biggl(q''+\frac{\mathcal{E}}{2E\sin\vartheta}\biggr)\biggl(\frac{4mE\sin\vartheta}{\hbar^2}\biggr)^{1/3}\Biggr]\times \\ \times\,\mathrm{Ai}\Biggl[-\biggl(q'+\frac{\mathcal{E}}{2E\sin\vartheta}\biggr)\biggl(\frac{4mE\sin\vartheta}{\hbar^2}\biggr)^{1/3}\Biggr], \end{aligned}\tag{2.34}$$

and $\mathrm{Ai}(z)$ denotes the Airy function $\left(\xi = \frac{2}{3}z^{3/2}\right)$:

$$\mathrm{Ai}(z) = \frac{1}{3}\sqrt{z}\Bigl[I_{-1/3}(\xi) - I_{1/3}(\xi)\Bigr] = \frac{1}{\pi}\sqrt{\frac{z}{3}}\,K_{1/3}(\xi), \tag{2.35}$$

$$\mathrm{Ai}(-z) = \frac{1}{3}\sqrt{z}\Bigl[J_{-1/3}(\xi) - J_{1/3}(\xi)\Bigr]. \tag{2.36}$$

2.3. The Path Integral in Displaced Parabolic Coordinates on D_{I}. In parabolic coordinates the Lagrangian and the Hamiltonian are given by

$$\mathcal{L}(\xi,\dot\xi,\eta,\dot\eta) = \frac{m}{2}(\xi^2-\eta^2+2a)(\xi^2+\eta^2)(\dot\xi^2+\dot\eta^2), \tag{2.37}$$

$$\mathcal{H}(\xi,p_\xi,\eta,p_\eta) = \frac{1}{2m}\frac{p_\xi^2+p_\eta^2}{(\xi^2-\eta^2+2a)(\xi^2+\eta^2)}. \tag{2.38}$$

The canonical momenta are

$$p_\xi = \frac{\hbar}{i}\biggl(\frac{\partial}{\partial\xi} + \frac{\xi}{\xi^2-\eta^2+2a} + \frac{\xi}{\xi^2+\eta^2}\biggr), \tag{2.39}$$

$$p_\eta = \frac{\hbar}{i}\biggl(\frac{\partial}{\partial\eta} - \frac{\eta}{\xi^2-\eta^2+2a} + \frac{\eta}{\xi^2+\eta^2}\biggr). \tag{2.40}$$

The quantum Hamiltonian has the form

$$H = -\frac{\hbar^2}{2m}\frac{1}{(\xi^2-\eta^2+2a)(\xi^2+\eta^2)}\biggl(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\biggr) = \tag{2.41}$$

$$= \frac{1}{2m}\frac{1}{\sqrt{(\xi^2-\eta^2+2a)(\xi^2+\eta^2)}}(p_\xi^2+p_\eta^2)\times$$

$$\times \frac{1}{\sqrt{(\xi^2-\eta^2+2a)(\xi^2+\eta^2)}}. \tag{2.42}$$

The path integral formulation is as follows (with the implemented time-transformation function $f(\xi,\eta) = (\xi^2-\eta^2+2a)(\xi^2+\eta^2) = \sqrt{g}$):

$$K(\xi'',\xi',\eta'',\eta';T) = \int\limits_{\xi(t')=\xi'}^{\xi(t'')=\xi''} \mathcal{D}\xi(t)\times$$

$$\times \int\limits_{\eta(t')=\eta'}^{\eta(t'')=\eta''} \mathcal{D}\eta(t)(\xi^2-\eta^2+2a)(\xi^2+\eta^2)\times$$

$$\times \exp\left[\frac{im}{2\hbar}\int\limits_0^T(\xi^2-\eta^2+2a)(\xi^2+\eta^2)(\dot\xi^2+\dot\eta^2)dt\right] =$$

$$= \int\frac{dE}{2\pi\hbar}\,\mathrm{e}^{-iET/\hbar}\int\limits_0^{s''}ds''\int\limits_{\xi(0)=\xi'}^{\xi(s'')=\xi''}\mathcal{D}\xi(s)\int\limits_{\eta(0)=\eta'}^{\eta(s'')=\eta''}\mathcal{D}\eta(s)\times$$

$$\times \exp\left\{\frac{i}{\hbar}\int\limits_0^{s''}\left[\frac{m}{2}(\dot\xi^2+\dot\eta^2) + E\Bigl(\xi^4-\eta^4+2a(\xi^2+\eta^2)\Bigr)\right]ds\right\}. \tag{2.43}$$

This is a path integral of a quartic anharmonic oscillator which cannot be solved.

3. DARBOUX SPACE D_{II}

In this section we consider the Darboux space D_{II} (1.2). We have the following four coordinate systems:

$$\begin{aligned}
&((u,v)\text{ coordinates:}) && x=\frac{1}{2}(v+iu),\quad y=\frac{1}{2}(v-iu), && (3.1)\\
&(\text{Polar:}) && u=\varrho\cos\vartheta, &&\\
& && v=\varrho\sin\vartheta\qquad(\varrho>0,\vartheta\in(-\pi/2,\pi/2)), && (3.2)\\
&(\text{Parabolic:}) && u=\xi\eta,\quad v=\frac{1}{2}(\xi^2-\eta^2)\qquad(\xi>0,\eta>0), && (3.3)\\
&(\text{Elliptic:}) && u=d\cosh\omega\cos\varphi, &&\\
& && v=d\sinh\omega\sin\varphi\ (\omega>0,\varphi\in(-\pi/2,\pi/2)). && (3.4)
\end{aligned}$$

The $2d$ is the interfocal distance in the elliptic system. Separation of variables is possible in all four coordinate systems. For convenience we also display in the following the special case of the parameters $a=-1$ and $b=1$ [38] (Table 1). The infinitesimal distance is given in these four cases:

$$\begin{aligned}
ds^2 &= \left(\frac{a}{(x-y)^2}+b\right)dxdy,\\
((u,v)\text{ coordinates:}) &= \frac{bu^2-a}{u^2}(du^2+dv^2)=\frac{u^2+1}{u^2}(du^2+dv^2), \qquad (3.5)\\
(\text{Polar:}) &= \frac{b\varrho^2\cos^2\vartheta-a}{\varrho^2\cos^2\vartheta}(d\varrho^2+\varrho^2d\vartheta^2)=\\
&= \frac{\varrho^2\cos^2\vartheta+1}{\varrho^2\cos^2\vartheta}(d\varrho^2+\varrho^2d\vartheta^2), \qquad (3.6)\\
(\text{Parabolic:}) &= \frac{b\xi^2\eta^2-a}{\xi^2\eta^2}(\xi^2+\eta^2)(d\xi^2+d\eta^2)=\\
&= \left[\left(b\xi^2-\frac{a}{\xi^2}\right)+\left(b\eta^2-\frac{a}{\eta^2}\right)\right](d\xi^2+d\eta^2)= \qquad (3.7)\\
&= \frac{1+\xi^2\eta^2}{\xi^2\eta^2}(\xi^2+\eta^2)(d\xi^2+d\eta^2), \qquad (3.8)
\end{aligned}$$

$$\begin{aligned}
(\text{Elliptic:}) &= \frac{bd^2\cosh^2\omega\cos^2\varphi-a}{\cosh^2\omega\cos^2\varphi}(\cosh^2\omega-\cos^2\varphi)\times\\
&\times\ (d\omega^2+d\varphi^2)==\left[\left(bd^2\cosh^2\omega+\frac{a}{\cosh^2\omega}\right)-\right.\\
&-\ \left.\left(bd^2\cos^2\varphi+\frac{a}{\cos^2\varphi}\right)\right](d\omega^2+d\varphi^2). \qquad (3.9)
\end{aligned}$$

Table 1. **Limiting cases of coordinate systems on D_{II}**

Metric:	D_{II}	$\Lambda^{(2)}$ $(a=-1, b=0)$	E_2 $(a=0, b=1)$
$\dfrac{bu^2-a}{u^2}(du^2+dv^2)$	(u,v)-system	Horicyclic	Cartesian
$\dfrac{b\varrho^2\cos^2\vartheta-a}{\varrho^2\cos^2\vartheta}(d\varrho^2+d\vartheta^2)$	Polar	Equidistant	Polar
$\dfrac{b\xi^2\eta^2-a}{\xi^2\eta^2}(\xi^2+\eta^2)(d\xi^2+d\eta^2)$	Parabolic	Semicircular–parabolic	Parabolic
$\dfrac{bd^2\cosh^2\omega\cos^2\varphi-a}{\cosh^2\omega\cos^2\varphi}\times$ $\times(\cosh^2\omega-\cos^2\varphi)(d\omega^2+d^2\varphi^2)$	Elliptic	Elliptic–parabolic	Elliptic

We can see that the case $a=-1$, $b=0$ leads to the case of the Poincaré upper half-plane [17, 23, 24], i.e., the two-dimensional hyperboloid $\Lambda^{(2)}$. In this case separation of variables is possible in nine coordinate systems [58]; this has been extensively discussed in [24, 29]. The parabolic case corresponds to the semicircular–parabolic system and the elliptic case to the elliptic–parabolic system on the two-dimensional hyperboloid. On the other hand, the case $a=0$, $b=1$ just gives the usual two-dimensional Euclidean plane with its four coordinate systems which allow separation of variables of the Laplace–Beltrami equation, i.e., the Cartesian, polar, parabolic, and elliptic system. Hence, the Darboux space II contains as special cases a space of constant zero curvature (Euclidean plane) and a space of constant negative curvature (the hyperbolic plane). A discussion of a more general case on the question of contracting Lie algebras corresponding to coordinate systems in spaces of constant curvature can be found in [36]. This includes the emerging of coordinate systems in flat space from curved spaces.
We find for the Gaussian curvature in the (u,v)-system

$$K=\frac{a(a-3bu^2)}{(a-2bu^2)^3}. \tag{3.10}$$

For $b=0$ we find $K=1/a$ which is indeed a space of constant curvature, and the quantity a measures the curvature. In particular, for the unit-two-dimensional hyperboloid we have $K=1/a$, with $a=-1$ as the special case of $\Lambda^{(2)}$. In the following we will assume that $a<0$ in order to assure the positive definiteness of the metric (1.2).

3.1. The Path Integral in (u,v) Coordinates on D_{II}. We start with the (u,v)-coordinate system. We formulate the classical Lagrangian and Hamiltonian, respectively:

$$\mathcal{L}(u,\dot{u},v,\dot{v}) = \frac{m}{2}\frac{bu^2-a}{u^2}(\dot{u}^2+\dot{v}^2), \tag{3.11}$$

$$\mathcal{H}(u,p_u,v,p_v) = \frac{1}{2m}\frac{u^2}{bu^2-a}(p_u^2+p_v^2). \tag{3.12}$$

The canonical momenta are

$$p_u = \frac{\hbar}{i}\bigg(\frac{\partial}{\partial u} + \frac{bu}{bu^2-a} - \frac{1}{u}\bigg), \quad p_v = \frac{\hbar}{i}\frac{\partial}{\partial v}. \tag{3.13}$$

The quantum Hamiltonian has the form

$$H = -\frac{\hbar^2}{2m}\frac{u^2}{bu^2-a}\bigg(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\bigg) = \tag{3.14}$$

$$= \frac{1}{2m}\frac{u}{\sqrt{bu^2-a}}(p_u^2+p_v^2)\frac{u}{\sqrt{bu^2-a}}. \tag{3.15}$$

We write down path integral, perform a time-transformation with $f(u) = (bu^2 - a)/u^2 = \sqrt{g}$ and insert the path integral solution for the free particle in the variable v and the radial harmonic oscillator [34,59] in the variable u ($\lambda^2 - 1/4 = 2maE/\hbar^2$):

$$\begin{aligned}
&K(u'',u',v'',v';T) = \\
&= \int\limits_{u(t')=u'}^{u(t'')=u''} \mathcal{D}u(t) \int\limits_{v(t')=v'}^{v(t'')=v''} \mathcal{D}v(t)\frac{bu^2-a}{u^2}\exp\Bigg[\frac{im}{2\hbar}\int\limits_0^T \frac{bu^2-a}{u^2}(\dot{u}^2+\dot{v}^2)dt\Bigg] = \\
&= \int\limits_{-\infty}^{\infty}\frac{dE}{2\pi\hbar}\,\mathrm{e}^{-iET/\hbar}\int\limits_0^{\infty} ds'' \int\limits_{u(0)=u'}^{u(s'')=u''}\mathcal{D}u(s)\int\limits_{v(0)=v'}^{v(s'')=v''}\mathcal{D}v(s)\times \\
&\qquad\times\exp\Bigg\{\frac{i}{\hbar}\int\limits_0^{s''}\bigg[\frac{m}{2}(\dot{u}^2+\dot{v}^2) - \frac{aE}{u^2}\bigg]ds + \frac{i}{\hbar}bEs''\Bigg\} = \\
&= \int\limits_{-\infty}^{\infty}\frac{dE}{2\pi\hbar}\,\mathrm{e}^{-iET/\hbar}\int\limits_0^{\infty}\frac{ds''}{2\pi}\int\limits_{-\infty}^{\infty}dk\,\mathrm{e}^{ik(v''-v')}\exp\bigg(\frac{i}{\hbar}bEs'' - \frac{i}{\hbar}\frac{\hbar^2k^2}{2m}s''\bigg)\times
\end{aligned}$$

$$\times \int\limits_{u(0)=u'}^{u(s'')=u''} \mathcal{D}u(s) \exp\left[\frac{i}{\hbar}\int\limits_0^{s''}\left(\frac{m}{2}\dot{u}^2 - \hbar^2\frac{\lambda^2-\frac{1}{4}}{2mu^2}\right)ds\right] =$$

$$= \int\limits_{-\infty}^{\infty}\frac{dE}{2\pi\hbar}\,\mathrm{e}^{-iET/\hbar}\int\limits_0^{\infty}ds''\int\limits_{-\infty}^{\infty}dk\,\mathrm{e}^{ik(v''-v')}\frac{m\sqrt{u'u''}}{i\hbar s''}\times$$

$$\times \exp\left[\frac{i}{\hbar}\left(bE - \frac{\hbar^2k^2}{2m}\right)s'' + \frac{i}{\hbar}\frac{m}{2s''}({u'}^2+{u''}^2)\right] I_\lambda\left(\frac{mu'u''}{i\hbar s''}\right). \quad (3.16)$$

We can see that the case $a = 0$ yields the solution of the free particle in $\mathbb{R}^2$ since $\lambda = \pm 1/2$, and a proper combination of $I_{\pm 1/2}$ gives exponentials. Together with the integral [15, p. 719]

$$\int\limits_0^{\infty}\mathrm{e}^{-a/x-bx}J_\nu(cx)\frac{dx}{x} = 2J_\nu\left[\sqrt{2a(\sqrt{b^2+d^2}-b)}\right]K_\nu\left[\sqrt{2a(\sqrt{b^2+d^2}+b)}\right] \quad (3.17)$$

we obtain for the Green function (λ resolved and taken on the cut, $a < 0$)

$$G(u'', u', v'', v'; E) = \frac{2m\sqrt{u'u''}}{i\hbar}\int\limits_{-\infty}^{\infty}dk\,\mathrm{e}^{ik(v''-v')}\times$$

$$\times I_\lambda\left(\sqrt{k^2-\frac{2mbE}{\hbar^2}}\,u_<\right)K_\lambda\left(\sqrt{k^2-\frac{2mbE}{\hbar^2}}\,u_>\right), \quad (3.18)$$

$$G(u'', u', v'', v'; E)\frac{\hbar}{\pi^3}\int\limits_{-\infty}^{\infty}dk\,\mathrm{e}^{ik(v''-v')}\int\limits_0^{\infty}\frac{p\sinh\pi p dp}{\frac{\hbar^2}{2m|a|}\left(p^2+\frac{1}{4}\right)-E}\times$$

$$\times K_{ip}\left(\sqrt{k^2-\frac{2mbE}{\hbar^2}}\,u'\right)K_{ip}\left(\sqrt{k^2-\frac{2mbE}{\hbar^2}}\,u''\right), \quad (3.19)$$

with

$$\lambda = \sqrt{\frac{1}{4}-\frac{2m|a|E}{\hbar^2}} \equiv ip. \quad (3.20)$$

The wave functions and the energy spectrum are read off:

$$\Psi(u,v) = \frac{\mathrm{e}^{ikv}}{\sqrt{2\pi}}\frac{\sqrt{2p\sinh\pi p}}{\pi}K_{ip}\left(\sqrt{k^2-\frac{2mbE}{\hbar^2}}\,u\right), \quad (3.21)$$

$$E = \frac{\hbar^2}{2m|a|}\left(p^2+\frac{1}{4}\right). \quad (3.22)$$

Here I have used the following identity as used in [32] utilizing [52, p. 194] and [15, p. 819; 732], respectively

$$I_\lambda(ax)K_\lambda(bx) = \frac{1}{\pi\sqrt{ab}}\int_0^\infty dt\,\mathcal{Q}_{\lambda-1/2}\bigg(\frac{a^2+b^2+t^2}{2ab}\bigg)\cos xt =$$

$$= \frac{1}{\pi\sqrt{ab}}\int_0^\infty dt\cos xt\int_0^\infty \frac{p\tanh\pi p\,dp}{\lambda^2+p^2}\mathcal{P}_{ip-1/2}\bigg(\frac{a^2+b^2+t^2}{2ab}\bigg) =$$

$$= \frac{2}{\pi^2}\int_0^\infty \frac{p\tanh\pi p\,dp}{\lambda^2+p^2}K_{ip}(ax)K_{ip}(bx). \quad (3.23)$$

$\mathcal{P}_\mu, \mathcal{Q}_\mu$ are Legendre functions of the first and second kind, respectively [15, p. 999]. The case $a=-1$, $b=1$ gives the case of [38], i.e., the Green function:

$$G(u'',u',v'',v';E) = \int_{-\infty}^\infty dk\,\mathrm{e}^{ik(v''-v')}\int_0^\infty \frac{dp}{\pi^2}\frac{2p\sinh\pi p}{\frac{\hbar^2}{2m}\Big(p^2+\frac{1}{4}\Big)-E}\times$$

$$\times K_{ip}\left(\sqrt{k^2-\Big(p^2+\frac{1}{4}\Big)}\,u''\right)K_{ip}\left(\sqrt{k^2-\Big(p^2+\frac{1}{4}\Big)}\,u'\right). \quad (3.24)$$

And the wave functions read

$$\Psi(u,v) = \frac{\mathrm{e}^{ikv}}{\sqrt{2\pi}}\frac{\sqrt{2p\sinh\pi p}}{\pi}K_{ip}\left(\sqrt{k^2-\Big(p^2+\frac{1}{4}\Big)}\,u\right). \quad (3.25)$$

If we take in (3.19) $a=-1$, $b=0$, we obtain the solution for the Poincaré upper half-plane [32]. We can evaluate (3.16) by means of [15, p. 719] yielding

$$G(u'',u',v'',v';E) = \frac{m}{\sqrt{2\pi}}\int_0^\infty \mathrm{e}^{-z\cosh d}I_{ip}(z)\frac{dz}{\sqrt{z}} = \frac{m}{\pi}\mathcal{Q}_{-\frac{1}{2}-ip}(\cosh d), \quad (3.26)$$

where

$$\cosh d = \frac{(v''-v')^2+u'^2+u''^2}{2u'u''}, \quad (3.27)$$

which is the Poincaré distance on the hyperboloid. For $b\neq 0$ such an expression cannot be found.

3.2. The Path Integral in Polar Coordinates on $D_{\rm II}$. In polar coordinates the classical Lagrangian and Hamiltonian are given by

$$\mathcal{L}(r,\dot r,\vartheta,\dot\vartheta) = \frac{m}{2}\biggl(b-\frac{a}{\varrho^2\cos^2\vartheta}\biggr)(\dot\varrho^2+\varrho^2\dot\vartheta^2), \tag{3.28}$$

$$\mathcal{H}(\varrho,p_\varrho,\vartheta,p_\vartheta) = \frac{1}{2m}\biggl(b-\frac{a}{\varrho^2\cos^2\vartheta}\biggr)^{-1}\biggl(p_\varrho^2+\frac{1}{\varrho^2}p_\vartheta^2\biggr). \tag{3.29}$$

The momentum operators are

$$p_\varrho = \frac{\hbar}{i}\biggl[\frac{\partial}{\partial\varrho}+\biggl(\frac{b\varrho\cos^2\vartheta}{b\cos^2\vartheta\varrho^2-a}-\frac{1}{2\varrho}\biggr)\biggr], \tag{3.30}$$

$$p_\vartheta = \frac{\hbar}{i}\biggl[\frac{\partial}{\partial\vartheta}+\biggl(\tan\vartheta-\frac{b\varrho^2\sin\vartheta\cos\vartheta}{b\varrho^2\cos^2\vartheta-a}\biggr)\biggr], \tag{3.31}$$

and the quantum Hamiltonian is given by

$$H = -\frac{\hbar^2}{2m}\biggl(b-\frac{a}{\varrho^2\cos^2\vartheta}\biggr)^{-1}\biggl(\frac{\partial^2}{\partial\varrho^2}+\frac{1}{\varrho}\frac{\partial}{\partial\varrho}+\frac{1}{\varrho^2}\frac{\partial^2}{\partial\vartheta^2}\biggr) = \tag{3.32}$$

$$= \frac{1}{2m}\biggl(b-\frac{a}{\varrho^2\cos^2\vartheta}\biggr)^{-1/2}\biggl(p_\varrho^2+\frac{1}{\varrho^2}p_\vartheta^2\biggr)\biggl(b-\frac{a}{\varrho^2\cos^2\vartheta}\biggr)^{-1/2} - \biggl(b-\frac{a}{\varrho^2\cos^2\vartheta}\biggr)^{-1}\frac{\hbar^2}{8m\varrho^2}. \tag{3.33}$$

Hence, we get for the path integral

$$K(\varrho'',\varrho',\vartheta'',\vartheta';T) = \int\limits_{\varrho(t')=\varrho'}^{\varrho(t'')=\varrho''}\mathcal{D}\varrho(t)\int\limits_{\vartheta(t')=\vartheta'}^{\vartheta(t'')=\vartheta''}\mathcal{D}\vartheta(t)\varrho\biggl(b-\frac{a}{\varrho^2\cos^2\vartheta}\biggr)\times$$
$$\times\exp\left\{\frac{i}{\hbar}\int_0^T\left[\frac{m}{2}\biggl(b-\frac{a}{\varrho^2\cos^2\vartheta}\biggr)(\dot\varrho^2+\varrho^2\dot\vartheta^2)+\biggl(b-\frac{a}{\varrho^2\cos^2\vartheta}\biggr)^{-1}\frac{\hbar^2}{8m\varrho^2}\right]dt\right\}. \tag{3.34}$$

However, this coordinate representation is not very well suited for our purposes, except that we recover for $a=0$ polar coordinate in $\mathbb{R}^2$. We introduce $\varrho=e^{\tau_2}$

and $\cos\vartheta = 1/\cosh\tau_1$. This gives the transformed path integral

$$\int\limits_{\tau_1(t')=\tau_1'}^{\tau_1(t'')=\tau_1''}\mathcal{D}\tau_1(t)\int\limits_{\tau_2(t')=\tau_2'}^{\tau_2(t'')=\tau_2''}\mathcal{D}\tau_2(t)\cosh\tau_1\bigg(\frac{b\,\mathrm{e}^{2\tau_2}}{\cosh^2\tau_1}-a\bigg)\times$$
$$\times\exp\Bigg\{\frac{i}{\hbar}\int\limits_0^T\Bigg[\frac{m}{2}\bigg(\frac{b\,\mathrm{e}^{2\tau_2}}{\cosh^2\tau_1}-a\bigg)(\dot\tau_1^2+\cosh^2\tau_1\dot\tau_2^2)\;-$$
$$-\bigg(\frac{b\,\mathrm{e}^{2\tau_2}}{\cosh^2\tau_1}-a\bigg)^{-1}\frac{\hbar^2}{8m}\bigg(1+\frac{1}{\cosh^2\tau_1}\bigg)\Bigg]dt\Bigg\}=\int\limits_{-\infty}^{\infty}\frac{dE}{2\pi\hbar}\,\mathrm{e}^{-iET/\hbar}\int\limits_0^\infty ds''\times$$
$$\times K(\tau_1'',\tau_1',\tau_2'',\tau_2';s'') \quad (3.35)$$

with the time-transformed path integral $K(s'')$ given by $\left(\lambda=\sqrt{\frac{1}{4}-2m|a|E/\hbar^2},\ a<0\right)$:

$$K(\tau_1'',\tau_1',\tau_2'',\tau_2';s'')=\int\limits_{\tau_1(0)=\tau_1'}^{\tau_1(s'')=\tau_1''}\mathcal{D}\tau_1(s)\int\limits_{\tau_2(0)=\tau_2'}^{\tau_2(s'')=\tau_2''}\mathcal{D}\tau_2(s)\cosh\tau_1\times$$
$$\times\exp\Bigg\{\frac{i}{\hbar}\int\limits_0^{s''}\Bigg[\frac{m}{2}(\dot\tau_1^2+\cosh^2\tau_1\dot\tau_2^2)+Eb\frac{\mathrm{e}^{2\tau_2}}{\cosh^2\tau_1}-aE-\frac{\hbar^2}{8m}\bigg(1+\frac{1}{\cosh^2\tau_1}\bigg)\Bigg]dt\Bigg\}. \quad (3.36)$$

The path integral in τ_2 has now the form of the path integral for Liouville quantum mechanics [32, 34] yielding

$$K(\tau_1'',\tau_1',\tau_2'',\tau_2';s'')=\sqrt{\cosh\tau_1'\cosh\tau_1'}\exp\bigg[\frac{i}{\hbar}\bigg(|a|E-\frac{\hbar^2}{8m}\bigg)s''\bigg]\times$$
$$\times\frac{2}{\pi^2}\int\limits_0^\infty dk\,k\sinh\pi k K_{ik}\bigg(\frac{\sqrt{-2mbE}}{\hbar}\,\mathrm{e}^{\tau_2'}\bigg)K_{ik}\bigg(\frac{\sqrt{-2mbE}}{\hbar}\,\mathrm{e}^{\tau_2''}\bigg)\times$$
$$\times\int\limits_{\tau_1(0)=\tau_1'}^{\tau_1(s'')=\tau_1''}\mathcal{D}\tau_1(s)\exp\Bigg[\frac{i}{\hbar}\int\limits_0^{s''}\Bigg(\frac{m}{2}\dot\tau_1^2-\frac{\hbar^2}{2m}\frac{k^2+\frac{1}{4}}{\cosh^2\tau_1}\Bigg)ds\Bigg]=$$
$$=\sqrt{\cosh\tau_1'\cosh\tau_1'}\exp\bigg[\frac{i}{\hbar}\bigg(|a|E-\frac{\hbar^2}{8m}\bigg)s''\bigg]\times$$

$$\times \frac{2}{\pi^2}\int_0^\infty dk\, k\sinh\pi k K_{ik}\left(\frac{\sqrt{-2mbE}}{\hbar}\,\mathrm{e}^{\tau_2'}\right)K_{ik}\left(\frac{\sqrt{-2mbE}}{\hbar}\,\mathrm{e}^{\tau_2''}\right)\times$$

$$\times \frac{1}{2}\sum_{\pm}\int_0^\infty \frac{p\sinh\pi p dp}{\cosh^2 k+\sinh^2\pi p}\mathrm{e}^{-ip^2\hbar s''/2m}P_{ik-1/2}^{ip}(\pm\tanh\tau_1')P_{ik-1/2}^{ip}(\pm\tanh\tau_1''), \tag{3.37}$$

where we have inserted the path integral solution for the special case of the modified Pöschl–Teller potential [24]. Performing the s'' integration gives the energy spectrum:

$$E = \frac{\hbar^2}{2m|a|}\left(p^2+\frac{1}{4}\right)$$

with the Green function

$$G(\tau_1'',\tau_1',\tau_2'',\tau_2';E) = \sum_{\pm}\int_0^\infty dp\int_0^\infty dk\frac{\Psi_{p,k,\pm}(\tau_1'',\tau_2'')\Psi^*_{p,k,\pm}(\tau_1',\tau_2')}{\dfrac{\hbar^2}{2m|a|}\left(p^2+\dfrac{1}{4}\right)-E}, \tag{3.38}$$

and the wave functions are given by

$$\Psi_{p,k,\pm}(\tau_1,\tau_2) = \frac{\sqrt{2\cosh\tau_1}}{\pi}\sqrt{k\sinh\pi k}\,K_{ik}\left(i\sqrt{\frac{b}{|a|}\left(p^2+\frac{1}{4}\right)}\,\mathrm{e}^{\tau_2}\right)\times$$
$$\times\sqrt{\frac{p\sinh\pi p dp}{\cosh^2 k+\sinh^2\pi p}}P_{ik-1/2}^{ip}(\pm\tanh\tau_1), \tag{3.39}$$

$$\Psi_{p,k,\pm}(\varrho,\vartheta) = \frac{\sqrt{2\sin\vartheta}}{\pi}\sqrt{k\sinh\pi k}\,K_{ik}\left(i\sqrt{\frac{b}{|a|}\left(p^2+\frac{1}{4}\right)}\,\varrho\right)\times$$
$$\times\sqrt{\frac{p\sinh\pi p dp}{\cosh^2 k+\sinh^2\pi p}}P_{ik-1/2}^{ip}(\pm\sin\vartheta). \tag{3.40}$$

Here, we have inserted the original (ϱ,ϑ)-coordinate system. For $b=0$ we obtain the equidistant coordinate system on the two-dimensional hyperboloid. For the coordinate τ_1, this is obvious. For the coordinate τ_2, we observe that the K_ν-Bessel function can be represented in this limit as [15, p. 1063]

$$K_\nu(x) = \sqrt{\pi}\,\mathrm{e}^{-x}(2x)^\nu\psi\left(\frac{1}{2}+\nu,1+2\nu,x\right)\propto$$
$$\propto\frac{\Gamma(-2ik)}{\Gamma\left(\dfrac{1}{2}-ik\right)}\sqrt{\pi}\,\mathrm{e}^{ik\tau_2}\to\sqrt{\frac{\pi}{2k\sinh\pi k}}\,\mathrm{e}^{ik\tau_2}, \tag{3.41}$$

which gives together with the normalization factors the final result of [24].

3.3. The Path Integral in Parabolic Coordinates on $D_{\rm II}$. The classical Lagrangian and Hamiltonian are given by

$$\mathcal{L}(\xi,\dot\xi,\eta,\dot\eta) = \frac{m}{2}\frac{b\xi^2\eta^2-a}{\xi^2\eta^2}(\xi^2+\eta^2)(\dot\xi^2+\dot\eta^2), \tag{3.42}$$

$$\mathcal{H}(\xi,p_\xi,\eta,p_\eta) = \frac{1}{2m}\frac{\xi^2\eta^2}{b\xi^2\eta^2-a}\frac{p_\xi^2+p_\eta^2}{\xi^2+\eta^2}. \tag{3.43}$$

The canonical momenta are given by

$$p_\xi = \frac{\hbar}{i}\left(\frac{\partial}{\partial\xi}+\frac{b\xi+a/\xi^3}{\sqrt{g}}\right), \tag{3.44}$$

$$p_\eta = \frac{\hbar}{i}\left(\frac{\partial}{\partial\eta}+\frac{b\eta+a/\eta^3}{\sqrt{g}}\right). \tag{3.45}$$

The quantum Hamiltonian has the form:

$$H = -\frac{\hbar^2}{2m}\left(b\xi^2+b\eta^2-\frac{a}{\xi^2}-\frac{a}{\eta^2}\right)^{-1}\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\right)= \tag{3.46}$$

$$= \frac{1}{2m}\left(b\xi^2+b\eta^2-\frac{a}{\xi^2}-\frac{a}{\eta^2}\right)^{-1/2}(p_\xi^2+p_\eta^2)\times$$
$$\times\ \left(b\xi^2+b\eta^2-\frac{a}{\xi^2}-\frac{a}{\eta^2}\right)^{-1/2}. \tag{3.47}$$

We obtain for the path integral in parabolic coordinates and a time transformation (the time-transformation function reads $f(\xi,\eta)=(b\xi^2\eta^2-a)(\xi^2+\eta^2)/\xi^2\eta^2=\sqrt{g}$):

$$K(\xi'',\xi',\eta'',\eta';T)=\int\limits_{\xi(t')=\xi'}^{\xi(t'')=\xi''}\mathcal{D}\xi(t)\int\limits_{\eta(t')=\eta'}^{\eta(t'')=\eta''}\mathcal{D}\eta(t)\frac{b\xi^2\eta^2-a}{\xi^2\eta^2}(\xi^2+\eta^2)\times$$
$$\times\exp\left[\frac{im}{2\hbar}\int\limits_0^T\frac{b\xi^2\eta^2-a}{\xi^2\eta^2}(\xi^2+\eta^2)(\dot\xi^2+\dot\eta^2)dt\right]=$$
$$=\int\limits_{-\infty}^{\infty}\frac{dE}{2\pi\hbar}\,\mathrm{e}^{-iET/\hbar}\int\limits_0^\infty ds''K(\xi'',\xi',\eta'',\eta';s''), \tag{3.48}$$

with the path integral $K(s'')$ given by

$$K(\xi'',\xi',\eta'',\eta';s'') = K_\xi(\xi'',\xi';s'')K_\eta(\eta'',\eta';s'') = \\ = \int\limits_{\xi(0)=\xi'}^{\xi(s'')=\xi''} \mathcal{D}\xi(s)\exp\left\{\frac{i}{\hbar}\int\limits_0^{s''}\left[\frac{m}{2}\dot{\xi}^2 + E\left(b\xi^2 - \frac{a}{\xi^2}\right)\right]ds\right\}\times \\ \times \int\limits_{\eta(0)=\eta'}^{\eta(s'')=\eta''} \mathcal{D}\eta(s)\exp\left\{\frac{i}{\hbar}\int\limits_0^{s''}\left[\frac{m}{2}\dot{\eta}^2 + E\left(b\eta^2 - \frac{a}{\eta^2}\right)\right]ds\right\}. \quad (3.49)$$

The Green function of the kernel $G_\eta(\mathcal{E})$ is given by $\left(\lambda^2 = \frac{1}{4} - 2m|a|E/\hbar^2,\right.$ $\left.\omega^2 = -2bE/m,\ a<0\right)$:

$$G_\xi(\xi'',\xi';\mathcal{E}) = \frac{\Gamma\left[\frac{1}{2}(1+\lambda-\mathcal{E}/\hbar\omega)\right]}{\hbar\omega\sqrt{\xi'\xi''}\Gamma(1+\lambda)}\times \\ \times W_{\mathcal{E}/2\hbar\omega,\lambda/2}\left(\frac{m\omega}{\hbar}\eta_>^2\right)M_{\mathcal{E}/2\hbar\omega,\lambda/2}\left(\frac{m\omega}{\hbar}\eta_<^2\right). \quad (3.50)$$

$M_{\mu,\nu}(z), W_{\mu,\nu}(z)$ are the Whittaker functions [15, p. 1059]. Inserting the solution for the radial kernel in ξ we get for $G(E)$ (we can also interchange ξ and η, however, the final result will be symmetric in ξ and η, so this is not necessary)

$$G(\xi'',\xi',\eta'',\eta';E) = \frac{m\sqrt{\xi'\xi''}}{i\hbar}\int\limits_0^\infty \frac{\omega ds''}{\sin\omega s''}\exp\left[-\frac{m\omega}{2i\hbar}({\xi'}^2+{\xi'}^2)\cot\omega s''\right]\times \\ \times I_\lambda\left(\frac{m\omega\xi'\xi''}{i\hbar\sin\omega s''}\right)\int\frac{d\mathcal{E}}{2\pi i}\,\mathrm{e}^{-i\mathcal{E}s''/\hbar}\frac{\Gamma\left[\frac{1}{2}(1+\lambda-\mathcal{E}/\hbar\omega)\right]}{\hbar\omega\sqrt{\eta'\eta''}\,\Gamma(1+\lambda)}W_{\mathcal{E}/2\hbar\omega,\lambda/2}\left(\frac{m\omega}{\hbar}\eta_>^2\right)\times \\ \times M_{\mathcal{E}/2\hbar\omega,\lambda/2}\left(\frac{m\omega}{\hbar}\eta_<^2\right) = \frac{\hbar}{i\pi^2}\int\frac{d\mathcal{E}}{2\pi i}\sqrt{\frac{\xi'\xi''}{\eta'\eta''}}\int\limits_0^\infty\frac{dp\,p\sinh\pi p}{\frac{\hbar^2}{2m|a|}\left(p^2+\frac{1}{4}\right)-E}\times \\ \times\frac{\Gamma\left[\frac{1}{2}(1+\lambda-\mathcal{E})\right]}{\Gamma(1+\lambda)}W_{\mathcal{E}/2,\lambda/2}\left(\frac{m\omega}{\hbar}\eta_>^2\right)M_{\mathcal{E}/2,\lambda/2}\left(\frac{m\omega}{\hbar}\eta_<^2\right)\times \\ \times\int\limits_0^\infty\frac{\omega ds''}{\sin\omega s''}\exp\left[-i\mathcal{E}\omega s'' - \frac{m\omega}{2i\hbar}({\xi'}^2+{\xi'}^2)\cot\omega s''\right]K_{ip}\left(\frac{m\omega\xi'\xi''}{i\hbar\sin\omega s''}\right), \quad (3.51)$$

where we have used the dispersion relation [16]

$$I_\lambda(z) = \frac{\hbar^2}{\pi^2 m}\int\limits_0^\infty \frac{dp\,p\sinh\pi p}{\dfrac{\hbar^2}{2m}\left(p^2+\dfrac{1}{4}\right)-E}K_{ip}(z), \tag{3.52}$$

and have redefined $\mathcal{E}\to\mathcal{E}\hbar\omega$. The s''-integral $I(p)$ is evaluated in the following way: We set $u=\omega s''$, followed by a Wick rotation, yielding

$$\int\limits_0^\infty \frac{du}{\sinh u}\exp\left[-u\mathcal{E}-\frac{m\omega}{2\hbar}({\xi'}^2+{\xi'}^2)\coth u\right]K_{ip}\left(\frac{m\omega\xi'\xi''}{\hbar\sinh u}\right)=$$

(subsitution $\sinh u = 1/\sinh v$:)

$$=\int\limits_0^\infty dv\left(\coth\frac{v}{2}\right)^{\mathcal{E}}\exp\left[-\frac{m\omega}{2\hbar}({\xi'}^2+{\xi'}^2)\cosh v\right]K_{ip}\left(\frac{m\omega}{\hbar}\xi'\xi''\sinh v\right)=$$

$$=\frac{\hbar}{2m\omega\xi'\xi''}\left|\Gamma\left[\frac{1}{2}(1+ip-\mathcal{E})\right]\right|^2 W_{\mathcal{E}/2,ip/2}\left(\frac{m\omega}{\hbar}{\xi''}^2\right)W_{\mathcal{E}/2,ip/2}\left(\frac{m\omega}{\hbar}{\xi'}^2\right), \tag{3.53}$$

where we have applied the integral representation [15, p. 729]:

$$\int\limits_0^\infty\left(\coth\frac{x}{2}\right)^{2\nu}\exp\left(-\frac{a_1+a_2}{2}t\cosh x\right)K_{2\mu}(t\sqrt{a_1a_2}\,\sinh x)dx=$$

$$=\frac{\Gamma\left(\dfrac{1}{2}+\mu-\nu\right)}{t\sqrt{a_1a_2}\Gamma(1+2\mu)}W_{\nu,\mu}(a_1t)W_{\nu,\mu}(a_2t). \tag{3.54}$$

Collecting terms, this gives for $G(E)$

$$G(\xi'',\xi',\eta'',\eta';E)=\frac{i\hbar}{4\pi^2}(\xi'\xi''\eta'\eta'')^{-1/2}\int d\mathcal{E}\times$$

$$\times\int\limits_0^\infty\frac{dp\,p\sinh\pi p}{\dfrac{\hbar^2}{2m|a|}\left(p^2+\dfrac{1}{4}\right)-E}\frac{\left|\Gamma\left[\dfrac{1}{2}(1+ip-\mathcal{E})\right]\right|^2}{\tilde p^2}\times$$

$$\times W_{\mathcal{E}/2,ip/2}\left(i\tilde p{\xi''}^2\right)W^*_{\mathcal{E}/2,ip/2}\left(i\tilde p{\xi'}^2\right)\times$$

$$\times\left|\Gamma\left[\frac{1}{2}(1+ip-\mathcal{E})\right]\right|^2 W_{\mathcal{E}/2,ip/2}\left(i\tilde p\eta_>^2\right)\frac{W_{\mathcal{E}/2,ip/2}\left(i\tilde p\eta_<^2\right)}{\Gamma\left[\dfrac{1}{2}(1-ip-\mathcal{E})\right]\Gamma(1+ip)}. \tag{3.55}$$

We have abbreviated $\omega = i\frac{\hbar}{m}\sqrt{b\left(p^2+\frac{1}{4}\right)/|a|} \equiv i\frac{\hbar}{m}\tilde{p}$. In order to evaluate this expression on the cut, we use the following representation as given in [52, p. 298]

$$W_{\chi,\mu(z)} = \frac{\pi}{\sin(2\pi\mu)}\times \\ \times\left[\frac{M_{\chi,-\mu}(z)}{\Gamma\left(\frac{1}{2}+\mu-\chi\right)\Gamma(1-2\mu)} - \frac{M_{\chi,\mu}(z)}{\Gamma\left(\frac{1}{2}-\mu-\chi\right)\Gamma(1+2\mu)}\right]. \quad (3.56)$$

This gives the final expression for the Green function:

$$G(\xi'',\xi',\eta'',\eta';E) = \hbar(\xi'\xi''\eta'\eta'')^{-1/2}\times \\ \times\int d\mathcal{E}\int_0^\infty \frac{dp\,p\sinh\pi p}{\frac{\hbar^2}{2m|a|}\left(p^2+\frac{1}{4}\right)-E}\frac{\left|\Gamma\left[\frac{1}{2}(1+ip-\mathcal{E})\right]\right|^4}{2\pi\tilde{p}^2}\times \\ \times W_{\mathcal{E}/2,ip/2}\left(i\tilde{p}\xi''^2\right)W^*_{\mathcal{E}/2,ip/2}\left(i\tilde{p}\xi'^2\right)W_{\mathcal{E}/2,ip/2}\left(i\tilde{p}\xi''^2\right)W^*_{\mathcal{E}/2,ip/2}\left(i\tilde{p}\eta'^2\right). \quad (3.57)$$

The wave functions can be easily read off from this expression:

$$\Psi_{\mathcal{E},p}(\xi,\eta) = \sqrt{\frac{p\sinh\pi p}{2\pi\xi\eta}}\frac{\left|\Gamma\left[\frac{1}{2}(1+ip-\mathcal{E})\right]\right|^2}{\tilde{p}}\times \\ \times W_{\mathcal{E}/2,ip/2}\left(i\tilde{p}\xi^2\right)W_{\mathcal{E}/2,ip/2}\left(i\tilde{p}\xi^2\right). \quad (3.58)$$

Note the simplifications in the case $a=-1, b=1$.

Recall that we have redefined $\mathcal{E}\to\hbar\omega\mathcal{E}$ in (3.53). In the limiting case $b=0$ and $\mathcal{E}/\hbar\omega$ reinserted, this means that $\mathcal{E}/2\hbar\omega\to\infty$ for $\omega\to 0$; however the product of the index and the argument of the Whittaker functions $(\mathcal{E}/2\hbar\omega)\cdot(m\omega/\hbar)$ is constant. In this limit the Whittaker functions yield K_ν- and $H_\nu^{(1)}$-Bessel functions, as it should be for the semicircular–parabolic coordinate system on the two-dimensional hyperboloid. Of course, the parabolic system in $\mathbb{R}^2$ can be recovered, e.g., staring from (3.49). This concludes the discussion.

3.4. The Path Integral in Elliptic Coordinates on $D_{\rm II}$. The classical Lagrangian and Hamiltonian are given by

$$\mathcal{L}(\omega,\dot\omega,\varphi,\dot\varphi) = \frac{m}{2}\frac{bd^2\cosh^2\omega\cos^2\varphi-a}{\cosh^2\omega\cos^2\varphi}(\cosh^2\omega-\cos^2\varphi)(\dot\omega^2+\dot\varphi^2) = \\ = \frac{m}{2}\left[\left(bd^2\cosh^2\omega+\frac{a}{\cosh^2\omega}\right)-\left(bd^2\cos^2\varphi+\frac{a}{\cos^2\varphi}\right)\right](\dot\omega^2+\dot\varphi^2), \quad (3.59)$$

$$\mathcal{H}(\omega, p_\omega, \varphi, p_\varphi) = \frac{1}{2m} \frac{\cosh^2\omega\cos^2\varphi}{(bd^2\cosh^2\omega\cos^2\varphi - a)(\cosh^2\omega - \cos^2\varphi)}(p_\omega^2 + p_\varphi^2). \tag{3.60}$$

In the following we use

$$\sqrt{g} = \frac{bd^2\cosh^2\omega\cos^2\varphi - a}{\cosh^2\omega\cos^2\varphi}(\cosh^2\omega - \cos^2\varphi).$$

For the momentum operators we obtain

$$p_\omega = \frac{\hbar}{i}\left[\frac{\partial}{\partial\omega} + \frac{\tanh\omega}{\sqrt{g}}\left(bd^2\cosh^2\omega - \frac{a}{\cosh^2\omega}\right)\right], \tag{3.61}$$

$$p_\varphi = \frac{\hbar}{i}\left[\frac{\partial}{\partial\varphi} + \frac{\tan\varphi}{\sqrt{g}}\left(bd^2\cos^2\varphi - \frac{a}{\cos^2\varphi}\right)\right]. \tag{3.62}$$

This gives for the quantum Hamiltonian

$$\begin{aligned} H = -\frac{\hbar^2}{2m}\frac{\cosh^2\omega\cos^2\varphi}{(bd^2\cosh^2\omega\cos^2\varphi - a)(\cosh^2\omega - \cos^2\varphi)}\left(\frac{\partial^2}{\partial\omega^2} + \frac{\partial^2}{\partial\varphi^2}\right) = \\ = \frac{1}{2m}\frac{1}{\sqrt[4]{g}}(p_\omega^2 + p_\phi^2)\frac{1}{\sqrt[4]{g}}. \end{aligned} \tag{3.63}$$

Therefore we obtain for the path integral (the time-transformation function reads $f(\omega, \varphi) = \sqrt{g}$)

$$\begin{aligned} &K(\omega'', \omega', \varphi'', \varphi'; T) = \\ &= \int\limits_{\omega(t')=\omega'}^{\omega(t'')=\omega''} \mathcal{D}\omega(t) \int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''} \mathcal{D}\varphi(t) \frac{bd^2\cosh^2\omega\cos^2\varphi - a}{\cosh^2\omega\cos^2\varphi}(\cosh^2\omega - \cos^2\varphi) \times \\ &\quad \times \exp\left[\frac{im}{2\hbar}\int_0^T \frac{bd^2\cosh^2\omega\cos^2\varphi - a}{\cosh^2\omega\cos^2\varphi}(\cosh^2\omega - \cos^2\varphi)(\dot\omega^2 + \dot\varphi^2)dt\right] = \\ &= \int_{-\infty}^{\infty}\frac{dE}{2\pi\hbar}\,\mathrm{e}^{-iET/\hbar}\int_0^\infty ds'' K(\omega'', \omega', \varphi'', \varphi'; s''), \end{aligned} \tag{3.64}$$

with the path integral $K(\omega'',\omega',\varphi'',\varphi';s'')$ given by $(a<0)$

$$K(\omega'',\omega',\varphi'',\varphi';s'') = \int\limits_{\omega(0)=\omega'}^{\omega(s'')=\omega''}\mathcal{D}\omega(s)\int\limits_{\varphi(0)=\varphi'}^{\varphi(s'')=\varphi''}\mathcal{D}\varphi(s)\times$$
$$\times\exp\left\{\frac{i}{\hbar}\int\limits_0^{s''}\left[\frac{m}{2}(\dot\omega^2+\dot\varphi^2)+Ebd^2(\cosh^2\omega-\cos^2\varphi)+\times\right.\right.$$
$$\left.\left.\times|a|E\left(\frac{1}{\cosh^2\omega}+\frac{1}{\cos^2\varphi}\right)\right]ds\right\}. \quad (3.65)$$

For $a=0$ we recover elliptic coordinates in $\mathbb{R}^2$. This path integral has the form of the spheroidal coordinate system. Actually, almost the same path integral was investigated in [24, p. 122] in connection with the elliptic-paraboloid coordinate system on the three-dimensional hyperboloid. Let us set:

$$\lambda^2=\frac{1}{4}-\frac{2m|a|E}{\hbar^2},\quad \tilde\kappa^2=-\frac{2mbd^2E}{\hbar^2}.$$

In [24] we have derived the following heuristic path integral identity:

$$\int\limits_{\mu(t')=\mu'}^{\mu(t'')=\mu''}\mathcal{D}\mu(t)\int\limits_{\nu(t')=\nu'}^{\nu(t'')=\nu''}\mathcal{D}\nu(t)d^2(\sinh^2\mu+\sin^2\nu)\times$$
$$\times\exp\left\{\frac{i}{\hbar}\int\limits_{t'}^{t''}\left[\frac{m}{2}d^2(\sinh^2\mu+\sin^2\nu)(\dot\mu^2+\dot\nu^2)-\frac{\hbar^2}{2md^2}\frac{\lambda^2-\frac{1}{4}}{\sinh^2\mu\sin^2\nu}\right]dt\right\}=$$
$$=d\sqrt{\sin\nu'\sin\nu''\sinh\mu'\sinh\mu''}\sum_{l=0}^{\infty}\frac{2l+1}{\pi}\frac{\Gamma(l-\lambda+1)}{\Gamma(l+\lambda+1)}\int\limits_0^{\infty}p^2dp\,\mathrm{e}^{-i\hbar p^2T/2m}\times$$
$$\times\operatorname{ps}_l^{\lambda\,*}(\cos\nu';p^2d^2)\operatorname{ps}_l^{\lambda}(\cos\nu'';p^2d^2)S_l^{\lambda\,(1)\,*}(\cosh\mu';pd)S_l^{\lambda\,(1)}(\cosh\mu'';pd), \quad (3.66)$$

where $S_l^{n\,(1)},\operatorname{ps}_l^n$ are prolate spheroidal wave functions [53]. By considering a proper analytic continuation and observing

$$\operatorname{ps}_\nu^\mu(x;0)=P_\nu^\mu(x)\quad(|x|\leqslant 1), \quad (3.67)$$

we found the solution ($a > 0$, $|\vartheta| < \pi/2$, $\varrho \in \mathbb{R}$):

$$\int\limits_{a(t')=a'}^{a(t'')=a''} \mathcal{D}a(t) \int\limits_{\vartheta(t')=\vartheta'}^{\vartheta(t'')=\vartheta''} \mathcal{D}\vartheta(t) \frac{\cosh^2 a - \cos^2\vartheta}{\cosh^3 a \cos^3\vartheta} \int\limits_{\varrho(t')=\varrho'}^{\varrho(t'')=\varrho''} \mathcal{D}\varrho(t) \times$$

$$\times \exp\left[\frac{im}{2\hbar}\int_{t'}^{t''} \frac{(\cosh^2 a - \cos^2\vartheta)(\dot{a}^2 + \dot{\vartheta}^2) + \dot{\varrho}^2}{\cosh^2 a \cos^2\vartheta} dt - \frac{3i\hbar T}{8m}\right] =$$

$$= \sqrt{\cosh a' \cosh a''}\, \cos\vartheta' \cos\vartheta'' \int\limits_{\mathbb{R}} \frac{d\kappa}{2\pi} \mathrm{e}^{i\kappa(\varrho''-\varrho')} \times$$

$$\times \int\limits_0^\infty dp\, \sinh\pi p \int\limits_0^\infty \frac{dk\, k \sinh\pi k}{(\cosh^2\pi k + \sinh^2\pi p)^2}\, \mathrm{e}^{-i\hbar T(p^2+1)/2m} \times$$

$$\times \sum_{\epsilon,\epsilon'=\pm1} S_{ip-1/2}^{ik\,(1)}(\epsilon\tanh a''; i\kappa) S_{ip-1/2}^{ik\,(1)\,*}(\epsilon\tanh a'; i\kappa) \times$$

$$\times \mathrm{ps}_{ik-1/2}^{ip}(\epsilon'\sin\vartheta''; -\kappa^2) \mathrm{ps}_{ik-1/2}^{ip\,*}(\epsilon'\sin\vartheta'; -\kappa^2). \quad (3.68)$$

Let us use (3.68): Of course, the variable ϱ is omitted, we use only the emerging parameter κ. The parameters λ in (3.66) and (3.65) are the same up to the factor $|a|$. The main difference is in the parameters κ and $\tilde{\kappa}$, the latter being imaginary. Inserting in $\tilde{\kappa}$ the energy spectrum $|a|E = \dfrac{\hbar^2}{2m}\left(p^2 + \dfrac{1}{4}\right)$ gives $\tilde{\kappa} = id\sqrt{b\left(p^2 + \dfrac{1}{4}\right) \Big/ |a|} \equiv i\tilde{p}$. Combining (3.66) and (3.68) yields for the path integral (3.65):

$$G(\omega'', \omega', \varphi'', \varphi'; E) =$$

$$= \sqrt{\cos\varphi' \cos\varphi''} \int\limits_0^\infty \frac{dp\, \sinh\pi p}{\dfrac{\hbar^2}{2m|a|}\left(p^2 + \dfrac{1}{4}\right) - E} \int\limits_0^\infty \frac{dk\, k \sinh\pi k}{(\cosh^2\pi k + \sinh^2\pi p)^2} \times$$

$$\times \sum_{\epsilon,\epsilon'=\pm1} S_{ip-1/2}^{ik\,(1)}(\epsilon\tanh\omega''; -\tilde{p}) S_{ip-1/2}^{ik\,(1)\,*}(\epsilon\tanh\omega'; -\tilde{p}) \times$$

$$\times \mathrm{ps}_{ik-1/2}^{ip}(\epsilon'\sin\varphi''; \tilde{p}^2) \mathrm{ps}_{ik-1/2}^{ip\,*}(\epsilon'\sin\varphi'; \tilde{p}^2)\,. \quad (3.69)$$

The wave functions have the form:

$$\Psi_{p,k;\epsilon,\epsilon'}(\omega, \varphi) = \sqrt{\cos\varphi} \times$$

$$\times \frac{\sqrt{\sinh\pi pk \sinh\pi k}}{\cosh^2\pi k + \sinh^2\pi p} S_{ip-1/2}^{ik\,(1)}(\epsilon\tanh\omega; -\tilde{p})\, \mathrm{ps}_{ik-1/2}^{ip}(\epsilon'\sin\varphi; \tilde{p}^2). \quad (3.70)$$

Of course, the cases $a=-1, b=1$ simplify the formulas a little bit. For $b=0$ the spheroidal wave functions give the limiting case (3.67), therefore the solution of the elliptic-parabolic system on the two-dimensional hyperboloid emerges [24]. This completes the discussion on $D_{\rm II}$.

4. DARBOUX SPACE $D_{\rm III}$

The coordinate systems to be considered in the Darboux space $D_{\rm III}$ are as follows:

$$\begin{aligned}
&((u,v)\text{ system}) && x=v+iu,\ y=v-iu, && (4.1)\\
&(\text{Polar:}) && \xi=\varrho\cos\varphi,\ \eta=\varrho\sin\varphi\ (\varrho>0,\varphi\in[0,2\pi]), && (4.2)\\
&(\text{Parabolic:}) && \xi=2\,\mathrm{e}^{-u/2}\cos\frac{v}{2}\ \eta=2\,\mathrm{e}^{-u/2}\sin\frac{v}{2}, &&\\
& && u=\ln\frac{4}{\xi^2+\eta^2},\ v=\arcsin\frac{2\xi\eta}{\xi^2+\eta^2}\ (\xi\in\mathbb{R},\eta>0), && (4.3)\\
&(\text{Elliptic:}) && \xi=d\cosh\omega\cos\varphi,\ \eta=d\sinh\omega\sin\varphi && (4.4)\\
& && (\omega>0,\varphi\in[-\pi,\pi]), &&\\
&(\text{Hyperbolic:}) && \xi=\frac{\mu-\nu}{2\sqrt{\mu\nu}}+\sqrt{\mu\nu},\ \eta=i\frac{\mu-\nu}{2\sqrt{\mu\nu}}-\sqrt{\mu\nu}\ (\mu,\nu>0). && (4.5)
\end{aligned}$$

For the line element we get (we also display, where the metric is rescaled in such a way that we set $a=b=1$ [38]):

$$\begin{aligned}
ds^2 &= \big(a\,\mathrm{e}^{-(x+y)/2}+b\,\mathrm{e}^{-(x+y)}\big)dxdy,\\
((u,v)\text{ coordinates:}) &= \mathrm{e}^{-2u}(b+a\,\mathrm{e}^{u})(du^2+dv^2)=\\
&= (\mathrm{e}^{-u}+\mathrm{e}^{-2u})(du^2+dv^2), && (4.6)\\
(\text{Polar:}) &= \left(a+\frac{b}{4}\varrho^2\right)(d\varrho^2+\varrho^2d\varphi^2)=\\
&= \left(1+\frac{1}{4}\varrho^2\right)(d\varrho^2+\varrho^2d\varphi^2), && (4.7)\\
(\text{Parabolic:}) &= \left(a+\frac{b}{4}(\xi^2+\eta^2)\right)(d\xi^2+d\eta^2)=\\
&= \left(1+\frac{1}{4}(\xi^2+\eta^2)\right)(d\xi^2+d\eta^2), && (4.8)\\
(\text{Elliptic:}) &= \left(a+\frac{b}{4}d^2(\sinh^2\omega+\cos^2\varphi)\right)\times\\
&\times\ d^2(\sinh^2\omega+\sin^2\varphi)(d\omega^2+d\varphi^2), && (4.9)\\
(\text{Hyperbolic:}) &= \left(a+\frac{b}{2}(\mu-\nu)\right)(\mu+\nu)\left(\frac{d\mu^2}{\mu^2}-\frac{d\nu^2}{\nu^2}\right). && (4.10)
\end{aligned}$$

For the Gaussian curvature we find

$$K = -\frac{ab\,\mathrm{e}^{-3u}}{(b\,\mathrm{e}^{-2u} + a\,\mathrm{e}^{-u})^4}. \tag{4.11}$$

For, e.g., $a = 1, b = 0$ we recover the two-dimensional flat space with the corresponding coordinate systems. To assure the positive definiteness of the metric (1.3), we can require $a, b > 0$.

4.1. The Path Integral in (u, v) Coordinates on $D_{\rm III}$. The classical Lagrangian and Hamiltonian are given by

$$\mathcal{L}(u, \dot{u}, v, \dot{v}) = \frac{m}{2}\frac{b+a\,\mathrm{e}^{u}}{\mathrm{e}^{2u}}(\dot{u}^2 + \dot{v}^2),\ \mathcal{H}(u, p_u, v, p_v) = \frac{1}{2m}\frac{\mathrm{e}^{2u}}{b+a\,\mathrm{e}^{u}}(p_u^2 + p_v^2). \tag{4.12}$$

The canonical momenta are given by

$$p_u = \frac{\hbar}{i}\left(\frac{\partial}{\partial u} - \frac{1}{2}\frac{a\,\mathrm{e}^{-u} + 2b\,\mathrm{e}^{-2u}}{a\,\mathrm{e}^{-u}+b\,\mathrm{e}^{-2u}}\right), \quad p_v = \frac{\hbar}{i}\frac{\partial}{\partial v}, \tag{4.13}$$

and for the quantum Hamiltonian we find

$$H = -\frac{\hbar^2}{2m}\frac{1}{a\,\mathrm{e}^{-u} + b\,\mathrm{e}^{-2u}}\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) = \tag{4.14}$$

$$= \frac{1}{2m}\sqrt{\frac{1}{a\,\mathrm{e}^{-u} + b\,\mathrm{e}^{-2u}}}\left(p_u^2 + p_v^2\right)\sqrt{\frac{1}{a\,\mathrm{e}^{-u} + b\,\mathrm{e}^{-2u}}}. \tag{4.15}$$

Therefore we obtain for the path integral

$$\begin{aligned} K(u'', u', v'', v'; T) &= \int\limits_{u(t')=u'}^{u(t'')=u''}\mathcal{D}u(t)\int\limits_{v(t')=v'}^{v(t'')=v''}\mathcal{D}v(t)(a\,\mathrm{e}^{-u} + b\,\mathrm{e}^{-2u})\times \\ &\times\exp\left[\frac{im}{2\hbar}\int\limits_0^T(a\,\mathrm{e}^{-u} + b\,\mathrm{e}^{-2u})(\dot{u}^2 + \dot{v}^2)dt\right] = \\ &= \int\limits_{-\infty}^{\infty}\frac{dE}{2\pi\hbar}\,\mathrm{e}^{-iET/\hbar}\int\limits_0^{\infty}ds''K(u'', u', v'', v'; s''), \end{aligned} \tag{4.16}$$

with the time-transformed path integral $K(s'')$ given by $(f(u) = (a\,\mathrm{e}^{-u} + b\,\mathrm{e}^{-2u}) = \sqrt{g})$

$$K(u'',u',v'',v';s'') = \int\limits_{u(0)=u'}^{u(s'')=u''} \mathcal{D}u(s) \int\limits_{v(0)=v'}^{v(s'')=v''} \mathcal{D}v(s) \times$$
$$\times \exp\left\{ \frac{i}{\hbar} \int\limits_0^{s''} \left[\frac{m}{2}(\dot{u}^2 + \dot{v}^2) + Eb\left(\mathrm{e}^{-2u} + \frac{a}{b}\mathrm{e}^{-u}\right)\right] ds \right\}. \quad (4.17)$$

We observe that the path integral in the variable u is a path integral for the Morse potential

$$V(x) = \frac{V_0^2\hbar^2}{2m}\left(\mathrm{e}^{-2x} - 2\alpha\,\mathrm{e}^{-x}\right)$$

with $V_0 = \sqrt{-2mbE}/\hbar$ and $\alpha = -a/2b$; the Green functions in u and v are given by [34]

$$G_u(u'';u';\mathcal{E}) = \frac{m\Gamma\left(\frac{1}{2} + \sqrt{-2m\mathcal{E}}/\hbar + a\sqrt{-2mbE}/2b\hbar\right)}{\hbar\sqrt{-2mbE}\,\Gamma(1 + 2\sqrt{-2m\mathcal{E}}/\hbar)}\,\mathrm{e}^{(u'+u'')/2} \times$$
$$\times W_{-a\sqrt{-2mbE}/2b\hbar,\sqrt{-2m\mathcal{E}}/\hbar}\left(\frac{\sqrt{-8mbE}}{\hbar}\,\mathrm{e}^{-u_<}\right) \times$$
$$\times M_{-a\sqrt{-2mbE}/2b\hbar,\sqrt{-2m\mathcal{E}}/\hbar}\left(\frac{\sqrt{-8mbE}}{\hbar}\,\mathrm{e}^{-u_>}\right), \quad (4.18)$$

$$G_v(v'';v';\mathcal{E}) = \frac{1}{2\pi}\sum_{l=-\infty}^{\infty} \frac{\mathrm{e}^{il(v''-v')}}{\hbar^2 l^2/2m - \mathcal{E}}. \quad (4.19)$$

This gives for the Green function in (u,v) coordinates the solution (note $\sqrt{-2m\mathcal{E}}/\hbar \to +l$)

$$G(u'',u',v'',v';E) = \sum_{l=-\infty}^{\infty} \frac{\mathrm{e}^{il(v''-v')}}{2\pi} \frac{m\Gamma\left(\frac{1}{2}+l+\frac{a\sqrt{-2mbE}}{2b\hbar}\right)}{\hbar\sqrt{-2mbE}\,\Gamma(1+2l)}\mathrm{e}^{(u'+u'')/2} \times$$
$$\times W_{-a\sqrt{-2mbE}/2b\hbar,l}\left(2\frac{\sqrt{-2mbE}}{\hbar}\,\mathrm{e}^{-u_<}\right) M_{-a\sqrt{-2mbE}/2b\hbar,l}\left(2\frac{\sqrt{-2mbE}}{\hbar}\,\mathrm{e}^{-u_>}\right). \quad (4.20)$$

In order to extract the wave-functions we use the following representation:

$$\frac{1}{\sin\alpha}\exp\big[-(x+y)\cot\alpha\big]I_{2\mu}\bigg(\frac{2\sqrt{xy}}{\sin\alpha}\bigg)=$$
$$=\frac{1}{2\pi\sqrt{xy}}\int_{-\infty}^{\infty}\frac{\Gamma\Big(\frac{1}{2}+\mu+ip\Big)\Gamma\Big(\frac{1}{2}+\mu-ip\Big)}{\Gamma^2(1+2\mu)}\times$$
$$\times\,\mathrm{e}^{-2\alpha p+\pi p}M_{+ip,\mu}(-2ix)M_{-ip,\mu}(+2iy)dp. \tag{4.21}$$

This relation can be derived by using an integral representation as given by Buchholz [4, p. 158]. Application of this representation by exploiting $\mathcal{E}=\hbar^2l^2/2m$ and evaluating the residuum at $E=\hbar^2p^2/2m$ ($\omega=\sqrt{-2E/m}$, we set $a=b=1$, and utilize the calculation of [16] by inserting the path integral solution of the radial harmonic oscillator in (4.17) together with an appropriate coordinate transformation) yields:

$$G(u'',u',v'',v';E)=\sum_{l=-\infty}^{\infty}\frac{\mathrm{e}^{il(v''-v')}}{2\pi}\frac{1}{\pi i}\int\frac{d\mathcal{E}}{\frac{\hbar^2l^2}{2m}-\mathcal{E}}\sqrt{-\frac{8mE}{\hbar^2}}\times$$
$$\times\int_0^{\infty}\frac{d\sigma}{\sin\omega\sigma}\exp\bigg[\frac{4iaE}{\hbar}\sigma-\frac{\sqrt{8mE}}{\hbar}\Big(\mathrm{e}^{-u'}+\mathrm{e}^{-u''}\Big)\cot\omega\sigma\bigg]\times$$
$$\times I_{\sqrt{8m\mathcal{E}}/\hbar}\bigg(\frac{\sqrt{8mE}\,\mathrm{e}^{-(u'+u'')/2}}{\hbar\sin\omega\sigma}\bigg)=\sum_{l=-\infty}^{\infty}\frac{\mathrm{e}^{il(v''-v')}}{2\pi}\mathrm{e}^{(u'+u'')/2}\times$$
$$\times\int_0^{\infty}\frac{\mathrm{e}^{\pi p/2}dp}{\frac{\hbar^2p^2}{2m}-E}\frac{\Big|\Gamma\Big(\frac{1}{2}+l+ip\Big)\Big|^2}{2\pi\Gamma^2(1+2l)}M_{ip/2,l}\Big(-2ip\,\mathrm{e}^{-u'}\Big)M_{-ip/2,l}\Big(2ip\,\mathrm{e}^{-u''}\Big). \tag{4.22}$$

And we can read off the wave functions

$$\Psi_{p,l}(u,v)=\frac{\mathrm{e}^{ilv}}{\sqrt{2\pi}}\frac{\mathrm{e}^{\pi p/4}}{\sqrt{2\pi}}\frac{\Gamma\Big(\frac{1}{2}+l+ip/2\Big)}{\Gamma(1+2l)}M_{ip/2,l}\Big(-2i\mathrm{e}^{-u}\Big), \tag{4.23}$$

respectively with (a,b) reinserted:

$$\Psi_{p,l}(u,v)=\frac{\mathrm{e}^{ilv}}{\sqrt{2\pi}}\frac{\mathrm{e}^{\pi p/4}}{\sqrt{2\pi}}\frac{\Gamma\Big(\frac{1}{2}+l+iap/2\sqrt{b}\Big)}{\Gamma(1+2l)}M_{iap/\sqrt{b}2,l}\Big(-2ip\sqrt{b}\,\mathrm{e}^{-u}\Big). \tag{4.24}$$

Note that this evaluation is almost the same as in the path integral for the two-dimensional Coulomb potential [34]. This concludes the discussion of the (u,v) system on D_{III}.

4.2. The Path Integral in Polar Coordinates on D_{III}. In the coordinates (ϱ,φ) the Lagrangian and Hamiltonian take on the form

$$\begin{aligned}\mathcal{L}(\varrho,\dot\varrho,\varphi,\dot\varphi) &= \frac{m}{2}\left(a+\frac{b}{4}\varrho^2\right)(\dot\varrho^2+\varrho^2\dot\varphi^2),\\ \mathcal{H}(\varrho,p_\varrho,\varphi,p_\varphi) &= \frac{1}{2m}\frac{1}{a+\dfrac{b}{4}\varrho^2}\left(p_\varrho^2+\frac{1}{\varrho^2}p_\varphi^2\right).\end{aligned} \tag{4.25}$$

The canonical momenta are given by

$$p_\varrho=\frac{\hbar}{i}\left(\frac{\partial}{\partial\varrho}+\frac{b\varrho}{4a+b\varrho^2}+\frac{1}{2\varrho}\right),\quad p_\varphi=\frac{\hbar}{i}\frac{\partial}{\partial\varphi}. \tag{4.26}$$

Therefore the quantum Hamiltonian is given by

$$H = -\frac{\hbar^2}{2m}\frac{1}{a+\dfrac{b}{4}\varrho^2}\left(\frac{\partial^2}{\partial\varrho^2}+\frac{1}{\varrho}\frac{\partial}{\partial\varrho}+\frac{1}{\varrho^2}\frac{\partial^2}{\partial\varphi^2}\right)= \tag{4.27}$$

$$= \frac{1}{2m}\sqrt{\frac{1}{a+\dfrac{b}{4}\varrho^2}}\left(p_\varrho^2+\frac{1}{\varrho^2}p_\varphi^2\right)\sqrt{\frac{1}{a+\dfrac{b}{4}\varrho^2}}-\left(a+\frac{b}{4}\varrho^2\right)\frac{\hbar^2}{8m\varrho^2}, \tag{4.28}$$

and in this case we have an additional quantum potential $\propto\hbar^2$. This gives for the path integral $\left(f(\varrho)=a+\dfrac{b}{4}\varrho^2=\sqrt{g}\right)$

$$\begin{aligned}K(\varrho'',\varrho',\varphi'',\varphi';T) &= \int\limits_{\varrho(t')=\varrho'}^{\varrho(t'')=\varrho''}\mathcal{D}\varrho(t)\int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''}\mathcal{D}\varphi(t)\left(a+\frac{b}{4}\varrho^2\right)\varrho\times\\ &\times\exp\left\{\frac{i}{\hbar}\int\limits_0^T\left[\frac{m}{2}\left(a+\frac{b}{4}\varrho^2\right)(\dot\varrho^2+\varrho^2\dot\varphi^2)+\left(a+\frac{b}{4}\varrho^2\right)^{-1}\frac{\hbar^2}{8m\varrho^2}\right]dt\right\}=\\ &= \int\limits_{-\infty}^{\infty}\frac{dE}{2\pi\hbar}\,\mathrm{e}^{-iET/\hbar}\int\limits_0^\infty ds''K(\varrho'',\varrho',\varphi'',\varphi';s''),\end{aligned} \tag{4.29}$$

with the time-transformed path integral $K(s'')$ given by

$$K(\varrho'',\varrho',\varphi'',\varphi';s'') = \int\limits_{\varrho(0)=\varrho'}^{\varrho(s'')=\varrho''}\mathcal{D}\varrho(s)\varrho\int\limits_{\varphi(0)=\varphi'}^{\varphi(s'')=\varphi''}\mathcal{D}\varphi(s)\times$$

$$\times\exp\left\{\frac{i}{\hbar}\int\limits_0^{s''}\left[\frac{m}{2}(\dot\varrho^2+\varrho^2\dot\varphi^2)+E\left(a+\frac{b}{4}\varrho^2\right)+\frac{\hbar^2}{8m\varrho^2}\right]ds\right\} =$$

$$= \sum_{l=-\infty}^{\infty}\frac{\mathrm{e}^{il(\varphi''-\varphi')}}{2\pi\sqrt{\varrho'\varrho''}}\int\limits_{\varrho(0)=\varrho'}^{\varrho(s'')=\varrho''}\mathcal{D}\varrho(s)\times$$

$$\times\exp\left\{\frac{i}{\hbar}\int\limits_0^{s''}\left[\frac{m}{2}\dot\varrho^2+E\frac{b}{4}\varrho^2-\hbar^2\frac{l^2-\frac{1}{4}}{2m\varrho^2}\right]ds''+\frac{i}{\hbar}aEs''\right\} =$$

$$= \sum_{l=-\infty}^{\infty}\frac{\mathrm{e}^{il(\varphi''-\varphi')}}{2\pi}\frac{m\omega}{i\hbar\sin\omega s''}\times$$

$$\times\exp\left[-\frac{m\omega}{2i\hbar}({\varrho'}^2+{\varrho''}^2)\cot\omega s''+\frac{i}{\hbar}aEs''\right]I_l\left(\frac{m\omega\varrho'\varrho''}{i\hbar\sin\omega s''}\right) \quad (4.30)$$

($\omega^2=-Eb/2m$), where I have separated off the φ-path integration and in the last step inserted the path integral solution for the radial harmonic oscillator [34, 59]. We use the integral representation [15, p. 729]:

$$\int\limits_0^{\infty}\left(\coth\frac{x}{2}\right)^{2\nu}\exp\left(-\frac{a_1+a_2}{2}t\cosh x\right)I_{2\mu}(t\sqrt{a_1a_2}\sinh x)dx =$$

$$= \frac{\Gamma\left(\frac{1}{2}+\mu-\nu\right)}{t\sqrt{a_1a_2}\Gamma(1+2\mu)}M_{\nu,\mu}(a_1t)W_{\nu,\mu}(a_2t). \quad (4.31)$$

Therefore we obtain for the entire Green function (rescaling $b\to 4b^2$)

$$G(\varrho'',\varrho',\varphi'',\varphi';E) = \sum_{l=-\infty}^{\infty}\frac{\mathrm{e}^{il(\varphi''-\varphi')}}{2\pi}\int\limits_0^{\infty}\frac{m\omega ds''}{i\hbar\sin\omega s''}\times$$

$$\times\exp\left[-\frac{m\omega}{2i\hbar}({\varrho'}^2+{\varrho''}^2)\cot\omega s''+\frac{i}{\hbar}aEs''\right]I_l\left(\frac{m\omega\varrho'\varrho''}{i\hbar\sin\omega s''}\right)$$

(substitution $u=\omega s''=\hbar pbs''/m$, $E=\hbar^2p^2/2m$, and Wick rotation)

$$=\sum_{l=-\infty}^{\infty}\frac{e^{il(\varphi''-\varphi')}}{2\pi}\int_0^\infty\frac{du}{\sinh u}\exp\biggl[-\frac{ipb}{2}({\varrho'}^2+{\varrho''}^2)\coth u+iaup\biggr]I_l\biggl(\frac{ibp\varrho'\varrho''}{\sinh u}\biggr)$$

(substitution $\sinh u=1/\sinh v$)

$$=\sum_{l=-\infty}^{\infty}\frac{e^{il(\varphi''-\varphi')}}{2\pi}\int_0^\infty dv\biggl(\coth\frac{v}{2}\biggr)^{iap}\times$$
$$\times\exp\biggl[-\frac{ibp}{2}({\varrho'}^2+{\varrho''}^2)\cosh v\biggr]I_l(ibp\varrho'\varrho''\sinh v)$$

(reinserting $p=\sqrt{2mE}/\hbar$)

$$=\sum_{l=-\infty}^{\infty}\frac{e^{il(\varphi''-\varphi')}}{2\pi b\sqrt{\varrho'\varrho''}}\sqrt{-\frac{m}{2E}}\frac{\Gamma\biggl[\dfrac{1}{2}(1+l-a\sqrt{-2mE/B}/\hbar)\biggr]}{\Gamma(1+l)}\times$$
$$\times W_{a\sqrt{-2mE/b}/2\hbar,l/2}\biggl(b\sqrt{-\frac{2mE}{\hbar^2}}\,\varrho_>\biggr)M_{a\sqrt{-2mE/b}/2\hbar,l/2}\biggl(b\sqrt{-\frac{2mE}{\hbar^2}}\,\varrho_<\biggr).\tag{4.32}$$

In order to extract the wave functions we use again representation (4.21) and obtain

$$G(\varrho'',\varrho',\varphi'',\varphi';E)=\sum_{l=-\infty}^{\infty}\frac{e^{il(\varphi''-\varphi')}}{2\pi}\frac{1}{2\pi b\sqrt{\varrho'\varrho''}}\int_0^\infty\frac{dp\,e^{\pi p}}{\dfrac{\hbar^2p^2}{2m}-E}\times$$
$$\times\frac{\biggl|\Gamma\biggl[\dfrac{1}{2}(1+l+iap)\biggr]\biggr|^2}{\Gamma^2(1+l)}M_{iap/2,l/2}(-ibp\varrho')M_{-iap/2,l/2}(ibp\varrho''),\tag{4.33}$$

and the wave functions have the form

$$\Psi_{p,l}(\varrho,\varphi)=\frac{e^{il\varphi}}{\sqrt{2\pi}}\frac{e^{\pi p/2}}{\sqrt{2\pi b\varrho}}\frac{\Gamma\biggl[\dfrac{1}{2}(1+l+iap)\biggr]]}{l!}M_{iap/2,l/2}(-ibp\varrho).\tag{4.34}$$

Note that this system is very similar to the (u,v) system, the principal difference being another counting in l and the replacement $\varrho=e^{-u}$.

4.3. The Path Integral in Parabolic Coordinates on D_{III}. The classical Lagrangian and Hamiltonian are given by

$$\begin{aligned}
\mathcal{L}(\xi,\dot{\xi},\eta,\dot{\eta}) &= \frac{m}{2}\left(a+\frac{b}{4}(\xi^2+\eta^2)\right)(\dot{\xi}^2+\dot{\eta}^2),\\
\mathcal{H}(\xi,p_\xi,\eta,p_\xi) &= \frac{1}{2m}\frac{1}{a+\frac{b}{4}(\xi^2+\eta^2)}(p_\xi^2+p_\eta^2).
\end{aligned} \tag{4.35}$$

The canonical momenta are given by

$$p_\xi=\frac{\hbar}{i}\left(\frac{\partial}{\partial\xi}+\frac{b\xi}{a+\frac{b}{4}(\xi^2+\eta^2)}\right),\quad p_\eta=\frac{\hbar}{i}\left(\frac{\partial}{\partial\eta}+\frac{b\eta}{a+\frac{b}{4}(\xi^2+\eta^2)}\right), \tag{4.36}$$

and for the quantum Hamiltonian we find

$$H = -\frac{\hbar^2}{2m}\frac{1}{a+\frac{b}{4}(\xi^2+\eta^2)}\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\right)= \tag{4.37}$$

$$= \frac{1}{2m}\sqrt{\frac{1}{a+\frac{b}{4}(\xi^2+\eta^2)}}(p_\xi^2+p_\eta^2)\sqrt{\frac{1}{a+\frac{b}{4}(\xi^2+\eta^2)}}. \tag{4.38}$$

Therefore we obtain for the path integral

$$\begin{aligned}
K(\xi'',\xi',\eta'',\eta';T) &= \int\limits_{\xi(t')=\xi'}^{\xi(t'')=\xi''}\mathcal{D}\xi(t)\int\limits_{\eta(t')=\eta'}^{\eta(t'')=\eta''}\mathcal{D}\eta(t)\left(a+\frac{b}{4}(\xi^2+\eta^2)\right)\times\\
&\quad\times\exp\left[\frac{im}{2\hbar}\int_0^T\left(a+\frac{b}{4}(\xi^2+\eta^2)\right)(\dot{\xi}^2+\dot{\eta}^2)dt\right]=\\
&= \int_{-\infty}^{\infty}\frac{dE}{2\pi\hbar}\,\mathrm{e}^{-iET/\hbar}\int_0^\infty ds''K(\xi'',\xi',\eta'',\eta';s''),
\end{aligned} \tag{4.39}$$

with the time-transformed path integral $K(s'')$ given by (the time-transformation function reads $f(\xi,\eta)=\left(a+\dfrac{b}{4}(\xi^2+\eta^2)\right)=\sqrt{g}$)

$$K(\xi'',\xi',\eta'',\eta';s'')=\int\limits_{\xi(0)=\xi'}^{\xi(s'')=\xi''}\mathcal{D}\xi(s)\int\limits_{\eta(0)=\eta'}^{\eta(s'')=\eta''}\mathcal{D}\eta(s)\times$$
$$\times\exp\left\{\frac{i}{\hbar}\int\limits_0^{s''}\left[\frac{m}{2}(\dot\xi^2+\dot\eta^2)+E\frac{b}{4}(\xi^2+\eta^2)\right]ds''+a\frac{i}{\hbar}Es''\right\}. \quad (4.40)$$

The path integrals in ξ and η are path integrals for the harmonic oscillator, with $\omega^2=-Eb/2m$. The Green function for the harmonic oscillator in the variable ξ is given by [34]

$$G_\xi(\xi'',\xi';\mathcal{E})=$$
$$=\sqrt{\frac{m}{\pi\omega\hbar^2}}\,\Gamma\left(\frac{1}{2}-\frac{\mathcal{E}}{\hbar\omega}\right)D_{-\frac{1}{2}+\mathcal{E}/\hbar\omega}\left(\sqrt{\frac{2m\omega}{\hbar}}\,\xi_>\right)D_{-\frac{1}{2}+\mathcal{E}/\hbar\omega}\left(-\sqrt{\frac{2m\omega}{\hbar}}\,\xi_<\right), \quad (4.41)$$

and similarly for $G_\eta(\eta'',\eta';\mathcal{E})$. The $D_\nu(z)$ are parabolic cylinder functions [15, p. 1064]. This gives ($b\to 4b^2$)

$$G(\xi'',\xi',\eta'',\eta';E)=$$
$$=\int d\mathcal{E}\,\frac{m}{\pi\hbar^2 b}\sqrt{-\frac{m}{2E}}\,\Gamma\left(\frac{1}{2}+\frac{aE-\mathcal{E}}{b\hbar}\sqrt{-\frac{m}{2E}}\right)\Gamma\left(\frac{1}{2}+\frac{\mathcal{E}}{b\hbar}\sqrt{-\frac{m}{2E}}\right)\times$$
$$\times D_{-\frac{1}{2}+\frac{aE-\mathcal{E}}{b\hbar}\sqrt{-\frac{m}{2E}}}\left(\sqrt[4]{-\frac{8mEb^2}{\hbar^2}}\xi_>\right)D_{-\frac{1}{2}+\frac{aE-\mathcal{E}}{b\hbar}\sqrt{-\frac{m}{2E}}}\left(-\sqrt[4]{-\frac{8mEb^2}{\hbar^2}}\xi_<\right)\times$$
$$\times D_{-\frac{1}{2}+\frac{\mathcal{E}}{b\hbar}\sqrt{-\frac{m}{2E}}}\left(\sqrt[4]{-\frac{8mEb^2}{\hbar^2}}\,\eta_>\right)D_{-\frac{1}{2}+\frac{\mathcal{E}}{b\hbar}\sqrt{-\frac{m}{2E}}}\left(-\sqrt[4]{-\frac{8mEb^2}{\hbar^2}}\,\eta_<\right). \quad (4.42)$$

Considering (4.40), we observe that it has the same form as the path integral for the Coulomb potential in two dimensions in parabolic coordinates which was solved in [8, 23, 27, 34]. In the present case the Coulomb coupling α is replaced by $aE/2$; and the energy E, by $bE/4$. Introducing the «Bohr»-radius $a_B=\hbar^2/m\alpha=2\hbar^2/maE$ we find for the solution of (4.39) as follows:

$$K(\xi'',\xi',\eta'',\eta';T)=\sum_{e,o}\int\limits_{\mathbb{R}}d\zeta\int\limits_{\mathbb{R}}dp\,\mathrm{e}^{-i\hbar p^2T/2m}\,\Psi^{(e,o)\,*}_{p,\zeta}(\xi',\eta')\Psi^{(e,o)}_{p,\zeta}(\xi'',\eta''), \quad (4.43)$$

and $\sum_{e,o}$ denotes the summation over even and odd states, respectively; the functions $\Psi^{(e,o)}_{p,\zeta}(\xi,\eta)$ are given by

$$\Psi^{(e,o)}_{p,\zeta}(\xi,\eta)=\frac{\mathrm{e}^{\pi/2ap}}{\sqrt{2}4\pi^2}\times$$
$$\times\left(\begin{array}{l}\left|\Gamma\left(\frac{1}{4}-\frac{i}{2p}(1/a_B+\zeta)\right)\right|^2 E^{(0)}_{-\frac{1}{2}+\frac{i}{p}(1/a_B+\zeta)}(\mathrm{e}^{-i\pi/4}\sqrt{2p}\,\xi)\times\\ \qquad\times E^{(0)}_{-\frac{1}{2}-\frac{i}{p}(1/a_B+\zeta)}(\mathrm{e}^{-i\pi/4}\sqrt{2p}\,\eta)\\ \\ \left|\Gamma\left(\frac{3}{4}-\frac{i}{2p}(1/a_B+\zeta)\right)\right|^2 E^{(1)}_{-\frac{1}{2}+\frac{i}{p}(1/a_B+\zeta)}(\mathrm{e}^{-i\pi/4}\sqrt{2p}\,\xi)\times\\ \qquad\times E^{(1)}_{-\frac{1}{2}-\frac{i}{p}(1/a_B+\zeta)}(\mathrm{e}^{-i\pi/4}\sqrt{2p}\,\eta)\end{array}\right),\qquad(4.44)$$

which are δ-normalized according to [51]

$$\int_0^\infty d\nu\int_{\mathbb{R}} d\xi(\xi^2+\eta^2)\Psi^{(e,o)\,*}_{p',\zeta'}(\xi,\eta)\Psi^{(e,o)}_{p,\zeta}(\xi,\eta)=\delta(p'-p)\delta(\zeta'-\zeta),\qquad(4.45)$$

and ζ is the parabolic separation constant. The functions $E^{(0)}_\nu(z)$ and $E^{(1)}_\nu(z)$ are even and odd parabolic cylinder functions in the variable z, respectively [4]:

$$E^{(0)}_\nu=\sqrt{2}\,\mathrm{e}^{-z^2/4}{}_1F_1\left(-\frac{\nu}{2},\frac{1}{2};\frac{z^2}{2}\right)=\sqrt{2\pi}\left(\frac{z^2}{2}\right)^{-1/4}\mathcal{M}_{\nu/2+1/4,-1/4}\left(\frac{z^2}{2}\right),$$
$$E^{(1)}_\nu=2z\,\mathrm{e}^{-z^2/4}{}_1F_1\left(\frac{1-\nu}{2},\frac{3}{2};\frac{z^2}{2}\right)=\sqrt{2\pi}\left(\frac{z^2}{2}\right)^{-1/4}\mathcal{M}_{\nu/2+1/4,1/4}\left(\frac{z^2}{2}\right),\qquad(4.46)$$

${}_1F_1(a;b;z)$ is the confluent hypergeometric function [15, p. 1057], and $\mathcal{M}_{\chi,\mu}(z)=M_{\chi,\mu}(z)/\Gamma(1+2\mu)$. Note the relation

$$D_\nu(z)=2^{\nu/2}\sqrt{\frac{\pi}{2}}\left[\frac{E^{(0)}_\nu(z)}{\Gamma((1-\nu)/2)}-\frac{E^{(1)}_\nu(z)}{\Gamma(-\nu/2)}\right].\qquad(4.47)$$

This concludes the discussion.

4.4. The Path Integral in Elliptic Coordinates on D_{III}. The classical Lagrangian and Hamiltonian are given by

$$\mathcal{L}(\omega,\dot\omega,\varphi,\dot\varphi)=\frac{m}{2}d^2\left(a+\frac{b}{4}d^2(\sinh^2\omega+\cos^2\varphi)\right)(\sinh^2\omega+\sin^2\varphi)(\dot\omega^2+\dot\varphi^2),\qquad(4.48)$$

$$\mathcal{H}(\omega, p_\omega, \varphi, p_\varphi) = \frac{1}{2m} \frac{p_\omega^2 + p_\varphi^2}{d^2\left(a + \frac{b}{4}d^2(\sinh^2\omega + \cos^2\varphi)\right)(\sinh^2\omega + \sin^2\varphi)}. \tag{4.49}$$

The canonical momenta have the form

$$p_\omega = \frac{\hbar}{i}\left(\frac{\partial}{\partial\omega} + \frac{bd^2\sinh\omega\cosh\omega}{4a + bd^2(\sinh^2\omega + \cos^2\varphi)} + \frac{\sinh\omega\cosh\omega}{\sinh^2\omega + \sin^2\varphi}\right), \tag{4.50}$$

$$p_\varphi = \frac{\hbar}{i}\left(\frac{\partial}{\partial\varphi} - \frac{bd^2\sin\varphi\cos\varphi}{4a + bd^2(\sinh^2\omega + \cos^2\varphi)} + \frac{\sin\varphi\cos\varphi}{\sinh^2\omega + \sin^2\varphi}\right), \tag{4.51}$$

and for the quantum Hamiltonian we find (we use $\sqrt{g} = d^2\left(a + \frac{b}{4}d^2(\sinh^2\omega + \cos^2\varphi)\right)(\sinh^2\omega + \sin^2\varphi)$)

$$H = -\frac{\hbar^2}{2m}\frac{1}{\sqrt{g}}\left(\frac{\partial^2}{\partial\omega^2} + \frac{\partial^2}{\partial\varphi^2}\right) = \frac{1}{2m}\frac{1}{\sqrt[4]{g}}(p_\omega^2 + p_\varphi^2)\frac{1}{\sqrt[4]{g}}. \tag{4.52}$$

This gives for the path integral

$$K(\omega'', \omega', \varphi'', \varphi'; T) == \int\limits_{\omega(t')=\omega'}^{\omega(t'')=\omega''} \mathcal{D}\omega(t) \int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''} \mathcal{D}\varphi(t)\sqrt{g} \times$$

$$\times \exp\left[\frac{imd^2}{2\hbar}\int\limits_0^T \left(a + \frac{b}{4}d^2(\sinh^2\omega + \cos^2\varphi)\right)(\sinh^2\omega + \sin^2\varphi)(\dot\omega^2 + \dot\varphi^2)dt\right] =$$

$$= \int\limits_{-\infty}^{\infty} \frac{dE}{2\pi\hbar}\,\mathrm{e}^{-iET/\hbar}\int\limits_0^\infty ds'' K(\omega'', \omega', \varphi'', \varphi'; s''), \tag{4.53}$$

the time-transformation function ($f(\omega, \varphi) = \sqrt{g}$) with the path integral $K(s'')$ given by

$$K(\omega'', \omega', \varphi'', \varphi'; s'') =$$

$$= \int\limits_{\omega(0)=\omega'}^{\omega(s'')=\omega''} \mathcal{D}\omega(s) \int\limits_{\varphi(0)=\varphi'}^{\varphi(s'')=\varphi''} \mathcal{D}\varphi(s)\exp\left\{\frac{i}{\hbar}\int\limits_0^{s''}\left[\frac{m}{2}(\dot\omega^2 + \dot\varphi^2) + \right.\right.$$

$$\left.\left. + E\left(a + \frac{b}{4}d^2(\sinh^2\omega + \cos^2\varphi)\right)(\sinh^2\omega + \sin^2\varphi)\right]ds\right\}. \tag{4.54}$$

For this kind of problem we do not have any theory of special functions to treat with and we leave this intractable path integral as it stands. $b = 0$ gives the elliptic system in $\mathbb{R}^2$ [24].

4.5. The Path Integral in Hyperbolic Coordinates on $D_{\rm III}$. The classical Lagrangian and Hamiltonian have the form

$$\mathcal{L}(\mu,\dot\mu,\nu,\dot\nu) = \frac{m}{2}\left(a+\frac{b}{2}(\mu-\nu)\right)(\mu+\nu)\left(\frac{\dot\mu^2}{\mu^2}-\frac{\dot\nu^2}{\nu^2}\right), \tag{4.55}$$

$$\mathcal{H}(\mu,p_\mu,\nu,p_\nu = \frac{1}{2m}\frac{\mu^2p_\mu^2-\nu^2p_\nu^2}{\left(a+\frac{b}{2}(\mu-\nu)\right)(\mu+\nu)}. \tag{4.56}$$

The canonical momentum operators are given by

$$p_\mu = \frac{\hbar}{i}\left[\frac{\partial}{\partial\mu}+\frac{1}{2}\left(+\frac{1}{\mu+\nu}+\frac{b}{a+\frac{b}{2}(\mu-\nu)}-\frac{1}{\mu}\right)\right], \tag{4.57}$$

$$p_\nu = \frac{\hbar}{i}\left[\frac{\partial}{\partial\mu}+\frac{1}{2}\left(+\frac{1}{\mu+\nu}-\frac{b}{a+\frac{b}{2}(\mu-\nu)}-\frac{1}{\nu}\right)\right], \tag{4.58}$$

and the quantum Hamiltonian has the form

$$H = -\frac{\hbar^2}{2m}\frac{1}{\left(a+\frac{b}{2}(\mu-\nu)\right)(\mu+\nu)}\times \times\left[\mu^2\left(\frac{\partial^2}{\partial\mu^2}-\frac{1}{\mu}\frac{\partial}{\partial\mu}\right)-\nu^2\left(\frac{\partial^2}{\partial\nu^2}-\frac{1}{\nu}\frac{\partial}{\partial\nu}\right)\right], \tag{4.59}$$

$$= \frac{1}{2m}\left[\frac{\mu}{\sqrt{\left(a+\frac{b}{2}(\mu-\nu)\right)(\mu+\nu)}}p_\mu^2\frac{\mu}{\sqrt{\left(a+\frac{b}{2}(\mu-\nu)\right)(\mu+\nu)}}- -\frac{\nu}{\sqrt{\left(a+\frac{b}{2}(\mu-\nu)\right)(\mu+\nu)}}p_\nu^2\frac{\nu}{\sqrt{\left(a+\frac{b}{2}(\mu-\nu)\right)(\mu+\nu)}}\right]. \tag{4.60}$$

Note that from each coordinate there comes a quantum potential $\Delta V = \hbar^2/8m$, however they are canceling each other due to the minus-sign in the metric in ν.

The path integral has the form

$$K(\mu'',\mu',\nu'',\nu';T)=\int\limits_{\mu(t')=\mu'}^{\mu(t'')=\mu''}\mathcal{D}\mu(t)\int\limits_{\nu(t')=\nu'}^{\nu(t'')=\nu''}\mathcal{D}\nu(t)\frac{\left(a+\frac{b}{2}(\mu-\nu)\right)(\mu+\nu)}{\mu\nu}\times$$
$$\times\exp\left[\frac{im}{2\hbar}\int\limits_0^T\left(a+\frac{b}{2}(\mu-\nu)\right)(\mu+\nu)\left(\frac{\dot\mu^2}{\mu^2}-\frac{\dot\nu^2}{\nu^2}\right)dt\right]=$$
$$=\int\limits_{-\infty}^{\infty}\frac{dE}{2\pi\hbar}\,\mathrm{e}^{-iET/\hbar}\int\limits_0^{\infty}ds''K(\mu'',\mu',\nu'',\nu';s''),\quad(4.61)$$

with the time-transformation function $f(\mu,\nu)=\left(a+\frac{b}{2}(\mu-\nu)\right)(\mu+\nu)$, and the path integral $K(s'')$ is given by

$$K(\mu'',\mu',\nu'',\nu';s'')=\int\limits_{\mu(0)=\mu'}^{\mu(s'')=\mu''}\mathcal{D}\mu(s)\int\limits_{\nu(0)=\nu'}^{\nu(s'')=\nu''}\mathcal{D}\nu(s)\frac{1}{\mu\nu}\times$$
$$\times\exp\left\{\frac{i}{\hbar}\int\limits_0^{s''}\left[\frac{m}{2}\left(\frac{\dot\mu^2}{\mu^2}-\frac{\dot\nu^2}{\nu^2}\right)+aE(\mu+\nu)+\frac{1}{2}bE(\mu^2-\nu^2)\right]ds\right\}.\quad(4.62)$$

Each of the last path integrals has a similar form as the one discussed in [16]. One can perform the transformation $\mu=\mathrm{e}^x$, $\nu=\mathrm{e}^y$. This gives, e.g., in the variable μ in the short-time element

$$\frac{im}{2\epsilon\hbar}\frac{(\Delta\mu^{(j)})^2}{\mu^{(j-1)}\mu^{(j)}}\simeq\frac{im}{2\epsilon\hbar}(\Delta y^{(j)})^2+\frac{im}{24\epsilon\hbar}(\Delta y^{(j)})^4\doteq\frac{im}{2\epsilon\hbar}(\Delta y^{(j)})^2-\frac{i\epsilon\hbar}{8m},\quad(4.63)$$

where use has been made of the identity $(\Delta y^{(j)})^4\doteq 3\left(\frac{i\epsilon\hbar}{m}\right)^2$, which is, of course, valid only in the sense of fluctuating paths. Note that a quantum potential $\Delta V=-\hbar^2/8m$ appears. However, the same potential arises in the transformation $\nu=\mathrm{e}^y$, but with the opposite sign, and both contributions cancel. Therefore the path-integration in (μ,ν) now gives a path-integration in (x,y) of the following form:

$$K(x'',x',y'',y';s'') = \int\limits_{x(0)=x'}^{x(s'')=x''} \mathcal{D}x(s) \int\limits_{y(0)=y'}^{y(s'')=y''} \mathcal{D}y(s) \times$$
$$\times \exp\left\{\frac{i}{\hbar}\int\limits_0^{s''}\left[\frac{m}{2}(\dot{x}^2-\dot{y}^2)+E\left(\frac{b}{2}\mathrm{e}^{2x}+a\,\mathrm{e}^x\right)-E\left(\frac{b}{2}\mathrm{e}^{2y}-a\,\mathrm{e}^y\right)\right]ds\right\}, \tag{4.64}$$

and we find the product of two path integrals for the Morse potential. Applying the same techniques as in [16] we obtain for the Green function $G(\mu',\mu',\nu'',\nu';E)$ (with the abbreviations $\omega=\sqrt{-bE/m}$ and rescaling $b\to 2b$):

$$G(\mu',\mu',\nu'',\nu';E) = \frac{\hbar}{2\pi i}\int d\mathcal{E}\left(\frac{\sqrt{-2mE}}{\hbar^2}\right)^2 \int\limits_0^\infty \frac{d\sigma}{\sin\omega\sigma}\times$$
$$\times \exp\left[\frac{4aiE}{\hbar}\sigma+\frac{\sqrt{-2mE}}{\hbar}({\mu'}^2+{\mu''}^2)\cot\omega\sigma\right] I_{\sqrt{8m\mathcal{E}}/\hbar}\left(\frac{2\sqrt{-2mE}\mu'\mu'}{i\hbar\sin\omega\sigma}\right)\times$$
$$\times \int\limits_0^\infty \frac{d\tau}{\sin\omega\tau}\exp\left[-\frac{4aiE}{\hbar}\tau+\frac{\sqrt{-2mE}}{\hbar}({\nu'}^2+{\nu''}^2)\cot\omega\tau\right]\times$$
$$\times I_{\sqrt{8m\mathcal{E}}/\hbar}\left(\frac{2\sqrt{-2mE}\nu'\nu'}{i\hbar\sin\omega\tau}\right). \tag{4.65}$$

Note that we have due to the minus-sign in ν in the metric an additional minus-sign in $\mathcal{E}$ in the Green function in the variable ν. Using now the same integral formula form Ref. 4 as before, we get (simplifying $a=b=1$)

$$G(\mu',\mu',\nu'',\nu';E) = \frac{1}{2\pi i}\frac{4^2}{\hbar^2}\int d\mathcal{E}\frac{2mE}{\hbar^2}\int\limits_0^\infty d\sigma\int\limits_0^\infty d\tau\int\limits_{-\infty}^\infty dp_1\int\limits_{-\infty}^\infty dp_2\times$$
$$\times \exp\left[i\left(\frac{4E}{\hbar}+2i\omega p_1\right)\sigma-i\left(\frac{4E}{\hbar}-2i\omega p_2\right)\tau\right]\times$$
$$\times \frac{\mathrm{e}^{\pi(p_1+p_2)}}{\mu'\mu''\nu'\nu''}\frac{\left|\Gamma\left(\frac{1}{2}+\tilde{\mathcal{E}}+ip_1\right)\right|^2}{\Gamma^2(1+2\tilde{\mathcal{E}})}\frac{\left|\Gamma\left(\frac{1}{2}+\tilde{\mathcal{E}}+ip_2\right)\right|^2}{\Gamma^2(1+2\tilde{\mathcal{E}})}\times$$
$$\times M_{ip_1,\tilde{\mathcal{E}}}\left(-2\frac{\sqrt{-2mE}}{\hbar}{\mu'}^2\right)M_{-ip_1,\tilde{\mathcal{E}}}\left(2\frac{\sqrt{-2mE}}{\hbar}{\mu''}^2\right)\times$$
$$\times M_{ip_2,\tilde{\mathcal{E}}}\left(-2\frac{\sqrt{-2mE}}{\hbar}{\nu'}^2\right)M_{-ip_2,\tilde{\mathcal{E}}}\left(2\frac{\sqrt{-2mE}}{\hbar}{\nu''}^2\right) \tag{4.66}$$

($\tilde{\mathcal{E}} = \sqrt{8m\mathcal{E}}/\hbar$). Performing the σ and τ integrations gives poles for p_1 and p_2 yielding $E = \hbar^2 p_1^2/2m = \hbar^2 p_2^2/2m$, as it should be, e.g., the p_2 integration evaluates the residuum and we obtain:

$$G(\mu', \mu', \nu'', \nu'; E) = \int \frac{\lambda d\lambda}{\mu'\mu''\nu'\nu''} \int_0^\infty \frac{\mathrm{e}^{2\pi p} dp}{\dfrac{\hbar^2 p^2}{2m} - E} \frac{\left|\Gamma\left(\dfrac{1}{2} + \lambda + ip\right)\right|^4}{4\pi p^2 \Gamma^4(1+2\lambda)} \times$$
$$\times M_{ip,\lambda}\Big(- 2ip\mu'^2\Big) M_{-ip,\lambda}\Big(2ip\mu''^2\Big) M_{ip,\lambda}\Big(- 2ip\nu'^2\Big) M_{-ip,\lambda}\Big(2ip\nu''^2\Big), \tag{4.67}$$

which gives the normalized wave functions:

$$\Psi_{p,\lambda}(\mu, \nu) = \sqrt{\frac{\lambda}{4\pi\mu\nu} \frac{\left|\Gamma\left(\dfrac{1}{2} + \lambda + ip\right)\right|^2}{\Gamma^2(1+2\lambda)}} \frac{\mathrm{e}^{\pi p}}{p} M_{ip,\lambda}\Big(-2ip\mu^2\Big) M_{ip,\lambda}\Big(-2ip\nu^2\Big). \tag{4.68}$$

This concludes the discussion on D_{III}.

5. DARBOUX SPACE D_{IV}

Finally, we consider the Darboux space D_{IV}. We have the coordinate systems:

((u, v) coordinates:) $x = v + iu, \quad y = v - iu \quad (u \in (0, \pi/2),\ v \in \mathbb{R})$, (5.1)

(Equidistant:) $u = \arctan(\mathrm{e}^\alpha), \quad v = \dfrac{\beta}{2} \quad (\alpha \in \mathbb{R}, \beta \in \mathbb{R})$, (5.2)

(Horospherical:) $x = \log\dfrac{\mu - i\nu}{2},\ y = \log\dfrac{\mu + i\nu}{2}\ (\mu, \nu > 0)$, (5.3)

(Elliptic:) $\mu = d\cosh\omega\cos\varphi, \quad \nu = d\sinh\omega\sin\varphi \quad (\omega > 0, \varphi \in (0, \pi/2))$. (5.4)

We obtain the following forms of the line-element ($a > 2b$, $a_\pm = (a \pm 2b)/4$):

$$ds^2 = -\frac{b[\mathrm{e}^{(x-y)/2} + \mathrm{e}^{(y-x)/2}] + a}{\left(\mathrm{e}^{(x-y)/2} - \mathrm{e}^{(y-x)/2}\right)^2} dxdy = -\frac{b[\mathrm{e}^{x-y} + \mathrm{e}^{y-x}] + a}{\left(\mathrm{e}^{x-y} - \mathrm{e}^{y-x}\right)^2} dxdy$$

$$
\begin{aligned}
((u,v)\text{ coordinates:}) \quad &= \quad \frac{2b\cos u + a}{4\sin^2 u}(du^2+dv^2) = \\
&= \quad \left(\frac{a_+}{\sin^2 u}+\frac{a_-}{\cos^2 u}\right)(du^2+dv^2) \\
& \qquad \left(\text{rescaling } \frac{u}{2}\to u,\ \frac{v}{2}\to v\text{:}\right), \qquad (5.5)\\
(\text{Equidistant:}) \quad &= \quad \frac{a-2b\tanh\alpha}{4}(d\alpha^2+\cosh^2\alpha d\beta^2), \qquad (5.6)\\
(\text{Horospherical:}) \quad &= \quad \left(\frac{a_+}{\nu^2}+\frac{a_-}{\mu^2}\right)(d\mu^2+d\nu^2), \qquad (5.7)\\
(\text{Elliptic:}) \quad &= \quad \left(\frac{a_-}{\cosh^2\omega\cos^2\varphi}+\frac{a_+}{\sinh^2\omega\sin^2\varphi}\right)\times \\
&\times \quad (\cosh^2\omega-\cos^2\varphi)(d\omega^2+d\varphi^2) = \\
&= \quad \left(\frac{a_+}{\sin^2\varphi}+\frac{a_-}{\cos^2\varphi}+\frac{a_+}{\sinh^2\omega}-\frac{a_-}{\cosh^2\omega}\right)\times \\
&\times \quad (d\omega^2+d\varphi^2). \qquad (5.8)
\end{aligned}
$$

We observe that the diagonal term in the metric corresponds to a Pöschl–Teller potential, a Rosen–Morse potential, an inverse-square radial potential, and a Pöschl–Teller and modified Pöschl–Teller, respectively. In particular, the (u,v) and the equidistant systems are the same, they just differ in the parameterization. The limiting cases $a=2b$ and $b=0$ give particular cases for the metric on the two-dimensional hyperboloid (Table 2).

Table 2. **Limiting Cases of Coordinate Systems on** D_{IV}

Metric:	D_{IV}	$\Lambda^{(2)}$ $(a=2b)$	$\Lambda^{(2)}$ $(b=0)$
$\frac{2b\cos u+a}{4\sin^2 u}(du^2+dv^2)$	(u,v) coordinates	Equidistant	Equidistant
$\frac{a-2b\tanh\alpha}{4}(d\alpha^2+\cosh^2\alpha d\beta^2)$	Equidistant	Equidistant	Equidistant
$\left(\frac{a_-}{\mu^2}+\frac{a_+}{\nu^2}\right)(d\mu^2+d\nu^2)$	Horospherical	Horicyclic	Semicircular–parabolic
$\left(\frac{a_-}{\cosh^2\omega\cos^2\varphi}+\frac{a_+}{\sinh^2\omega\sin^2\varphi}\right)$ $\times(\cosh^2\omega-\cos^2\varphi)(d\omega^2+d\varphi^2)$	Elliptic	Elliptic–parabolic	Hyperbolic–parabolic

For the Gaussian curvature we obtain, e.g., in the (u, v) system

$$K = -\frac{\dfrac{a_+^2}{\sin^6 u} + \dfrac{a_-^2}{\cos^6 u} + \dfrac{a_- a_+}{\sin^4 u \cos^4 u}}{\left(\dfrac{a_+}{\sin^2 u} + \dfrac{a_-}{\cos^2 u}\right)^3}. \tag{5.9}$$

The case $a = 2b$ yields $a_- = 0$, and

$$K = -\frac{1}{b}, \tag{5.10}$$

and therefore again a space of constant curvature, the hyperboloid $\Lambda^{(2)}$ is given for $b > 0$. We have set the sign in the metric (1.4) in such a way that from $a = 2b > 0$ the hyperboloid $\Lambda^{(2)}$ emerges. We could also choose the metric (1.4) with the opposite sign, then $a = 2b < 0$ would give the same result. In the following it is understood that we make this restriction of positive definiteness of the metric and we do not dwell into the problem of continuation into nonpositive definiteness. Because the (u, v) coordinates and the equidistant system are the same, we do not evaluate the path integral in the equidistant system. In the following we assume $a_+ > 0$ and $a_+ > a_-$.

5.1. The Path Integral in (u, v) Coordinates on D_{IV}. The classical Lagrangian and Hamiltonian are given by

$$\mathcal{L}(u, \dot{u}, v, \dot{v}) = \frac{m}{2}\frac{2b\cos 2u + a}{\sin^2 2u}(\dot{u}^2 + \dot{v}^2), \tag{5.11}$$

$$\mathcal{H}(u, p_u, v, p_v) = \frac{1}{2m}\frac{\sin^2 2u}{2b\cos 2u + a}(p_u^2 + p_v^2). \tag{5.12}$$

The canonical momentum operators are given by

$$p_u = \frac{\hbar}{i}\left(\frac{\partial}{\partial u} + 2\cot 2u - \frac{2b\sin 2u}{2b\cos 2u + a}\right), \quad p_v = \frac{\hbar}{i}\frac{\partial}{\partial v}, \tag{5.13}$$

and the Hamiltonian operator has the form

$$H = -\frac{\hbar^2}{2m}\frac{\sin^2 2u}{2b\cos 2u + a}\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) = \tag{5.14}$$

$$= \frac{1}{2m}\frac{\sin 2u}{\sqrt{2b\cos 2u + a}}(p_u^2 + p_v^2)\frac{\sin 2u}{\sqrt{2b\cos 2u + a}}. \tag{5.15}$$

We obtain for the path integral

$$K(u'',u',v'',v';T)=\int\limits_{u(t')=u'}^{u(t'')=u''}\mathcal{D}u(t)\int\limits_{v(t')=v'}^{v(t'')=v''}\mathcal{D}v(t)\left(\frac{a_+}{\sin^2 u}+\frac{a_-}{\cos^2 u}\right)\times$$
$$\times\exp\left[\frac{im}{2\hbar}\int\limits_0^T\left(\frac{a_+}{\sin^2 u}+\frac{a_-}{\cos^2 u}\right)(\dot u^2+\dot v^2)dt\right]. \qquad (5.16)$$

This formulation in (u,v) coordinates is unconvenient. Without the term $a_+/\sin^2 u$ (5.16) would be identical with the path integral in the hyperbolic strip [17], which is actually a reformulation of equidistant coordinates on the two-dimensional hyperboloid. Following [17] we perform the coordinate transformation $\cos u=\tanh\tau$. Further, we separate off the v-path integration, and additionally we make a time transformation with the time-transformation function $f=a_+/\sin^2 u+a_-/\cos^2 u$. Due to the coordinate transformation $\cos u=\tanh\tau$ additional quantum terms appear according to

$$\exp\left(\frac{im}{2\epsilon\hbar}\frac{\big(\Delta u^{(j)}\big)^2}{\cos u^{(j-1)}\cos u^{(j)}}\right)\doteq\exp\left[\frac{im}{2\epsilon\hbar}\big(\Delta\tau^{(j)}\big)^2-i\frac{\hbar}{8m}\left(1+\frac{1}{\cosh^2\tau^{(j)}}\right)\right]. \qquad (5.17)$$

We get for the path integral (5.16)

$$K(u'',u',v'',v';T)=$$
$$=\int\limits_{-\infty}^{\infty}\frac{dE}{2\pi\hbar}\,\mathrm{e}^{-iET/\hbar}\int\limits_0^\infty ds''\exp\left[\frac{i}{\hbar}\left(a_+E-\frac{\hbar^2}{8m}\right)s''\right]K(\tau'',\tau',v'',v';s''), \qquad (5.18)$$

and the time-transformed path integral $K(s'')$ is given by

$$K(\tau'',\tau',v'',v';s'')=\int\limits_{-\infty}^{\infty}dk_v\frac{\mathrm{e}^{ik_v(v''-v')}}{2\pi}(\cosh\tau'\cosh\tau'')^{-1/2}\times$$
$$\times\int\limits_{\tau(0)=\tau'}^{\tau(s'')=\tau''}\mathcal{D}\tau(s)\exp\left[\frac{i}{\hbar}\int\limits_0^{s''}\left(\frac{m}{2}\dot\tau^2+\frac{a_-E}{\sinh^2\tau}-\frac{\hbar^2}{2m}\frac{k_v^2+\frac14}{\cosh^2\tau}\right)ds\right]. \qquad (5.19)$$

The special case $a_-=0$ gives the wave functions on the two-dimensional hyperboloid in equidistant coordinates, respectively on the hyperbolic strip. Inserting the solution for the modified Pöschl–Teller potential and evaluating the Green function on the cut yields for the path integral solution on $D_{\rm IV}$ as follows $(K(u'',u',v'',v';T) = K(\tau'',\tau',v'',v';T))$:

$$K(u'',u',v'',v';T) = \int_{-\infty}^{\infty} dk_v \int_0^{\infty} dp\, \mathrm{e}^{-iTE_p/\hbar} \Psi_{p,k_v}(\tau'',v'')\Psi^*_{p,k_v}(\tau',v'), \quad (5.20)$$

$$\Psi_{p,k_v}(\tau,v) = \frac{\mathrm{e}^{ik_v v}}{\sqrt{2\pi a_+ \cosh\tau}}\, \Psi_p^{(\eta,ik)}(\tau), \quad (5.21)$$

$$E_p = \frac{\hbar^2}{2ma_+}\left(p^2 + \frac{1}{4}\right), \quad (5.22)$$

where $\eta^2 = \dfrac{1}{4} - 2ma_e E/\hbar^2$ and the wave functions for the modified Pöschl–Teller functions as given in Appendix B. Reinserting $\cos u = \tanh\tau$ gives the solution in terms of the variable u.

5.2. The Path Integral in Horospherical Coordinates on $D_{\rm IV}$. The classical Lagrangian and Hamiltonian are given by

$$\mathcal{L}(\mu,\dot\mu,\nu,\dot\nu) = \frac{m}{2}\left(\frac{a_+}{\nu^2} + \frac{a_-}{\mu^2}\right)(\dot\mu^2 + \dot\nu^2), \quad (5.23)$$

$$\mathcal{H}(\mu,p_\mu,\nu,p_\nu) = \frac{1}{2m}\frac{\mu^2\nu^2(p_\mu^2 + p_\nu^2)}{a_+\mu^2 + a_-\nu^2}. \quad (5.24)$$

For the canonical momentum operators we have

$$p_\mu = \frac{\hbar}{i}\left(\frac{\partial}{\partial\mu} - \frac{\nu^2 a_-/\mu}{a_+\mu^2 + a_-\nu^2}\right), \quad (5.25)$$

$$p_\nu = \frac{\hbar}{i}\left(\frac{\partial}{\partial\nu} - \frac{\mu^2 a_+/\nu}{a_+\mu^2 + a_-\nu^2}\right), \quad (5.26)$$

and for the quantum Hamiltonian we get

$$H = -\frac{\hbar^2}{2m}\frac{\mu^2\nu^2}{a_+\mu^2 + a_-\nu^2}\left(\frac{\partial^2}{\partial\mu^2} + \frac{\partial^2}{\partial\nu^2}\right), \quad (5.27)$$

$$= \frac{1}{2m}\sqrt{\frac{\mu^2\nu^2}{a_+\mu^2 + a_-\nu^2}}\,(p_\mu^2 + p_\nu^2)\sqrt{\frac{\mu^2\nu^2}{a_+\mu^2 + a_-\nu^2}}. \quad (5.28)$$

For the path integral we obtain (time-transformation function $f(\mu,\nu) = a_+/\mu^2 + a_-/\nu^2 = \sqrt{g}$)

$$K(\mu'',\mu',\nu'',\nu';T) = \int\limits_{\mu(t')=\mu'}^{\mu(t'')=\mu''} \mathcal{D}\mu(t) \int\limits_{\nu(t')=\nu'}^{\nu(t'')=\nu''} \mathcal{D}\nu(t)\left(\frac{a_+}{\nu^2}+\frac{a_-}{\mu^2}\right)\times$$
$$\times \exp\left[\frac{im}{2\hbar}\int\limits_0^T \left(\frac{a_+}{\nu^2}+\frac{a_-}{\mu^2}\right)(\dot\mu^2+\dot\nu^2)dt\right] =$$
$$= \int\limits_{-\infty}^{\infty} \frac{dE}{2\pi\hbar}\,\mathrm{e}^{-iET/\hbar}\int\limits_0^\infty ds''K(\mu'',\mu',\nu'',\nu';s''), \quad (5.29)$$

and the time-transformed path integral $K(s'')$ is given by

$$K(\mu'',\mu',\nu'',\nu';s'') = \int\limits_{\mu(0)=\mu'}^{\mu(s'')=\mu''} \mathcal{D}\mu(s) \int\limits_{\nu(0)=\nu'}^{\nu(s'')=\nu''} \mathcal{D}\nu(s)\times$$
$$\times \exp\left\{\frac{i}{\hbar}\int\limits_0^{s''}\left[\frac{m}{2}(\dot\mu^2+\dot\nu^2)+E\left(\frac{a_+}{\nu^2}+\frac{a_-}{\mu^2}\right)\right]ds\right\}. \quad (5.30)$$

These two path integrals can be solved by means of the solution for the inverse-square radial potential, by following the approach of [24] for the semicircular–parabolic system on the two-dimensional hyperboloid. We insert the path integral solution for the inverse-square radial potential (kernel and the Green function) and obtain

$$G(\mu'',\mu',\nu'',\nu';E) = \int\limits_0^\infty ds''K(\mu'',\mu',\nu'',\nu';s'') = \frac{4m^2}{\hbar^3}\sqrt{\mu'\mu''\nu'\nu''}\times$$
$$\times \int \frac{\mathrm{d}\mathcal{E}}{2\pi i} I_\lambda\left(\sqrt{2m\mathcal{E}}\frac{\nu_<}{\hbar}\right)K_\lambda\left(\sqrt{2m\mathcal{E}}\frac{\nu_>}{\hbar}\right)I_{\tilde\lambda}\left(\sqrt{-2m\mathcal{E}}\frac{\mu_<}{\hbar}\right)K_{\tilde\lambda}\left(\sqrt{-2m\mathcal{E}}\frac{\mu_>}{\hbar}\right) =$$
$$= \frac{m^2}{\hbar^3}\sqrt{\mu'\mu''\nu'\nu''}\int\limits_0^\infty \frac{ds''}{s''}\int\frac{d\mathcal{E}}{2\pi i}\,\mathrm{e}^{i\mathcal{E}s''/\hbar}\times$$
$$\times \exp\left[\frac{m}{2i\hbar s''}({\nu'}^2+{\nu''}^2)\right] I_\lambda\left(\frac{im\nu'\nu''}{\hbar s''}\right)I_{\tilde\lambda}\left(\sqrt{-2m\mathcal{E}}\frac{\mu_<}{\hbar}\right)K_{\tilde\lambda}\left(\sqrt{-2m\mathcal{E}}\frac{\mu_>}{\hbar}\right). \quad (5.31)$$

I have used the abbreviations (assuming $a_+ > a_-$)

$$\lambda^2 = \frac{1}{4} - \frac{2mE}{\hbar^2}a_+, \quad \tilde{\lambda}^2 = \frac{1}{4} - \frac{2mE}{\hbar^2}a_-, \quad \tilde{p}^2 = \frac{a_-}{a_+}\left(p^2 + \frac{1}{4}\right) - \frac{1}{4}. \tag{5.32}$$

For the μ-dependent part one uses the dispersion relation (3.52) together with the integral representation [15, p. 725]

$$\int_0^\infty \mathrm{e}^{-x/2-(z^2+w^2)/2x} K_\nu\left(\frac{zw}{x}\right)\frac{dx}{x} = 2K_\nu(z)K_\nu(w). \tag{5.33}$$

In order to analyze the ν-dependent part, we first rewrite the ν-dependent part of the Green function according to

$$I_\lambda(-ik\mu_<)K_\lambda(-ik\mu_>) = \frac{i\pi}{2}J_\lambda(k\mu_<)H_\lambda^{(1)}(k\mu_>), \tag{5.34}$$

and then the wave functions on the cut are obtained by using

$$\begin{aligned}\Psi_{k,p}(\mu'')\Psi^*_{k,p}(\mu') \quad &\propto \quad \Big[J_{-ip}(k\mu'')H_{-ip}^{(1)}(k\mu') - J_{ip}(k\mu'')H_{ip}^{(1)}(k\mu')\Big] = \\ &= \quad \sinh\pi p H_{ip}^{(1)}(k\mu'')H_{-ip}^{(1)}(k\mu'),\end{aligned} \tag{5.35}$$

($\lambda = -ip$) and the relation of the Hankel function, i.e., $H_\nu^{(1)}(z) = i\big[\mathrm{e}^{-i\nu\pi}J_\nu(z) - J_{-\nu}(z)\big]/\sin\pi\nu$. Therefore with $\mathcal{E} = \hbar^2\kappa/2m$:

$$\begin{aligned}G(\mu'',\mu',\nu'',\nu';E) &= \frac{\sqrt{\mu'\mu''\nu'\nu''}}{2\pi^2}\Bigg\{\int\frac{d\kappa}{2\pi i}I_{\tilde{\lambda}}\big(-i\sqrt{\kappa}\,\mu_<\big)K_{\tilde{\lambda}}\big(-i\sqrt{\kappa}\,\mu_>\big)\times \\ &\times\int_0^\infty\frac{dp\,p\sinh\pi p}{\dfrac{\hbar^2}{2ma_+}\left(p^2+\dfrac{1}{4}\right)-E}K_{ip}\big(\sqrt{\kappa}\,\nu'\big)K_{ip}\big(\sqrt{\kappa}\,\nu'\big) + (\mu\leftrightarrow\nu)\Bigg\} = \\ &= \frac{1}{8\pi^2}\sqrt{\mu'\mu''\nu'\nu''}\int\mathrm{d}\kappa\int_0^\infty\frac{dp\,p\sinh\pi p\sinh\pi\tilde{p}}{\dfrac{\hbar^2}{2ma_+}\left(p^2+\dfrac{1}{4}\right)-E}\times \\ &\times\Big[K_{ip}\big(\sqrt{\kappa}\,\nu'\big)K_{ip}\big(\sqrt{\kappa}\,\nu'\big)H_{-i\tilde{p}}^{(1)}\big(\sqrt{\kappa}\,\mu'\big)H_{-i\tilde{p}}^{(1)}\big(\sqrt{\kappa}\,\mu''\big) + (\mu\leftrightarrow\nu)\Big].\end{aligned} \tag{5.36}$$

I have taken into account that the final result must be symmetrical in μ and ν which also accounts for the additional factor $1/2$. The wave functions thus have the form

$$\Psi_{p,\kappa}(\mu,\nu)=\frac{\sqrt{\mu\nu}}{2\sqrt{2}\,\pi}\sqrt{p\sinh\pi p\sinh\pi\tilde{p}}\Big[K_{ip}\big(\sqrt{\kappa}\,\nu\big)H^{(1)}_{-i\tilde{p}}\big(\sqrt{\kappa}\,\mu\big)+(\mu\leftrightarrow\nu)\Big]. \tag{5.37}$$

Note that $a_-=0$ gives the horicyclic path integral on the two-dimensional hyperboloid. Then $\widetilde{\lambda}=\pm 1/2$, and the corresponding Bessel functions give exponentials, therefore we obtain the wave functions of the horicyclic system. This result completes the calculation.

5.3. The Path Integral in Elliptic Coordinates on $D_{\rm IV}$. The classical Lagrangian and Hamiltonian are given by

$$\mathcal{L}(\omega,\dot\omega,\varphi,\dot\varphi)=$$
$$=\frac{m}{2}\left(\frac{a_-}{\cosh^2\omega\cos^2\varphi}+\frac{a_+}{\sinh^2\omega\sin^2\varphi}\right)(\cosh^2\omega-\cos^2\varphi)(\dot\omega^2+\dot\varphi^2), \tag{5.38}$$

$$\mathcal{H}(\omega,p_\omega,\varphi,p_\varphi)=$$
$$=\frac{1}{2m}\left(\frac{a_-}{\cosh^2\omega\cos^2\varphi}+\frac{a_+}{\sinh^2\omega\sin^2\varphi}\right)^{-1}\frac{p_\omega^2+p_\varphi^2}{\cosh^2\omega-\cos^2\varphi}. \tag{5.39}$$

For the canonical momentum operators we get

$$p_\omega = \frac{\hbar}{i}\left[\frac{\partial}{\partial\omega}+g^{-1/2}\left(\frac{a_-\tanh\omega}{\sinh^2\omega}-\frac{a_+\coth\omega}{\cosh^2\omega}\right)\right], \tag{5.40}$$

$$p_\varphi = \frac{\hbar}{i}\left[\frac{\partial}{\partial\varphi}+g^{-1/2}\left(\frac{a_-\tan\varphi}{\sin^2\varphi}-\frac{a_+\cot\varphi}{\cos^2\varphi}\right)\right], \tag{5.41}$$

the Hamiltonian operator is given by

$$H = -\frac{\hbar^2}{2m}\left(\frac{a_-}{\cosh^2\omega\cos^2\varphi}+\frac{a_+}{\sinh^2\omega\sin^2\varphi}\right)^{-1}\times \tag{5.42}$$
$$\times\frac{1}{\cosh^2\omega-\cos^2\varphi}\left(\frac{\partial^2}{\partial\omega^2}+\frac{\partial^2}{\partial\varphi^2}\right)=$$
$$=\frac{1}{2m}\frac{1}{\sqrt[4]{g}}(p_\omega^2+p_\varphi^2)\frac{1}{\sqrt[4]{g}}. \tag{5.43}$$

For the path integral in elliptic coordinates we obtain (note that for $a_-=0$ we get the path integral in elliptic–parabolic coordinates on the two-dimensional hyperboloid)

$$K(\omega'',\omega',\varphi'',\varphi';T)=\int\limits_{\omega(t')=\omega'}^{\omega(t'')=\omega''}\mathcal{D}\omega(t)\int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''}\mathcal{D}\varphi(t)\sqrt{g}\times$$
$$\times\exp\left[\frac{im}{2\hbar}\int\limits_0^T\left(\frac{a_+}{\sin^2\varphi}+\frac{a_-}{\cos^2\varphi}+\frac{a_+}{\sinh^2\omega}-\frac{a_-}{\cosh^2\omega}\right)(\dot\omega^2+\dot\varphi^2)dt\right].\tag{5.44}$$

In order to obtain a convenient form to evaluate (5.44) we perform the coordinate transformation $\cos\varphi=\tanh\tau$ in the same way as in (5.16). Performing the time transformation with $f(\omega,\varphi)=\sqrt{g}$, the time-transformed path integral $K(s'')$ is given by

$$K(\omega'',\omega',\tau'',\tau';T)=$$
$$=\int\limits_{-\infty}^{\infty}\frac{dE}{2\pi\hbar}\,\mathrm{e}^{-iET/\hbar}\int\limits_0^\infty ds''\exp\left[\frac{i}{\hbar}\left(a_+E-\frac{\hbar^2}{8m}\right)\right]K(\omega'',\omega',\tau'',\tau';s''),\tag{5.45}$$

and the time-transformed path integral $K(s'')$ is given by

$$K(\omega'',\omega',\tau'',\tau';s'')=\int\limits_{\omega(0)=\omega'}^{\omega(s'')=\omega''}\mathcal{D}\omega(s)\int\limits_{\tau(0)=\tau'}^{\tau(s'')=\tau''}\mathcal{D}\tau(s)\cosh\tau\times$$
$$\times\exp\left\{\frac{i}{\hbar}\int\limits_0^{s''}\left[\frac{m}{2}(\dot\tau^2+\cosh^2\tau\dot\omega^2)+\frac{a_+E}{\sinh^2\tau}+\right.\right.$$
$$\left.\left.+\frac{1}{\cosh^2\tau}\left(\frac{a_+E}{\sinh^2\omega}-\frac{a_-E}{\cosh^2\omega}-\frac{\hbar^2}{8m}\right)\right]ds\right\}.\tag{5.46}$$

The ω-path integration is separated by means of a modified Pöschl–Teller potential with $\eta^2=1/4+2mEa_+/\hbar^2$, $\nu^2=1/4+2mEa_-/\hbar^2$, and the τ-path integration is the same as in (5.19). This gives the solution:

$$K(\omega'',\omega',\tau'',\tau';T) = \int_{-\infty}^{\infty} dk_v \int_0^{\infty} dp\, \mathrm{e}^{-iTE_p/\hbar} \Psi^*_{p,k}(\omega',\tau')\Psi_{p,k}(\omega'',\tau''), \quad (5.47)$$

$$\Psi_{p,k}(\omega,\tau) = (\cosh\tau)^{-1/2}\, \Psi_k^{(\eta,\nu)}(\omega)\Psi_p^{(\eta,ik)}(\tau), \quad (5.48)$$

$$E_p = \frac{\hbar^2}{2ma_+}\left(p^2+\frac{1}{4}\right), \quad (5.49)$$

where $\eta_2^2 = 1/4 - 2ma_eE/\hbar^2$, and we can reinsert $\cos\varphi = \tanh\tau$. This concludes the discussion on the Darboux space D_{IV}.

6. SUMMARY AND DISCUSSION

In this paper I have discussed path integration on Darboux spaces, labeled by D_{I} to D_{IV}. We set up the metrics following Kalnins et al. [38, 39]. In each of these spaces the Schrödinger equation, respectively the path integral, were separable in several coordinate systems. Our results are summarized in Table 3.

In the Darboux space D_{I} we found the solutions in the (u,v) and rotated (u,v) coordinates. A closed expression for the Green functions could be found,

Table 3. **Solutions of the path integration in Darboux spaces**

Space and coordinate system		Solution in terms of the wave functions
D_{I}	(u,v) coordinates	Product of Airy functions
	Rotated (u,v) coordinates	Product of Airy functions
	Displaced parabolic	No solution
D_{II}	(u,v) coordinates	Exponential times K-Bessel function
	Polar	Legendre times K-Bessel function
	Parabolic	Product of W-Whittaker functions
	Elliptic	Spheroidal wave functions
D_{III}	(u,v) coordinates	Exponential times M-Whittaker functions
	Polar	Exponential times M-Whittaker functions
	Parabolic	Product of parabolic cylinder functions
	Elliptic	No solution
	Hyperbolic	Product of M-Whittaker functions
D_{IV}	(u,v) coordinates	Exponential times Legendre function
	Equidistant	Exponential times Legendre function
	Horospherical	Product of K- and $H^{(1)}$-Bessel functions
	Elliptic	Product of Legendre functions

however, the wave function is only implicitly known because of the boundary conditions which must be imposed on the system. A solution in displaced parabolic coordinates was impossible due to its quartic anharmonic structure of the transformed dynamics.

In the Darboux space D_{II}, I succeeded in writing down the Green functions and the corresponding expansions into the wave functions. I found the expressions in the four coordinate systems, i.e., (u, v) coordinates, polar, parabolic and elliptic coordinates. Several path integral techniques from former studies were indispensable tools in the considerations. We stressed the limiting case of the hyperbolic plane, i.e., the two-dimensional hyperboloid. The Green function and the wave functions were determined in the soluble systems with the general feature that the energy spectrum has the form: $E = \dfrac{\hbar^2 p^2}{2m|a|}\left(p^2 + \dfrac{1}{4}\right)$. The additional zero-point energy $E_0 = \dfrac{\hbar^2 p^2}{8m|a|}$ is a characteristic feature of the quantum motion on spaces with negative curvature [33].

In the Darboux space D_{III}, I found the solutions in the (u, v) system, the closely related polar system, the (u, v) system, and the hyperbolic coordinate system. In elliptic coordinates no solution could be found. The Green function and the wave functions were determined in the soluble systems with the general feature that the energy spectrum has the form: $E = \dfrac{\hbar^2 p^2}{2m}$, which is different in its zero-point valued form D_{II}.

In the Darboux space D_{IV}, we found solutions in (u, v) coordinates, horospherical, and elliptic coordinates. Here also the limiting case to the two-dimensional hyperboloid was shortly mentioned. The Green functions and the wave functions were calculated in the separable coordinate system. The energy spectrum has the form: $E = \dfrac{\hbar^2 p^2}{2ma_+}\left(p^2 + \dfrac{1}{4}\right)$, similarly as on D_{II}.

We were able to solve the various path integral representations, because we have now to our disposal not only the basic path integrals for the harmonic oscillator, the linear oscillator, the radial harmonic oscillator, and the modified Pöschl–Teller potential, but also path integral identities derived from path integration on harmonic spaces like the elliptic and spheroidal path integral representations with its more complicated special functions [24, 26, 34]. This includes also numerous transformation techniques to find a particular solution based on one of the basic solutions. Various Green-function analysis techniques can be applied to find not only an expression for the Green function but also for the wave functions and the energy spectrum.

The present study continues the analysis of path integrals on curved space [24] with the simple case of the two-dimensional Euclidean space with its four separating coordinate systems (Cartesian, polar, elliptic and parabolic) up to the complicated case of the three-dimensional hyperboloid with its 34 separating coordinate systems.

In our papers [27–30] we have studied superintegrable potentials on spaces of constant curvature, i.e., flat space [27], spheres [28] and two- and three-dimensional hyperboloids [27, 28]. In the Euclidean flat spaces $\mathbb{R}^2$ and $\mathbb{R}^3$ a complete list was given, and they were called Smorodinski–Winternitz potentials [61]. We have extended this study by introducing corresponding potentials on spaces with (nonzero) constant curvature, i.e., on spheres and hyperboloids. Further studies along these lines for superintegrability on spaces with constant curvature were performed by Kalnins et al. [41] on the complex 2-sphere (five coordinate systems which separate the Laplace–Beltrami equation), on the complex Euclidean space $E_{2,C}$ [42] (six coordinate systems which separate the Laplace–Beltrami equation), in $\mathbb{R}^2$ and on the two-dimensional sphere (with emphasis on the polynomial solutions of the superintegrable potentials) [43], and on the two-dimensional hyperboloid [44, 45]. In the latter also two potentials were studied which have not until then been considered. The focus in those studies was of course on the harmonic oscillator (with its deformations and generalizations) and the Coulomb potential. As it turns out, in all those spaces a harmonic oscillator and a Coulomb potential could be defined and solved in various coordinate representations. In particular, the three-dimensional Coulomb potential problem separates into spherical, conical, parabolic, and prolate spheroidal coordinates. These features translated also into the three-dimensional sphere and the three-dimensional hyperboloid. This particular feature of the Coulomb system has its origin in its superintegrability, i.e., beside the energy and the angular momentum conservation we have an additional conserved quantity, the Lenz–Runge vector.

Therefore we have explicitly shown that path integral calculations are not only possible in flat space (with several potential problems), or in spaces with nonvanishing constant curvature, but are also applicable in spaces of nonconstant curvature.

The most serious drawback of the path integral method in comparison to the operator method is that in the operator method we can investigate the more complicated parametric coordinate systems in terms of Lamé polynomials. For this kind of coordinate systems a path integral approach exists for the spheroidal wave functions (elliptic wave functions in $\mathbb{R}^2$ and spheroidal wave functions in $\mathbb{R}^3$) based on the theory of Meixner and Schäfke [53], and for the three-dimensional sphere [26].

It is therefore quite natural the raise of the question of superintegrable system in Darboux spaces, in fact one of the intentions of [38,39]. And indeed, analogies of an oscillator and a Coulomb potential can be found. However, the freedom of choice of free parameters seems somewhat limited in comparison with the spaces of constant curvature. This can be understood by the feature of the Darboux spaces that the corresponding metric includes already a complicated «potential» term, i.e., the metric almost equals a superintegrable potential in $\mathbb{R}^2$. The time-transformation function is in almost all cases equal to $\sqrt{g}$ which is most obvious in the case of $D_{\rm IV}$. These issues will be discussed in more detail in a future publication [31].

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Appendix A
FORMULATION OF THE PATH INTEGRAL IN CURVED SPACES

In order to set up our notation for path integrals on curved manifolds we proceed in a canonical way. To avoid unnecessary overlap with our Table of Path Integrals [34] I give in the following only the essential information required for the path integral representation on curved spaces. For more details concerning ordering prescriptions, transformation techniques, perturbation expansions, point interactions, and boundary conditions I refer to [34], where also listings of the application of Basic Path Integrals will be presented. In the following $\mathbf{q}$ denote some D-dimensional coordinates. We start by considering the classical Lagrangian corresponding to the line element $ds^2 = g_{ab}dq^a dq^b$ of the classical motion in some D-dimensional Riemannian space

$$\mathcal{L}_{\rm cl}(\mathbf{q}, \dot{\mathbf{q}}) = \frac{m}{2}\left(\frac{ds}{dt}\right)^2 - V(\mathbf{q}) = \frac{m}{2} g_{ab}(\mathbf{q})\dot{q}^a\dot{q}^b - V(\mathbf{q}). \tag{A.1}$$

The quantum Hamiltonian is *constructed* by means of the Laplace–Beltrami operator

$$H = -\frac{\hbar^2}{2m}\Delta_{\rm LB} + V(\mathbf{q}) = -\frac{\hbar^2}{2m}\frac{1}{\sqrt{g}}\frac{\partial}{\partial q^a}g^{ab}\sqrt{g}\frac{\partial}{\partial q^b} + V(\mathbf{q}) \tag{A.2}$$

as a *definition* of the quantum theory on a curved space. Here are $g = \det(g_{ab})$ and $(g^{ab}) = (g_{ab})^{-1}$. The scalar product for wave functions on the manifold reads $(f, g) = \displaystyle\int {\rm d}\mathbf{q}\sqrt{g} f^*(\mathbf{q})g(\mathbf{q})$, and the momentum operators which are Hermitian with respect to this scalar product are given by

$$p_a = \frac{\hbar}{i}\left(\frac{\partial}{\partial q^a} + \frac{\Gamma_a}{2}\right), \qquad \Gamma_a = \frac{\partial \ln\sqrt{g}}{\partial q^a}. \tag{A.3}$$

In terms of the momentum operators (A.3) we can rewrite $\underline{\rm H}$ by using a product according to $g_{ab} = h_{ac}h_{cb}$ [34]. Then we obtain for the Hamiltonian (A.2) (PF —

Product Form)

$$\underline{\mathrm{H}} = -\frac{\hbar^2}{2m}\Delta_{\mathrm{LB}} + V(\mathbf{q}) = \frac{1}{2m}h^{ac}p_a p_b h^{cb} + \Delta V_{\mathrm{PF}}(\mathbf{q}) + V(\mathbf{q}), \tag{A.4}$$

and for the path integral

$$\begin{aligned} K(\mathbf{q}'',\mathbf{q}';T) = & \int\limits_{\mathbf{q}(t')=\mathbf{q}'}^{\mathbf{q}(t'')=\mathbf{q}''} \mathcal{D}_{\mathrm{PF}}\mathbf{q}(t)\sqrt{g(\mathbf{q})}\times \\ & \times \exp\left\{\frac{i}{\hbar}\int\limits_{t'}^{t''}\left[\frac{m}{2}h_{ac}(\mathbf{q})h_{cb}(\mathbf{q})\dot{q}^a\dot{q}^b - V(\mathbf{q}) - \Delta V_{\mathrm{PF}}(\mathbf{q})\right]dt\right\} = \\ & = \lim_{N\to\infty}\left(\frac{m}{2\pi i\epsilon\hbar}\right)^{ND/2}\prod_{k=1}^{N-1}\int \mathrm{d}\mathbf{q}_k\sqrt{g(\mathbf{q}_k)}\times \\ & \times\exp\left\{\frac{i}{\hbar}\sum_{j=1}^{N}\left[\frac{m}{2\epsilon}h_{bc}(\mathbf{q}_j)h_{ac}(\mathbf{q}_{j-1})\Delta q_j^a\Delta q_j^b - \epsilon V(\mathbf{q}_j) - \epsilon\Delta V_{\mathrm{PF}}(\mathbf{q}_j)\right]\right\}. \end{aligned} \tag{A.5}$$

ΔV_{PF} denotes the well-defined quantum potential

$$\begin{aligned} \Delta V_{\mathrm{PF}}(\mathbf{q}) = & \frac{\hbar^2}{8m}\Big[g^{ab}\Gamma_a\Gamma_b + 2(g^{ab}\Gamma_b)_{,b} + g^{ab}{}_{,ab}\Big] + \\ & + \frac{\hbar^2}{8m}\Big(2h^{ac}h^{bc}{}_{,ab} - h^{ac}{}_{,a}h^{bc}{}_{,b} - h^{ac}{}_{,b}h^{bc}{}_{,a}\Big) \end{aligned} \tag{A.6}$$

arising from the specific lattice formulation (A.5) of the path integral or the ordering prescription for position and momentum operators in the quantum Hamiltonian, respectively. We have used the abbreviations $\epsilon = (t'' - t')/N \equiv T/N$, $\Delta\mathbf{q}_j = \mathbf{q}_j - \mathbf{q}_{j-1}$, $\mathbf{q}_j = \mathbf{q}(t' + j\epsilon)$ $(t_j = t' + \epsilon j, j = 0, \ldots, N)$ and we interpret the limit $N \to \infty$ as equivalent to $\epsilon \to 0$, T fixed. The lattice representation can be obtained by exploiting the composition law of the time-evolution operator $U = \exp(-iHT/\hbar)$, respectively its semigroup property.

Note that the first summand on ΔV corresponds to the quantum potential of the Weyl-ordered Hamiltonian, respectively a midpoint prescription of the path integral. Note also that in the case that the metric tensor is diagonal to the unit tensor, i.e., $(g_{ab} = f^2\delta_{ab})$ we obtain

$$\Delta V(\mathbf{q}) = \hbar^2\frac{D-2}{8m}\sum_a \frac{(4-D)f_{,a}^2 + 2f\cdot f_{,aa}}{f^4}. \tag{A.7}$$

This gives the important special case that for $D = 2$: $\Delta V = 0$, a property which is quite useful for the considered two-dimensional Darboux spaces.

The path integral representation (A.5) is not explicitly evaluable in many cases, in particular if coordinate-dependent metric terms are explicitly present, or potentials like the Coulomb potential. Here the so-called «time transformation» comes into play which leads in combination with «coordinate transformation» to general «space-time transformations» (also «Duru–Kleinert transformation» [7, 8, 46]) in path integrals. The time transformation is implemented [46] by introducing a new «pseudotime» s''. In order to do this, one first makes use of the operator identity (one-dimensional case)

$$\frac{1}{H-E} = f_r(x,t)\frac{1}{f_l(x,t)(H-E)f_r(x,t)}f_l(x,t), \tag{A.8}$$

where H is the Hamiltonian corresponding to the path integral $K(t'',t')$, and $f_{l,r}(x,t)$ are functions in q and t, multiplying from the left or from the right, respectively, onto the operator $(H-E)^{-1}$. Secondly, one introduces a new «pseudotime» s'' and assumes that the constraint

$$\int_0^{s''} ds f_l\big(F(q(s),s)\big)\cdot f_r\big(F(q(s),s)\big) = T = t'' - t' \tag{A.9}$$

has for all admissible paths a unique solution $s'' > 0$ given by

$$s'' = \int_{t'}^{t''} \frac{dt}{f_l(x,t)f_r(x,t)} = \int_{t'}^{t''} \frac{ds}{F'^2(q(s),s)}. \tag{A.10}$$

Here one has made the choice $f_l\big(F(q(s),s)\big) = f_r\big(F(q(s),s)\big) = F'\big(q(s),s\big)$ in order that in the final result the metric coefficient in the kinetic energy term is equal to one. A convenient way to derive the corresponding transformation formulae uses the energy-dependent Green function $G(E)$ of the kernel $K(T)$ defined by

$$G(q'',q';E) = \left\langle q''\left|\frac{1}{H-E-i\epsilon}\right|q'\right\rangle = \frac{i}{\hbar}\int_0^\infty dT e^{i(E+i\epsilon)T/\hbar}K(q'',q';T). \tag{A.11}$$

For the one-dimensional path integral one obtains the following transformation formula:

$$K(x'',x';T) = \int_{\mathbb{R}} \frac{dE}{2\pi i} \mathrm{e}^{-iET/\hbar} G(q'',q';E), \tag{A.12}$$

$$G(q'',q';E) = \frac{i}{\hbar}\Big[F'(q'')F'(q')\Big]^{1/2} \int_0^\infty ds'' \hat{K}(q'',q';s''), \tag{A.13}$$

with the transformed path integral $\hat{K}$ having the form

$$\hat{K}(q'',q';s'') = \lim_{N\to\infty} \left(\frac{m}{2\pi i\epsilon\hbar}\right)^{1/2} \prod_{k=1}^{N-1} \int dq_k \times$$
$$\times \exp\left\{\frac{i}{\hbar}\sum_{j=1}^{N}\left[\frac{m}{2\epsilon}(\Delta q_j)^2 - \epsilon F'^2(\bar{q}_j)\Big(V(F(\bar{q}_j)) - E\Big) - \epsilon\Delta V(\bar{q}_j)\right]\right\}, \tag{A.14}$$

$$\equiv \int_{q(0)=q'}^{q(s'')=q''} \mathcal{D}q(s) \times$$
$$\times \exp\left\{\frac{i}{\hbar}\int_0^{s''}\left[\frac{m}{2}\dot{q}^2 - F'^2(q)\Big(V(F(q) - E\Big) - \Delta V(q)\right] ds\right\}, \tag{A.15}$$

and with the quantum potential ΔV given by

$$\Delta V(q) = \frac{\hbar^2}{8m}\left(3\frac{F''^2}{F'^2} - 2\frac{F'''}{F'}\right). \tag{A.16}$$

Note that ΔV has the form of a Schwarz derivative of F. A rigorous lattice derivation is far from being trivial and has been discussed elsewhere [13, 34, 46].

Let us consider a pure time transformation in a path integral. Let ($\mathbf{q}$ — a D-dimensional coordinate)

$$G(\mathbf{q}'',\mathbf{q}';E) = \sqrt{f(\mathbf{q}')f(\mathbf{q}'')}\frac{i}{\hbar}\int_0^\infty ds'' \Big\langle \mathbf{q}'' \Big| \exp\Big(-is''\sqrt{f}\,(H-E)\sqrt{f}/\hbar\Big)\Big|\mathbf{q}'\Big\rangle, \tag{A.17}$$

which corresponds to the introduction of the «pseudotime» $s'' = \int_{t'}^{t''} ds/f(\mathbf{q}(s))$

and we assume that the Hamiltonian H is the product ordered. Then

$$G(\mathbf{q}'', \mathbf{q}'; E) = \frac{i}{\hbar}(f'f'')^{\frac{1}{2}(1-D/2)} \int_0^\infty \tilde{K}(\mathbf{q}'', \mathbf{q}'; s'')ds'', \tag{A.18}$$

with the path integral

$$\tilde{K}(\mathbf{q}'', \mathbf{q}'; s'') = \int_{\mathbf{q}(0)=\mathbf{q}'}^{\mathbf{q}(s'')=\mathbf{q}''} \mathcal{D}\mathbf{q}(s)\sqrt{\tilde{g}}\times$$

$$\times \exp\left\{\frac{i}{\hbar}\int_0^{s''}\left[\frac{m}{2}\tilde{h}_{ac}\tilde{h}_{cb}\dot{q}^a\dot{q}^b - f\Big(V(\mathbf{q}) + \Delta V_{PF}(\mathbf{q}) - E\Big)\right]ds\right\}. \tag{A.19}$$

Here $\tilde{h}_{ac} = h_{ac}/\sqrt{f}$, $\sqrt{\tilde{g}} = \det(\tilde{h}_{ac})$ and (A.19) is of the canonical product form. Note that for $D = 2$ the prefactor gives unity.

This latter path integral technique of «time-transformation» is used in this paper in almost all cases in order to solve the corresponding path integrals in the various coordinate systems on Darboux spaces. Of course, the time-transformation is used in such a way that the metric term (g_{ab}) is transformed to unity.

In our calculations we have in all cases a metric which is diagonal, and in almost all cases is of the form $g_{ab} = f^2\delta_{ab}$. This has the consequence that the quantum potential $\Delta V = 0$ and the term $\tilde{h}_{ac}$ can be transformed to unity. This simplifies the calculations significantly.

Appendix B
SOME IMPORTANT PATH INTEGRAL SOLUTIONS AND IDENTITIES

In this Appendix we cite some important path integral solutions, in particular for the (radial) harmonic oscillator, the linear potential, and for the modified Pöschl–Teller potential.

B.1. The Path Integral for the Radial Harmonic Oscillator. The calculation of the path integral for the radial harmonic oscillator has first been performed by Peak and Inomata [59]. For a comprehensive bibliography, see [34]. We have the path integral representation ($r > 0$)

$$\int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t) \exp\left[\frac{i}{\hbar}\int\limits_{t'}^{t''}\left(\frac{m}{2}(\dot r^2-\omega^2 r^2)-\hbar^2\frac{\lambda^2-\frac{1}{4}}{2mr^2}\right)dt\right] =$$

$$= \frac{m\omega\sqrt{r'r''}}{i\hbar\sin\omega T}\exp\left[-\frac{m\omega}{2i\hbar}(r'^2+r''^2)\cot\omega T\right] I_\lambda\left(\frac{m\omega r'r''}{i\hbar\sin\omega T}\right), \quad \text{(B.20)}$$

$I_\lambda(z)$ is a modified Bessel function. The energy-dependent kernel (the Green function $G(E)$) is given by

$$\frac{i}{\hbar}\int\limits_0^\infty dE\, \mathrm{e}^{iET/\hbar} \int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t) \exp\left[\frac{i}{\hbar}\int\limits_{t'}^{t''}\left(\frac{m}{2}(\dot r^2-\omega^2 r^2)-\hbar^2\frac{\lambda^2-\frac{1}{4}}{2mr^2}\right)dt\right] =$$

$$= \frac{\Gamma\left[\frac{1}{2}(1+\lambda-E/\hbar\omega)\right]}{\hbar\omega\sqrt{r'r''}\Gamma(1+\lambda)} W_{E/2\hbar\omega,\lambda/2}\left(\frac{m\omega}{\hbar}r_>^2\right) M_{E/2\hbar\omega,\lambda/2}\left(\frac{m\omega}{\hbar}r_<^2\right). \quad \text{(B.21)}$$

Here $M_{\mu,\nu}(z)$ and $W_{\mu,\nu}(z)$ are Whittaker functions.

B.2. The Green Function for the Linear Potential. The energy-dependent kernel for the linear potential is given by

$$\frac{i}{\hbar}\int\limits_0^\infty dE\, \mathrm{e}^{iET/\hbar} \int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \exp\left[\frac{i}{\hbar}\int\limits_0^T\left(\frac{m}{2}\dot x^2-kx\right)dt\right] =$$

$$= \frac{4}{3}\frac{m}{\hbar^2}\left[\left(x'-\frac{E}{k}\right)\left(x''-\frac{E}{k}\right)\right]^{1/2}\times$$

$$\times K_{1/3}\left[\frac{\sqrt{8mk}}{3\hbar}\left(x_>-\frac{E}{k}\right)^{3/2}\right] I_{1/3}\left[\frac{\sqrt{8mk}}{3\hbar}\left(x_<-\frac{E}{k}\right)^{3/2}\right]. \quad \text{(B.22)}$$

B.3. The Green Function for the Harmonic Oscillator. The energy-dependent kernel for the harmonic oscillator is given by

$$\frac{i}{\hbar}\int\limits_0^\infty dE\, \mathrm{e}^{iET/\hbar} \int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \exp\left[\frac{im}{2\hbar}\int\limits_0^T(\dot x^2-\omega^2x^2)dt\right] =$$

$$= \sqrt{\frac{m}{\pi\hbar^3\omega}}\Gamma\left(\frac{1}{2}-\frac{E}{\hbar\omega}\right) D_{-\frac{1}{2}+\frac{E}{\hbar\omega}}\left(\sqrt{\frac{2m\omega}{\hbar}}\,x_>\right) D_{-\frac{1}{2}+\frac{E}{\hbar\omega}}\left(-\sqrt{\frac{2m\omega}{\hbar}}\,x_<\right). \quad \text{(B.23)}$$

The $D_\nu(z)$ are parabolic cylinder functions.

B.4. The Modified Pöschl–Teller Potential. The path integral solution for the modified Pöschl–Teller potential can be achieved by means of the $SU(1,1)$-path integral. For a comprehensive bibliography, see [34]. We have [1, 6, 13, 47]

$$\int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t) \exp\left\{ \frac{i}{\hbar} \int\limits_{t'}^{t''} \left[\frac{m}{2}\dot{r}^2 - \frac{\hbar^2}{2m} \left(\frac{\eta^2 - \frac{1}{4}}{\sinh^2 r} - \frac{\nu^2 - \frac{1}{4}}{\cosh^2 r} \right) \right] dt \right\} =$$

$$= \sum_{n=0}^{N_M} \Psi_n^{(\eta,\nu)\,*}(r') \Psi_n^{(\eta,\nu)}(r'') \exp\left\{ \frac{i\hbar T}{2m} \Big[2(k_1 - k_2 - n) - 1 \Big]^2 \right\} \times$$

$$\times \int\limits_0^\infty dp\, \Psi_p^{(\eta,\nu)\,*}(r') \Psi_p^{(\eta,\nu)}(r'') \exp\left(-\frac{i\hbar T}{2m} p^2 \right). \quad \text{(B.24)}$$

Let us introduce the numbers k_1, k_2 defined by: $k_1 = 1/2(1 \pm \nu)$, $k_2 = 1/2(1 \pm \eta)$, where the correct sign depends on the boundary conditions for $r \to 0$ and $r \to \infty$, respectively. In particular for $\eta^2 = 1/4$, i.e., $k_2 = 1/4, 3/4$, we obtain wave functions with even and odd parity, respectively. The number N_M denotes the maximal number of states with $0, 1, \ldots, N_M < k_1 - k_2 - 1/2$. The bound state wave functions read as ($\kappa = k_1 - k_2 - n$)

$$\Psi_n^{(\eta,\nu)}(r) = N_n^{(\eta,\nu)} (\sinh r)^{2k_2 - \frac{1}{2}} (\cosh r)^{-2k_1 + \frac{3}{2}} \times$$
$$\times\, {}_2F_1(-k_1 + k_2 + \kappa, -k_1 + k_2 - \kappa + 1; 2k_2; -\sinh^2 r), \quad \text{(B.25)}$$

$$N_n^{(\eta,\nu)} = \frac{1}{\Gamma(2k_2)} \left[\frac{2(2\kappa - 1)\Gamma(k_1 + k_2 - \kappa)\Gamma(k_1 + k_2 + \kappa - 1)}{\Gamma(k_1 - k_2 + \kappa)\Gamma(k_1 - k_2 - \kappa + 1)} \right]^{1/2}. \quad \text{(B.26)}$$

The scattering states are given by

$$\Psi_p^{(\eta,\nu)}(r) = N_p^{(\eta,\nu)} (\cosh r)^{2k_1 - \frac{1}{2}} (\sinh r)^{2k_2 - \frac{1}{2}} \times$$
$$\times\, {}_2F_1(k_1 + k_2 - \kappa, k_1 + k_2 + \kappa - 1; 2k_2; -\sinh^2 r), \quad \text{(B.27)}$$

$$N_p^{(\eta,\nu)} = \frac{1}{\Gamma(2k_2)} \sqrt{\frac{p \sinh \pi p}{2\pi^2}} \Big[\Gamma(k_1 + k_2 - \kappa)\Gamma(-k_1 + k_2 + \kappa) \times$$
$$\times\, \Gamma(k_1 + k_2 + \kappa - 1)\Gamma(-k_1 + k_2 - \kappa + 1) \Big]^{1/2}, \quad \text{(B.28)}$$

$[\kappa = 1/2(1+ip)]$. ${}_2F_1(a,b;c;z)$ is the hypergeometric function. The Green function has the form

$$\frac{i}{\hbar}\int_0^\infty dE\, \mathrm{e}^{iET/\hbar} \int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t)\times$$
$$\times\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{r}^2 - \frac{\hbar^2}{2m}\left(\frac{\eta^2-1/4}{\sinh^2 r} - \frac{\nu^2-1/4}{\cosh^2 r}\right)\right]dt\right\} =$$
$$= \frac{m}{\hbar^2}\frac{\Gamma(k_1-L_\nu)\Gamma(L_\nu+k_1+1)}{\Gamma(k_1+k_2+1)\Gamma(k_1-k_2+1)}\times$$
$$\times(\cosh r'\cosh r'')^{-(k_1-k_2)}(\tanh r'\tanh r'')^{k_1+k_2+1/2}\times$$
$$\times\,{}_2F_1\left(-L_\nu+k_1, L_\nu+k_1+1; k_1-k_2+1; \frac{1}{\cosh^2 r_<}\right)\times$$
$$\times\,{}_2F_1\left(-L_\nu+k_1, L_\nu+k_1+1; k_1+k_2+1; \tanh^2 r_>\right). \quad \text{(B.29)}$$

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