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WAVE PACKETS IN QUANTUM THEORY OF COLLISIONS

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Two methodological troubles of the quantum theory of collisions are considered. The first is the undesirable interference of the incident and scattered waves in the stationary approach to scattering. The second concerns the nonstationary approach to the theory of collisions of the type $a+b \to c+d$. In order to calculate the cross section one uses the matrix element $\langle cd|S|ab\rangle$ of the S matrix. The element is proportional to δ function expressing the energy conservation. The corresponding probability $|\langle cd|S|ab\rangle|^2$ contains δ^2 which is mathematically senseless. The known regular way to overcome the difficulty seems to be unsatisfactory. In this paper, both the troubles are resolved using wave packets of incident particles.

Рассматриваются две методологические трудности квантовой теории столкновений. Первая касается трактовки интерференции падающей и рассеянной волн в стационарном подходе к рассеянию. Вторая возникает в нестационарном подходе к теории столкновений типа $a + b \to c + d$. В этом подходе для вычисления поперечного сечения используется S-матричный элемент $\langle cd|S|ab\rangle$. Он пропорционален δ-функции, выражающей сохранение энергии. Соответствующая вероятность $|\langle cd|S|ab\rangle|^2$ содержит δ^2, что математически бессмысленно. Известная физическая трактовка δ^2 представляется неудовлетворительной. Обе трудности преодолеваются в настоящей работе посредством использования волновых пакетов падающих частиц.

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INTRODUCTION

An approach to the theory of scattering is known which may be called stationary. The scattering is considered as a stationary process: there is a steady flux of particles incident on a potential V (the target). Scattered particles are also described by a steady flux. The state of the system is described by a vector which is constant in time. The vector is an eigenstate of the total Hamiltonian $H = H_0 + V$ which belongs to the continuous spectrum of H: $H\psi_k = E_k\psi_k$. The eigenstate is known to be the superposition of the incident wave $I_{\mathbf{k}}(\mathbf{x})$ and scattered wave $S_{\mathbf{k}}(\mathbf{x})$, their asymptotic behavior being

$$\Psi_{\mathbf{k}}(\mathbf{x}) = I_{\mathbf{k}}(\mathbf{x}) + S_{\mathbf{k}}(\mathbf{x}), \quad I_{\mathbf{k}}(\mathbf{x}) = \mathrm{e}^{i\mathbf{kx}}, \quad S_{\mathbf{k}}(\mathbf{x}) = A(\vartheta,\varphi)\ \mathrm{e}^{ikr}/r. \quad (1)$$

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Here $k = \sqrt{2mE_k}$ and $\mathbf{x} = (r, \vartheta, \varphi)$, ϑ being the angle between $\mathbf{k}$ and $\mathbf{x}$. The axis z is chosen to be parallel to the momentum $\mathbf{k}$ of the incident particle ($\mathbf{x}$ being its position).

The stationary approach uses the probability flux or its density

$$\mathbf{j}(\mathbf{x}) = \frac{i}{2m} \left[\nabla\psi^*(\mathbf{x})\psi(\mathbf{x}) - \psi^*(\mathbf{x})\nabla\psi(\mathbf{x}) \right] \tag{2}$$

instead of the usual probability $|\psi(\mathbf{x})|^2$, e.g., see [3, Ch. IV, § 29]; [5, Ch. II, § 15, Ch. XI, § 95]; [10, Ch. II.7]. The probability flux generated by $\Psi_\mathbf{k} = I_\mathbf{k} + S_\mathbf{k}$ (see Eq. (1)) is

$$\mathbf{j}_\mathbf{k} = \frac{i}{2m} (\nabla I^*_\mathbf{k} \cdot I_\mathbf{k} - \text{c.c.}) + \frac{i}{2m} (\nabla S^*_\mathbf{k} \cdot S_\mathbf{k} - \text{c.c.}) + \\ + \frac{i}{2m} (\nabla S^*_\mathbf{k} \cdot I_\mathbf{k} - \text{c.c.}) + \frac{i}{2m} (\nabla I^*_\mathbf{k} \cdot S_\mathbf{k} - \text{c.c.}) . \tag{3}$$

The cross section is determined by the ratio of the scattered flux (the second term in Eq. (3)) to the incident one (the first term in Eq. (3)), see Sec. 1 below. Besides these fluxes, the total flux $\mathbf{j}_\mathbf{k}$ contains the interference terms (the third and fourth in Eq. (3)). Their physical sense is unclear. It is unknown what contribution the interference terms may bring in the cross section. One may conjecture that they must vanish if one replaces the plane incident wave $I_\mathbf{k}$ by an incident wave packet, see [9, Ch. X.5]; [10, Ch. V, end of § 18]. This conjecture is confirmed in Sec. 2. The wave packet is used which tends to the plane wave $I_\mathbf{k} = \exp(i\mathbf{k}\mathbf{x})$ when packet dimension increases. Other ways of the packet introduction are possible. For example, Messiah [9] used a classical ensemble of small packets which have different impact parameters. Finally, averaging over the ensemble is carried out, see [9, Ch. X].

The stationary theory of scattering is nonrelativistic and is inapplicable, e. g., to the photon scattering (photon position operator and density of flux have no satisfactory definitions). The nonstationary approach is applicable to any process of the type $a + b \to c + d$. It is based on the solution of the Schroedinger equation for the operator $U(t, t_0)$ of evolution in time. Initially, at the moment t_0, the system is in a state $\Psi(t_0) = |ab\rangle$. At the moment t, the system state is described by the vector $\Psi(t) = U(t, t_0)\Psi(t_0)$. The probability to find a final state $|cd\rangle$ at the moment t is equal to $|\langle cd|U(t, t_0)|ab\rangle|^2$. The limit $t_0 \to -\infty$, $t \to +\infty$, i. e., S matrix, is usually considered.

The approach has the following trouble. The matrix element $\langle cd|S|ab\rangle$ is known to be proportional to the δ function which expresses the total energy conservation: the total initial energy $E_a + E_b$ is equal to the final energy $E_c + E_d$

$$\langle cd|S|ab\rangle \sim \delta(E), \quad E = (E_c + E_d) - (E_a + E_b). \tag{4}$$

The probability $|\langle cd|S|ab\rangle|^2$ is proportional to the square δ^2 of this δ function. This quantity does not exist mathematically (see [1]). Physicists gave to δ^2 an interpretation, see the end of Sec. 3, but it cannot be recognized as satisfactory. Another resolution of this trouble is known, e. g., see [7, Ch. I4]; [10, Ch. VIII]. It is presented in Sec. 3 using the packet description of the initial state $|ab\rangle$.

So a packet description of the incident particle allows us to resolve two troubles of the collision theory stated above.

1. DEFINITIONS OF CROSS SECTION

The density of the incident flux F is defined as the number of incident particles crossing per unit time a unit surface placed perpendicular to the direction of propagation. Let ρ be the number of particles per unit volume and $\mathbf{v}$ be the velocity of the incident particles. Then $\mathbf{F} = \rho\mathbf{v}$. If there is one particle in 1 cm^3, then $\mathbf{F} = \mathbf{v}$.

Let us assume that the coordinate origin is in the center of a target. Let j_r be the density of the probability flux of the scattered particles. The probability (or the number of particles) going through the area element $r d\varphi r \sin\vartheta d\vartheta \equiv r^2 d\Omega$ during one unit of time is equal to $j_r r^2 d\Omega$. This is the probability ΔN to detect the particle in the solid angle $\Delta\Omega$ during one unit of time (e. g., one second)

$$\Delta N = j_r r^2 \Delta\Omega. \tag{5}$$

The quantity ΔN may be related to the probability $\Delta W(t)$ of the particle detection in the solid angle at the moment t. One may assume

$$\Delta N = \Delta W(t + 1\,\mathrm{s}) - \Delta W(t) \cong \frac{d}{dt}\Delta W(t) 1\,\mathrm{s}. \tag{6}$$

Usually another relation is assumed

$$\Delta N = \frac{W(t)}{t}. \tag{7}$$

Relations (6) and (7) coincide if time derivative $d\Delta W(t)/dt$ is constant. One has the relation

$$j_r r^2 \Delta\Omega = \Delta N = \frac{\Delta W(t)}{t}. \tag{8}$$

Therefore, the definition

$$\Delta\sigma = \frac{j_r}{F} r^2 d\Omega \tag{9}$$

of the cross section (see [3, Ch. XIII]; [5, Ch. XI, § 95]) is equivalent to

$$\Delta\sigma = \frac{\Delta N}{F}, \tag{10}$$

cf. [9, Ch. 10].

2. PACKETS IN STATIONARY APPROACH

There is the conjecture that the description of an incident particle by a packet (instead of the plane wave) will turn into zero interference terms in the total flux (the third and fourth terms in Eq. (3)). Let us give a confirmation of this conjecture.

The introduction of a packet implies that scattering is no longer a stationary process. Even if the scattering potential is absent, the (free) packet shifts and spreads.

One possible setting of the problem of a packet scattering will be considered here (for another approach, see, e. g., [9, Ch. X]). The definition of the cross section as the ratio of fluxes (see Eq. (9)) will be retained though fluxes will not be stationary. The natural requirement is assumed (and ensured): in the limit when the packet turns into a plane wave the result should go to the usual stationary one.

Consider a superposition

$$\Psi(\mathbf{x}) = \int d^3k \tilde{I}_{\mathbf{k}} \Psi_{\mathbf{k}}(\mathbf{x}) \tag{11}$$

of the H eigenfunctions $H\Psi_{\mathbf{k}} = E_k \Psi_{\mathbf{k}}$, for $\tilde{I}_{\mathbf{k}}$ see below. The superposition is not H eigenfunction, but the vector

$$\Psi(\mathbf{x}, t) = \int d^3k \ \mathrm{e}^{-iE_k t} \ \Psi_{\mathbf{k}}(\mathbf{x}) I_{\mathbf{k}} \tag{12}$$

is a solution of the equation $i\partial\Psi(t)/\partial t = H\Psi(t)$.

The vector $\Psi(\mathbf{x}, t)$ consists of two parts

$$\Psi(\mathbf{x}, t) = I(\mathbf{x}, t) + S(\mathbf{x}, t), \tag{13}$$

$$I(\mathbf{x}, t) = \int d^3k \ \mathrm{e}^{-iE_k t} \tilde{I}_{\mathbf{k}} \ \mathrm{e}^{i\mathbf{k}\mathbf{x}}, \tag{14}$$

$$S(\mathbf{x}, t) = \int d^3k \ \mathrm{e}^{-iE_k t} \tilde{I}_{\mathbf{k}} A(\vartheta, \varphi) \mathrm{e}^{ikr}/r. \tag{15}$$

The vector $I(\mathbf{x}, t)$ is the known description of the moving free packet (it is assumed that the spectrum of $H = H_0 + V$ is the same as the H_0 spectrum). In order to calculate $S(\mathbf{x}, t)$ and the cross section, the following program is accepted.

(a) The initial wave packet $I(\mathbf{x}, 0)$ will be chosen.

(b) This determines the coefficients $\tilde{I}_{\mathbf{k}}$ in Eqs. (11), (15) and, therefore, $S(\mathbf{x}, t)$ may be calculated.

(c) Absence of the interference terms will be verified.

(d) Incident and scattering fluxes may then be found as well as the cross section.

(a) Choice of $I(\mathbf{x}, 0)$.

Consider the auxiliary wave function $f(\mathbf{x})$ which is concentrated in a ball of the radius R_I (V_I being the ball volume). The ball will be named «support of $f(\mathbf{x})$».

Note. In other words «support» is defined here as the volume outside which the function practically vanishes (or is unobservably small). In mathematics a different definition is accepted: the support is the volume outside which the function is exactly zero.

If $f(\mathbf{x})$ is spherically symmetric, then the average position $\int d^3x \mathbf{x} f^*(\mathbf{x}) f(\mathbf{x})$ is zero. Fourier transform of $f(\mathbf{x})$ is also spherically symmetric and, therefore, the average momentum also equals zero. The average position of the shifted function $f(\mathbf{x} - \mathbf{a}) \equiv I_{\mathbf{a}}(\mathbf{x})$ is equal to $\mathbf{a}$, average momentum being zero as before. One may verify that the function $\mathrm{e}^{i\mathbf{p}\mathbf{x}} f(\mathbf{x}) \equiv I_{\mathbf{p}}(\mathbf{x})$ has average momentum $\mathbf{p}$. At last, consider the shifted function $I_{\mathbf{p}}(\mathbf{x})$, i.e., the function

$$I_{\mathbf{p}}(\mathbf{x} - \mathbf{a}) = \mathrm{e}^{i\mathbf{p}(\mathbf{x}-\mathbf{a})} f(\mathbf{x} - \mathbf{a}) \equiv I_{\mathbf{pa}}(\mathbf{x}). \tag{16}$$

Its average position is $\mathbf{a}$ and average momentum is $\mathbf{p}$ (the factor $\exp(-i\mathbf{pa})$ may be omitted).

Let us assume

$$f(\mathbf{x}) = \begin{cases} 0, & \text{outside the ball } V_I, \\ 1, & \text{inside } V_I. \end{cases} \tag{17}$$

This means that $I_{\mathbf{p}}(\mathbf{x}) = f(\mathbf{x}) \exp(i\mathbf{p}\mathbf{x})$ is equal to $\exp(i\mathbf{p}\mathbf{x})$ inside V_I. When $R_I \to \infty$, the vector $I_{\mathbf{p}}(\mathbf{x})$ tends to the plane wave whose wave function is equal to $\exp(i\mathbf{p}\mathbf{x})$ everywhere. We have $\int d^3x |I_{\mathbf{p}}(\mathbf{x})|^2 = V_I$: there is one particle in the unit volume.

The initial (at the moment $t = 0$) wave function of the incident packet is chosen to be equal to

$$I_{\mathbf{p}}(\mathbf{x}) = \mathrm{e}^{i\mathbf{p}\mathbf{x}} f(\mathbf{x}). \tag{18}$$

Here $\mathbf{p}$ is the packet average momentum. At the moment $t = 0$ the packet centre $\mathbf{a}$ is in the point 0 on the z axis, i. e., in the centre of potential V.

(b) Choosing $I_{\mathbf{p}}(\mathbf{x})$ one may calculate $S(\mathbf{x}, t)$ (see (15)) using the Fourier transform $\tilde{f}(\mathbf{p} - \mathbf{k})$ of $I_{\mathbf{p}}(\mathbf{x})$. If the packet $I_{\mathbf{p}}(\mathbf{x})$ has a macroscopical dimension (e. g., 1 cm), then $\tilde{f}(\mathbf{p} - \mathbf{k})$ has a sharp maximum at $\mathbf{k} \cong \mathbf{p}$. Therefore, $E_k \cong E_p$ and $|\mathbf{k}| \cong |\mathbf{p}|$. Let us calculate $S(\mathbf{x}, t)$ approximately using the expansions of E_k and $|\mathbf{k}|$ in Taylor series about the point $\mathbf{p}$:

$$E_k \cong E_p + (\mathbf{k} - \mathbf{p})\mathbf{v}, \quad |\mathbf{k}| \cong |\mathbf{p}| + (\mathbf{k} - \mathbf{p})\mathbf{u}, \tag{19}$$

$$\mathbf{v} = \frac{\mathbf{p}}{m}, \quad \mathbf{u} = \frac{\mathbf{p}}{p}, \quad \mathbf{v} = \mathbf{u}v. \tag{20}$$

The scattering amplitude $A(\vartheta_k, \varphi_k)$ is simply replaced by $A(\vartheta_p, \varphi_p)$, ϑ_k and ϑ_p being the angles between $\mathbf{x}$ and $\mathbf{k}$, $\mathbf{p}$, respectively, cf. Eq. (1). One obtains

$$S(\mathbf{x}, t) = A(\vartheta_p, \varphi_p)\frac{1}{r}\exp\, i(pr - E_p t) \int d^3k\, \mathrm{e}^{i(\mathbf{k}-\mathbf{p})(\mathbf{u}r-\mathbf{v}t)} \tilde{f}(\mathbf{p}-\mathbf{k}). \tag{21}$$

After the change $\mathbf{k}' = \mathbf{p} - \mathbf{k}$ of variables in $\int d^3k \ldots$ one gets

$$S(\mathbf{x}, t) = A\frac{1}{r}\exp i(pr - E_p t) \int d^3k' \tilde{f}(\mathbf{k}')\, \mathrm{e}^{i(\mathbf{k}'\mathbf{b})},$$
$$\mathbf{b} \equiv -\mathbf{u}(r - vt).$$

Here $\int d^3k' \tilde{f}(\mathbf{k}') \exp(i\mathbf{k}'\mathbf{b})$ is Fourier representation of $f(\mathbf{b})$. Therefore

$$S(\mathbf{x}, t) = A\frac{1}{r}\exp i(pr - E_p t) f(-\mathbf{u}(r - vt)). \tag{22}$$

Remind that $f(\mathbf{b}) = 1$ when $|\mathbf{b}| \leqslant R_I$, i.e., when

$$|-\mathbf{u}(r - vt)| \leqslant R_I. \tag{23}$$

Here $|\mathbf{u}| = 1$ and f is spherically symmetric. Therefore Eq. (23) may be rewritten as

$$|r - vt| \leqslant R_I. \tag{24}$$

Equation (24) determines the movement of the support of the scattered wave $S(\mathbf{x}, t)$. Let us compare it with the movement of the initial incident packet $I_\mathbf{p}(\mathbf{x}) = f(\mathbf{x}) \exp i\mathbf{p}\mathbf{x}$ (see Eq. (18)). The latter is the shift $\mathbf{x} \to \mathbf{x} - \mathbf{v}t$ along the z axis if the spreading is neglected (e.g., see [12, Ch. 10.4]):

$$I_\mathbf{p}(\mathbf{x}, t) = I_\mathbf{p}(\mathbf{x} - \mathbf{v}t, 0) = f(\mathbf{x} - \mathbf{v}t) \exp i\mathbf{p}(\mathbf{x} - \mathbf{v}t), \quad \mathbf{v} = \frac{\mathbf{p}}{m}. \tag{25}$$

This means the shift of the packet support, i.e., the ball of the radius R_I (the phase factor $\exp\, i\mathbf{p}\mathbf{v}t$ is inessential). The shifted ball is described by the inequality

$$|\mathbf{x} - \mathbf{v}t| \leqslant R_I. \tag{26}$$

The shifted ball is not a spherically symmetric region (excluding the case when the packet centre coincides with the coordinate origin, i.e., potential centre). Meanwhile, inequality (24) describes a spherically symmetric region at all times: (24) does not contain angles ϑ, φ of the vector $\mathbf{x}$ but contains only $|\mathbf{x}| = r$. At fixed t, the support region is the spherical layer between the spheres of the radii $vt - R_I$ and $vt + R_I$. The thickness of the layer is equal to $2R_I$. As t increases, this layer «inflates» preserving its thickness.

(c) We obtain that the scattered packet is in the spherical layer described above, while the support of the incident packet moves along the z axis. The moving-packets' supports do not practically intersect if $r \gg R_I$. Therefore, $S(\mathbf{x},t)I_{\mathbf{p}}(\mathbf{x},t) = 0$: S is zero, where I is nonzero and vice versa. So the interference of the incident and scattered waves is absent (with the exception of small values of the angle ϑ).

(d) The used expressions (25) and (22) for incident and scattered wave packets differ from the corresponding I_k and S_k waves in Eq. (1) only in one respect: the former have an additional factor f which is equal to unity inside the moving packets and vanishes outside them. Therefore, the incident and scattered fluxes *inside* packets are the same as in the stationary case. However, these fluxes are nonstationary: their supports move in space. One may retain the previous definition (9) for the cross section having in mind that the fluxes F and j_r in (9) are intrapacket ones.

Note once more that the used packet's description of scattering turns into the ordinary stationary one when $R_I \to \infty$ (the condition $r \gg R_I$ being implied).

3. PACKETS IN NONSTATIONARY APPROACH

The nonstationary approach to the collision theory uses the evolution operator $U(t,t_0)$ (interaction or Dirac picture is in mind). For the reaction $a+b \to c+d$ one must calculate the matrix element of the type $\langle cd|U(t,t_0)|ab\rangle$. Let us assume that the initial state $|i\rangle = |ab\rangle$ is the product of packets $|a\rangle$, $|b\rangle$ (see Sec. 2), for example,

$$|a\rangle = \int d^3k|\mathbf{k}\rangle a(\mathbf{k}).$$

The packets have finite supports in the coordinate space. The supports are supposed to be of macroscopically large dimensions and, therefore, packet spreading may be neglected, see [6, Ch. 3.1]; [12, Ch. 10.4].

The particles do not interact if their supports do not intersect. So the interaction lasts during a finite interval T of time. In the following I let $t_0 = -T/2$, $t = T/2$.

Let us consider the matrix element of the expansion of $U(T/2, -T/2)$ in the perturbation series

$$U_{fi}(T) \equiv \langle f|U\left(\frac{T}{2} - \frac{T}{2}\right)|i\rangle = \langle f|i\rangle + i\int\limits_{-T/2}^{T/2} dt\ \mathrm{e}^{i(E_f-E_i)t}\langle f|H^s_{\mathrm{int}}|i\rangle +$$

$$+ i^2\sum_m \int\limits_{-T/2}^{T/2} dt_1\ \mathrm{e}^{i(E_f-E_m)t_1} \int\limits_{-T/2}^{t_1} dt_2\ \mathrm{e}^{i(E_m-E_i)t_2}\langle f|H^s_{\mathrm{int}}|m\rangle\langle m|H^s_{\mathrm{int}}|i\rangle, \quad (27)$$

cf. [8, Chs. 1.2 and 1.3]; [11, Ch. 11.6]. Here E_f denotes the final total energy $E_f \equiv E_c + E_d$. Analogously, $E_i \equiv E_a + E_b$ (E_a and E_b being average energies); E_m is the total energy of the intermediate state $|m\rangle$; $H^s_{\rm int}$ is the interaction Hamiltonian in the Schroedinger picture. One may suppose that $\langle f|i\rangle = 0$.

The second term in Eq. (27) is proportional to

$$2\pi\delta_T(E) \equiv \int\limits_{-T/2}^{T/2} dt\ \mathrm{e}^{iEt} = \frac{2\sin ET/2}{E}, \quad E \equiv E_f - E_i. \tag{28}$$

The third term contains the integral over t_2

$$\int\limits_{-T/2}^{t_1} dt_2\, \mathrm{e}^{i(E_m-E_i)t_2} = \frac{\mathrm{e}^{i(E_m-E_i)t_1} - \mathrm{e}^{i(E_m-E_i)(-T/2)}}{i(E_m - E_i)}. \tag{29}$$

The contribution to (29) originating from the lower limit $-T/2$ tends to zero as $T \to \infty$ due to fast oscillations of $\exp i(E_m - E_i)T/2$. For a strict proof of this statement one must use the packet description of $|i\rangle$ and Riemann–Lebesque lemma, see, e.g., [11, Ch. 11 after Eq. (11.165)]. Neglecting this contribution one obtains that the remaining integral over t_1 is equal to $2\pi\delta_T(E)$, Eq. (28). Analogously, one may argue that in all orders of the expansion (27) $U_{fi}(T)$ is proportional to $\delta_T(E_f - E_i)$

$$U_{fi}(T) \cong \delta_T(E_f - E_i)\langle f|R|i\rangle, \tag{30}$$

where $\langle f|R|i\rangle$ ceases to depend upon T as $T \to \infty$. Note that $\delta_T(E) \to \delta(E)$ as $T \to \infty$. In this limit Eq. (27) turns into

$$S_{fi} = 2\pi\delta(E_f - E_i)\langle f|R|i\rangle, \tag{31}$$

where S is the S matrix. One gets that the probability $|\langle f|U(\infty, -\infty)|i\rangle|^2$ is proportional to δ^2. The square of the δ function has no mathematical sense, see [11, part III, Sec. 12.5].

However, for the cross section we need probability in unit of time, see Sec. 1. It may be defined as $|U_{fi}(T)|^2/T$. It follows from Eqs. (30) and (28) that

$$\frac{|U_{fi}(T)|^2}{T} \sim \frac{4\sin^2 ET/2}{TE^2}. \tag{32}$$

The r.h.s. of Eq. (32) tends to $2\pi\delta(E)$ as $T \to \infty$, not to $\delta^2(E)$, see [7, Ch. 2, Eq. (8.19)]. So we obtain the following value for the probability in unit time:

$$\lim_{T\to\infty} \frac{|U_{fi}(T)|^2}{T} = 2\pi\delta(E_f - E_i)|\langle f|R|i\rangle|^2. \tag{33}$$

The probability in unit time may be defined in a different way, namely as $d|U_{fi}(T)|^2/dT$. We have

$$\frac{d}{dT}|U_{fi}(T)|^2 \sim \frac{d}{dT}\frac{4\sin^2 ET/2}{E^2} = \frac{2}{E}\sin ET. \qquad (34)$$

In the limit $T \to \infty$ one gets in the r.h.s. of Eq. (33) the $\delta(E)$ function as above, cf. [11, Ch. 11, Eq. (11.91)].

The usual way to calculate the cross section is to start with the S-matrix element $\langle f|U(\infty, -\infty)|i\rangle$. The reason is that it is relativistic invariant, allows renormalization, etc., unlike $\langle f|U(t, t_0)|i\rangle$. But the probability $|\langle f|S|i\rangle|^2$ is proportional to $\delta^2(E_f - E_i)$ and this is senseless. The trouble is usually overcome in the following manner (see, e.g., [8, Ch. 1.2]; [11, Ch. I4.1]; [2, Ch. 5, § 37]; [4]). In the product $\delta(E)\delta(E)$ one of the δ functions is replaced by $\delta(0)$ because of the presence of the other δ function. Basing on the representation

$$\delta(E) = \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T/2}^{T/2} \mathrm{e}^{iEt} dt \qquad (35)$$

$\delta(0)$ is replaced by $T/2\pi$. In order to obtain the probability in unit of time one divides $|\langle f|S|i\rangle|^2$ by T. So one gets

$$\frac{|\langle f|S|i\rangle|^2}{T} \sim \delta(E)$$

which is a sensible result.

However, the argumentation is not satisfactory: $\delta(E)$ does not depend on T unlike $\delta_T(E)$, see Eq. (28) ($\delta(E)$ is the limit of (28) as $T \to \infty$). Instead of (35) one may use the representation

$$\delta(E) = \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} \mathrm{e}^{iEt} dt$$

and analogously obtain $\delta(0) = 2T/2\pi$ instead of $\delta(0) = T/2\pi$.

Nevertheless, the resulting «probability in unit time» obtained in the books cited above coincides with the r.h.s. of Eq. (33) obtained here.

CONCLUSION

As has been expected, the interference of the incident and scattered waves in the stationary theory of scattering is absent if the waves are described by packets. The cause is the nonintersection of the packets' supports (excepting the limitingly small scattering angles).

Consideration of the δ^2 trouble arising in the nonstationary approach needs the determination of the «probability in unit time» $\Delta N(T)$. In Sec. 3, for this purpose the evolution operator $U(T/2, -T/2)$ was used, $\Delta N(T)$ being defined as

$$\Delta N(T) = \frac{|\langle f|U(T/2, -T/2)|i\rangle|^2}{T}. \tag{36}$$

In the limit $T \to \infty$, one gets (see Sec. 3),

$$\lim \Delta N(T) \sim \delta(E), \quad E = E_f - E_i. \tag{37}$$

The result is free of the δ^2 trouble.

Another way to get «probability in unit time» is given in books on quantum field theory, e.g., see [2, 4, 8, 11]. [8, 11, 2, 4]. At first, one considers the $\lim U(T/2, -T/2)$ as $T \to \infty$, i.e., the S matrix (in our way the limit $T \to \infty$ is carried out later, see (36), (37)). The S-matrix elements $\langle f|S|i\rangle$ are proportional to $\delta(E)$. The corresponding probability $|\langle f|S|i\rangle|^2$ is proportional to $\delta^2(E)$, which is senseless. In the books δ^2 is treated in an unsatisfactory manner presented and criticized at the end of Sec. 3.

Both the ways give the same result which is represented by the r.h.s. of Eq. (33). Here the satisfactory way of getting the result is considered.

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