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We address the problem of exactly describing stochastic nonequilibrium systems that are widely used to model one-dimensional transport in biology, traffic flow and others. We review the matrix product states ansatz to interacting multiparticle systems and its extension to a tridiagonal (generalized Onsager) algebra approach. The stationary probability distribution is expressed as a matrix product state with respect to a quadratic algebra defined by the dynamics of the process. The states involved in the matrix elements are determined by the boundary conditions. This reflects the intriguing feature of open systems that the bulk behaviour in the steady state strongly depends on the boundary rates. The importance of the boundary conditions manifests itself in the fact that the boundary operators are generators of a tridiagonal algebra whose irreducible modules are the Askey–Wilson polynomials. The matrices of the matrix product ansatz obey the tridiagonal algebraic relations for particular values of the structure constants. Previously known representations, both infinite-dimensional and finite-dimensional ones, are recovered within the tridiagonal framework. The boundary Askey–Wilson and tridiagonal symmetry is the deep algebraic property of driven diffusive systems allowing for the exact solvability in the steady state and the exact description of the stochastic dynamics.

В обзоре рассматриваются матричный подход к стохастическим неравновесным системам и его обобщение до трехдиагонального алгебраического подхода. Веса стационарных состояний выражаются как матричные элементы матриц, удовлетворяющих квадратичной алгебре. Состояния, входящие в матричные элементы, сильно зависят от граничных условий. Значимость граничных процессов проявляется в том, что граничные операторы порождают алгебру Аски–Уилсона или обобщенную алгебру Онсагера в качестве граничной симметрии. Все представления квадратичной алгебры приложений матричного подхода восстанавливаются из теории представлений алгебры Аски–Уилсона через полиномы Аски–Уилсона. Граничная симметрия указывает на глубокие алгебраические свойства стохастических систем, которые приводят к точному решению в стационарном состоянии и к точному описанию динамики процессов взаимодействия.

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INTRODUCTION

The rich behaviuor of the large variety of phenomena in nonequilibrium conditions is far from being well understood. Many-particle systems [1–5], interacting with stochastic dynamics, have received a lot of attention. In particular, reaction–diffusion processes are of both theoretical and experimental interest not only because they describe various mechanisms in physics and chemistry [1], but they also provide a way of modelling phenomena like traffic flow [6], kinetics of biopolymerization [7], interface growth [8]. Among these, the asymmetric simple exclusion process (ASEP) has become a paradigm in nonequilibrium physics due to its simplicity, rich behaviour and wide range of applicability. It is an exacltly solvable model of an open many-particle stochastic system interacting with hard core exclusion. At large time the ASEP exibits relaxation to a steady state, and even after the relaxation it has a nonvanishing current. An intriguing feature is the occurrence of boundary induced phase transitions [9] and the fact that the bulk properties depend strongly on the boundary rates. In addition, reaction–diffusion processes provide a very good playground to increase utility of quantum groups [10] since the deformation parameter acquires a direct physical meaning, i.e., for diffusion processes it is ratio of left-to-right probability rates. The ASEP is the fundamental model of nonequilibrium physics, just as in its turn, is the Ising model for equilibrium statistical mechanics.

A stochastic process is described in terms of a master equation for the probability distribution $P(s_i, t)$ of a stochastic variable $s_i = 0, 1, 2, \ldots, n-1$ at a site $i = 1, 2, \ldots, L$ of a linear chain. A state on the lattice at a time t is determined by the occupation numbers s_i, and a transition to another configuration s'_i during an infinitesimal time step dt is given by the probability $\Gamma(s, s')dt$. The rates $\Gamma \equiv \Gamma^{ik}_{jl}$, $i, j, k, l = 0, 1, 2, \ldots, n-1$ are assumed to be independent of the position in the bulk. At the boundaries, i.e., sites 1 and L, additional processes can take place with rates L^j_i and R^j_i. These are Markov processes [11, 12]. Due to probability conservation

$$\Gamma(s, s) = -\sum_{s' \neq s} \Gamma(s', s). \tag{1}$$

The master equation for the time evolution of a stochastic system

$$\frac{dP(s,t)}{dt} = \sum_{s'} \Gamma(s, s') P(s', t) \tag{2}$$

can be mapped to a Schrödinger equation in imaginary time for a quantum Hamiltonian with the nearest-neighbour interaction in the bulk and single-site boundary terms

$$\frac{dP(t)}{dt} = -HP(t), \tag{3}$$

where

$$H = \sum_j H_{j,j+1} + H^{(L)} + H^{(R)}. \tag{4}$$

The ground state of the Hamiltonian, in general non-Hermitian, corresponds to the steady state of the stochastic dynamics where all probabilities are stationary. The mapping provides a connection with integrable quantum spin chains [13–15] and allows for exact results of the stochastic dynamics with the formalism of quantum mechanics. However, it is important to emphasize that the correspondence is only formal. There are distinguishing features due to the stochastic nature of the Markov processes, the most important of which are discussed in [3]. We just point out the fact that the components of the probability distribution as a state vector are probabilities and must be positive. In quantim mechanics the components of the state vector are probability amplitudes whose modulus squared are probabilities.

The matrix product ansatz [16,17], inspired by the inverse scattering method in the study of integrable systems, was developed for derivation of exact results for the steady state properties of interacting many-particle systems. The idea is that the stationary probability distribution is expressed as a product of (or a trace over) matrices that form a representation of a quadratic algebra $\Gamma_{jl}^{ik} D_i D_k = x_l D_j - x_j D_l; i, j, k = 0, 1, \ldots, n-1$. The algebra of the operators D_i is determined by the dynamics of the process, while the states involved in the calculation of the matrix elements are determined by the boundary conditions. The algebraic approach provides an economic and convenient technique for the derivation of solvable recursion relations for the steady state weights and hence for the calculation of the current and the correlation functions. The recursions have been obtained in earlier works [18,19] for the two species model, however they were not readily generalized to other models. Besides the formulation for n-species models [17,20], there was also a generalization of the matrix product anzatz to the full dynamic stochastic problem [21].

This review is an attempt to better understanding the symmetry properties underlying the algebraic relations of the matrix approach to one-dimensional stochastic exclusion processes and to explore the consequences for the exact description of the dynamics. We put an emphasis on the fact that the states involved in the expressions for the matrix elements are determined by the boundary conditions, while the operator algebra is determined by the bulk dynamics. This reflects the property of open stochastic systems that, in contrast to equilibrium mechanics, the boundary conditions are of major importance. We consider the boundary operators of the open asymmetric exclusion process as generators of an Askey–Wilson and a tridiagonal (deformed Onsager) algebra whose irreducible modules are given in terms of the Askey–Wilson polynomials. The boundary algebra and its representations depend on the boundary parameters. A special case of the boundary algebra is *a tridiagonal algebra generated by the operators* $D_i, i = 0, 1$ which suggests a formulation of the matrix ansatz as a tridiagonal algebra. Previously known representations used in various applications of the matrix product approach to different cases of the open ASEP are reconstructed within the tridiagonal algebraic formalism. The boundary AW and deformed Onsager symmetries are the deep algebraic properties of the interacting many-body system which allow for extending the exact solvability beyond the stationary state.

1. MATRIX-PRODUCT-STATE APPROACH TO DIFFUSION MODELS

For diffusion processes with n species on a chain of L sites with the nearest-neighbour interaction with exclusion, a site can be either empty or occupied by a particle of a given type. In the set of occupation numbers $(s_1, s_2, \ldots, s_L)$ specifying a configuration of the system $s_i = 0$ if a site i is empty, $s_i = 1$ if there is a first-type particle at a site $i, \ldots, s_i = n-1$ if there is an $(n-1)$th-type particle at a site i. Diffusion or Brownian motion is the oldest physical example of a Markov process corresponding to a probability rate matrix $\Gamma_{ki}^{ik} = g_{ik}$, with $i, k = 0, 1, 2, \ldots, n-1$. On successive sites the species i and k exchange places with probability $g_{ik}dt$, where $i, k = 0, 1, 2, \ldots, n-1$. With $i < k$, g_{ik} are the probability rates of hopping to the left, and g_{ki} to the right. The event of exchange happens if out of two adjacent sites one is a vacancy and the other is occupied by a particle, or each of the sites is occupied by a particle of a different type. The n-species symmetric simple exclusion process is known as the lattice gas model of particle hopping between the nearest-neighbour sites with a constant rate $g_{ik} = g_{ki} = g$ [22, 23]. The n-species asymmetric simple exclusion process with hopping in a preferred direction is the driven diffusive lattice gas of particles moving under the action of an external field. The driving force imposes a bias [24] on the hopping rates so that the process is totally asymmetric if all jumps occur in one direction only (forward), and partially asymmetric if there is a nonzero probability of hopping also backward, the rates g_{ik} for moving to the left being different from the rates g_{ki} for moving to the right. The number of particles n_i of each species in the bulk is conserved $\sum\limits_{i=0}^{n-1} n_i = L$ and this is the case of periodic boundary conditions where one treats the site index as cyclic, $s_{L+i} = s_i$ to obtain a system on a ring. In the case of open systems, the lattice gas is coupled to external reservoirs of particles of fixed density. In most studied examples [17,20], one considers phase transitions inducing boundary processes [9] when a particle of type k, $k = 1, 2, \ldots, n-1$ is added with a rate L_k^0 and/or removed with a rate L_0^k at the left end of the chain, and it is removed with a rate R_0^k and/or added with a rate R_k^0 at the right end of the chain.

Through different physical interpretaion, the diffusion processes [25–30] cover a wide range of phenomena. The symmetric simple exclusion process is the lattice gas model for reptation [31] of single polymer chains in a random enviroment of other polymers. The diffusive motion of polymer segments (defects) is similar to lattice Brownian motion (particle–hole exchange) with an exclusive interaction and even though one-dimensional it is used to describe a three-dimensional system. Physical application of the two-species asymmetric exclusion process are systems where it is important to understand the current of particles through channels of finite length like diffusion driven lattice gas models of kinetics of biopolymerization, fluctuations of interfaces, traffic flow. The multispecies asymmetric exclusion process provides a simplified model of traffic flow [32–35] and by joining many open boundary systems into a network appears to describe realistic traffic phenomena [36]. The intensive study of the asymmetric exclusion process is further motivated due to its connection to interface dynamics since it can be exactly mapped to a $1+1$ model of interface growth [37] where a particle at a site corresponds to a step downwards of one unit of growth and a hole at a site corresponds to a step upwards.

For diffusion processes with n species the quadratic algebra, known as diffusion algebra, was studied in [38]. It has the form

$$g_{ik} D_i D_k - g_{ki} D_k D_i = x_k D_i - x_i D_k, \tag{5}$$

where g_{ik} and g_{ki} are positive (or zero) probability rates, $i, k = 0, 1, \ldots, n-1$, and x_i are representation dependent parameters. (No summation over repeated indices in Eq. (5).) The algebra generated by the n elements D_k obeying the $n(n-1)/2$ relations (5) is an associative algebra with a unit e and with a Poincare–Birkhoff–Witt basis given by the ordered monomials

$$D_{s_1}^{n_1} D_{s_2}^{n_2} \cdots D_{s_l}^{n_l}, \tag{6}$$

where $s_1 < s_2 < \ldots s_l$, $l \geqslant 1$ and $n_1, n_2, \ldots n_l$ are non-negative integers.

The quadratic algebra has a representation in an auxiliary Hilbert space where the n generators act as operators.

Matrix Product Ansatz (MPA). The idea is that stationary probability distribution is related to products of matrices satisfying the diffusion algebra. For systems with periodic boundary conditions, the (unnormalized) stationary weight of a configuration is given by the expression

$$P(s_1, \ldots, s_L) = \mathrm{Tr}\,(D_{s_1} D_{s_2} \cdots D_{s_L}). \tag{7}$$

When boundary processes are considered, the (unnormalized) stationary weight is a matrix element in the auxiliary vector space

$$P(s_1, \ldots, s_L) = \langle w | D_{s_1} D_{s_2} \cdots D_{s_L} | v \rangle \tag{8}$$

with respect to the vectors $|v\rangle$ and $\langle w|$, defined by the boundary conditions

$$\langle w | (L_i^k D_k + x_i) = 0, \quad (R_i^k D_k - x_i) | v \rangle = 0, \tag{9}$$

where the x sum up to zero, because of the form of the boundary rate matrices

$$L_i^i = -\sum_{j=0}^{L-1} L_j^i, \quad R_i^i = -\sum_{j=0}^{L-1} R_j^i, \quad \sum_{i=0}^{n-1} x_i = 0. \tag{10}$$

The vectors $\langle w|$ and $|v\rangle$ belong to the auxiliary Hilbert space and obey the condition $\langle w|v\rangle \neq 0$ in order that the steady states (8) be nonzero. A normalization factor Z_L is obtained by summing the weights over all n^L configurations of the L-site chain. The above relations simply mean that one associates with an occupation number s_i at position i a matrix $D_{s_i} = D_k$ ($i = 1, 2, \ldots, L$; $k = 0, 1, \ldots, n-1$) if a site i is occupied by a k-type particle. In various applications of the matrix product ansatz mostly infinite-dimensional representations are used. In the case of the totally asymmetric exclusion process there exist no finite-dimensional representations with dimension bigger than one. Finite-dimensional representations [15, 39] are defined by a relation between the boundary parameters and the bulk parameter. They correspond to an invariant subspace of the infinite-dimensional matrices and give exact results only on some special curves of the phase diagram. Due to the constraint on the model parameters, they restrict the physical properties of the nonequilibrium system in consideration. An example is the three species ASEP with shock profiles [25] where for a constraint on the model parameters the representation of the quadratic algebra is finite-dimensional (dimension two) and the stationary state becomes a Bernoulli measure. Finite-dimensional representations are appropriate for the ASEP on a ring [20] as they were suggested [40, 41] for the MPA relation to Bethe ansatz. The matrix product solution to the multispecies partially asymmetric exclusion process was found in [42] by defining the matrices hierarchically, namely by expressing the matrices of the n-species system in terms of those of the $(n-1)$-species system.

2. THE NONCOMMUTATIVE SPACE OF THE MANY-BODY SYSTEM

The quadratic algebra is an associative algebra generated by a unit e and n elements D_k obeying $n(n-1)/2$ relations (5). The alphabetically ordered monomials

$$D_{s_1}^{n_1} D_{s_2}^{n_2} \cdots D_{s_l}^{n_l}, \tag{11}$$

where $s_1 < s_2 < \ldots, s_l$, $l \geqslant 1$, and $n_1, n_2, \ldots, n_l$ are non-negative integers, are a linear basis in the algebra. The possibility of alphabetical ordering is achieved using the relations (5). It is sufficient to verify coincidence of two different ways of ordering for cubic monomials only which gives a relation for the rates g_{ik}. The ordering of higher order monomials does not give rise to any further relation. The linear independence of alphabetically ordered monomials is also sufficient to verify for cubic monomials only [43–45].

The algebra (5) admits an involution through the mapping $D_i \to D_i^+$ which with real parameters $x_i^+ = -x_i$ defines Hermitian elements $D_i = D_i^+$

provided

$$g_{ij}^{+} = g_{ji} \tag{12}$$

(or anti-Hermitian $D_i = -D_i^{+}$ if $g_{ij}^{+} = -g_{ji}$).

A Poincare–Birkhoff–Witt property of an algebra is important when one considers a quantum group of transformations of a quantum plane and a deformation of a universal enveloping as its dual. The quantum transformations and duality properties are written in a compact form with an R-matrix operator satisfying a quantum Yang–Baxter equation. Triangle Yang–Baxter relations are coded expressions for hidden symmetries of integrable models.

It is readily seen that the relations (5) with all x_k equal to zero are the defining relations of a $SL_q(n)$ quantum plane to which a multiparameter R matrix corresponds. In our case with g_{ik} real it is the $SU_q(n)$ quantum plane. The presence of the linear terms on the right-hand side of (5) with different from zero c-numbers x_k will break the quantum invariance yet not completely. Due to the requirement that the braid associativity condition is also fulfilled with the linear terms the symmetry will be only reduced. On the other hand, when all the rates g_{ik} are equal, the relations (5) are of a Lie-algebra type. The n generators D_i can be mapped to the generators of $GL(n)$.

The origin of the hidden symmetries in the matrix algebra might have important consequences as a step to integrability of the diffusion stochastic dynamics. A model with $GL_q(n)$ symmetry has an R-matrix operator, a constant solution of the Yang–Baxter equation. Through a «Baxterization» procedure, originally introduced by Jones [46], one can construct the matrix $\check{R}(\lambda)$, a parameter-dependent solution of the Yang–Baxter equation. The spectral parameter λ is a suitably chosen function of the temperature, field strengths or other physical parameters of the model. For integrable quantum spin chain models one can build a one-parameter family of commuting transfer matrices $[T(\lambda), T(\mu)] = 0$, a property directly implied by the Yang–Baxter equation. The importance of this commutativity becomes clear from the expansion of the transfer matrix $\ln T(\lambda) = \sum_n (\lambda^n/n!) Q_n$ which yields an infinite set of mutually commuting conserved charges Q_n including the Hamiltonian.

The first steps on the way to exact description of many-body systems within the MPA concerns a realization of the algebra (5) consistent with the boundary conditions and revealing the hidden symmetry algebras of the corresponding diffusion processes. The quadratic algebra associated with the n-species symmetric and asymmetric diffusion processes defines in general the comodule structure of $GL(n)$ and $GL_q(n)$. The hidden symmetries imply a possibility for a mapping to the $GL(n)$ integrable quantum spin chain associated with an n-state vertex model [47], and in particular, the six-vertex model for $n = 2$. The number of species n are the spin states n of a spin variable S related by $n = 2S + 1$. The most important constraint imposed by the symmetry is the charge (spin momentum) conservation interpreted as a particle number conservation. The introduction of boundary processes reduces the $GL_q(n)$ symmetry in the bulk to either $GL_q(n-1)$ or $GL_q(n-2)$, thus breaking the charge (particle number) conservation law.

The n-generator diffusion algebras have been considered in [38] where the quadratic algebras of the MPA for models with three species were completely classified. In describing the symmetries of the n-species processes [48] we follow the classification of [38].

1. In the case of a Lie-algebra-type diffusion algebras the n generators D_i and e can be mapped to the generators J_{jk} of $SU(n)$ and the mapping is invertible. The universal enveloping algebra generated by D_i belongs to the universal enveloping algebra of the Lie algebra of $SU(n)$.

2. It is known that the multiparameter quantized noncommutative space [44, 49–51, 79] can be realized equivalently as a q-deformed Heisenberg algebra [44, 50] of n oscillators depending on $n(n-1)/2+1$ parameters (or in general on $n(n-1)/2+n$ parameters [79]). The universal enveloping algebra (UEA) of the elements D_i in the case of a diffusion algebra with all coefficients x_i on the RHS of Eq. (5) equal to zero belongs to the UEA of a multiparameter deformed Heisenberg algebra to which a consistent multiparameter $GL_q(n)$ quantization corresponds.

3. It has been shown in [38] that for a relation with nonzero x terms on the RHS of (5) only then is braid associativity satisfied if out of the coefficients x_i, x_k, x_l corresponding to an ordered triple $D_iD_kD_l$ either one coefficient x is zero or two coefficients x are zero and the rates are respectively related. We present a solution for the corresponding algebraic relations in terms of deformed oscillators. We argue that the appearance of the nonzero linear terms in the RHS of the quantum plane relations leads to a lower dimensional noncommutative space and a reduction of the $U_q(n)$ invariance. We show that the diffusion algebras in this case can be obtained by either a change of basis in the n-dimensional noncommutative space or by a suitable change of a subsystem of the basis of the lower dimensional quantum space. The presence of one x-dependent term due to a boundary process reduces the $U_q(n)$ invariance in the bulk to $U_q(n-1) \times U(1)$ invariance. In the presence of two x linear terms the symmetry in the bulk is reduced to $U_q(n-2) \times SU_q(2)$, each being deformed with a different deformation parameter and this is the hidden symmetry algebra.

The advantage of the matrix product state method is that important physical properties and quantities like multiparticle correlation functions, currents, density profiles, phase diagrams can be obtained from the representations of the matrix quadratic algebra. We have constructed in [48] the infinite- and/or finite-dimensional representations for the cases listed above. The representations are consistent with the corresponding boundary conditions. We give the solutions of the algebraic relations for general n and comment the invariance properties. The boundary vectors are obtained explicitly for $n = 2$ and $n = 3$ and a generalization to higher n if not straightforward is discussed in each case.

3. THE ASEP WITH ONLY INCOMING PARTICLES AT THE LEFT BOUNDARY AND ONLY OUTGOING AT THE RIGHT ONE

We comment this two-species solution for the case of particular boundary conditions for the reason of enhancing the utility of the q-deformed oscillator coherent states [53–55] which provide the most simple and convenient approach to an unified solution (see [56]) of both the partially and the totally asymmetric processes. In the partially asymmetric case, the probability rate of hopping to the left is $g_{01} = q$ while the right probability rate is $g_{10} = 1$. The totally asymmetric exclusion process of particles hopping to the right only is obtained for $q = 0$. At the left boundary a particle can be added with a probability αdt and it can be removed at the right boundary with a probability βdt. The quadratic algebra is generated by a unit and two generators obeying the relation:

Case A — the partially asymmetric simple exclusion process ($0 < q < 1$)

$$D_1 D_0 - q D_0 D_1 = D_0 + D_1. \tag{13}$$

Case B — the totally asymmetric simple exclusion process ($q = 0$)

$$D_1 D_0 = D_0 + D_1 \tag{14}$$

with the same boundary conditions defining in both cases the boundary vectors $\langle w|$ and $|v\rangle$:

$$\langle w|D_0 = \langle w|\frac{1}{\alpha}, \quad D_1|v\rangle = \frac{1}{\beta}|v\rangle. \tag{15}$$

The algebraic solutions (with the corresponding boundary problems (15)) for the partially and for the totally asymmetric cases are of the form of shifted deformed oscillators for a real parameter $0 < q < 1$ and for $q = 0$, respectively.

Case A

$$D_0 = \frac{1}{1-q} + \frac{a^+}{\sqrt{1-q}}, \quad D_1 = \frac{1}{1-q} + \frac{a}{\sqrt{1-q}}. \tag{16}$$

To solve the boundary problem we choose the vector $|v\rangle$ to be the (unnormalized!) eigenvector of the annihilation operator a for a real value of the parameter v and the vector $\langle w|$ to be the eigenvector (unnormalized and different from the conjugated one) of the creation operator for the real parameter w:

$$|v\rangle = \mathrm{e}_q^{-\frac{1}{2}vw}\, \mathrm{e}_q^{va^+} |0\rangle \quad \langle w| = \langle 0|\, \mathrm{e}_q^{wa}\, \mathrm{e}_q^{-\frac{1}{2}wv}. \tag{17}$$

The factor $e_q^{-\frac{1}{2}vw}$ in (17) is due to the condition $\langle w|v\rangle = 1$, which is a convenient choice in physical applications. According to the algebraic solution, these are also eigenvectors of the shifted operators with the corresponding relations of the eigenvalues

$$\frac{1}{\alpha} = \frac{1}{1-q} + \frac{w}{\sqrt{1-q}}, \quad \frac{1}{\beta} = \frac{1}{1-q} + \frac{v}{\sqrt{1-q}}. \tag{18}$$

Hence the boundary vectors $|v\rangle$ and $\langle w|$ are a subset of the coherent states of the q-deformed Heisenberg algebra, labelled by the positive real parameters $v(\alpha, q)$ and $w(\beta, q)$ defined in (18). The relation of the boundary vectors to the coherent states simplifies the calculation of the stationary probability distribution. Since, according to the algebraic solution:

$$(D_0 + D_1)^L = \left(\frac{2}{1-q} + \frac{a^+ + a}{\sqrt{1-q}}\right)^L =$$
$$= \sum_{m=0}^{L} \frac{L!}{m!(L-m)!} \frac{2^{L-m}}{(1-q)^{L-m}(\sqrt{1-q})^m}(a^+ + a)^m \tag{19}$$

in order to find the expectation values with respect to the coherent states, one has to normally order the mth power of the linear combination $a + a^+$, using $aa^+ - qa^+a = 1$. This is achieved with the help of the Stirling numbers

$$(a^+ + a)^m = \sum_{k=0}^{[m/2]} S_m^{(k)} \sum_{l=0}^{m-2k} \frac{[m-2k]!}{[l]![m-2k-l]!}(a^+)^l a^{m-2k-l}, \tag{20}$$

where the q-deformed Stirling numbers $S_m^{(k)}$ satisfy the recurrence relation

$$S_{m+1}^{(k)} = [k]S_m^{(k)} + S_m^{(k-1)} \tag{21}$$

with $S_m^{(0)} = \delta_{0m}$, $S_m^{(1)} = S_m^{(m)} = 1$ and $S_m^{(m-1)} = \sum\limits_{i=1}^{i=m-1}[i]$. For the correlation functions one also needs the expressions

$$a^k a^+ = q^k a^+ a^k + [k]a^{k-1}, \quad a(a^+)^k = q^k (a^+)^k a + [k](a^+)^{k-1}. \tag{22}$$

Using these relations one can easily find the relevant physical quantities of the system. Thus for the normalization factor Z^L one obtains

$$\langle w|(D_0 + D_1)^L|v\rangle = \sum_{m=0}^{L} \frac{L!}{m!(L-m)!} \frac{2^{L-m}}{(1-q)^{L-\frac{m}{2}}} \times$$
$$\times \sum_{k=0}^{[m/2]} \sum_{l=0}^{m-2k} S_m^{(k)} \frac{[m-2k]!}{[l]![m-2k-l]!} w^l v^{m-2k-l}. \tag{23}$$

It can be verified, after rescaling the parameters v and w by $1/\sqrt{1-q}$, that this expression coincides with the one evaluated in [57] up to the factor $\langle w|v\rangle$, which is chosen there to be $\langle w|v\rangle \neq 1$.

Case B

$$D_0 = 1 + a^+_{q=0}, \quad D_1 = 1 + a_{q=0}. \tag{24}$$

As the algebra itself, the solution (24) and the boundary vectors are also obtained as the limit $q \to 0$ of the q-dependent solution and eigenvectors where the representation of the oscillator operators in (24) is found from Eqs. (13) with $q = 0$, namely $a^+|n\rangle = |n+1\rangle$, $a|n\rangle = |n-1\rangle$ and

$$w = \frac{1-\alpha}{\alpha}, \quad v = \frac{1-\beta}{\beta}. \tag{25}$$

Hence the boundary vectors have the form

$$\begin{aligned} \langle w| &= \langle n| \sum_{n=0}^{\infty} \left(\frac{1-\alpha}{\alpha}\right)^n \left(\frac{1}{\alpha} + \frac{1}{\beta} - \frac{1}{\alpha\beta}\right)^{1/2}, \\ |v\rangle &= \left(\frac{1}{\alpha} + \frac{1}{\beta} - \frac{1}{\alpha\beta}\right)^{1/2} \sum_{n=0}^{\infty} \left(\frac{1-\beta}{\beta}\right)^n |n\rangle. \end{aligned} \tag{26}$$

The physical quantities of the model are readily obtained from the partially asymmetric case in the limit $q \to 0$. Equation (20) becomes simply

$$(a + a^+)^L|_{q=0} = \sum_{k=0}^{[m/2]} S_m^{(k)}|_{q=0} \sum_{l=0}^{m-2k} (a^+_{q=0})^l (a_{q=0})^{m-2k-l}, \tag{27}$$

where now $S_{m+1}^{(k)}|_{q=0} = S_m^{(k)}|_{q=0} + S_m^{(k-1)}|_{q=0}$ and $S_m^{(m-1)}|_{q=0} = m-1$. The expression for Z_L becomes

$$\langle w|(D_0 + D_1)^L|v\rangle = \sum_{m=0}^{L} \frac{2^{l-m} L!}{m!(L-m)!} \sum_{k=0}^{[m/2]} \sum_{l=0}^{m-2k} S_m^{(k)}|_{q=0} w^l v^{m-2k-l}. \tag{28}$$

Inserting in Eq. (28) the expressions for v and w in terms of α and β from (25), it can be verified, after some algebra, that it coincides with the expression for the normalization factor obtained in [16] (as the current and the correlation functions do coincide, too). The coherent-state description thus povides a unified solution of the partially and fully asymmetric simple exclusion models.

The q-deformed coherent states were used in [58] as an intermediate step in the calculation of the normalization factor to the stationary probability distribution of the ASEP with only incoming and only outgoing particles at both ends of the chain. The calculation was performed with the help of the complete set of q-Hermite orthogonal polynomials. The boundary vectors $\langle w|$ and $|v\rangle$ were chosen to be q-boson coherent states. After inserting the resolution of unity for the q-Hermite polynomials into $\langle w|(D_0 + D_1)^L|v\rangle$, the q-coherent states contributed to the weight function in such a way that it pointed out to making use of the Al-Salam–Chihara polynomials. The exact solution of the open ASEP (for the particular boundary conditions $\gamma = \delta = 0$) with detailed calculation of all the relevant physical quantities was elaborated in [58] with the help of the Al-Salam–Chihara polynomials.

4. THE QUADRATIC ALGEBRA OF THE INTERACTING MANY-BODY SYSTEM AS A TRIDIAGONAL ALGEBRA

We consider now the two-species partially asymmetric simple exclusion process with incoming and outgoing particles at both boundaries. We simplify the notations, namely, at the left boundary a particle can be added with probability αdt and removed with probability γdt, and at the right boundary it can be removed with probability βdt and added with probability δdt. The system is described by the configuration set $s_1, s_2, \ldots, s_L$, where $s_i = 0$ if a site $i = 1, 2, \ldots, L$ is empty and $s_i = 1$ if a site i is occupied by a particle. The particles hop with a probability $g_{01}dt$ to the left and with a probability $g_{10}dt$ to the right, where without loss of generality we can choose the right probability rate $g_{10} = 1$ and the left probability rate $g_{01} = q$. The model depends on five parameters — the bulk probability rate q and the four boundary rates. The asymmetric exclusion process has a particle–hole symmetry

$$\alpha \leftrightarrow \gamma, \quad \beta \leftrightarrow \delta, \quad q \leftrightarrow q^{-1} \tag{29}$$

and a left–right symmetry

$$\alpha \leftrightarrow \delta, \quad \beta \leftrightarrow \gamma, \quad q \leftrightarrow q^{-1}. \tag{30}$$

The totally asymmetric process corresponds to $q = 0$. The quadratic algebra of the matrix product approach

$$D_1 D_0 - q D_0 D_1 = x_0 D_1 - D_0 x_1, \quad x_0 + x_1 = 0 \tag{31}$$

has been derived from the bulk dynamics and can be solved [57] by a pair of deformed oscillators [53, 54] $aa^+ - qa^+a = 1$

$$D_0 = \frac{x_0}{1-q} + \frac{x_0 a^+}{\sqrt{1-q}}, \quad D_1 = \frac{-x_1}{1-q} + \frac{-x_1 a}{\sqrt{1-q}}. \tag{32}$$

The boundary conditions have the form

$$\begin{aligned}(\beta D_1 - \delta D_0)|v\rangle &= x_0|v\rangle,\\ \langle w|(\alpha D_0 - \gamma D_1) &= \langle w|(-x_1).\end{aligned} \tag{33}$$

We stress the one-parameter dependence of the MPA due to $x_0 = -x_1 = \zeta$ with $0 < \zeta < \infty$, which follows from $x_0 + x_1 = 0$. In the most known applications it is restricted to the choice $\zeta = 1$. In our opinion, the relation $x_0 + x_1 = 0$ implies an Abelian symmetry with a conserved quantity $D_0 + D_1$, following from $D_0 \to D_0 + x_0, D_1 \to D_1 + x_1$.

For a given configuration $(s_1, s_2, \dots, s_L)$, the stationary probability is given by the expectation value

$$P(s) = \frac{\langle w|D_{s_1} D_{s_2} \cdots D_{s_L}|v\rangle}{Z_L}, \tag{34}$$

where $D_{s_i} = D_1$ if a site $i = 1, 2, \dots, L$ is occupied and $D_{s_i} = D_0$ if a site i is empty and

$$Z_L = \langle w|(D_0 + D_1)^L|v\rangle \tag{35}$$

is the normalization factor to the stationary probability distribution. Once the representation of the diffusion algebra of the matrix-product ansatz and the boundary vectors are known, one can evaluate all the relevant physical quantities such as the mean density at a site i

$$\langle s_i\rangle = \frac{\langle w|(D_0 + D_1)^{i-1} D_1 (D_0 + D_1)^{L-i}|v\rangle}{Z_L}, \tag{36}$$

the two-point correlation function

$$\langle s_i s_j\rangle = \frac{\langle w|(D_0 + D_1)^{i-1} D_1 (D_0 + D_1)^{j-i-1} D_1 (D_0 + D_1)^{L-j}|v\rangle}{Z_L}, \tag{37}$$

and higher correlation functions. The current J through a bond between site i and site $i + 1$ is given by

$$\begin{aligned} J &= \langle s_i(1 - s_{i+1}) - q(1 - s_i)s_{i+1}\rangle = \\ &= \frac{\langle w|(D_0 + D_1)^{i-1}(D_1 D_0 - q D_0 D_1)(D_0 + D_1)^{L-i-1}|v\rangle}{Z_L}, \end{aligned} \tag{38}$$

which has a very simple form

$$J = \zeta \frac{Z_{L-1}}{Z_L}. \tag{39}$$

Within the matrix product approach an exact solution of the partially asymmetric exclusion process with only incoming particles at the left boundary and only outgoing particles at the right one ($\delta = \gamma = 0$ in (33)) was achieved through relation to q-Hermit polynomials [57] and in a more general case to Al-Salam–Chihara polynomials [58]. An alternative solution was shown in the previous section by choosing the boundary vectors $|v\rangle$ and $\langle w|$ to be deformed coherent states of the pair of q-boson oscillators. In the case of general boundary conditions (33), an exact solution related to Askey–Wilson polynomials was proposed and studied [59, 60].

We consider [61] the algebra generated by three generators D_0, D_1 and their q commutator $D_2 = [D_0, D_1]_q$, where for any X, Y

$$[X, Y]_q = q^{1/2}XY - q^{-1/2}YX. \tag{40}$$

Proposition I. The operators D_0, D_1 of the asymmetric exclusion process and their q commutator $[D_0, D_1]_q$ form a closed linear algebra

$$\begin{aligned} [D_0, D_1]_q &= D_2, \\ [D_1, [D_0, D_1]_q]_q &= q^{-1/2}x_1(q^{1/2} - q^{-1/2})\{D_0, D_1\} - \\ &- q^{-1}x_1^2 D_0 + q^{-1}x_0x_1D_1 - x_0q^{-1/2}(q^{1/2} - q^{-1/2})D_1^2, \\ [[D_0, D_1]_q, D_0]_q &= -x_0q^{-1/2}(q^{1/2} - q^{-1/2})\{D_0, D_1\} - \\ &- x_0^2q^{-1}D_1 + x_0x_1q^{-1}D_0 - x_1q^{-1/2}(q^{1/2} - q^{-1/2})D_0^2. \end{aligned} \tag{41}$$

The proposition is readily verified by using the explicit form of the MPA quadratic relation (31). The algebra can equivalently be described as a two-relation algebra for the pair D_0, D_1

$$\begin{aligned} D_0D_1^2 - (q + q^{-1})D_1D_0D_1 + D_1^2D_0 + x_1q^{-1/2}(q^{1/2} - q^{-1/2})\{D_0, D_1\} = \\ = x_1^2q^{-1}D_0 - x_0x_1q^{-1}D_1 + x_0q^{-1/2}(q^{1/2} - q^{-1/2})D_1^2, \\ D_0^2D_1 - (q + q^{-1})D_0D_1D_0 + D_1D_0^2 - x_0q^{-1/2}(q^{1/2} - q^{-1/2})\{D_0, D_1\} = \\ = x_0^2q^{-1}D_1 - x_0x_1q^{-1}D_0 + x_1q^{-1/2}(q^{1/2} - q^{-1/2})D_0^2. \end{aligned} \tag{42}$$

Relations (42) are the well-known Askey–Wilson relations

$$\begin{aligned} A^2A^* - (q + q^{-1})AA^*A + A^*A^2 - \gamma(AA^* + A^*A) = \\ = \rho A^* + \gamma^*A^2 + \omega A + \eta, \\ A^{*2}A - (q + q^{-1})A^*AA^* + AA^{*2} - \gamma^*(AA^* + A^*A) = \\ = \rho^*A + \gamma A^{*2} + \omega A^* + \eta^*. \end{aligned} \tag{43}$$

The algebra (43) was first considered in the works of Zhedanov [62] (see also [63, 64]) who showed that the Askey–Wilson polynomials give pairs of infinite-dimensional matrices satisfying the Askey–Wilson (AW) relations. It is recently discussed in a more general framework of a tridiagonal algebra [65, 66], that is an associative algebra with a unit generated by a tridiagonal pair (see Definition 1.1 on p. 2 in [67]) of operators A, A^* and defining relations

$$\begin{aligned} [A, A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \rho A^*] = 0, \\ [A^*, A^{*2}A - \beta A^*AA^* + AA^{*2} - \gamma^*(AA^* + A^*A) - \rho^* A] = 0. \end{aligned} \tag{44}$$

In the general case a tridiagonal pair is determined by the sequence of scalars $\beta, \gamma, \gamma^*, \rho, \rho^*$ from a field K. (We keep the conventional notations, used in the literature, for the scalars of a tridiagonal pair; β and γ should not be confused with the ASEP boundary rates.) Tridiagonal pairs have been studied according to their dependence on the scalars [65, 66]. Examples are the q-Serre relations with $\beta = q + q^{-1}$ and $\gamma = \gamma^* = \rho = \rho^* = 0$

$$\begin{aligned} [A, A^2A^* - (q + q^{-1})AA^*A + A^*A^2] = 0, \\ [A^*, A^{*2}A - (q + q^{-1})A^*AA^* + AA^{*2}] = 0, \end{aligned} \tag{45}$$

and the Dolan–Grady relations [68] with $\beta = 2, \gamma = \gamma^* = 0, \rho = k^2, \rho^* = k^{*2}$

$$[A, [A, [A, A^*]]] = k^2[A, A^*], \quad [A^*, [A^*, [A^*, A]]] = k^{*2}[A^*, A]. \tag{46}$$

The AW and tridiagonal relations are determined up to an affine transformation

$$A \to tA + c, \qquad A^* \to t^*A^* + c^*, \tag{47}$$

where t, t^*, c, c^* are some scalars. The affine transformation can be used to bring a tridiagonal or Askey–Wilson relation in a reduced form with $\gamma = \gamma^* = 0$.

As seen from the Askey–Wilson relations (42) for the ASEP matrices D_0 and D_1, they form a Leonard pair (i.e., a tridiagonal pair of operators A, A^* for which all the eigenspaces of A and A^* have dimension 1 [66]) with

$$\rho = x_1^2 q^{-1}, \quad \rho^* = x_0^2 q^{-1}, \quad \omega = -x_0 x_1 q^{-1}, \tag{48}$$

$$\gamma = -x_1 q^{-1/2}(q^{1/2} - q^{-1/2}), \quad \gamma^* = x_0 q^{-1/2}(q^{1/2} - q^{-1/2}) \tag{49}$$

and $\eta = \eta^* = 0$. Besides $\gamma = \gamma^*$, $\rho = \rho^*$ due to $x_0 + x_1 = 0$. We can now rescale the operators D_0, D_1 to set $\gamma = \gamma^* = 0$. This is achieved with the help of the transformations, which is consistent with the Abelian symmetry behind the bulk quadratic algebra of the matrix product ansatz

$$D_0 \to D_0 + \frac{x_0 q^{-1/2}}{q^{1/2} - q^{-1/2}}, \quad D_1 \to D_1 - \frac{x_1 q^{-1/2}}{q^{1/2} - q^{-1/2}}. \tag{50}$$

However, the shift of the generators amounts to a tridiagonal pair with sequence of scalars $\beta = -(q+q^{-1}), \gamma = \gamma^* = 0,\ \rho = \rho^* = 0$. Thus the operators of the ASEP matrix product ansatz obey the relations of a tridiagonal algebra

$$\begin{aligned}
&[D_1, D_0 D_1^2 - (q+q^{-1}) D_1 D_0 D_1 + D_1^2 D_0] = 0, \\
&[D_0, D_1 D_0^2 - (q+q^{-1}) D_0 D_1 D_0 + D_0^2 D_1] = 0,
\end{aligned} \tag{51}$$

which is a special case of the tridiagonal relations of the ASEP boundary operators. Relations (51) are equivalent to the level zero q-Serre relations of $U_q(\hat{su}(2))$. *Thus, within the tridiagonal approach to the asymmetric simple exclusion process, the $U_q(\hat{su}(2))$ quantum affine symmetry arises as the hidden symmetry of the bulk dynamics.*

The bulk Askey–Wilson algebra of the symmetric exclusion process (see [61]) follows immediately as the limit $q \to 1$ of the AW algebra of the bulk asymmetric exclusion process. Hence we have:

Proposition II. The operators D_0, D_1 of the symmetric exclusion process and their commutator $[D_0, D_1]$ form a closed linear algebra

$$\begin{aligned}
[D_0, D_1] &= D_2, \\
[D_1, [D_0, D_1]] &= x_1\{D_0, D_1\} - x_1^2 D_0 + x_0 x_1 D_1 - x_0 D_1^2, \\
[[D_0, D_1], D_0] &= -x_0\{D_0, D_1\} - x_0^2 D_1 + x_0 x_1 D_0 - x_1 D_0^2.
\end{aligned} \tag{52}$$

The proposition can be independently directly verified by using the explicit form of the MPA quadratic relation for the case of symmetric diffusion. The algebra can equivalently be described as a two-relation algebra for the pair D_0, D_1

$$\begin{aligned}
D_0 D_1^2 - 2 D_1 D_0 D_1 + D_1^2 D_0 + x_1\{D_0, D_1\} &= x_1^2 D_0 - x_0 x_1 D_1 + x_0 D_1^2, \\
D_0^2 D_1 - 2 D_0 D_1 D_0 + D_1 D_0^2 - x_0\{D_0, D_1\} &= x_0^2 D_1 - x_0 x_1 D_0 + x_1 D_0^2.
\end{aligned} \tag{53}$$

As seen from the explicit form of the algebra, the matrices D_0, D_1 of the symmetric exclusion process satisfy the Askey–Wilson relations with the sequence of scalars

$$\beta = 2, \quad \rho = x_2^2, \quad \rho^* = x_0^2, \quad \omega = -x_0 x_1, \quad \gamma = -x_1, \quad \gamma^* = -x_0. \tag{54}$$

We can transform the generators D_0, D_1 by the affine shifts

$$D_0 \to D_0 + x_0, \quad D_1 \to D_1 - x_1. \tag{55}$$

We observe again the same property that the shift of the generators amounts to a tridiagonal pair with sequence of scalars $\gamma = \gamma^* = 0,\ \rho = \rho^* = 0$. Thus the operators of the symmetric matrix product ansatz obey the relations of a tridiagonal algebra

$$\begin{aligned}
&[D_1, D_0D_1^2 - 2D_1D_0D_1 + D_1^2D_0] = 0,\\
&[D_0, D_1D_0^2 - 2D_0D_1D_0 + D_0^2D_1] = 0,
\end{aligned} \tag{56}$$

which are the Dolan–Grady relations with $k = k^* = 0$, i.e., a special case of the tridiagonal relations of the boundary operators of the symmetric process. *Thus within the tridiagonal approach to the symmetric simple exclusion process, the* $\hat{su}(2)$ *quantum affine symmetry arises as the hidden symmetry of the bulk dynamics.*

The bulk Askey–Wilson algebra of the totally asymmetric exclusion process cannot be obtained directly as the limit $q = 0$ of the partially asymmetric process. The procedure is more involved. In the later section we will derive both, the bulk and the boundary algebras starting with the quadratic algebra of the totally asymmetric process.

5. THE TRIDIAGONAL BOUNDARY ALGEBRA OF THE OPEN ASEP

We consider now the general case of incoming and outgoing particles at both boundaries. With all the boundary parameters nonzero, there are four operators $\beta D_1, -\delta D_0, -\gamma D_1, \alpha D_0$ and one needs an addition rule to form two linearly independent boundary operators acting on the dual boundary vectors. To proceed with a solution, we first note that the quadratic algebra is invariant with respect to the following transformations:

$$D_0 \leftrightarrow D_1, \quad q \leftrightarrow q^{-1}, \quad x_1 \leftrightarrow q^{-1}x_0, \quad x_0 \leftrightarrow q^{-1}x_1. \tag{57}$$

This symmetry together with

$$\alpha \leftrightarrow q^{-1}\delta, \quad \beta \leftrightarrow q^{-1}\gamma, \quad \gamma \leftrightarrow q^{-1}\beta, \quad \delta \leftrightarrow q^{-1}\alpha, \tag{58}$$

$$\alpha \leftrightarrow q^{-1}\gamma, \quad \beta \leftrightarrow q^{-1}\delta, \quad \gamma \leftrightarrow q^{-1}\alpha, \quad \delta \leftrightarrow q^{-1}\beta \tag{59}$$

leaves invariant both the quadratic algebra and the boundary conditions and reflects the left–right and the particle–hole symmetry of the physical system. This results in an isomorphic algebra

$$D_0D_1 - q^{-1}D_1D_0 = q^{-1}x_0D_1 - q^{-1}D_0x_1, \tag{60}$$

which can be solved by an equivalent set [53, 54] of deformed oscillators

$$\tilde{a}\tilde{a^+} - q^{-1}\tilde{a^+}\tilde{a} = 1. \tag{61}$$

With only two boundary parameters α, β one can use either quadratic algebra to obtain an exactly solvable model through the matrix state method. The situation is different in the four boundary parameter cases. One can consider two cases:

A — two relations of the same form

$$\beta D_1 \alpha D_0 - q\alpha D_0 \beta D_1 = x_1 \beta\alpha D_0 - \alpha\beta D_1 x_0 \tag{62}$$

and

$$\gamma D_1 \delta D_0 - q\delta D_0 \gamma D_1 = x_1 \gamma\delta D_0 - \delta\gamma D_1 x_0. \tag{63}$$

B — one relation of the form (62) and another one of the equivalent form

$$\delta D_0 \gamma D_1 - q^{-1}\gamma D_1 \delta D_0 = q^{-1} x_0 \delta\gamma D_1 - q^{-1}\gamma\delta D_0 x_1. \tag{64}$$

These are the two independent relations for the boundary operators corresponding to either case. Any other relation will depend on the representation used for the solution of (62), (63), and (64) in order to be consistent with the solution. In both cases one needs an addition rule to form two linearly independent boundary operators $B^R = \beta D_1 - \delta D_0, B^L = -\gamma D_1 + \alpha D_0$. A solution to this problem within the matrix product ansatz is obtained by using the $U_q(sl(2))$ algebra in the form of a deformed (u, v) algebra. Some special cases are the $U_q(su(2))$ ($(u, -u)$, $u < 0$), a particular q-oscillator algebra $cu_q(2)$ ((u, u), $u > 0$) and two isomorphic oscillator algebras $eu_q^{\pm}(2)$ ($uv = 0$). The (u, v) deformed algebra is convenient to including all the applications to the solution of the MPA quadratic algebra (31). The (u, u) algebra appears to be more convenient for a solution of the algebraic relations in case B as we have explored it in [A19]. For the A case and in order to emphasize the equivalence of the ASEP to the integrable spin $1/2$ XXZ chain we will apply the $U_q(su(2))$ algebra. It is generated by three elements with the defining commutation relations

$$[N, A_\pm] = \pm A_\pm, \quad [A_+, A_-] = \frac{q^N - q^{-N}}{q^{1/2} - q^{-1/2}} \tag{65}$$

and a central element

$$Q = A_+ A_- - \frac{q^{N-1/2} - q^{-N+1/2}}{(q^{1/2} - q^{-1/2})^2}. \tag{66}$$

The representations are labelled by the values of the Casimir

$$Q(\kappa) = \frac{q^\kappa + q^{1-\kappa}}{q^{-1/2}(1-q)^2} \tag{67}$$

for some fixed parameter κ. Given a basis $|n, \kappa\rangle$, a representation is defined by $N|n, \kappa\rangle = (\kappa + n)|n, \kappa\rangle$, $A_-|n, \kappa\rangle = r_n|n-1, \kappa\rangle$, $A_+|n, \kappa\rangle = r_{n+1}|n+1, \kappa\rangle$, where

$$r_n^2 = \frac{(1-q^n)q^{1/2}(q^\kappa - q^{1-n-\kappa})}{(1-q)^2}. \tag{68}$$

The state $|0,\kappa\rangle$ is the vacuum with $r_0 = 0$. The representation is infinite-dimensional if for all n

$$-q^{\kappa} + q^{1-n-\kappa} > 0 \tag{69}$$

which is fulfilled for $U_q(sl_2)$ ($\kappa > 0$), $cu_q(2), eu_q^{\pm}$ (arbitrary real κ), and finite-dimensional of dimension $l+1$ if (46) is fulfilled for $n < l$, and for some $n = l$

$$q^{\kappa} - q^{-l-\kappa} = 0 \tag{70}$$

which is the case of $U_q(su_2)$.

The relations (62), (63) can be solved by choosing the boundary operators in the form

$$\begin{aligned}
&\beta D_1 - \delta D_0 = \\
&= -\frac{x_1\beta}{\sqrt{1-q}} q^{N/2} A_+ - \frac{x_0\delta}{\sqrt{1-q}} A_- q^{N/2} - \frac{x_1\beta q^{1/2} + x_0\delta}{1-q} q^N - \frac{x_1\beta + x_0\delta}{1-q}, \\
&\alpha D_0 - \gamma D_1 = \\
&= \frac{x_0\alpha}{\sqrt{1-q}} q^{-N/2} A_+ + \frac{x_1\gamma}{\sqrt{1-q}} A_- q^{-N/2} + \frac{x_0\alpha q^{-1/2} + x_1\gamma}{1-q} q^{-N} + \frac{x_0\alpha + x_1\gamma}{1-q}.
\end{aligned} \tag{71}$$

We separate the shift parts from the boundary operators. Denoting the corresponding rest operator parts by A and A^* we write the left and right boundary operators in the form

$$\begin{aligned}
\beta D_1 - \delta D_0 &= A - \frac{x_1\beta + x_0\delta}{1-q}, \\
\alpha D_0 - \gamma D_1 &= A^* + \frac{x_0\alpha + x_1\gamma}{1-q}.
\end{aligned} \tag{72}$$

Proposition III. The shifted boundary operators A and A^* defined by

$$\begin{aligned}
A &= \beta D_1 - \delta D_0 + \frac{x_1\beta + x_0\delta}{1-q}, \\
A^* &= \alpha D_0 - \gamma D_1 - \frac{x_0\alpha + x_1\gamma}{1-q}
\end{aligned} \tag{73}$$

and their q commutator

$$[A, A^*]_q = q^{1/2} A A^* - q^{-1/2} A^* A \tag{74}$$

form a closed linear algebra, namely the Askey–Wilson algebra

$$\begin{aligned}
[[A, A^*]_q, A]_q &= -\rho A^* - \omega A - \eta, \\
[A^*, [A, A^*]_q]_q &= -\rho^* A - \omega A^* - \eta^*,
\end{aligned} \tag{75}$$

where the operator-valued structure constants are given by

$$-\rho = x_0x_1\beta\delta q^{-1/2}(q^{1/2}+q^{-1/2})^2, \quad -\rho^* = x_0x_1\alpha\gamma q^{-3/2}(q^{1/2}+q^{-1/2})^2, \quad (76)$$

$$-\omega = (x_1\beta q^{1/2} + x_0\delta)(x_1\gamma + x_0\alpha) - (x_1^2\beta\gamma + x_0^2\alpha\delta)(q^{1/2} - q^{-1/2})Q,$$

$$\eta = q^{1/2}(q^{1/2} + q^{-1/2})\times$$
$$\times\left(x_0x_1\beta\delta(x_1\gamma + x_0\alpha)Q - \frac{(x_1\beta q^{1/2} + x_0\delta)(x_1^2\beta\gamma + x_0^2\alpha\delta)}{q^{1/2} - q^{-1/2}}\right), \quad (77)$$

$$\eta^* = q^{1/2}(q^{1/2} + q^{-1/2})\times$$
$$\times\left(x_0x_1\alpha\gamma(x_1\beta q^{1/2} + x_0\delta)Q + \frac{(x_0\alpha + x_1\gamma)(x_0^2\alpha\delta + x_1^2\beta\gamma)}{q^{1/2} - q^{-1/2}}\right).$$

The proposition is straightforward to verify by using the representation for A and A^* on the RHS of formula (71). We have given the explicit expressions for the structure constants for the $U_q(su(2))$ which is relevant for case A. It is characteristic of this algebra that the structure constants are representation-dependent. Analogous expressions are obtained by using any other form of the deformed (u, v) algebra. In particular, for the (u, u) algebra in case B the expressions for ρ, ρ^* differ in sign and we skip the long formulae for ω, η, η^*. The considered realization of the AW-algebra generators is the Granovskii and Zhedanov realization [63] and corresponds to the evaluation representation for the $U_q(\hat{su}(2))$ generators $\pi_\nu(E_1^\pm) = A^\pm, \pi_\nu(E_0^\pm) = \nu^{\pm1}A^\mp, \pi_\nu(q^{H_1}) = q^N, \pi_\nu(q^{H_0}) = q^{-N}$, where $A^\pm, N$ are the $U_q(sl(2))$ generators.

The tridiagonal algebra of the open ASEP has the form of deformed Dolan–Grady relations for the shifted boundary operators

$$[A, [A, [A, A^*]_q]_{q^{-1}} = \rho[A, A^*], \quad [A^*, [A^*, [A^*, A]_q]_{q^{-1}} = \rho^*[A^*, A], \quad (78)$$

with ρ, ρ^* given by (76). It follows from (75) through the natural homomorphism TD $\rightarrow$ AW. As readily seen from the definition (72), the (shifted) boundary operators of the ASEP obeying the AW algebra (75) form a Leonard pair with $\beta = q + q^{-1}, \gamma = \gamma^* = 0,\ \rho, \rho^*, \omega, \eta, \eta^*$ given by (76), (77), as well as a tridiagonal pair determined by $\beta = q + q^{-1}, \rho, \rho^*$. The TD algebra of the bulk generators D_0, D_1 is a special form of the boundary algebra (78) with $\rho, \rho^* = 0$.

We have derived the boundary operators using the $U_q(su(2))$ invariance property due to the known equivalence of the ASEP to the integrable spin $1/2$ XXZ chain with most general boundary terms. The importance of the tridiagonal algebraic approach is that it allows for an independent study of stochastic dynamics. The interpretation of the bulk tridiagonal algebra is that quantum affine $U_q(\hat{su}(2))$ arises as the hidden symmetry of the bulk dynamics. Once the bulk stochastic dynamics is specified, the boundary operators, which define the boundary conditions, should be determined as covariant objects with respect to the bulk symmetry. Thus the proposed representation of the boundary operators in the form of Eq. (71) is naturally interpreted from the point of view of the general homomorphism of the Askey–Wilson algebra with two generators as a coideal subalgebra of $U_q(\hat{su}(2))$.

6. ASKEY–WILSON ALGEBRA AND REFLECTION EQUATION

As mentioned in the previous section, the boundary generators of the ASEP are determined through the evaluation representation of $U_q(\hat{su}(2))$ and thus form a coideal subalgebra of the quantum affine group. The driven diffusive lattice system exhibits the properties of a large class of models, namely the affine Toda field theories, with quantum affine symmetry in the bulk and integrable boundary conditions.

The main idea of integrability of lattice systems (within the inverse scattering method [69]) is the existence of a family of commuting transfer matrices, depending on a spectral parameter. For quantum spin chains, the transfer matrices give rise to infinitely many mutually commuting conservation laws. This is the Abelian symmetry of the system. The infinitely many commuting conserved charges can be diagonalized simultaneously and their common eigenspace is finite-dimensional in most cases. Thus the Abelian symmetry reduces the degeneracies of the spectrum from infinite to finite which is the reason for integrability. In addition many systems possess non-Abelian symmetries. They determine the R-matrix operator, a solution of the Yang–Baxter equation, up to an overall scalar factor and are identified as the quantum bulk symmetries. In the presence of general boundaries the quantum symmetry, and the intergrability of the model as well, are broken. However with suitably chosen boundary conditions [70, 71] a remnant of the bulk symmetry may survive and the system possesses hidden boundary symmetries, which determine a K matrix, a solution of a boundary Yang–Baxter equation and allow for the exact solvability. Such nonlocal boundary symmetry charges were originally obtained for the sine-Gordon model [72] and generalized to affine Toda field theories [73], and derived from spin chain point of view as commuting with the transfer matrix for a special choice of the boundary conditions [74] or analogously as the one boundary Temperley–Lieb algebra centralizer in the «nondiagonal» spin $1/2$ representation [75].

We argue (for details, see [78]) that one can construct a K matrix in terms of the Askey–Wilson algebra generators, which satisfies a boundary Yang–Baxter equation (known as a reflection equation).

We consider models of statistical physics in which the spin variable is associated with the site i of a one-dimensional lattice. An example of a model with quantum affine symmetry is the spin $1/2$ XXZ model with Hamiltonian defined on an infinite-dimensional chain

$$H = -\frac{1}{2}\sum_i (\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \Delta \sigma_i^z \sigma_{i+1}^z), \tag{79}$$

where the Pauli matrices $\sigma_i^x, \sigma_i^y, \sigma_i^z$ act on the ith component of the infinite tensor product $\ldots \otimes V_{i-1} \otimes V_i \otimes V_{i+1} \otimes \ldots$, with $V = \mathbf{C}^2$. This model is known to be integrable [76] within the representation theory of the affine quantized algebra $U_q(\hat{sl}(2))$. Namely, given the $U_q(\hat{sl}(2))$ R-matrix operator $R(z_1/z_2) \in \mathrm{End}_{\mathbf{C}} V_{z_1} \otimes V_{z_2}$, where V_z is the two-dimensional $U_q(\hat{sl}(2))$ evaluation module, satisfying the Yang–Baxter equation

$$R_{12}(z_1/z_2) R_{13}(z_1) R_{23}(z_2) = R_{23}(z_2) R_{13}(z_1) R_{12}(z_1/z_2), \tag{80}$$

then the Hamiltonian is written as $H = \sum H_{ii+1}$, where the two-site Hamiltonian density is obtained as

$$H_{ii+1} = \frac{d}{du} P R_{ii+1}|_{u=0}, \tag{81}$$

with P — the permutation operator and $z_1/z_2 = \mathrm{e}^u$. The generators act on the quantum space by means of the infinite coproduct and the invariance with respect to the affine $U_q(\hat{sl}(2))$ manifests in the property

$$[H, \Delta^\infty(G_k)] = 0 \tag{82}$$

for any of the generators G_k of $U_q(\hat{sl}(2))$. If we introduce for finite chain a boundary of a particular form, such as diagonal boundary terms, the symmetry is reduced to $U_q(sl(2))$ and the invariant Hamiltonian has the form [77]

$$H_{XXZ}^{\mathrm{QGr}} = -\frac{1}{2}\sum_{i=1}^{L-1} (\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \Delta_q \sigma_i^z \sigma_{i+1}^z + h(\sigma_{i+1}^z - \sigma_i^z) + \Delta_q), \tag{83}$$

where

$$\Delta_q = -\frac{1}{2}(q + q^{-1}), \quad h = \frac{1}{2}(q - q^{-1}). \tag{84}$$

In the presence of a boundary in addition to the R matrix there is one more matrix $K(z)$ which satisfies the boundary Yang–Baxter equation, also known as a reflection equation

$$\begin{aligned} &R(z_1/z_2)(K(z_1) \otimes I) R(z_1 z_2)(I \otimes K(z_2)) - \\ &\qquad - (I \otimes K(z_2)) R(z_1 z_2)(K(z_1) \otimes I) R(z_1/z_2) = 0. \end{aligned} \tag{85}$$

Within the quantum inverse scattering method the K matrix is related to the quantum current $L = L^+(L^-)^{-1}$, where $L^\pm \in \mathrm{End}\, V \otimes U_q(\hat{sl}(2))$. In [78], the two generators of the Askey–Wilson algebra were constructed as linear covariant objects with the coproduct properties of two-sided coideals of the quantum affine symmetry $U_q(\hat{sl}(2))$. It is suggestive to construct the K matrix in terms of the AW algebra generators. We note that in [79] a K matrix was constructed by using an AW algebra in a slightly different form.

Let $R(z)$ be the symmetric trigonometric R matrix with deformation parameter $q^{1/2}$

$$R(z) = \\ = \begin{pmatrix} q^{1/2}z - q^{-1/2}z^{-1} & 0 & 0 & 0 \\ 0 & z - z^{-1} & q^{1/2} - q^{-1/2} & 0 \\ 0 & q^{1/2} - q^{-1/2} & z - z^{-1} & 0 \\ 0 & 0 & 0 & q^{1/2}z - q^{-1/2}z^{-1} \end{pmatrix} \tag{86}$$

acting on the auxiliary tensor product space $V_{z_1} \otimes V_{z_2}$ which carries the fundamental representations of the covariance algebra. Then one can construct an operator $L(z)$ [80] in terms of the $U_q(sl(2))$ generators

$$L(z) = \begin{pmatrix} zq^{J_3} - z^{-1}q^{-J_3} & (q^{1/2} - q^{-1/2})J_- \\ (q^{1/2} - q^{-1/2})J_+ & zq^{-J_3} - z^{-1}q^{J_3} \end{pmatrix} \tag{87}$$

acting on the tensor product $V_0 \otimes V_Q$ of the auxiliary space V_0 and the quantum space V_Q which in the general case carry finite-dimensional inequivalent $U_q(sl(2))$ representations. The L operator satisfies

$$R(z_1/z_2)L_1(z_1)L_2(z_2) = L_2(z_2)L_1(z_1)R(z_1/z_2), \tag{88}$$

where $L_1 = L \otimes I$ and $L_2 = I \otimes L$. As is known, this relation together with reflection equation (85) constitute the basic algebraic relations of the inverse scattering method to integrable models.

We are now going to construct a solution to Eq. (85) in terms of the operators A, A^*.

Proposition IV. Let A, A^* generate the AW algebra, the linear covariance algebra for $U_q(sl(2))$. Then, there exists a reflection matrix $K(z) = K^{\mathrm{op}}(z) + K^c(z)$, constructed in terms of the AW algebra generators, where the part K^{op} has the form

$$K^{\mathrm{op}}(z) = \\ = \begin{pmatrix} q^{1/2}zA - q^{-1/2}z^{-1}\frac{\sqrt{\rho}}{\sqrt{\rho^*}}A^* & -\frac{\sqrt{\rho}}{\sqrt{\rho^*}}(q^{1/2} - q^{-1/2})[A^*, A]_q \\ -\rho^{-1}\frac{\sqrt{\rho}}{\sqrt{\rho^*}}(q^{1/2} - q^{-1/2})[A, A^*]_q & -q^{-1/2}z^{-1}A + q^{1/2}z\frac{\sqrt{\rho}}{\sqrt{\rho^*}}A^* \end{pmatrix}, \tag{89}$$

and the part $K^c(z)$ is

$$K^c_{11} = \frac{q^{1/2}z\eta^* - q^{-1/2}z^{-1}\eta}{\rho(q^{1/2}+q^{-1/2})}, \qquad K^c_{22} = \frac{q^{-1/2}z\eta - q^{1/2}z^{-1}\eta^*}{\rho(q^{1/2}+q^{-1/2})},$$
$$K^c_{12} = -\rho\frac{q^{1/2}z^2+q^{-1/2}z^{-2}}{q^{1/2}+q^{-1/2}} - \frac{\sqrt{\rho}}{\sqrt{\rho^*}}\omega,\ K^c_{21} = -\frac{q^{1/2}z^2+q^{-1/2}z^{-2}}{q^{1/2}+q^{-1/2}} - \rho^{-1}\frac{\sqrt{\rho}}{\sqrt{\rho^*}}\omega. \tag{90}$$

The matrix $K(z)$ is a solution of the boundary Yang–Baxter equation (85) provided the operators A, A^* obey the tridiagonal algebraic relations of the AW algebra in the reduced general form (75) with all structure constants $\rho, \rho^*, \omega, \eta, \eta^*$ nonzero. We denote this solution $K(z,\rho)$.

The proof of this proposition is rather long but straightforward. It is directly verified using the explicit form of the R matrix (86) and the AW algebraic relations (75) that the boundary matrix K from (89), (90) solves reflection equation (85).

We emphasize on the factor $\sqrt{\rho}/\sqrt{\rho^*}$ to A^* and ω in the K matrix. This factor is due to the fact that the solution of the boundary Yang–Baxter equation (85) in terms of the AW algebra generators requires $\rho = \rho^*$. This is not a problem since given the AW algebra in the general form (75) we can relate it to an algebra with $\rho = \rho^*$ rescaling $A^* \to \dfrac{\sqrt{\rho}}{\sqrt{\rho^*}}A^*$. Alternatively we can rescale $A \to \dfrac{\sqrt{\rho^*}}{\sqrt{\rho}}A$ to obtain an AW algebra with $\rho = \rho^*$. This gives a second solution $K(z,\rho^*)$ of the reflection equation. Its $K^{\mathrm{op}}(z,\rho^*)$ part has the form

$$K^{\mathrm{op}}(z) = $$
$$= \begin{pmatrix} q^{1/2}z\dfrac{\sqrt{\rho^*}}{\sqrt{\rho}}A - q^{-1/2}z^{-1}A^* & -\dfrac{\sqrt{\rho^*}}{\sqrt{\rho}}(q^{1/2}-q^{-1/2})[A^*,A]_q \\ -\rho^{*-1}\dfrac{\sqrt{\rho^*}}{\sqrt{\rho}}(q^{1/2}-q^{-1/2})[A,A^*]_q & -q^{-1/2}z^{-1}\dfrac{\sqrt{\rho^*}}{\sqrt{\rho}}A + q^{1/2}zA^* \end{pmatrix}. \tag{91}$$

The matrix elements of the $K^c(z,\rho^*)$ part are obtained from (90) by the interchange $\rho \leftrightarrow \rho^*$. The solution $K(z,\rho^*)$ can be implemented to construct a solution $K^*(z)$ of the dual reflection equation [70, 81] Namely, the matrix $K^*(z) = K^t(z^{-1},\rho^*)$ solves the dual reflection equation (which is obtained from Eq. (85) by changing $z_{1,2} \to q^{-1/2}z_{1,2}^{-1}$ and $K \to K^t$).

Setting $\rho = \rho^*$ and $\eta = \eta^* = 0$ in (89) and (90), we obtain the K matrix considered in [82–84] for such a very particular case of an AW algebra and for the spin 1/2 quantum space representation. We note that an AW algebra in the general form with a sequence of scalars $-(q+q^{-1}), \gamma, \gamma^*, \omega, \eta, \eta^*$ cannot be reduced to such a particular algebra with structure constants $-(q+q^{-1}), \rho, \rho, 0, 0, \omega, 0, 0$. There exists unique affine transformation [85] to only set $\gamma = \gamma^* = 0$ (and simultaneously either $\rho = \eta^* = 0$ or $\eta = \rho^* = 0$).

7. INTERPRETATION OF THE ASEP BOUNDARY OPERATORS

The mapping of the stochastic master equation to a Schrödinger equation in imaginary time provides a connection to integrable models. As is known, the open ASEP is related to the integrable spin $1/2$ XXZ quantum chain through the similarity transformation $\Gamma = -qU_\mu^{-1} H_{XXZ} U_\mu$ (for details, see [15]). H_{XXZ} is the Hamiltonian of the $U_q(su(2))$ invariant quantum spin chain H_{XXZ}^{QGr} (83) with anisotropy Δ_q and with added nondiagonal boundary terms B_1 and B_L. Namely,

$$H_{XXZ} = H_{XXZ}^{\mathrm{QGr}} + B_1 + B_L, \tag{92}$$

where the transition rates of the ASEP are related to the boundary terms in the following way (μ is a free parameter, irrelevant for the spectrum):

$$\begin{aligned} B_1 &= \frac{1}{2q}\left(\alpha + \gamma + (\alpha - \gamma)\sigma_1^z - 2\alpha\mu\sigma_1^- - 2\gamma\mu^{-1}\sigma_1^+\right), \\ B_L &= \frac{\left(\beta + \delta - (\beta - \delta)\sigma_L^z - 2\delta\mu q^{L-1}\sigma_L^- - 2\beta\mu^{-1}q^{-L+1}\sigma_L^+\right)}{2q}. \end{aligned} \tag{93}$$

It has been shown by Sandow and Schuetz [86] that the bulk driven diffusive system with reflecting boundaries can be mapped to the spin $1/2$ $U_q(su(2))$-invariant quantum spin chain. The $U_q(su(2))$ generators satisfying Eqs. (65) and (66) act on the tensor product representation space $(V^2)^{\otimes L}$ as

$$\begin{aligned} q^{\pm N} &= q^{\pm\frac{\sigma_3}{2}} \otimes q^{\pm\frac{\sigma_3}{2}} \otimes \ldots \otimes q^{\pm\frac{\sigma_3}{2}}, \\ A_\pm &= \sum_i q^{\frac{\sigma_3}{4}} \otimes \ldots \otimes q^{\frac{\sigma_3}{4}} \otimes \sigma_i^\pm \otimes q^{-\frac{\sigma_3}{4}} \otimes \ldots \otimes q^{\frac{-\sigma_3}{4}}, \end{aligned} \tag{94}$$

where $\sigma_3, \sigma^\pm$ are the Pauli matrices and the index i means that the matrix is associated with the ith site of the chain (ith position in the tensor product). The representation is completely reducible, the product of L spin $1/2$ representations decomposes into a direct sum of spin j irreducible representations with the maximal highest weight $j = L/2$ decreasing by 1 to $j = 0$ or $j = 1/2$ for even L or odd L. Within the matrix product approach the bulk process with reflecting boundary conditions is described by a quadratic algebra

$$D_1 D_0 - q D_0 D_1 = 0, \tag{95}$$

which defines a two-dimensional noncommutative plane with the $SU_q(2)$ action as its symmetry. The operators associated with the bulk ASEP form the two-dimensional comodule of $SU_q(2)$. As a consequence of Eq. (95), for generic q, a spin j representation of $U_q(su(2))$ can be realized in the space of the q-symmetrized product of $L = 2j$ two-dimensional representations D_μ, $\mu = 0, 1$, with basis $D_0^{L-k} D_1^k$, $k = 0, 1, \ldots, L$. The stationary probability distribution, i.e., the ground state of the $U_q(su(2))$ invariant Hamiltonian H_{XXZ}^{QGr}, corresponds to the q symmetrizer of the Young diagram with one row and L boxes [87]. The presence of the boundary processes (i.e., the nondiagonal boundary terms in the Hamiltonian) reduces the $U_q(su(2))$ bulk invariance and amounts to the appearance of linear terms in the quadratic algebra. The boundary conditions define the boundary operators which carry a residual symmetry of the process. It is expressed in the fact that the boundary operators are constructed in terms of the $U_q(su(2))$ generators, as seen from the explicit formulae (71). With $A_\pm, N$ being the generators of a finite-dimensional $U_q(su(2))$ representation, it can be verified from Eq. (71) that $\alpha D_0 - \gamma D_1$ commutes with $H(q)^{\mathrm{QGr}}$ and $\beta D_1 - \delta D_0$ commutes with $H(-q^{-1})^{\mathrm{QGr}}$, where according to [77]

$$H^{\mathrm{QGr}}(-q^{-1}) = -U H^{\mathrm{QGr}}(q) U^{-1} \tag{96}$$

and

$$U = \exp\left(i \frac{\pi}{2} \sum_{m=1}^{L} m \sigma_m^3 \right). \tag{97}$$

Thus the boundary operators constructed as the linear covariant objects of the bulk $U_q(su(2))$ symmetry acquire a very important physical meaning — they can be interpreted as the two nonlocal conserved charges of the open ASEP. Such nonlocal boundary symmetry charges were originally obtained for the sine-Gordon model [103] and generalized to affine Toda field theories [73], and derived from spin chain point of view as commuting with the transfer matrix for a special choice of the boundary conditions [74]. In particular, the left boundary operator $\alpha D_0 - \gamma D_1$ in the finite-dimensional representation of (75) is analogous to the one boundary Temperley–Lieb algebra centralizer in the «nondiagonal» spin $1/2$ representation [75].

8. REPRESENTATIONS OF THE ASEP BOUNDARY ALGEBRA

The Askey–Wilson algebra is known to possess some important properties which allow one to obtain its ladder representations, spectra, overlap functions. We briefly sketch these properties (for details, see [62, 63]). Let f_r be an eigenvector of A with eigenvalue λ_r:

$$A f_r = \lambda_r f_r. \tag{98}$$

Then we can construct a new eigenstate

$$f_s = (Ag(A) + A^* h(A) + A_0 k(A)) f_r, \tag{99}$$

where A_0 denotes the q commutator $[A, A^*]_q$, and

$$Af_s = \lambda_s f_s. \tag{100}$$

It follows from the algebra that f_s will also be an eigenvector of A, if for the new eigenvalue the quadratic relation holds

$$\lambda_r^2 + \lambda_s^2 - (q + q^{-1})\lambda_r \lambda_s - \rho = 0. \tag{101}$$

This yields for each state f_r two neighbouring states ($r' = r - 1$ and $r'' = r + 1$) whose eigenvalues are the roots of the above quadratic equation. In this parameterizations the operator A is diagonal and the operator A^* is tridiagonal

$$Af_r = a_{r+1} f_{r+1} + b_r f_r + c_{r-1} f_{r-1}. \tag{102}$$

The expressions for the spectrum and the matrix coefficient can be obtained explicitly. The quadratic equation is

$$\lambda_{r+1}^2 + \lambda_r^2 - (q + q^{-1})\lambda_r \lambda_{r+1} - \rho = 0 \tag{103}$$

which yields the spectrum

$$\lambda_r = q^{-r} + \frac{\rho q^r}{(q - q^{-1})^2}. \tag{104}$$

Depending on the sign of ρ it is hyperbolic of the form «sinh» or «cosh» and «exp» if $\rho = 0$. The algebra possesses a duality property. Due to the duality property the dual basis exists in which the operator A^* is diagonal and the operator A is tridiagonal. We have

$$A^* f_p^* = \lambda_p^* f_p^*, \quad Af_s^* = a_{s+1}^* f_{s+1}^* + b_s^* f_s^* + c_{s-1}^* f_{s-1}^*, \tag{105}$$

where λ_p^* satisfies quadratic equation (103) with $-\rho$ replaced by $-\rho^*$. The overlap function of the two bases $\langle s|r\rangle = \langle f_s^*|f_r\rangle$ can be expressed in terms of the Askey–Wilson polynomials. It is a long procedure to find the matrix elements, the eigenvalues, and the eigenvectors of the operators A, A^* in the ladder representations. They are equivalent for the AW algebra (75) and the TD algebra (78) in the space with the AW polynomials as the basis. The explicit form of the ASEP boundary algebra infinite-dimensional representations is obtained in [88]. Two different realizations of the deformed (u, v) algebra are considered for the two different sets of algebraic relations satisfied by the operators $\alpha D_0, \beta D_1, \gamma D_1, \delta D_0$, namely, the $(u, -u)$ algebra for the case A and the (u, u) algebra for the case B. For convenience one denotes

$$\pm\frac{\gamma}{\alpha} = ac, \quad \pm\frac{\delta}{\beta} = bd, \tag{106}$$

where the $+$ sign corresponds to the (u, u) algebra and the – sign — to $(u, -u)$ algebra. Besides we set $x_0 = -x_1 = \zeta$, where ζ is a free parameter from the algebraic relation $x_0 + x_1 = 0$. We rescale the generators as follows:

$$A \to (1-q)\frac{q^{-1/2}}{\zeta\beta}A, \quad A^* \to (1-q)\frac{q^{-1/2}}{\zeta\alpha}A^*. \tag{107}$$

The representations, we have found, are isomorphic to the basic representation of the Askey–Wilson algebra [89] in the space of symmetric Laurent polynomials $f[y]$ with a basis $(p_0, p_1, \ldots)$, where $p_n = p_n(x; a, b, c, d)$ denotes the nth Askey–Wilson polynomial [90] depending on four parameters a, b, c, d

$$p_n = {}_4\Phi_3\left(\begin{matrix} q^{-n}, abcdq^{n-1}, ay, ay^{-1} \\ ab, ac, ad \end{matrix}\ |q; q\right) \tag{108}$$

with $p_0 = 1$, $x = y + y^{-1}$ and $0 < q < 1$. Let $\mathcal{A}$ denote the matrix whose matrix elements enter the three-term recurrence relation for the Askey–Wilson polynomials

$$xp_n = b_n p_{n+1} + a_n p_n + c_n p_{n-1}, \quad p_{-1} = 0, \tag{109}$$

$$\mathcal{A} = \begin{pmatrix} a_0 & c_1 & & \\ b_0 & a_1 & c_2 & \\ & b_1 & a_2 & \cdot \\ & & \cdot & \cdot \end{pmatrix} \tag{110}$$

The explicit form of the matrix elements of $\mathcal{A}$ reads

$$a_n = a + a^{-1} - b_n - c_n, \tag{111}$$

$$b_n = \frac{(1-abq^n)(1-acq^n)(1-adq^n)(1-abcdq^{n-1})}{a(1-abcdq^{2n-1})(1-abcdq^{2n})}, \tag{112}$$

$$c_n = \frac{a(1-q^n)(1-bcq^{n-1})(1-bdq^{n-1})(1-cdq^{n-1})}{(1-abcdq^{2n-2})(1-abcdq^{2n-1})}. \tag{113}$$

The basis is orthogonal with the orthogonality condition for the Askey–Wilson polynomials $P_n = a^{-n}(ab, ac, ad; q)_n p_n$

$$\int_{-1}^{1} \frac{w(x)}{2\pi\sqrt{1-x^2}} P_m(x; a, b, c, d|q)\, P_n(x; a, b, c, d|q)\, dx = h_n \delta_{mn}, \tag{114}$$

where $w(x) = \dfrac{h(x,1)h(x,-1)h(x,q^{1/2})h(x,-q^{1/2})}{h(x,a)h(x,b)h(x,c)h(x,d)}$ and $h(x,\mu) = \prod_{k=0}^{\infty}[1 - 2\mu x q^k + \mu^2 q^{2k}]$, and

$$h_n = \frac{(abcdq^{n-1};q)_n(abcdq^{2n};q)_\infty}{(q^{n+1},abq^n,acq^n,adq^n,bcq^n,bdq^n,cdq^n;q)_\infty}. \tag{115}$$

We summarize the results for the representation of the ASEP boundary operators:

Case A. There is a representation π in a space with basis

$$(p_0,p_1,p_2,\ldots)^t \tag{116}$$

with respect to which the right boundary operator $D_1 - (\delta/\beta)D_0 \equiv D_1 + bdD_0$ is diagonal. The representing matrix is $\mathrm{diag}\,(\lambda_0,\lambda_1,\lambda_2,\ldots)$ with the eigenvalues λ_n given by

$$\lambda_n = \frac{q^{1/2}}{1-q}\left(bq^{-n} + dq^n + 1 + bd\right). \tag{117}$$

The left boundary operator $D_0 - \gamma/\alpha D_1 \equiv D_0 + acD_1$, its tridiagonal and its representing matrix has the form

$$\pi(D_0 + acD_1) = \frac{q^{1/2}}{1-q}\left(b\mathcal{A}^t + 1 + ac\right). \tag{118}$$

The dual representation π^* has a basis

$$(p_0,p_1,p_2,\ldots) \tag{119}$$

with respect to which the left boundary operator $\pi^*(D_0 + acD_1)$ is diagonal $(\lambda_0^*,\lambda_1^*,\lambda_2^*,\ldots)$ with diagonal elements

$$\lambda_n^* = \frac{q^{1/2}}{1-q}\left(aq^{-n} + cq^n + 1 + ac\right). \tag{120}$$

The right boundary operator is represented by a tridiagonal matrix

$$\pi^*(D_1 + bdD_0) = \frac{q^{1/2}}{1-q}\left(a\mathcal{A} + 1 + bd\right). \tag{121}$$

The formulae (117), (118) and (120), (121) define the ladder representation (resp. the dual representation) of the boundary operators in a Hilbert space with an inner product. In the above formulae the Askey–Wilson parameters are assumed to be arbitrary functions of the boundary parameters except for the relations (106). The form of these functions is uniquely determined by the eigenvalue equations with the choice for the left and right boundary vectors

$$\langle w| = h_0^{-1/2}(p_0, 0, 0, \ldots), \quad |v\rangle = h_0^{-1/2}(p_0, 0, 0, \ldots)^t, \tag{122}$$

where h_0 is a normalization from the orthogonality relation. These vectors belong to the two dual representations of the tridiagonal boundary algebra and are the eigenvectors of the corresponding diagonal operator $\pi(D_1 + bdD_0)$ and $\pi^*(D_0 + acD_1)$. The eigenvalue equations

$$\begin{aligned} \left(D_1 - \frac{\delta}{\beta}D_0\right)|v\rangle - s|v\rangle = 0, \\ \langle w|\left(D_0 - \frac{\gamma}{\alpha}D_1\right) - \langle w|s = 0 \end{aligned} \tag{123}$$

are solved by the functions

$$a = \kappa_+^*, \quad b = \kappa_+, \quad c = \kappa_-^*, \quad d = \kappa_-, \tag{124}$$

where

$$\begin{aligned} \kappa_\pm &= \frac{-(\beta - \delta - (1-q)) \pm \sqrt{(\beta - \delta - (1-q))^2 + 4\beta\delta}}{2\beta}, \\ \kappa_\pm^* &= \frac{-(\alpha - \gamma - (1-q)) \pm \sqrt{(\alpha - \gamma - (1-q))^2 + 4\alpha\gamma}}{2\alpha}. \end{aligned} \tag{125}$$

It is important to emphasize that the expressions (125) are the functions of the parameters that define the phase space of the model. In previous applications they have always been taken for granted. It is quite remarkable that here they follow from the properties of the boundary Askey–Wilson algebra representations.

Case B. There exists a representation π with basis $(p_0, p_1, \ldots)^t$ with respect to which the right boundary operator $D_1 - (\delta/\beta)D_0 \equiv \pi(D_1 - bdD_0)$ is diagonal with eigenvalues

$$\lambda_n = \frac{q^{1/2}}{1-q}\left(bq^{-n} + dq^n + 1 - bd\right) \tag{126}$$

and the left boundary operator $D_0 - (\gamma/\alpha)D_1 \equiv \pi(D_0 - acD_1)$ is tridiagonal with

$$\pi(D_0 - acD_1) = \frac{q^{1/2}}{1-q}\left(b\mathcal{A}^t + 1 - ac\right). \tag{127}$$

The dual representation has a basis $(p_0, p_1, \ldots)$ with respect to which $\pi^*(D_0 - acD_1)$ is diagonal with diagonal elements

$$\lambda_n^* = \frac{q^{1/2}}{1-q}\left(aq^{-n} + cq^n + 1 - ac\right) \tag{128}$$

and $\pi^*(D_1 - bdD_0)$ is tridiagonal

$$\pi^*(D_1 - bdD_0) = \frac{q^{1/2}}{1-q}\,(a\mathcal{A} + 1 - bd)\,. \tag{129}$$

Once again the relation of the Askey–Wilson parameters to the boundary parameters are uniquely determined by the eigenvalue equations with the choice of the boundary vectors as in (122). The explicit expressions are different from the case A and are denoted

$$a = \kappa'^*_+, \quad b = \kappa'_+, \quad c = \kappa'^*_-, \quad d = \kappa'_-, \tag{130}$$

where

$$\begin{aligned} \kappa'_\pm &= \frac{-(\beta+\delta-(1-q)) \pm \sqrt{(\beta+\delta-(1-q))^2 - 4\beta\delta}}{2\beta}, \\ \kappa'^*_\pm &= \frac{-(\alpha+\gamma-(1-q)) \pm \sqrt{(\alpha+\gamma-(1-q))^2 - 4\alpha\gamma}}{2\alpha}. \end{aligned} \tag{131}$$

We want to once again emphasize the representation dependence of the Askey–Wilson algebra (as well as of the MPA bulk quadratic algebra (31)). Using any of the particular forms of the deformed (u, v) algebra, we obtain the AW algebra as its linear covariance algebra. The functional-dependent structure constants $\rho, \rho^*, \omega, \eta, \eta^*$ carry the information of the corresponding algebra and, in particular, this reflects in different spectra of the diagonal (tridiagonal) operators and different Askey–Wilson polynomials. This is the formal mathematical difference between $U_q(su(2))$ in case A and the deformed general oscillator algebra $c_qu(2)$ used in case B. Namely, the spectrum of the diagonal operators for $c_qu(2)$ with positive structure constants ρ, ρ^* is of the form $\sim \cosh$, while for $U_q(su(2))$ with negative structure constants ρ, ρ^*, it is $\sim \sinh$. Hence one has different identifications of the AW four parameters with the boundary rates which, in our opinion, may enrich the variety of physical applications and is worth considering.

9. THE ISOMORPHIC TRIDIAGONAL ALGEBRAS OF THE TRANSFER MATRIX

The next step is to obtain a representation for the matrix $C = D_0 + D_1$ in terms of which the normalization factor is defined. It enters all the expressions for the physical quantities and plays the role of the transfer matrix. In the tridiagonal formulation the importance of this operator is that it forms an isomorphic tridiagonal pair with each of the boundary operators. This is shown by returning to the set of operators which provide a solution of the quadratic algebra (31). These imply that we can write the matrix $D_0 + D_1$ in two equivalent forms

$$D_0 + D_1 = \frac{x_0}{\sqrt{1-q}} q^{-N/2} A_+ + \frac{-x_1}{\sqrt{1-q}} A_- q^{-N/2} + \frac{x_0 - x_1}{1-q}(q^{-N}+1), \quad (132)$$

$$D_0 + D_1 = \frac{x_0}{\sqrt{1-q}} q^{N/2} A_+ + \frac{-x_1}{\sqrt{1-q}} A_- q^{N/2} + \frac{x_0 - x_1}{1-q}(q^{N}+1). \quad (133)$$

Proposition V. The right boundary operator $B^R = D_1 + bdD_0$ and the operator $D_0 + D_1$ from Eq. (132), namely the pair

$$\begin{aligned} D_1 + bdD_0 &= A - \frac{1}{1-q}\left(x_1 + x_0\frac{\delta}{\beta}\right), \\ D_0 + D_1 &= A^* + \frac{1}{1-q}(x_0 - x_1), \end{aligned} \quad (134)$$

where now A and A^* obey the tridiagonal algebra relations

$$\begin{aligned} [A, A^2A^* - (q+q^{-1})AA^*A + A^*A^2 + bdq^{-1}(q^{1/2} - q^{-1/2})^2 A^*] = 0, \\ [A^*, A^{*2}A - (q+q^{-1})A^*AA^* + AA^{*2} + (q-q^{-1})^2 A] = 0, \end{aligned} \quad (135)$$

form a tridiagonal pair which is isomorphic to the boundary operators' one. Hence the representations of this tridiagonal pair are readily obtained from the isomorphic representations of the boundary operators in the basis p_n. Namely, with respect to the basis $(p_0, p_1, p_2, \ldots)^t$ the operator $D_0 + D_1$ is tridiagonal

$$(D_0 + D_1)p_n = \frac{1}{1-q}(2+x)p_n, \quad xp_n = Ap_n, \quad \pi(A) = \mathcal{A}^t \quad (136)$$

and it is diagonal with respect to the dual basis $(p_0, p_1, p_2, \ldots)$ with spectrum

$$\lambda^* = \frac{1}{1-q}(2 + q^n - q^{-n}). \quad (137)$$

And, of course, the matrices representing the right boundary operator are the same as in (117) and (121).

Proposition VI. The left boundary operator $B^L = D_0 - (\gamma/\alpha)D_1$ and the operator $D_0 + D_1$ in Eq. (133)

$$\begin{aligned} D_0 + D_1 &= A + \frac{1}{1-q}(x_0 - x_1), \\ D_0 - \frac{\gamma}{\alpha}D_1 &= A^* + \frac{1}{1-q}\left(x_0 + x_1\frac{\gamma}{\alpha}\right) \end{aligned} \quad (138)$$

with

$$\begin{gathered}[A, A^2A^* - (q+q^{-1})AA^*A + A^*A^2 + (q-q^{-1})^2A^*] = 0,\\ [A^*, A^{*2}A - (q+q^{-1})A^*AA^* + AA^{*2} + acq^{-1}(q-q^{-1})^2A] = 0\end{gathered} \tag{139}$$

form a tridiagonal pair which is isomorphic to the boundary operators' one. Hence the representations of this tridiagonal pair are readily obtained from the isomorphic representations of the boundary operators in the basis p_n. Accordingly, the operator $D_0 + D_1$ is diagonal with respect to the basis $(p_0, p_1, p_2, \ldots)^t$ with eigenvalues

$$\lambda = \frac{1}{1-q}(2 + q^n - q^{-n}) \tag{140}$$

and tridiagonal with respect to the dual basis $(p_0, p_1, p_2, \ldots)$

$$(D_0 + D_1)p_n = \frac{1}{1-q}(2+x)p_n, \quad xp_n = A^*p_n, \quad \pi^*(A^*) = \mathcal{A}. \tag{141}$$

The diagonal and tridiagonal representations of the left boundary operator are given in (118) and (120).

Analogous considerations apply for the operator $D_0 + \xi D_1$ (ξ is a fugacity) considered in [59, 60].

Thus each boundary operator forms a tridiagonal pair with the transfer matrix $D_0 + D_1$. The tridiagonal representation of the transfer matrix with respect to the basis, the AW polynomials and corresponding to the boundary algebra representation of case A with $\kappa_\pm, \kappa_\pm^{(*)}$ from Eq. (125), has been used in [59] for the exact solution of the open ASEP. It was considered in the form of an eigenvalue equation for the transfer matrix with the AW polynomials as eigenfunctions, however with no connection to the Askey–Wilson algebra nor to the tridiagonal one.

10. FINITE-DIMENSIONAL REPRESENTATIONS OF THE ASKEY–WILSON ALGEBRA

The condition for the ladder representations in terms of the Askey–Wilson polynomials to be finite-dimensional is $b_n = b_{n_0+n_f} = 0$, where n_f is the dimension of the representation and n_0 is some parameter that is characteristic for the representation. It is appropriate to take n_0 to be equal to k, the fixed value of the Casimir characterizing the representation series of the $U_q(sl(2))$ algebra. Without loss of generality we take $n_0 = 0$ for simplicity. Then from the explicit form of b_n it follows that the representation is finite-dimensional if for some $n = n_f$ any of the factors

$$1 - abq^n, \quad 1 - acq^n, \quad 1 - adq^n, \quad 1 - abcdq^{n-1} \tag{142}$$

in the numerator of b_n is zero. In particular, the first factor in (142) $1 - abq^n$, expressed in terms of the boundary ASEP parameters, gives

$$\kappa_+^*(\alpha, \gamma)\kappa_+(\beta, \delta) = q^{-n_f}, \tag{143}$$

and relabelling $n = 0, 1, 2, \ldots, n_f$ to $l = 1, 2, 3, \ldots, l_f$ we have

$$\kappa_+(\alpha, \gamma)\kappa_+(\beta, \delta) = q^{1-l_f}. \tag{144}$$

The latter relation coincides with the defining condition of a finite-dimensional representation of the operators D_0, D_1 found by Mallick and Sandow in the equivalent realization of a diagonal and an upper diagonal matrices, as they were used in known applications of the matrix product ansatz $[15, 39]$.

As is already pointed out, the ladder representations of the Askey–Wilson and the tridiagonal ASEP algebra can be obtained on the basis of either infinite-dimensional representations of some of the different forms of the deformed (u, v) algebra or on the basis of the finite-dimensional representations of $U_q(su(2))$. If one uses the (u, u) algebra or the $(u, 0)$ $((0, u))$ algebra whose representations are infinite-dimensional, then it is Eq. (142) only that defines the finite-dimensional representations of the AW or tridiagonal algebra. In the case of a finite-dimensional representation of $U_q(su(2))$ besides the condition for a finite-dimensional representation of the Askey–Wilson algebra following from Eq. (142) there is one more additional constraint (given by (Eq. 70)), which is defining the dimension of the $U_q(su(2))$ representation.

11. SPECIAL CASES OF THE TRIDIAGONAL BOUNDARY ALGEBRA

From the tridiagonal boundary algebra in the case of general boundary conditions (and the two isomorphic ones involving the transfer matrix operator) one can obtain as special cases the algebras with $\rho = \rho^* = 0$. This is the tridiagonal algebra generated by the MPA matrices D_0, D_1, and also the two isomorphic algebras for the Leonard pairs $D_1, D_0 + D_1$ and $D_0 + D_1, D_0$. Such algebras correspond to the open ASEP with only injected particles at the left boundary and only removed particles at the right boundary. It is here important to emphasize that a Leonard pair admits a realization as diagonal and tridiagonal matrices and *an equivalent realization as upper and lower bidiagonal matrices* (see [65] for details). In this equivalent form the previously known representations of the MPA matrices D_0, D_1 have been used and are now recovered as special cases of the ASEP boundary algebra.

1. $\rho = \rho^* = 0$ due to $ac = 0, bd = 0$, and $a \neq 0, b \neq 0$
There is a representation (and its dual) with basis of the Al-Salam–Chihara polynomials $P_n(a, b; x)$

$$P_n(a,b;x) = a^{-n}(ab;q)_n \times {}_3\Phi_2 \left(\begin{matrix} q^{-n}, & a\exp^{i\theta}, & a\exp^{-i\theta} \\ & ab, & 0 \end{matrix} |q;q \right) \tag{145}$$

with $P_0 = 1$ and $x = \cos\theta$ and the three-terms recurrence relation

$$P_{n+1}(a,b,;x) + (a+b)q^n P_n(a,b;x) + (1-q)(1-abq^{n-1})P_{n-1}(a,b;x) = \\ = 2xP_n(a,b;x). \tag{146}$$

Let $\mathcal{A}_1$ denote the matrix

$$\mathcal{A}_1 = \begin{pmatrix} a_0 & c_1 & & \\ c_1 & a_1 & c_2 & \\ & c_2 & a_2 & \cdot \\ & & \cdot & \cdot \end{pmatrix} \tag{147}$$

satisfying $\mathcal{A}p_n(a,b;x) = 2xp_n(a,b;x)$, where $P_n(a,b;x) = (ab;q)_n p_n(a,b;x)$ and $p_n(a,b;x) = p_n$. Then with respect to $(p_0, p_1, p_2, \ldots)$ $\pi(D_1)$ is diagonal $(\lambda_0, \lambda_1, \lambda_2, \ldots)$

$$\lambda_n = \frac{q^{1/2}}{1-q}(1 + bq^n) \tag{148}$$

and D_0 is tridiagonal

$$\pi(D_0) = \frac{q^{1/2}}{1-q}\left(2 + \mathcal{A}_1\right). \tag{149}$$

In the dual basis D_0 is diagonal with

$$\lambda_n^* = \frac{q^{1/2}}{1-q}(1 + aq^n) \tag{150}$$

and D_1 is tridiagonal

$$\pi^*(D_1) = \frac{q^{1/2}}{1-q}\left(2 + \mathcal{A}_1\right). \tag{151}$$

For the isomorphic algebras we write only the matrices representing the transfer matrix $D_0 + D_1$. Namely, both the diagonal and the tridiagonal matrices in the representation π and in the dual representation π^* have the same form

$$\lambda_n = \lambda_n^* = \frac{1}{1-q}(2 + (a+b)q^N), \tag{152}$$

$$\pi(D_0 + D_1) = \pi^*(D_0 + D_1) = \frac{1}{1-q}(2 + \mathcal{A}_1) \tag{153}$$

with $\mathcal{A}_1$ from (147). As mentioned above, there is an equivalent representation of a Leonard pair as a lower and upper bidiagonal matrices. In this form the operators D_0, D_1 and with $D_0 + D_1$ as a tridiagonal matrix satisfying the eigenvalue equation

$$(D_0 + D_1)p_n = 2(1 + x)p_n \tag{154}$$

were applied in [58] for the exact solution of the ASEP with only incoming particles at the left boundary and outgoing particle at the right boundary.

2. $\rho = \rho^* = 0$ and $a = b = c = d = 0$

For $a = b = c = d = 0$ the Askey–Wilson polynomials reduce to the q-Hermite polynomials

$$H_n(x|q) = P_n(x; 0, 0, 0, 0|q) \tag{155}$$

with the three-term recurrence relation

$$P_{n+1}(x) + (1 - q^n)P_{n-1}(x) = 2xP_n(x). \tag{156}$$

Let $\mathcal{A}_2$ denote the corresponding matrix

$$\mathcal{A}_2 = \begin{pmatrix} 0 & c_1 & & \\ c_1 & 0 & c_2 & \\ & c_2 & 0 & \cdot \\ & & \cdot & \cdot \end{pmatrix}, \tag{157}$$

where $c_n = \sqrt{[n]}$ with $[n] = \dfrac{1 - q^n}{1 - q}$. The diagonal representation of D_1 and the dual diagonal D_0 are proportional to $\mathrm{diag}\,(1, q^{-1}, q^{-2}, \ldots)$ accordingly shifted by infinite-dimensional identity matrices with coefficients either depending on x_0, x_1 for the bulk algebra or determined by the boundary conditions. Hence the corresponding representation for the diagonal transfer matrix follows. The tridiagonal representations of D_0, D_1 and $D_0 + D_1$ have the form

$$\pi(D_0) = \pi^*(D_1) = \frac{1}{1 - q}(1 + \mathcal{A}_2), \tag{158}$$

$$\pi(D_0 + D_1) = \pi^*(D_0 + D_1) = \frac{1}{1 - q}(2 + \mathcal{A}_2). \tag{159}$$

As is already pointed out, a Leonard pair is determined by the sequence of scalars by means of which one can construct the representations as a diagonal and a tridiagonal matrices or equivalently as a lower bidiagonal and an upper bidiagonal matrices. In most of the applications of the MPA to different models, the representation in the latter form has been used. In the case of the q-Hermite polynomials this equivalent representation corresponds to shifted q-deformed oscillators as they were applied in [57] for the solution of the ASEP with only injected particles on the left boundary and only removed particles on the right one.

12. EXACT STATIONARY SOLUTION OF THE ASEP

We have shown that within the matrix-product-state ansatz to the open asymmetric exclusion process the two linear independent boundary operators form a Leonard pair obeying the AW algebra (and tridiagonal algebra as well) whose irreducible modules are expressed in terms of the Askey–Wilson polynomials. The four boundary parameters are related to the four parameters of the Askey–Wilson polynomials, the relation being uniquely determined in a given representation by the boundary eigenvalue equations. Each boundary operator forms an isomorphic Leonard pair with the operator $C = D_0 + D_1$ with representations described in terms of the Askey–Wilson polynomials as well. For particular values of the structure constants, the matrices D_0, D_1 of the matrix product ansatz also obey the tridiagonal algebraic relations. Within the tridiagonal framework the known representations of previously considered MPA applications to the ASEP are recovered as special cases. The Askey–Wilson algebra and the tridiagonal one provide remarkable insight into deep algebraic properties of stochastic processes. The rich representation theory and known structure of the polynomials in the Askey–Wilson scheme yield a generalization of the matrix product approach to a larger variety of possible applications.

The usefulness of the tridiagonal algebraic approach manifests in the simplified calculations of the relevant physical quantities for the ASEP. It is best shown in the two isomorphic algebras involving the transfer matrix $C = D_0 + D_1$. This operator has a special role because it enters all the expressions for the steady weights, the correlation functions, etc., as given by Eqs. (34)–(39). In paper [59], for the study of the open ASEP with most general boundary conditions, two infinite-dimensional tridiagonal matrices D_0, D_1, apparently very complicated, were used to solve the MPA quadratic algebra. The transfer matrix, constructed out of them, satisfies the eigenvalue equation for the AW polynomials,

$$(D_0 + D_1)p_n = (2 + x)p_n. \tag{160}$$

By using the orthogonality relation in the form

$$1 = h_0^{-1} \int dz\, w(y + y^{-1})|p(y + y^{-1})\rangle\langle p(y + y^{-1})| \tag{161}$$

one obtains (omitting the long technical details) the normalization factor and the current

A. $a > 1$, $a > b$

$$Z_L^a \simeq \left(\frac{(1+a)(1+a^{-1})}{1-q}\right)^L, \quad J \simeq (1-q)\frac{a}{(1+a)^2}.$$

B. $b > 1$, $b > a$

$$Z_L^b \simeq \left(\frac{(1+b)(1+b^{-1})}{1-q}\right)^L, \quad J \simeq (1-q)\frac{b}{(1+b)^2}$$

and analogously, the correlation functions, the density profile, etc. We refer to the exact stationary solution in Sec. 6 of [59]. From our point of view, Eq. (160) coincides with the tridiagonal representation of either the shifted operator A in (136) or the dual operator A^* in (141), as generators of the Askey–Wilson (tridiagonal) algebra. The procedure to compute the current, the steady weights, the correlation functions simplifies by using the orthogonality condition of the Askey–Wilson polynomials and the fact that they form a complete set in the space of Laurent polynomials. Analogous argumentation applies to the study of the ASEP with only incoming particles at the left boundary and outgoing ones at the right one, whose solution was related to the q-Hermite and Al-Salam–Chihara polynomials [57, 58]. As is already shown in Sec. 12, the representations of the MPA matrices in the form of upper and lower bidiagonal matrices, as they were applied within the MPA in these cases, can be reconstructed as the special cases of the AW (and tridiagonal) algebra. Thus the well-developed representation theory of the Askey–Wilson algebra provides a very elegant and convenient framework which allows for the exact solvability of the open ASEP in the stationary state.

13. DETAILED BALANCE

Detailed balance (DB) means that the probability $P(\{s\})$ for a transition from a configuration $\{s\}$ to another configuration $\{s'\}$ is equal to the probability of the reverse transition. Thus, for the event of hopping between site i and $i+1$ one has

$$P(s_1, \ldots, 0, 1, \ldots, s_L) = q^{-1} P(s_1, \ldots, 1, 0, \ldots, s_L) \tag{162}$$

for an event at the left boundary

$$P(1, s_2, \ldots, s_L) = \frac{\alpha}{\gamma} P(0, s_2, \ldots, s_L) \tag{163}$$

and correspondingly at the right boundary

$$P(s_1, \ldots, s_{L-1}, 1) = \frac{\delta}{\beta} P(s_1, \ldots, s_{L-1}, 0). \tag{164}$$

Starting from a given configuration and using Eqs. (162) and (163), one can always calculate the weights of all configurations by removing particles at the left boundary, and the consistency [91] with Eq. (164) requires

$$\alpha\beta = q^{L-1}\gamma\delta \tag{165}$$

which is the DB condition.

We have shown that the constraint (165) is very natural from the point of view of the tridiagonal algebra appoach. Namely, it defines the finite-dimensional representation of dimension L of the boundary algebra. Within the matrix product approach it has been discussed [15,91,105] that with this constraint the MPA fails to produce the stationary state.

The crucial point in shedding light on this is, in our opinion, to understand the importance of the parameter ζ behind $x_0 + x_1 = 0$ in the MPA quadratic algebra, (Eq. (31)). As we pointed out, this parameter has been fixed to $\zeta = 1$. Then the boundary conditions become

$$(\beta D_1 - \delta D_0)|v\rangle = |v\rangle, \quad \langle w|(\alpha D_0 - \gamma D_1) = \langle w| \tag{166}$$

and with the simple argument given in Sec. IV of [15] it is very easy to see that for

$$\alpha\beta = \gamma\delta \tag{167}$$

the MPA algebra as given in (31) (with $x_0 = -x_1 = 1$) and (166) does not have any representations (if $q \neq 1$). This conclusion is only true if $x_0 = -x_1 = \zeta$, for fixed nonzero ζ. As is readily seen, if the RHS of Eqs. (166) vanish, there exists a solution for the algebraic relations given by $\langle w|D_0|v\rangle \neq 0$ and $\langle w|D_1|v\rangle \neq 0$. From the point of the boundary algebra, this solution corresponds to the finite-dimensional representation of the latter.

As can be readily verified, the above detailed balance relations with P expressed as matrix elements of the type $\langle w|D_{s_1}D_{s_2}\cdots D_{s_L}|v\rangle$ are consistent with the limit $x_0 = -x_1 = 0$ of the MPA defining relations (31) and (166). The presence of the parameter-dependent linear terms in the bulk quadratic algebra is due to the boundary processes and these quadratic-linear algebraic relations provide recursive expressions for matrix elements of the steady weights, the current, the correlation functions. The quadratic algebra induces a reordering property and an element of length L can be brought in a linear combination of elements of length $L-1$ with positive coefficients. Hence one can compute all matrix elements of length L if $\langle w|D_0^k D_1^{L-k}|v\rangle$, $k = 0, 1, \ldots, L$, and all $(L-1)$ elements are known. From the boundary conditions (32) (i.e., with x-dependent RHS of (166)) and after reordering one has

$$\begin{aligned}\beta\langle w|D_0^k D_1^{L-k}|v\rangle - q^{L-k-1}\delta\langle w|D_0^{k+1}D_1^{L-k-1}|v\rangle &= x_0 P_{L-1},\\ -q^k\gamma\langle w|D_0^k D_1^{L-k}|v\rangle + \alpha\langle w|D_0^{k+1}D_1^{L-k-1}|v\rangle &= -x_1 P'_{L-1},\end{aligned}$$

where P_{L-1} and P'_{L-1} are positive linear combinations of matrix elements of length $L-1$. The conclusion is straightforward: If $\alpha\beta - q^{L-1}\gamma\delta \neq 0$, then the recursions are consistent. If

$$\alpha\beta - q^{L-1}\gamma\delta = 0, \tag{168}$$

then in order that a matrix element of length L exists, the zero determinant

$$x_0\alpha P_{L-1} - x_1 q^{L-k-1}\delta P'_{L-1} = 0 \tag{169}$$

implies

$$x_0 = -x_1 = 0. \tag{170}$$

All matrix elements of length $L-k$, with $k = 0, 1, \ldots, L-1$ are nonzero if $\alpha\beta - q^{L-k-1}\gamma\delta = 0$ for some $\alpha, \beta, \gamma, \delta$. This has the consequence that the current

$$J = \zeta \frac{\langle w|(D_0 + D_1)^{i-1}(D_1 D_0 - qD_0 D_1)(D_0 + D_1)^{L-i-1}|v\rangle}{Z_L} \tag{171}$$

vanishes and the probabilities satisfy the detailed balance condition. Thus, with the constraint on the boundary parameters even though the boundary processes are present, they become irrelevant and the nonequilibrium behaviour of the system is no longer maintained. The MPA in the limit $x_0 = x_1 = 0$ produces a stationary state whose probability weights satisfy detailed balance.

One may wonder that the boundary Askey–Wilson algebra has finite-dimensional representations corresponding to the constraint $\alpha\beta - q^{L-1}\gamma\delta = 0$ for nonzero $x_0 = -x_1 = \zeta$ unlike the quadratic MPA algebra for which the constraint is consistent with $x_0 = -x_1 = 0$. The reason for this lies in the fact that the representations of the quadratic algebra are contained among the representations of the tridiagonal algebra whose representation theory is richer (see [89] for details).

The boundary operators of the detailed balance case satisfy an Askey–Wilson algebra with $\eta = \eta^* = 0$ and $\rho = \rho^*$, $\omega \simeq Q$. Without writing the explicit form of the representation, related now to q-Hahn polynomials [92], we note that in the limit $x_0 = -x_1 = 0$ from the boundary equations it follows $b = d$ and $a = c$, with $b = \sqrt{-\beta/\delta}$ and $a = \sqrt{-\gamma/\alpha}$. Inserting it in the condition $abcd = a^2b^2 = q^{1-L}$ we obtain the DB condition. Formally $a^2b^2 = q^{1-L}$ for some parameters $a' = -a^2, b' = -b^2$, with $L = 1$, resembles the limit $q \to 1$ and we recover the detailed balance case at equal density $\rho_{a'} = \rho_{b'}$, with $\rho_{a'} = \dfrac{\alpha}{\alpha+\gamma}$, $\rho_{b'} = \dfrac{\beta}{\beta+\delta}$ [30]. The system will be in a product measure state with uniform density and zero current when $a'b' = 1$, valid for the one-dimensional representation. We can identify the one-dimensional representation of the boundary operators with $\langle w|D_1|v\rangle$ and $\langle w|D_0|v\rangle$, which are nonzero in the limit $x_0 = -x_1 = 0$. A nonvanishing matrix element of length L can be formed as the product of single-site matrix elements $\langle w|D_1|v\rangle^{L-k}$ and $\langle w|D_0|v\rangle^k$. Given the detailed balance condition $\alpha\beta = q^{L-1}\gamma\delta$, we can always find corresponding $\alpha'\beta' = \gamma'\delta'$ which define the one-dimensional representation of the boundary algebra and hence determine the steady state as a Bernoulli product measure at equal density for both reservoirs.

The MPA defining algebraic relations (31) and boundary conditions (32) provides solvable recursions for the stationary state, the current, the correlation functions of an open L-site nonequilibrium system for all values of the parameters in the range $0 \leqslant \alpha, \beta, \gamma, \delta, \leqslant 1$, $\alpha\gamma \neq 0$, $\beta\delta \neq 0$, $0 < q < 1$ and $x_0 = -x_1 = \zeta > 0$. The recursions remain valid with finite-dimensional matrices of dimension L determined by (one of) the conditions $\kappa_+(\alpha,\gamma)\kappa_+(\beta,\delta) = \kappa_+(\alpha,\gamma)\kappa_-(\beta,\delta) = q^{1-L}$, which also define L-dimensional representations of the boundary AW algebra. Exceptional subrange of parameters is given by the constraints $\alpha\beta = q^{L-1}\gamma\delta$ and $\zeta = 0$ when the MPA produces a stationary state satisfying detailed balance. Within the tridiagonal algebra approach the detailed balance constraint arises naturally as the condition for the finite-dimensional representation of the boundary algebra.

14. BOUNDARY ALGEBRA OF THE SYMMETRIC EXCLUSION PROCESS

We consider the symmetric simple exclusion process (SSEP) with most general boundary conditions of incoming and outgoing particle at both ends of the chain. Within the matrix product ansatz the quadratic bulk algebra of the SSEP is the $q = 1$ limit of the deformed quadratic algebra of the ASEP. In Sec. 5 we derived the bulk tridiagonal algebra of the symmetric process and it turned to be the $q = 1$ limit of the bulk tridiagonal algebra of the asymmetric process.

We have obtained [61] the boundary algebra of the symmetric process as the $q = 1$ limit of the tridiagonal boundary algebra of the asymmetric simple exclusion process. For the purpose we explore the natural homomorphism of the tridiagonal algebra (TD) generated by the pair A, A^* and the Askey–Wilson algebra (AW) of the ASEP defined by Eq. (75), namely $\mathrm{TD} \to \mathrm{AW}$. As already mentioned, this is readily verified by taking the commutator with A, respectively A^* of the first line, respectively the second line of Eq. (75), which gives

$$\begin{aligned} &[A,[A,[A,A^*]_q,]_{q^{-1}}]] = \rho[A,A^*], \\ &[A^*,[A^*[A^*,A]_q]_{q^{-1}}] = \rho^*[A^*,A] \end{aligned} \tag{172}$$

with ρ, ρ^* depending on the five parameters of the ASEP, as given by (76). The operators A, A^* were introduced as the shifted boundary operators of the ASEP. The limit $q = 1$ of the tridiagonal algebra (172) provides a way to determine the boundary algebra of the SSEP.

The boundary operators of the SSEP can be represented in the form (i.e., linear combinations in terms of the level zero affine $su(2)$ generators)

$$\begin{aligned} &\beta D_1 - \delta D_0 = -x_1\beta A_+ - x_0\delta A_- - (x_1\beta + x_0\delta)N - x_1\beta - x_0\delta, \\ &\alpha D_0 - \gamma D_1 = x_0\alpha A_+ + x_1\gamma A_- + (x_0\alpha + x_1\gamma)N + x_0\alpha + x_1\gamma, \end{aligned} \tag{173}$$

where $A_{\pm}, N$ are corresponding operators in the limit $q \to 1$ of Eq. (65). We separate the shift parts from the boundary operators. Denoting the corresponding rest operator parts by A and A^* we write the left and right boundary operators in the form

$$\begin{aligned} \beta D_1 - \delta D_0 &= A - x_1\beta - x_0\delta, \\ \alpha D_0 - \gamma D_1 &= A^* + x_0\alpha + x_1\gamma. \end{aligned} \tag{174}$$

Then we have the following:

Proposition VII. The operators A and A^* defined by the corresponding shifts of the boundary operators of the open symmetric exclusion process

$$\begin{aligned} A &= \beta D_1 - \delta D_0 + (x_1\beta + x_0\delta), \\ A^* &= \alpha D_0 - \gamma D_1 - (x_0\alpha + x_1\gamma) \end{aligned} \tag{175}$$

and their commutator

$$[A, A^*] = AA^* - A^*A \tag{176}$$

form a closed linear algebra, the boundary tridiagonal algebra of the SSEP

$$\begin{aligned} [A, [A, [A, A^*]]] &= \rho[A, A^*], \\ [A^*, [A^*, [A^*, A]]] &= \rho^*[A^*, A], \end{aligned} \tag{177}$$

where the structure constants are given by

$$-\rho = x_0 x_1 \beta\delta, \quad -\rho^* = x_0 x_1 \alpha\gamma. \tag{178}$$

This proposition is straightforward to verify by taking the $q \to 1$ limit in the chain of homomorphisms $TD \to AW \to U_q(\hat{s}u(2))$.

As is readily seen from the definition (174), the (shifted) boundary operators of the symmetric exclusion process obeying the algebra (177) form a tridiagonal pair with $\beta = 2, \gamma = \gamma^* = 0$, and ρ, ρ^* given by (178). The tridiagonal boundary algebra of the symmetric process is the limit $q = 1$ of the deformed boundary algebra of the ASEP, as the irreducible modules of the algebra in the symmetric model, i.e., the Wilson polynomials, are the $q = 1$ limit of the Askey–Wilson polynomials. The important properties of the deformed boundary algebra (both TD and AW) remain valid in the proper limit $q \to 1$. Stated more precisely which will correspond to the historical development of these algebras, for generic q the deformed tridiagonal algebra (deformed Golan–Grady relations) is the q-generalization of the Onsager algebra in the equivalent form of the Dolan–Grady relations. Its irreducible modules, i.e., the Askey–Wilson polynomials are q-counterpart of the Wilson polynomials. For applications, however one can even use the limit cases of the Wilson polynomials. These are the continuous Hahn and dual Hahn polynomials (see [92] and [115] for details on these polynomials). In both cases there exist limiting procedures to further obtain the Meixner–Pollaczek polynomials from which the Laguerre polynomials can be obtained. Let $P_n^{\mu}(x,\phi)$ denote the nth Meixner–Pollaczek polynomial

$$P_n^{\mu}(x,\phi) = \frac{(2\mu)_n}{n!}\,\mathrm{e}^{in\phi}{}_2F_1\left(\begin{array}{cc} -n, & \mu+ix \\ & 2\mu \end{array} |1-\mathrm{e}^{-2i\phi}\right). \tag{179}$$

The Laguerre polynomials can be obtained by the substitution $\mu = 1/2(\lambda+1)$, $x \to -1/2\phi^{-1}(x)$ and letting $\phi \to 0$

$$\lim_{\phi\to 0} P_n^{1/2\mu+1/2}\left(-\frac{x}{2\phi};\phi\right) = L_n^{(\lambda)}(x). \tag{180}$$

By definition the Laguerre polynomials have the form

$$L_n^{(\lambda)}(x) = \frac{(\lambda+1)_n}{n!}{}_1F_1\left(\begin{array}{c} -n \\ \lambda+1 \end{array} |x\right) \tag{181}$$

with orthogonality condition

$$\int_0^{\infty} \mathrm{e}^{-x}x^{\lambda}L_m^{\lambda}L_n^{\lambda}\,dx = \frac{\Gamma(n+\lambda+1)}{n!}\delta_{mn}, \quad \lambda > -1 \tag{182}$$

and recurrence relation

$$(n+1)L_{n+1}^{(\lambda)}(x) - (2n+\lambda+1-x)L_n^{(\lambda)}(x) + (n+\lambda)L_{n-1}^{(\lambda)}(x) = 0, \tag{183}$$

where $L_{-1}^{(\lambda)}(x) = 0$ and $L_0^{(\lambda)}(x) = 1$. One can identify

$$\lambda = \frac{\alpha+\beta+\gamma+\delta}{(\alpha+\gamma)(\beta+\delta)} - 1. \tag{184}$$

Denoting

$$l_n(x) = (-1)^n\left(\frac{n!\Gamma(\lambda+1)}{\Gamma(\lambda+n+1)}\right)^{1/2} L_n^{(\lambda)}(x), \tag{185}$$

we rewrite the orthogonality condition in the form

$$1 = \frac{1}{\Gamma(\lambda+1)}\int_0^{\infty} \mathrm{e}^{-x}x^{\lambda}|l(x)\rangle\langle l(x)|\,dx. \tag{186}$$

The vectors $|l(x)\rangle = (l_0(x), l_1(0), \ldots)^t$ and $\langle l(x)| = (l_0(x), l_1(0), \ldots)$ form the basis for the tridiagonal and the diagonal representation of the generators (and for the dual one). As it was proved to be the case for the ASEP in [88], each boundary operator of the symmetric process, together with the transfer matrix operator $D_0 + D_1$, forms an isomorphic AW algebra whose tridiagonal representation follows from the three-term recurrence relation, consistent with the orthogonality condition

$$(D_0 + D_1)|l(x)\rangle = x|l(x)\rangle, \qquad \langle l(x)|(D_0 + D_1) = \langle l(x)|x. \tag{187}$$

These properties can be used to exactly calculate the physical quantities [59] in the stationary state. The results [47] for the partition function Z_l and the current are reconstructed

$$Z_L = \frac{\Gamma(\lambda + L + 1)}{\Gamma(\lambda + 1)}, \quad J = \frac{1}{\lambda + L}. \tag{188}$$

The one-point function [47]

$$\langle \tau_i \rangle = \frac{\alpha}{\alpha + \gamma} - \frac{1}{\lambda + L} \frac{\alpha\beta - \gamma\delta}{(\alpha + \gamma)(\beta + \delta)} \left(\frac{1}{\alpha + \gamma} + i - 1 \right), \quad i = 1, 2, \ldots, L \tag{189}$$

shows that the particle density has a linear profile. Without reference to the AW and TD algebra, eigenvalue equation (187) was used in [59] to calculate the physical quantities of the symmetric exclusion process, as known from the matrix product approach. In our opinion, it is the boundary tridiagonal algebra of the symmetric exclusion process that allows for the exact solvability of the symmetric process in the stationary state and leads to a generalization of the matrix product method.

15. NONLOCAL CONSERVED CHARGES OF THE SYMMETRIC EXCLUSION PROCESS

In the previous section we have shown that the boundary symmetry of the symmetric exclusion process is the tridiagonal algebra (177) with the sequence of scalars

$$\beta = 2, \rho, \rho^*. \tag{190}$$

This algebra is the $q = 1$ limit of the tridiagonal algebra, mapped through the natural homomorphism to the Zhedanov algebra $AW(3)$ defined for $0 < q < 1$. The exact calculation of the physical quantities in the case of the symmetric exclusion process has been obtained in terms of the Laguerre polynomials, by implementing [59] the three-term recurrence relation to define the tridiagonal representation of the transfer matrix $D_0 + D_1$. As we pointed out, the ultimate relation of the exact solution in the stationary state to the Laguerre polynomials was possible due to the $q = 1$ boundary hidden symmetry of the SSEP with general boundary conditions.

The defining relations of the $q = 1$ SSEP boundary algebra, found in the form

$$\begin{gathered} [A, [A, [A, A^*]]] = \rho[A, A^*], \\ [A^*, [A^*, [A^*, A]]] = \rho^*[A^*, A], \end{gathered} \tag{191}$$

are the well-known Dolan–Grady relations for the shifted boundary operators. The importance of the Dolan–Grady [68] relations is that given a self-dual Hamiltonian

$$H = fA + f^* A^*, \tag{192}$$

where f, f^* are some coupling constants, and A, A^* satisfy the relations (191), then one can construct the (infinite) set of conserved commuting charges (see [61] for the case of SSEP)

$$Q_{2n} = f(R_{2n} - \tilde{R}_{2n-2}) + f^*(\tilde{R}_{2n} - R_{2n-2}) \tag{193}$$

in terms of the quantities

$$R_{2n} = -\frac{2}{\rho}[A[A^*, R_{2n-2}]] - \tilde{R}_{2n-2} \tag{194}$$

with

$$R_0 \equiv A, \quad Q_0 \equiv H. \tag{195}$$

The power of this result is that it is an operator statement and does not depend on the dimension of the system, or the nature of space-time manifold, i.e., lattice, continuum or loop space. It is elucidating the hidden symmetries of the system in consideration whose consequence is exact integrability.

In the case of the symmetric exclusion process from the boundary conditions we have

$$\begin{aligned} \langle w|A - (x_1\beta + x_0\delta)|v\rangle &= x_0\langle w|v\rangle, \\ \langle w|A^* + (x_0\alpha + x_1\gamma)|v\rangle &= -x_1\langle w|v. \end{aligned} \tag{196}$$

Hence

$$\langle w|x_0(A^* + (x_0\alpha + x_1\gamma)) + x_1(A - (x_0\beta + x_1\delta))|v\rangle = 0 \tag{197}$$

and we can interpret the quantity

$$x_0(A^* + (x_0\alpha + x_1\gamma)) + x_1(A - (x_0\beta + x_1\delta)) \tag{198}$$

as the Hamiltonian of the symmetric simple exclusion process in the auxiliary space. As we know, the generators of the tridiagonal algebra are determined up to shift transformations. In view of this property, it seems more convenient to consider the shifted Hamiltonian

$$H_s = x_0 A^* + x_1 A \tag{199}$$

which is self-dual, if we define $x_1^* = x_0$. Then we can straightforwardly apply the prescription of Dolan and Grady to obtain the conserved nonlocal charges. Taking into account the shifts we find the result

$$R_0 = A - x_1\beta - x_0\delta, \tag{200}$$

$$\tilde{R}_0 = A^* + x_0\alpha + x_1\gamma, \tag{201}$$

$$R_2 = -\frac{2}{\rho}[A - x_1\beta - x_0\delta, [A^* + x_0\alpha + x_1\gamma, A - x_1\beta - x_0\delta]] - A^* + x_0\alpha + x_1\gamma \quad (202)$$

and so on, according to formula (194). The expressions $A - x_1\beta - x_0\delta$ and $A^* + x_0\alpha + x_1\gamma$ are the right and left boundary operators

$$B^R = \beta D_1 - \delta D_0, \qquad B^L = \alpha D_1 - \gamma D_0, \quad (203)$$

respectively, which acquire a very important physical meaning. The boundary operators satisfying

$$\begin{aligned} &[B^R, [B^R, [B^R, B^L]]] = -x_0 x_1 \beta\delta [B^R, B^L], \\ &[B^L, [B^L, [B^L, B^R]]] = -x_0 x_1 \alpha\gamma [B^L, B^R] \end{aligned} \quad (204)$$

are the nonlocal conserved charges of the symmetric exclusion process with the help of which, as a consequence from the Dolan–Grady relations, the (unknown for the SSEP) set of conserved quantities of the process are constructed [61]. By properly rescaling the operators one achieves equal coefficient factors on the RHS of Eqs. (204). We note that quantum integrals of motion for the XXX Heisenberg infinite chain (known to be related to the symmetric exclusion process) were first obtained in [93]. For the open symmetric simple exclusion process, the existence of nonlocal conserved quantities will result in the exact solvability of the system beyond the stationary state. Thus the boundary hidden tridiagonal symmetry of the symmetric simple exclusion process is the deep algebraic property allowing for the exact steady state solution and resulting in the exact solvability beyond the stationary state.

16. THE ASKEY–WILSON ALGEBRA OF THE TOTALLY ASYMMETRIC EXCLUSION PROCESS

The totally asymmetric exclusion process (TASEP) is obtained from the ASEP by taking the limit $q = 0$ and setting $\gamma = \delta = 0$ in the boundary conditions. The MPA was applied for the first time for the exact solvability of the open two-species TASEP in [16]. It was then generalized to the three-species process on a ring in [25] which was also studied in [20,94] (see [40] and [5] for reviews). The MPA was applied in [32] for the exact solution of the open N-species TASEP. The stationary measure for the multispecies process on a ring was constructed in [95] by using MPA.

To study the boundary properties of the open TASEP we first denote the left and right boundary operators of the open TASEP by

$$B^L = \alpha A^* + \alpha, \quad B^R = \beta A + \beta. \quad (205)$$

The tridiagonal and the AW algebra of the totally asymmetric exclusion process cannot be obtained directly as the limit $q = 0$ of the partially asymmetric process. The procedure is more involved [96]. We derive this algebra [61] from the quadratic algebra of the totally asymmetric process.

We start with the quadratic algebra

$$D_1 D_0 = D_1 + D_0. \tag{206}$$

From this algebra the following relations follow:

$$D_1 D_0 D_1 = D_1^2 + D_0 D_1, \quad D_0 D_1 D_0 = D_0 D_1 + D_0^2 \tag{207}$$

and

$$D_1^2 D_0 = D_1^2 + D_1 + D_0, \quad D_1 D_0^2 = D_1 + D_0 + D_0^2 \tag{208}$$

which can alternatively be written as

$$\begin{aligned} D_1 D_0 D_1 - D_1^2 D_0 &= [D_0, D_1], \\ D_0 D_1 D_0 - D_1 D_0^2 &= [D_0, D_1]. \end{aligned} \tag{209}$$

The first and second lines of the LHS of (209) are, respectively,

$$[D_1 D_0, D_1], \quad [D_0, D_1 D_0]. \tag{210}$$

Hence we have

$$D_1 [D_0, D_1] = [D_0, D_1], \quad [D_0, D_1] D_0 = [D_0, D_1]. \tag{211}$$

Examples of matrices obeying the above relations are given by Eqs. (33) and (36), (38) in [17]. We can shift the operators D_0, D_1 by 1 (or respectively by the constants $c_0 = a$, $c_1 = b$)

$$D_0 \to D_0 + 1, \quad D_1 \to D_1 + 1. \tag{212}$$

Then we can write subsequently

$$D_1 D_0 = Z, \tag{213}$$

where either $Z = 1$ or $Z = ab$. (In the case of the totally asymmetric exclusion process within the matrix product approach $ab = \alpha\beta$, where α and β are the probability rates for the particles to be added and removed at both sides of the linear chain.) Hence

$$D_1 D_0 D_1 = Z D_1, \quad D_0 D_1 D_0 = Z D_0, \tag{214}$$

$$D_1^2 D_0 = Z D_1, \qquad D_1 D_0^2 = Z D_0. \tag{215}$$

Relations (209) become

$$D_1D_0D_1 - D_1^2D_0 = 0,\qquad D_0D_1D_0 - D_1D_0^2 = 0 \tag{216}$$

and consequently

$$D_1[D_0, D_1] = 0, \quad [D_0, D_1]D_0 = 0. \tag{217}$$

We need to emphasize that from Eqs. (216) the relations follow:

$$D_1D_0D_1^2 - D_1^2D_0D_1 = 0,\qquad D_0^2D_1D_0 - D_0D_1D_0^2 = 0, \tag{218}$$

which define the $q = 0$ limit of the q-Serre relations of $U_q(\hat{sl}(2))$, i.e., the $q = 0$ limit of the level zero $U_q(\hat{sl}(2))$ adjoint representation. This is consistent with the definition of the Askey–Wilson algebra, which is such that it yields a deformation of the level zero $U_q(\hat{sl}(2))$ q-Serre relations (for details, see [78]). However, we have now the additional relation

$$([D_0, D_1])^2 = Z[D_0, D_1]. \tag{219}$$

It is important to emphasize that the matrices obeying (207)–(211) and (214)–(217) are upper bidiagonal and lower bidiagonal. In our case, the analogue of the diagonal and tridiagonal matrices in the $q = 0$ limit of the basic representation (of the AW algebra) are given by $[D_0, D_1]$ and $D_1 + D_0$, respectively.

Proposition VIII. The $q = 0$ AW algebra depending on only two constants a, b, (i.e., $e_1 = a + b, e_2 = ab$) in the basic representation is defined by

$$D_1D_0 = e_2, \quad D_1[D_0, D_1] = 0, \quad [D_0, D_1]D_0 = 0, \tag{220}$$

$$[D_0, D_1](D_0 + D_1)[D_0, D_1] = 0, \quad ([D_0, D_1])^2 = e_2[D_0, D_1]. \tag{221}$$

The matrix $D_0 + D_1$ plays the role of the tridiagonal D; and $[D_0, D_1]$, of the diagonal D^* in the basic representation. Thus we have

$$[D_0, D_1](D_0 + D_1)^2[D_0, D_1] = e_2^2[D_0, D_1], \tag{222}$$

$$[D_0, D_1](D_0 + D_1)[D_0, D_1] = 0, \tag{223}$$

$$([D_0, D_1])^2 = e_2[D_0, D_1]. \tag{224}$$

We can now shift the matrix $D \to D + a + b$ to obtain a tridiagonal matrix with entries on the main diagonal, too.

Definition I. The $q = 0$ limit of the AW algebra depending on only two constants a, b, where

$$e_1 = a + b, \quad e_2 = ab, \tag{225}$$

is generated by D and D^* with defining relations in the basic representation

$$D^*DD^* = e_1e_2D^*, \qquad (D^*)^2 = e_2D^*. \tag{226}$$

One can alternatively consider defining relations for $q = 0$ limit of the AW algebra in a representation associated with the $q = 0$ limit of the level zero adjoint $U_q\hat{sl}(2))$ (Eq. (218)).

Definition II. The $q = 0$ limit of the AW algebra, generated by upper diagonal and lower diagonal matrices D_1 and D_0 and depending on only two constants a, b, is defined by

$$D_1D_0D_1 = abD_1, \quad D_0D_1D_0 = abD_0. \tag{227}$$

The boundary tridiagonal AW algebra of the totally asymmetric exclusion process corresponds to the values $a = \alpha, b = \beta$, while the bulk AW algebra is obtained for $a = b = 1$.

We note that the two representation-dependent definitions of the algebras considered above have in common the first relation in formula (226) and both relations in (227), which can be unified as

$$ABA = \tilde{c}(a,b)A, \tag{228}$$

where A, B are the generators of these algebras and $\tilde{c}(a,b)$ is a constant depending on the parameters a, b. We can multiply Eq. (228) by B subsequently from the left and from the right to obtain

$$ABAB - BABA = 0 \tag{229}$$

if $\tilde{c}(a,b) = 0$ or

$$ABAB - BABA = \tilde{c}(a,b)(AB - BA) \tag{230}$$

if $\tilde{c}(a,b) \neq 0$. We can consider Eq. (230) as an alternative definition of the $q = 0$ limit of the Askey–Wilson algebra, with structure constant $\tilde{c}(a,b)$. These relations can be very useful for applications. Namely, with the additional condition $A^2 = A$, which is a Hecke-type relation, Eqs. (229) and (230) have the form of a reflection equation and a modified boundary Yang–Baxter equation, respectively. The interpretation of Eqs. (229) and (230) as reflection equations should be associated with the proper R-matrix operator depending on a parameter $t \neq q$.

Thus, for the totally asymmetric exclusion process we obtain [61] the bulk algebra

$$D_1D_0D_1D_0 - D_0D_1D_0D_1 = D_1D_0 - D_0D_1 \tag{231}$$

and the boundary algebra generated by the right B^R and left B^L boundary operators

$$B^R = \beta D_1, \qquad B^L = \alpha D_0 \tag{232}$$

subject to the relation

$$B^R B^L B^R B^L - B^L B^R B^L B^R = \alpha\beta(B^R B^L - B^L B^R). \qquad (233)$$

It is worth to study the connection of the exact solvability of the totally asymmetric exclusion process in the stationary state to the integrability properties based on the boundary Yang–Baxter equation.

17. EXACT SOLUTION OF THE ASYMMETRIC EXCLUSION PROCESS FROM BOUNDARY ALGEBRA

We have pointed out that the boundary operators of the open ASEP are interpreted as the nonlocal conserved charges of the asymmetric diffusive system. The boundary Askey–Wilson algebra reveals deep algebraic properties of the stochastic system and allows for the exact solvability in the stationary state. The AW algebra operator-valued solution to the boundary YB equation puts into perspective to exactly describe its dynamical behaviour. The generalization of the Dolan–Grady rule to construct the set of conserved quantities is not so easy to extend to the deformed relations. Therefore we are going to construct the spectrum of the diffusion system exploring the representation theory of the Askey–Wilson and tridiagonal boundary algebras.

We consider a diagonalization procedure in the auxiliary space, which is the representation space of the boundary algebra of the process. The boundary operators of the ASEP are shifted generators of the generalized Onsager algebra. In the basic representation one of the generators is identified with the second order difference Askey–Wilson operator which is exactly solvable. We implement the algebraic Bethe ansatz based on the Bethe equation for the zeros of the Askey–Wilson polynomials. We find a complete set of 2^L eigenvectors with distinct eigenvalues including the unique ground state of eigenvalue zero.

Exact results for the dynamics of the ASEP have been derived using Bethe ansatz (BA). The power law scaling behaviour $L^{3/2}$ for the relaxation time to the steady state of the process on a lattice of L sites was first found by Dhar [97]. The spectral gap for the Markov evolution operator of the TASEP was calculated from the mapping to the Heisenberg spin chain for the half-filling case [24, 98, 99] and for arbitrary density [100]. The relation between the MPA and BA is in a process of study [2,21,41]. It has been proved that the Matrix Product (MP) representation following from the quadratic algebra for the TASEP on a ring can be deduced [101] by applying algebraic Bethe ansatz [100]. The Bethe ansatz solution for the open ASEP with general boundary conditions [102] was recently found as a consequence of the mapping to the integrable XXZ quantum spin chain. It was inferred from the integrability condition of Nepomechie [103] and Cao et al. [104] for the XXZ chain which in terms of the ASEP notations reads (k is an integer such that $|k| \leqslant L/2$):

$$(q^{L+2k} - 1)(\alpha\beta - q^{L-2k-2}\gamma\delta) = 0. \tag{234}$$

This ASEP BA solution is based on the first factor in (234) and is valid for a chain of *even* number of sites only. A detailed study of the exact spectral gap for the symmetric, the totally and partially asymmetric processes on a chain of even sites was presented in [105, 106].

In its turn, the algebraic tridiagonal approach based on the q-deformed Onzager algebra was applied in [84] for the exact spectrum of the XXZ chain. However, we specially stress the difference in the boundary symmetry algebras of the XXZ and the ASEP as an important point in the study of the two models. The boundary operators of the ASEP are shifted generators of the AW (and TD) algebra in the infinite-dimensional representation, where A^* is the second order difference operator for the AW polynomials and A is multiplication by x. In this representation all the structure constants in (76) and (77) are nonzero with no relations among them. The AW algebra of the XXZ spin chain, in [82], is a particular case of the ASEP boundary AW algebra due to $\rho = \rho^*$, $\eta = \eta^* = 0$. The TD algebra as a coideal subalgebra of the $U_q(\hat{s}l(2))$, explored for the exact spectrum of the XXZ chain with general boundary terms in [107] has equal structure constants $\rho = \rho^*$, depending on the boundary parameters at the left end of the chain. The ASEP and the XXZ spin chains are only formally equivalent through a similarity transformation but they describe different physics. A relation among the structure constants is unacceptable for a model of nonequilibrium physics because it will restrict the physics of the system.

The important distinguishing features due to the ASEP stochastic nature (for details see [3]) point out to an independent study of the stochastic process.

The transition rate matrix Γ which gives the time evolution of a stochastic process in the configuration space is an intensity matrix (i.e., it columns sum up to zero). Consequently Γ has a zero eigenvalue with a trivial left eigenvector and a nontrivial right eigenvector which gives the probabilities P_s in the stationary state [108]

$$\begin{aligned} &\langle 0|\Gamma = 0, \qquad \langle 0| = (1, 1, \ldots, 1), \\ &\Gamma|0\rangle = 0, \qquad |0\rangle = \sum_s P_s|s\rangle, \quad P_s = \lim_{t\to\infty} P_s(t), \end{aligned} \tag{235}$$

where $P_s(t)$ is the unnormalized probability for the system to be in a configuration s at time t.

The master equation can be formally solved $|P(t)\rangle = \exp(\Gamma t)|P(0)\rangle$. As noted in [3], if one can diagonalize the transition rate matrix, then one can obtain all probabilities at all times. Γ is non-Hermitian, in general, and has different left and right eigenvectors. As a stochastic matrix, it has a trivial «bra» eigenvector $\langle 0|\Gamma = 0$ and a nontrivial one which is the stationary state. One has $\Gamma|\psi_\lambda\rangle = \lambda|\psi_\lambda\rangle$, $\langle\phi_\mu|\Gamma = \mu\langle\phi_\mu|$, with the bi-orthogonality condition $\langle\phi_\mu|\psi_\lambda\rangle = \delta_{\mu,\lambda}$. Assuming that they form a complete system $1 = \sum\limits_\lambda |\psi_\lambda\rangle\langle\phi_\lambda|$, then $\Gamma = \sum\limits_\lambda \lambda|\psi_\lambda\rangle\langle\phi_\lambda|$. Hence $|P(t)\rangle = \exp(\lambda t)|\psi_\lambda\rangle\langle\phi_\lambda|P(0)\rangle$. Note that the transition matrix is a positive Markov matrix with real eigenvalues or, if complex, they appear in conjugate pairs. According to Perron–Frobenius theorem it has one maximum eigenvalue zero and negative real parts of all the eigenvalues.

The matrix product approach translates the steady state solution of the ASEP to the auxiliary Hilbert space where the right boundary operator has a right zero eigenvector and the left boundary operator has a zero eigenvector in the dual space. These operators acquire the interpretation of conserved nonlocal charges since they commute with the $U_q(su(2))$ Hamiltonian. They can be implemented for constructing a complete set of eigenstates forming an irreducible representation of the tridiagonal algebra in the auxiliary space of polynomials of given degree as a basis to obtain the dynamics of the process.

Such a representation is finite-dimensional (L-site chain). Recall that a finite L-dimensional representation of the Askey–Wilson algebra, the linear covariance algebra for $su_q(2)$, is defined by the conditions (in terms of the ASEP parameters)

$$q^{L+2\kappa} - 1 = 0, \qquad \alpha\beta = q^{L-1}\gamma\delta. \tag{236}$$

The second condition restricts the physics of the systems and, in particular, the nonequilibrium behavior (as shown in Sec. 13, it correspondes to detailed balance). The first condition in (236) is the condition for the finite-dimensional spin $j \equiv |\kappa| = L/2$ representation of $U_q(su(2))$ (according to Eq. (70)). It is known [109] that a finite-dimensional representation of $U_q(su(2))$ can be realized in a linear space of polynomials $F(z)$ of fixed degree defined by the first condition in (236). Expressed in the weight basis the generators have the form

$$\begin{aligned} q^{\pm N/2}F(z) &= q^{\mp j/2}F(q^{\pm 1/2}z), \\ A_+F(z) &= -\frac{z}{q^{1/2}-q^{-1/2}}(q^{-j}F(q^{1/2}z) - q^jF(q^{-1/2}z)), \\ A_-F(z) &= \frac{z}{q^{1/2}-q^{-1/2}}(F(q^{1/2}z) - F(q^{-1/2}z)). \end{aligned}$$

(The lowest weight is $F_0 = 1$ and the highest weight is $F_{2j} = z^{2j}$ with j integer or half integer.) This finite-dimensional representation of $U_q(su(2))$ in the space of polynomials of fixed degree yields an *infinite-dimensional representation* of the Askey–Wilson algebra. We cannot use it to obtain a second order Askey–Wilson q-difference operator (to be identified with the diagonal A^* or A) acting on polynomials of fixed degree. To obtain the AW second order q-difference operator in the space of polynomials of a given degree directly as a bilinear form, one uses a modified type of $U_q(su(2))$ algebra [110], which is a degeneration of the Sklyanin algebra [109]. It has been known [110] for quite a long time that the standard quantum algebra $U_q(sl(2))$ can be obtained as a contraction of a degenerate Sklyanin algebra [109], and the diagonalization problem for a general quadratic form in the generators (commonly interpreted as the Hamiltonian of a proper physical system) is equivalent to the difference equation for the AW polynomials [111]. The identification of the generator A^* with the second order difference operator for the AW polynomials in the basic representation is a strong motivation to study the AW algebra related spectral problem.

17.1. The Zeros of the AW Polynomials and Truncation of the Three-Term Recurrence Relation. We emphasize that the basic representation of the AW algebra is the irreducible infinite-dimensional representation of the tridiagonal algebra with two generators A, A^* in the space of symmetric Laurent polyomials [66, 89]. We will obtain the corresponding finite-dimensional representation of the TD algebra directly from the basic representation and consider a related spectral problem in a space of polynomials of a given degree [111]. Recall that for the operators A, A^* in the basic representation of the AW algebra we have

$$Ap_n[y] = (y + y^{-1})p_n[y], \quad A^*p_n[y] = \mathcal{D}p_n[y], \tag{237}$$

and for the second order q-difference operator $\mathcal{D}$ we have

$$\begin{aligned}\mathcal{D}p_n[y] = (1 + abcdq^{-1})p_n[y] + &\\ + \frac{(1-ay)(1-by)(1-cy)(1-dy)}{(1-y^2)(1-qy^2)}(p_n[qy] - p_n[y]) + &\\ + \frac{(a-y)(b-y)(c-y)(d-y)}{(1-y^2)(q-y^2)}(p_n[q^{-1}y] - p_n[y])&\end{aligned} \tag{238}$$

with $\mathcal{D}(1) = 1 + abcdq^{-1}$. The eigenvalue equation for the joint eigenfunctions p_n reads

$$\mathcal{D}p_n = \lambda_n^* p_n, \quad \lambda_n^* = q^{-n} + abcdq^{n-1}, \tag{239}$$

and the operator A^* is represented by an infinite-dimensional matrix $\mathrm{diag}\,(\lambda_0^*, \lambda_1^*, \lambda_2^*, \ldots)$. We denote

$$\varphi(y) = \frac{\prod_{\nu=1}^{4}(1 - w_\nu y)}{(1-y^2)(1-qy^2)}, \quad w_1 = a, \quad w_2 = b, \quad w_3 = c, \quad w_4 = d. \tag{240}$$

Following [111] we consider the eigenvalue equation (239) for a polynomial of a given finite degree n

$$\varphi(y)(p_n(qy) - p_n(y)) + \varphi(y^{-1})(p_n(q^{-1}y) - p_n(y)) = \\ = (q^{-n} - 1)(1 - q^{n-1}abcd)p_n(y) \quad (241)$$

and use the procedure of algebraic Bethe ansatz. Expanding the functions p_n as a product of its zeros

$$p_n(y) = \prod_{m=1}^{n} (y - y_m)(y - y_m^{-1}), \quad (242)$$

we plug it in Eq. (242) dividing thereafter both sides by $p_n(y)$. As commented in [111], one needs to cancel the singularities which at LHS are located at $y = 0, \infty, y_m$. The singular part at $y = 0$ vanishes identically. (It is assumed that $y_m \neq 0$.) The second order AW difference operator is exactly solvable [112] and the LHS is regular at ∞ from the beginning so that no restriction for the degree of the polynomial follows. Annihilation of poles at $y = y_m$ gives the Bethe-ansatz equation [111] for the zeros of the Askey–Wilson polynomials

$$\prod_{\nu=1}^{4} \frac{y_k - w_\nu}{w_\nu y_k - 1} = \prod_{l=1, l\neq k}^{L} \frac{(qy_k - y_l)(qy_k y_l - 1)}{(y_k - qy_l)(y_k y_l - q)}. \quad (243)$$

These equations are valid for any $L < 2j+1$, so that for any L there is exactly one polynomial (242). This means that for each $L(< 2j+1)$ the Bethe equations have exactly one solution for the set $y_k, k = 1, \ldots, L$. It is not obvious that (243) has a unique solution. For comments and details on the proof we refer to [111, 113].

We will use the unique solution for the AW zeros Bethe-ansatz equation to construct a finite-dimensional representation of the TD algebra in the space of Laurent polynomials of a given degree.

The operator $Ap_n = xp_n$ is represented by a tridiagonal matrix whose matrix elements are obtained from the three-term recurrence relation for the Askey–Wilson polynomials

$$xp_n = b_n p_{n+1} + a_n p_n + c_n p_{n-1}, \quad p_{-1} = 0. \quad (244)$$

We need find a way to terminate this sequence at p_L without using conditions, such that the parameters a, b, c, d become q^L because this imposes a restriction on the physics of the system.

Proposition IX. For any finite $n = L$, the tridiagonal algebra has an irreducible finite-dimensional representation in the space of Laurent polynomials of degree $L - 1$ with a basis, the discrete set of AW polynomials.

In the following we present our argumentation for the proof of this proposition.

The theory of the orthogonal polynomials, and in particular the Askey–Wilson polynomials, is based on the three-term recurrence relation, Eq. (244) with the initial conditions $p_0(x) = 1$, $p_{-1}(x) = 0$. There is a characterization theorem (see [114, 115]) concerning the orthogonality of the polynomials with respect to some measure. Namely, when the matrix elements are real, then the measure can be chosen real-valued and nondecreasing and the integration in the orthogonality condition is the real line (as it is the case with Askey–Wilson polynomials with real parameters, or if complex, in conjugate pairs). Then all zeros $x_s, s = 0, 1, \dots, L-1$ of any polynomial are real and simple [90]. Hence, these zeros can be used to construct a discrete orthogonality relation for polynomials of degree lower than L. In the finite discrete case the three-term recurrence relation is a discrete analogue of the Sturm–Liouville two-point boundary value problem with boundary conditions $p_{-1}(x) = 0, p_L(x) = 0$. If all the zeros $x_0, \dots, x_{L-1}$ are real and distinct (see a theorem by Atkinson [116] for a complete proof), then the orthogonality condition can be written in the from

$$\sum_{s=0}^{L-1} P_m(x_s)P_n(x_s)w_s = h_n\delta_{mn}, \qquad m, n = 0, 1, \dots, L-1, \tag{245}$$

and the weight function is

$$w_s = \frac{h_{L-1}}{p_{L-1}(x_s)p'_L(x_s)}, \tag{246}$$

where the prime indicates the first derivative. The theorem was proved in [116] for real a_n and positive b_n, c_n.

As mentioned above, to terminate the three-term recurrence relation at any finite $(n+1) \equiv L$ for a discrete set of AW polynomials ($p[y] = p[y^{-1}]$) due to

$$p_L[y] = 0 \tag{247}$$

we have to find a way to set $b_n = 0$ in the matrix representing the operator A without imposing a restrictive conditions on the model parameters. We note that by directly setting one of the factors $(1 - abq^n)$, $(1 - adq^n)$ or $(1 - abcdq^{n-1})$ in the numerator of b_n in (112) to be equal to zero, we restrict the physics of the system. In particular, $1 - abcdq^{n-1} = 0$ yields the second factor in the BA condition [102] for the XXZ spin chain (with the proper identification of the AW parameters a, b, c, d with the parameters of the boundary terms in (92)). For the ASEP we recall the presence of a parameter $\zeta \equiv x_0$ in the stationary state which is associated with an Abelian symmetry of the bulk operators D_0, D_1. Making use of this parameter we can obtain a discrete set of AW polynomials in the following steps:

1. We first set $p_L = 0$ obtaining the Bethe-ansatz equation (243) for the L zeros $y_0, y_1, \dots, y_{L-1}$.

2. Then we rescale

$$a \to \zeta a, \quad b \to \zeta b, \quad c \to \zeta c, \quad d \to \zeta d. \tag{248}$$

With $|\zeta| \leqslant 1$ we have a representation in terms of a', b', c', d', which has no effect on the Bethe equations and will not change the identification of the parameters a, b, c, d with the boundary probability rates. In the numerator of the matrix elements b_n we can set to zero any factor, e.g., $((1 - \zeta^2 abq^n) = 0$. The condition to terminate the AW algebra ladder representation due to $b_n = 0$ becomes

$$\zeta^2 abq^{L-1} = 1. \tag{249}$$

We thus obtain a discrete set of AW polynomials

$$p_n(x_k, a, b, c, d|q), \quad n = 0, 1, \dots, L-1 \tag{250}$$

with parameters rescaled according to (248) and such that

$$\sum_{k=0}^{L-1} w_k \, p_n(x_k) p_m(x_k) = 0 \tag{251}$$

for distinct n, m. Then one has

$$xp_{L-1}(x) = a_{L-1}p_{L-1}(x) + c_{L-1}p_{L-2}(x), \qquad \text{if} \quad x = x_k. \tag{252}$$

For general x, the relation

$$xp_{L-1}(x) - (a_{L-1}p_{L-1}(x) + c_{L-1}p_{L-2}(x)) \tag{253}$$

defines a polynomial

$$p_L(x) = \text{const} \prod_{k=0}^{L-1} (x - x_k). \tag{254}$$

Tridiagonal Pair Representation. The condition (249) determines an irreducible finite-dimensional representation, denoted W, of the tridiagonal algebra for $x = x_k$, with discrete basis obeying (245). The matrices, representing A, A^*, in the tridiagonal, diagonal representation are finite $L^2 \times L^2$ square matrices. They are block-tridiagonal and block-diagonal, respectively, where each block is an $L \times L$ square matrix. The representation W of the tridiagonal algebra in the space with basis, the discrete set of AW polynomials (251), is such, that the spectrum of the diagonal operator A^* is degenerate. Each eigenvalue λ^* has an eigenspace $p_n(x_k)$, with $k = 0, \ldots, L-1$ of dimension L.

Leonard Pair Representation. For each fixed x_k, there is a finite-dimensional subrepresentation V, with basis $p_n(x_k)$, $n = 0, \ldots, L-1$, which is not an invariant subspace of W. The vectors $|\nu_n\rangle = |p_n\rangle$ form an orthogonal basis for this representation $\langle \nu_m | \nu_n \rangle = \delta_{mn}$. The tridiagonal matrix representing A is irreducible tridiagonal, while the diagonal is such that each eigenvalue λ_n has dimension one. We have

$$\begin{aligned} [A, [A, [A, A^*]_q]_{q^{-1}}] &= \rho[A, A^*], \\ [A^*, [A^*, [A^*, A]_q]_{q^{-1}}] &= \rho^*[A^*, A] \end{aligned} \tag{255}$$

with ρ, ρ^* given by (76).

We want to relate this representation to a highest-weight irreducible finite-dimensional representation of the $U_q(\hat{s}u(2))$ with deformation parameter q. (Note the change of the deformation parameter from $q^{1/2}$ to q. We have defined the Askey–Wilson algebra as a coideal subalgebra of $U_{q^{1/2}}\hat{s}u(2)$ [78], however we relate its finite-dimensional representation with a discrete set of AW polynomials to $U_q(\hat{s}u(2))$.)

To proceed further we first need some definitions from [117] (see also [118–120]). Let $V_n(a) \equiv V_n(a; \nu_0, \ldots, \nu_n)$ $(a \subset \mathbf{C}^{\mathbf{x}})$ denote the finite-dimensional representation (dimension $(n+1)$) of $U_q(\hat{s}u(2))$ with basis $\nu_i, i = 0, \ldots, n$, and highest-weight vector ν_0 (and $V_n(qa; \nu_0, \ldots, \nu_n)$ denotes its dual). $V_n(a)$ may be regarded as representations of $U_q(L(su(2))$, where $U_q(L(su(2))$ denotes the quotient of $U_q(\hat{s}u(2))$ by the two-sided ideal generated by the central element C. $U_q(L(su(2))$ is a Hopf algebra which is a deformation of the universal enveloping algebra of the loop algebra $L(su(2)) = su(2)[s, s^{-1}]$ (with $a = s$ for $V_n(a)$). There is a unique polynomial with constant coefficient 1 associated to any finite-dimensional highest-weight representation $V_n(a, \nu_i)$ of $U_q(\hat{s}u(2))$, given by

$$P(u) = (1 - q^{n-1}au)(1 - q^{n-2}au) \cdots (1 - q^{-n+1}au). \tag{256}$$

The tensor product of an r-tuple of $V_1(a_r)$ irreducible highest-weight finite-dimensional representations

$$V = V_1(a_1) \otimes V_1(a_2) \otimes \cdots V_1(a_r) \tag{257}$$

has a highest-weight vector which is the tensor product of the highest vectors in each factor. The associated polynomial is

$$\prod_{i=1}^{r}(1 - a_i u) = \prod_{i=1}^{r}(1 - \xi_i^{-1} u) = P(u), \tag{258}$$

where $\xi_1, \xi_2, \ldots, \xi_r$ are the roots of the polynomial $P(u)$. A nonempty finite set of (complex) numbers is said to be a q string if it is of the form $\xi, q^{-2}\xi, q^{-4}\xi, \ldots, q^{-2r}\xi, r \in Z_{>0}$. The roots of the polynomial associated to an evaluation representation $V_n(a)$ are of the form $\xi, q^{-1}\xi, \ldots, q^{-(n-1)}\xi$ with $\xi = q^{n-1}a$ and form a q string $S_n(a)$ with $r = n-1$. Two q strings $S_n(a)$ and $S_n(b)$ are said not to be in general position iff

$$\frac{b}{a} = q^{\pm(m+n-2p-2)} \tag{259}$$

for some $0 < p \leqslant \min\{m, n\}$. A theorem by Chari and Pressley [117] states that a tensor product $V_{n_1} \otimes \cdots \otimes V_{n_r}$ is irreducible iff the q strings $S_{n_1}(a_1) \cdots S_{n_r}(a_{n_r})$ are in general position.

17.2. Tensor Product Representation of the ASEP Tridiagonal Algebra. For each fixed zero $y_i, i = 1, 2, 3 \ldots, L$, from the zeros of the solution to the Bethe-ansatz equation we identify the set $p_0(x_i), p_1(x_i)$ (where $x_i = y_i + y_i^{-1}$) with the irreducible highest-weight evaluation module $V_1(\nu_0(x_i), \nu_1(x_i))$ defined by [117]

$$\begin{gathered} q^H p_0 = q p_0, \quad q^H p_1 = q^{-1} p_1, \\ E^+ p_0 = 0, \quad E^+ p_1 = p_0, \quad E^- p_0 = p_1, \quad E^- p_1 = 0. \end{gathered} \tag{260}$$

The polynomial associated with the representation $V_1(p_0(x_i), p_1(x_i))$ is

$$P(u) = (1 - x_i u). \tag{261}$$

Proposition X. The tensor product of L irreducible highest-weight evaluation modules $V_1(x_i)$, each of dimension 2

$$V_1(x_1) \otimes V_1(x_2) \otimes \ldots \otimes V_1(x_L) \tag{262}$$

is a finite-dimensional irreducible highest-weight $U_q(\hat{s}u(2))$ evaluation module of dimension 2^L.

The proof is straightforward, following the theorem by Chari and Pressley [117] which states that the tensor product is irreducible iff the q strings $S_1(x_1), \ldots, S_1(x_L)$ are in general position. For the tensor product of any two evaluation representations $V_1(x_i) \otimes V_1(x_j)$, if we assume that the two q strings $S_1(x_i)$ and $S_1(x_j)$ are not in general position, we will have

$$\frac{x_i}{x_j} = q^2. \tag{263}$$

Hence it follows

$$y_i + y_i^{-1} = q^2(y_j + y_j^{-1}) \tag{264}$$

which contradicts the Bethe-ansatz equation for the zeros y_j of the AW polynomials. Therefore the tensor product of any two irreducible highest-weight evaluation modules $V_1(x_i)$ is irreducible. Since any two q strings $S_1(x_i)$ and $S_1(x_j)$ for $i \neq j$ are in general position, the proof can be generalized to the set of the q strings $S_1(x_1), \dots, S_1(x_L)$, which means that the tensor product (264) of L irreducible highest-weight evaluation modules, each of dimension 2, is a finite-dimensional irreducible highest-weight $U_q\hat{s}u(2)$ evaluation module of dimension 2^L. The tensor product contains up to a scalar factor, a unique highest-weight vector Ω of weight L,

$$K \equiv q^H \Omega = q^L \Omega. \tag{265}$$

The subrepresentation generated by Ω is the $L+1$-dimensional irreducible representation of the $U_q(su(2))$ subalgebra of $U_q(L(su(2)))$. The polynomial $P(u)$, associated with the subrepresentation generated by Ω, with $x_k \to x_k^{-1}$ and $u \equiv x$, coincides with the polynomial (254), for the choice $\mathrm{const} = (-1)^L \prod_{k=0}^{L-1} x_k^{-1}$. (To simplify notations in what follows we keep x_k to denote $x_k = (y_k + y_k^{-1})^{-1}$.)

Proposition XI. The module (257) forms a basis for the irreducible finite-dimensional tensor product representation of the tridiagonal algebra.

We recall, that for suitable choice of the structure constants in the AW algebra with two generators A, A^*, the Askey–Wilson polynomial $p_n(x)$ is kernel of the intertwining operator between a representation by a difference operators on the space of polynomials in x and a representation by tridiagonal operators on the space of infinite sequences $(c_n)_{n=1,2,\dots}$ [62, 89]. In the first representation A is multiplication by x and A^* is the second order q-difference operator for which the Askey–Wilson polynomials are eigenfunctions with explicit eigenvalues λ_n^*. In the second representation A^* is the diagonal operator with diagonal elements λ_n^* and A is the tridiagonal operator corresponding to the three-term recurrence relation for the Askey–Wilson polynomials.

Let us consider the action of A, A^* on the module $V_1(x_k)$. In the representation where the operator A^* is diagonal (and A is tridiagonal) we have $A^* p_n(x_k) = \lambda_n^* p_n(x_k)$. Hence

$$\begin{aligned} A^* p_0(x_k) &= (1 + q^{-1}abcd)\, p_0(x_k), \\ A^* p_1(x_k) &= (q^{-1} + abcd)\, p_1(x_k). \end{aligned} \tag{266}$$

In the (dual) representation of the AW algebra where A^* is the AW second order difference operator yielding the Bethe-ansatz equation for the zeros, the operator A is multiplication by x. We have

$$A p_n(x_k) = x_k p_n(x_k) \tag{267}$$

which means that each of the highest-weight irreducible modules $V_1(x_k)$ is an eigenstate of the operator A

$$A V_1(x_k) = x_k V_1(x_k). \tag{268}$$

On the tensor product of two irreducible modules $V_1(x_i) \otimes V_1(x_k)$ the operator A will act by means of the coproduct

$$\Delta(A) = A_{i_1} \otimes I + I \otimes A_{k_2} + A_{i_1} \otimes A_{k_2}. \tag{269}$$

Iterating the coproduct we obtain the action of the operator A on the tensor product. (We denote the n-fold iteration by $\Delta^{(1)} = \Delta, \Delta^{(n)} = (\Delta \otimes I^{(n-2)})\Delta^{(n-1)}$ with $I^{(n-2)} = I \otimes \ldots \otimes I(n-2$ times).) To make the formulae more transparent we denote the first two terms in (269) by $\Delta_P(A)$. We have

$$\Delta_P^{(n)}(A) = \sum_{k=1}^{n} I^{(k-1)} \otimes A_{i_k} \otimes I^{(n-k)}. \tag{270}$$

The complete set of eigenvalues of A on the tensor product will be given by the action of the

$$\Delta^{(n)} A = \Delta_P^{(n)} A + A \otimes \Delta_P^{(n-1)} A + \Delta_P^{(n-1)} A \otimes A + \ldots + A^{(k)} \otimes \Delta_P^{(n-k)} A^{(k)} + \\ + \Delta_P^{(n-k)} A^{(k)} \otimes A^{(k)} + A^{(n)}, \tag{271}$$

where $k = 1, \ldots, n-1$ and $A^{(l)} \equiv A \otimes A \otimes \ldots \otimes A$ (l times), for $l = k$ or $l = n$.

Proposition XII. The operators

$$A^{*(L)} = A^* \otimes I^{L-1} \tag{272}$$

and the operator $A^{(L)} = \Delta^{(L)} A$ as given in (271) for $n = L$ satisfy

$$[A^{*(L)}, [A^{*(L)}, [A^{*(L)}, A^{(L)}]_q]_{q^{-1}}] = \rho^* [A^{*(L)}, A^{(L)}]. \tag{273}$$

The proposition can be verified by direct computation. We observe the peculiarity of the Dolan–Grady algebra for the Ising model [68], namely in the deformed case only one of the relations is satisfied in a given representation. The other one is constructed by using duality properties.

Proposition XIII. The operators

$$A^{(L)} = I^{L-1} \otimes A \tag{274}$$

and the operator

$$A^{(L)} = \Delta_P^{(L)} A + A \otimes \Delta_P^{(n-1)} A + \Delta_P^{(l-1)} A \otimes A, \ldots, A^{(k)} \otimes \Delta_P^{(L-k)} A^{(k)} + \\ + \Delta_P^{(L-k)} A^{(k)} \otimes A^{(k)} + A^{(L)}, \tag{275}$$

where $k = 1, \ldots, L-1$ and $A^{(l)} \equiv A \otimes A \otimes \ldots \otimes A$ (l times), for $l = k$ or $l = L$, satisfy

$$[A^{(L)}, [A^{(L)}, [A^{(L)}, A^{*(L)}]_q]_{q^{-1}}] = \rho [A^{(L)}, A^{*(L)}]. \tag{276}$$

17.3. The Eigenvalues of the Generators of the TD Algebra Tensor-Product Representation. To obtain a compete set of 2^L eigenvectors with 2^L eigenvalues for any finite L, we associate with each lattice site i a basis vector $p_0(x_i)$ if a site is empty (occupation number $s_i = 0$) or $p_1(x_i)$ if there is a particle on the site (occupation number $s_i = 1$). Let $\psi(x_1, x_2, \ldots, x_L)$ denote the state of the ASEP on the lattice of L sites depending on the set $s_{i_1}, s_{i_2}, \ldots, s_{i_L}$ and belonging to the $U_q(\hat{s}u(2))$ irreducible tensor product representation

$$\psi(x_1, x_2, \ldots, x_L) = V_1(x_1) \otimes \ldots \otimes V_1(x_L) \tag{277}$$

with the highest-weight vector generating the $2j = L$ subrepresentation.

By definition, the highest-weight vector of the tensor product obeys $E^+\Omega = 0$, with $\Omega = p_0(x_1)p_0(x_2)\cdots p_0(x_L)$. The discrete set of AW polynomials satisfy the three-term recurrence relation (244) with $p_0(x) = 1$ for $x = x_k$. Hence the highest-weight vector Ω is a constant vector and is an eigenvector of the operator A^* with the eigenvalue determined by the condition $\mathcal{D}(1) = 1 + abcdq^{-1}$

$$A^*\Omega = (1 + abcdq^{-1})\Omega. \tag{278}$$

This property is related in a proper way to the ground state of the system. Namely, a corresponding shift of A^* produces a unique ground state with eigenvalue zero.

Let now ψ_0 denote the lowest weight vector of the tensor product representation, $\psi_0 = p_1(x_1)p_1(x_2)\cdots p_1(x_L)$. A state $\psi(x_1, x_2, \ldots, x_L)$ corresponding to any configuration on the lattice will be generated from ψ_0 by the action of the operators

$$E^+_{n_1}E^+_{n_2}\cdots E^+_{n_r}, \tag{279}$$

and we have

$$\psi(x_1, x_2, \ldots, x_L) \equiv \langle E_r| = \\ = \sum_{1\leqslant n_1<\ldots<n_r\leqslant L} a(n_1, n_2, \ldots, n_r)\langle\psi_0|E^+_{n_1}E^+_{n_2}\cdots E^+_{n_r}, \tag{280}$$

where coefficients $a(n_1, n_2, \ldots, n_r)$ depend on x_i. In order to determine the coefficients $a(n_1, n_2, \ldots, n_r)$ in the conventional Bethe ansatz, one defines a wave-number counting function and takes into account all $r!$ permutations of the numbers $(1, 2, \ldots, r)$.

By construction the state $\psi(x_1, x_2, \ldots, x_L)$ becomes an eigenvector of the operator A to be interpreted as playing the role of the Hamiltonian in the auxiliary space of the physical system. It acts on it by means of the coproduct which takes into account all the permutations of the partition $n_1, n_2, \ldots, n_r$ in (280). Namely, the action of the iterated coproduct according to (271) gives the eigenvalues $\sum_{k=1}^{L} x_k$, in the one occupation number *zero* $s_i = 0$ (one spin down) sector, the second-type operator terms in (271) give the values $\sum\limits_{i<j} x_i x_j$ in the two occupation numbers *zero* (two spin down) sector and so on, which yields all the eigenvalues whose number is

$$\sum_{n=1}^{L} \frac{L!}{n!(L-n)!} = 2^L - 1. \tag{281}$$

We thus obtain the $2^L - 1$ distinct eigenvalues of the operator A according to the action

$$A\psi(x_1, x_2, \ldots, x_L) = (\Delta^{(L)}A + A \otimes \Delta^{(L-1)}A + \\ + \Delta^{(L-1)}A \otimes A + \ldots + A^{(L)})\psi(x_1, x_2, \ldots, x_L) \tag{282}$$

from which the eigenvalue equation with the corresponding eigenvalues for the state $\psi(x_1, x_2, \ldots, x_L)$ follows

$$A\psi(x_1, x_2, \ldots, x_L) = \\ = \left(\sum_{i=1}^{L} x_i + \sum_{i<j} x_i x_j + \ldots + x_1 x_2 \cdots x_L \right) \psi(x_1, x_2, \ldots, x_L). \tag{283}$$

With the interpretation of the operator A as the Hamiltonian, Eq. (283) yields the energy eigenvalues.

As we already pointed out in this algebraic scheme there is no need of counting function since the right number of states and distinct eigenvalues is encoded in the polynomial representation of the AW algebra. The scheme works for any L, even for $L = 0$. In the latter case there are no zeros since p_0 is the constant term and hence $p_{-1} = 0$ which is the initial condition for the AW polynomials.

The considered algebraic Bethe ansatz based on the unique solution of the Bethe equations (243) yields for any $n = L$ an exactly solvable two-boundary value spectral problem with the identification of L with the spin value $2j$ of the finite-dimensional highest-weight evaluation representation of $U_q(\hat{s}u(2))$. There are two limit cases. The first one is

$$n \to \infty, \quad L \quad \text{finite}. \tag{284}$$

This limit is obtained by treating the product ab dependent on a, b and $1 - abq^n \neq 1$ is recovered, so that $b_n \neq 0$ when the infinite-dimensional representation of the AW algebra is restored corresponding to a finite-dimensional representation of $U_{q^{1/2}}(\hat{s}u(2))$. The thermodynamic limit for finite lattice systems with added boundary terms is conventionally obtained by letting

$$L \to \infty. \tag{285}$$

In our scheme we start from the very beginning with models in the infinite volume/infinite chain with quantum affine $U_{q^{1/2}}(\hat{s}l(2))(U_{q^{1/2}}(\hat{s}u(2)))$ symmetry which is manifest. Boundary conditions break the infinite volume symmetry. However with suitably imposed boundary conditions a remnant of this symmetry survives and is encoded in the nonlocal conserved charges, elements of the coideal AW subalgebra of $U_{q^{1/2}}(\hat{s}u(2))$, defined through the homomorphism to the quantized affine $U_{q^{1/2}}(\hat{s}u(2))$.

17.4. Exact Spectrum of the ASEP Transition Rate Matrix. We can now use the discussed prescription to obtain the complete set of eigenvectors with distinct eigenvalues for the transition matrix of the open ASEP. We recall that the left boundary operator and the right boundary operator are shifted TD algebra generators $D^L = A^* + \alpha - \gamma$ and $D^R = A + \beta - \delta$. The transition rate matrix of the ASEP is diagonalized in the auxiliary space of the finite-dimensional representation of the deformed Onsager boundary algebra. The basis in this representation is the $U_q(\hat{s}u(2))$ irreducible highest-weight tensor product evaluation module which forms the complete set of eigenvectors for the transition rate matrix. The ground state vector of the ASEP is related with the unique highest-weight vector.

We identify the transition matrix Γ_M of the open ASEP in the auxiliary space of symmetric Laurent polynomials p_n, $0 \leqslant n \leqslant L-1$ with the representation of the right boundary operator $A + \beta - \delta$ and the left boundary operator $A^* + \alpha - \gamma$ in the dual representation. For the chain of L sites, there is a representation of dimension 2^L, depending on the zeros of p_L, for any finite L. In this representation, the transition matrix Γ_M has a unique eigenstate $(\Omega, 0, 0, \ldots, 0)$ of eigenvalue zero which is the eigenstate of the left boundary operator, to be identified with the ASEP stationary state and $2^L - 1$ eigenstates of the right boundary operator with eigenvalues given by

$$\begin{aligned} E &= \beta - \delta + 1 - q + \\ &+ \left((1-q) \sum_{i=1}^{L} \hat{x}_i + (1-q)^2 \sum_{i<j} \hat{x}_i \hat{x}_j + \ldots + (1-q)^L \hat{x}_1 \hat{x}_2 \cdots \hat{x}_L \right), \end{aligned} \tag{286}$$

where $\hat{x}_i^{-1} = \hat{y}_i + \hat{y}_i^{-1}$. The zeros $\hat{y}_i$ satisfy the Bethe-ansatz equation with rescaled parameters according to (248), namely

$$\begin{aligned} &\frac{(\hat{y}_i - \zeta k_+(\alpha,\gamma))(\hat{y}_i - \zeta k_+(\beta,\delta))(\hat{y}_i - \zeta k_-(\alpha,\gamma))(\hat{y}_i - \zeta k_-(\beta,\delta))}{(\zeta k_+(\alpha,\gamma)\hat{y}_k - 1)(\zeta k_+(\beta,\delta)\hat{y}_k - 1)(\zeta k_-(\alpha,\gamma)\hat{y}_k - 1)(\zeta k_-(\beta,\delta)\hat{y}_k - 1)} = \\ &\qquad = \prod_{k=1, k\neq i}^{L} \frac{(qy_i - y_k)(qy_i y_k - 1)}{(y_i - qy_k)(y_i y_k - q)} \end{aligned} \tag{287}$$

with $\zeta = \dfrac{q^{(1-L)/2}}{\sqrt{k_+(\alpha,\gamma)k_+(\beta,\delta)}}$ and $k_\pm(u,v)$ given by

$$k_\pm(u,v) = \frac{v - u + (1-q) \pm \sqrt{(u - v - (1-q))^2 + 4uv}}{2u}. \tag{288}$$

There is a dual representation of dimension 2^L in the auxiliary space of symmetric Laurent polynomials p_n, $0 \leqslant n \leqslant L-1$. In the dual representation, the transfer matrix Γ_M has a unique eigenstate $(p_0, 0, 0, \ldots, 0)^t$ of eigenvalue zero which is the eigenstate of the right boundary operator and $2^L - 1$ distinct eigenvalues, the eigenvalues of the left boundary operator are given by

$$\begin{aligned} E &= \alpha - \gamma + 1 - q + \\ &+ \left((1-q)\sum_{i=1}^{L} \hat{x}_i + (1-q)^2 \sum_{i<j} \hat{x}_i \hat{x}_j + \ldots + (1-q)^L \hat{x}_1 \hat{x}_2 \cdots \hat{x}_L \right), \end{aligned} \tag{289}$$

where the $\hat{x}_i$ satisfy the Bethe-ansatz equation (287). This set of eigenvalues (289) is obtained from (286) by a shift and should not be considered as a different one.

We note that we can shift respectively by $\alpha + \gamma + 2\delta$ the right boundary operator $A + \beta - \delta$, and by $-\alpha - \gamma - 2\delta$ the left boundary operator $A^* + \alpha - \gamma$. Then we have a ground state with negative energy $-2\gamma - 2\delta$ and a constant term $\alpha + \beta + \gamma + \delta$ in Eq. (287). It will correspond to the result in [102, 105] for even number of lattice sites, in a different basis, for the energy eigenvalues in the one-spin sector, with $(\hat{y}_i + \hat{y}_i^{-1}) \to \dfrac{1}{1-Q^2}(-Q(z_i + z_i^{-1}) + Q^2 + 1)$, $Q^2 = q$. However, due to the ultimate relation [60] of the ASEP to the Askey–Wilson polynomials, already manifest in the exact solvability at the stationary state, the obtained diagonalization seems to be the most appropriate for this model of nonequilibrium physics.

We stress once again that the finite-dimensional representation of the tridiagonal algebra needed for the Bethe ansatz is obtained by using the general scheme with *no relation* among the model parameters so that we can apply it to models of nonequilibrium physics. The condition for the finite-dimensional representation of the AW and tridiagonal algebra which directly follows from the three-term recurrence relation coincides for the XXZ chain with the previously found Bethe-ansatz condition [103, 104].

We emphasize that the procedure for the ASEP is analogous but different from the one for obtaining the exact spectrum of XXZ chain in [107] where the starting point is a tridiagonal algebra with equal structure constants. It cannot be applied for the ASEP for the reason of restricting the nonequilibrium behaviour.

18. CONCLUSIONS AND DISCUSSION

We have reviewed the matrix-product approach to steady-state stochastic behaviour and its consistent extension to a tridiagonal algebra approach to driven diffusive systems. It is based on the quantum affine symmetry in the bulk and the boundary algebras as coideal subalgebras of the bulk symmetry:

1. The matrices of the MPA which determine the weights of each configuration in the steady state of the ASEP obey the level zero q-Serre relations of quantum affine $U_q(\hat{su}(2))$.

2. The boundary operators $D^L = \alpha D_0 - \gamma D_1$ and $D^R = \beta D_1 - \delta D_0$ are coideal elements of the bulk quantum affine $U_q(\hat{su}(2))$. Namely, $D^L = A^* + \alpha - \gamma$, $D^R = A + \beta - \delta$, where A^*, A generate the Askey–Wilson algebra [62,63] (as a coideal subalgebra).

3. The shifted boundary operators $A^* = D^L - \alpha + \gamma$ and $A = D^R - \beta + \delta$ generate the tridiagonal algebra [65, 66] (generalized Onsager algebra) which follows from the AW algebra, through the natural homomorphism, in the form of deformed Dolan–Grady relations.

The above points are formalized through the chain of homomorphisms

$$TD \to AW \to U_q(\hat{su}(2)).$$

The tridiagonal method is a means for the exact solvability of the simple exclusion process, analogously to the quantum inverse scattering method to integrable models.

The boundary AW symmetry is the algebraic property behind the exact steady state solution to the ASEP in terms of the AW polynomials although it was obtained [59] without reference to it. The consequence of the boundary AW algebra is the explicit construction of the operator-valued reflection K matrix, a solution to the boundary Yang–Baxter equation, which puts the Bethe ansatz solution into perspective. The applied algebraic Bethe ansatz based on Bethe equations for the roots of the Askey–Wilson polynomials p_n yields, for any $n = L$, a complete set of 2^L eigenvectors with distinct eigenvalues, including a unique ground state of the transition rate matrix (the formal Hamiltonian). The nontrivial point in the Bethe ansatz procedure is the truncation of the infinite-dimensional representation of the Askey–Wilson algebra by using the zeros of the Askey–Wilson polynomials to construct finite-dimensional representations of the tridiagonal algebra in such a way that no restriction on the nonequilibrium behaviour occurs. With the diagonalization of the transition matrix one can in principle calculate all the probabilities of the process at any time [5]. The boundary symmetry turns to be the deep algebraic property allowing for the exact description of the stochastic dynamics.

The tridiagonal boundary algebra in the form of deformed Dolan–Grady relations is a q generalization of the Onsager algebra. The algebraic scheme provides a unified description of the simple exclusion process. It allows for exact solution in the stationary state and for description of the stochastic dynamics. It is worth considering the generalization to many species processes and discrete-time processes [121–124]. There is a recent progress in the matrix product steady state solution of the multispecies models [42]. In systems with time-discrete dynamics where in one time step all sites are updated, the matrix product formalism is even more involved: the algebraic relations can be cubic or quartic. However, many-species stochastic systems and systems with discrete-time updating schemes have remained beyond the scope of this review.

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