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PECULIAR FEATURES OF THE RELATIONS BETWEEN POLE AND RUNNING HEAVY-QUARK MASSES AND ESTIMATES OF THE $O(\alpha_s^4)$ CONTRIBUTIONS

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Perturbative relations between pole and running heavy-quark masses, defined in the Minkowski regions, are considered. Special attention is paid to the appearance of the kinematic π^2 effects, which exist in the coefficients of these series. The estimates of order $O(\alpha_s^4)$ QCD corrections are presented.

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INTRODUCTION

Among important parameters of QCD are the masses of c, b and t quarks, which are more heavy than $N_L = 3$, 4, 5 number of lighter ones. They can be defined either as the poles of the corresponding renormalized heavy-quark propagators at $q^2 = M^2_{(N_L+1)}$ in the Minkowski space-like region or as the running masses $\overline{m}_{(N_L+1)}(\mu^2)$ in the $\overline{\rm MS}$-scheme. Their scale-dependence is described by the solution of the following equation:

$$\frac{\overline{m}_{(N_L+1)}(s)}{\overline{m}_{(N_L+1)}(\mu^2)} = \exp\left[\int\limits_{a_s(\mu^2)}^{a_s(s)} \frac{\gamma_{m_{(N_L+1)}}(x)}{\beta(x)}dx\right], \tag{1}$$

where $a_s(s) = \alpha_s(s)/\pi$ and $\alpha_s(s)$ is the QCD coupling constant of the $\overline{\rm MS}$-scheme, fixed in the **Minkowski** reference point $s > \overline{m}^2_{(N_L+1)}$, and the renormalization group functions $\gamma_{m_{(N_L+1)}}(x)$ and $\beta(x)$ are defined as

$$\gamma_{m_{(N_L+1)}}(a_s) = \frac{d\ln \overline{m}_{(N_L+1)}(\mu^2)}{d\ln \mu^2} = -\sum_{i\geqslant 0}\gamma_i(N_L)a_s^{i+1}, \tag{2}$$

$$\beta(a_s) = \frac{da_s(\mu^2)}{d\ln \mu^2} = -\sum_{i\geqslant 0}\beta_i(N_L)a_s^{i+2}. \tag{3}$$

The coefficients $\beta_i(N_L)$ and $\gamma_i(N_L)$ (apart from the coefficient γ_0) depend on $N_L + 1$ number of active flavours. Note, that for the $\overline{\rm MS}$-scheme running heavy-quark masses $\overline{m}_{(N_L+1)}(\mu^2)$ the **Minkowskian** normalization point $\mu^2 = \overline{m}^2_{(N_L+1)}$ is frequently used (see, e.g., [1]). In this case, the definition of $\overline{m}_{(N_L+1)}(\overline{m}^2_{(N_L+1)})$ may be geometrically illustrated by finding the intersection of the curve, which represents the inverse logarithmic scale-dependence of the squared running mass, with the bisectrix of the angle, formed by positive axes $0 \leqslant \overline{m}^2_{(N_L+1)} \leqslant \infty$ and $0 \leqslant \mu^2 \leqslant \infty$*. The relations between pole and running heavy-quark masses we will be interested read

$$M_{(N_L+1)} = \overline{m}_{(N_L+1)}(\overline{m}^2_{(N_L+1)}) \sum_{n=0}^{4} t_n^M(N_L) a_s^n(\overline{m}^2_{(N_L+1)}). \tag{4}$$

Note, that in the process of comparison of theoretical predictions for the e^+e^--annihilation Euclidean time-like characteristic, namely Adler D-function, with its experimental-motivated behaviour [2], pole heavy-quark masses were defined in the MOM-scheme, while running heavy-quark masses were defined at the **Euclidean** scale $\mu^2 = Q^2$. The similar mixed MOM–$\overline{\rm MS}$-scheme prescriptions are also widely used to analyze heavy-quark mass dependent effects in characteristics of deep inelastic scattering (see, e.g., [3, 4]). However, the processes, which may be observed at LHC, are described by theoretical predictions in the time-like region of energies. In view of this, it is important to study relations between different most commonly used definitions of heavy-quark masses and to derive the relations between pole and running heavy-quark masses, tied to the **Euclidean** and **Minkowski** regions of momentum transferred. This problem was analyzed in [5] with the help of the special Källén–Lehmann type representation. Here we will consider this approach in more detail, presenting additional arguments in favour of theoretical background of the investigations, performed in the work mentioned above. We will also update estimates of the order $O(\alpha_s^4)$ terms in the relation of Eq. (4), which were obtained in [5] using the extended to the mass-dependent case effective-charges inspired massless approach, elaborated in [6].

1. COMMENTS ON APPLICATION OF THE DISPERSION RELATIONS

Let us discuss the subject of applicability of the Källén–Lehmann type spectral representations within the context of perturbative QCD. The well-defined dispersion relation for the e^+e^--annihilation Adler function is well known

$$D_V(Q^2) = -Q^2 \frac{d\Pi_V(Q^2)}{dQ^2} = Q^2 \int\limits_0^\infty \frac{R(s)}{(s+Q^2)^2}\, ds, \tag{5}$$

*We are grateful to G. B. Pivovarov for the discussion of this topic.

where $\Pi_V(Q^2)$ is the photon vacuum polarization function and $R(s) \sim \mathrm{Im}\,\Pi_V$. The two-point function of the scalar quark currents $m_{(N_L+1)}\overline{\psi}_q\psi_q$ has the imaginary part, which defines the scalar Higgs boson decay width into quark–antiquark pairs. In this case, it is possible to write down the following representation [7]:

$$D_S(Q^2) = -Q^2 \frac{d}{dQ^2}\left[\frac{\Pi(Q^2)}{Q^2}\right] = Q^2 \int\limits_0^\infty \frac{R_S(s)}{(s+Q^2)^2}\, ds, \tag{6}$$

which faces no problems in the region where the asymptotic freedom property of QCD holds. The same equation was used in [5] to extend the massless procedure of the estimates of higher-order perturbative corrections to the Euclidean quantities [6] to the case of Eq. (6), which contains the dependence on the square of running mass $\overline{m}_{(N_L+1)}(Q^2)$ defined in the **Euclidean** region. However, as was shown in [8], the dispersive relation of Eq. (6) is valid within perturbative sector only and can not be proved on the level of rigour, considered in [9]. Indeed, it was shown in [8] that in the low-energy region Eq. (6) is ill-defined and contains fictitious $\Lambda^2_{\mathrm{QCD}}/Q^2$-term. It reflects the failure to remove the infinities from $\Pi_S(0)$. The well-defined dispersive relation, which does not contain this term, can be written down through the *second* derivative of the scalar correlator [10]. It leads to the following Euclidean function:

$$\overline{D}_S(Q^2) = 2Q^2 \int\limits_0^\infty \frac{sR_S(s)}{(s+Q^2)^3}\, ds. \tag{7}$$

Note, however, that its perturbative expansion differs from the one, which corresponds to the Euclidean part of perturbative series for $\Gamma(H^0 \to \overline{q}q)$, generated by the ill-defined in nonperturbative sector expression of Eq. (6). Moreover, the application of the «approximate» dispersion relation from Eq. (6) fixes the kinematic π^2 contributions to the coefficients of the perturbative series for $\Gamma(H^0 \to \overline{q}q)$ both in the expanded [5] and summed up [8, 11] forms. Note, that the idea of the summation of π^2 terms at lowest order of QCD was proposed and used over thirty-five years ago in the works of [7, 12, 13].

2. DISPERSION RELATIONS FOR THE POLE AND RUNNING HEAVY-QUARK MASSES

Consider now the following «approximate» dispersion model of [5] for the pole heavy-quark masses

$$M_{(N_L+1)} = \frac{1}{2\pi i} \int\limits_{-\overline{m}_{(N_L+1)}(\overline{m}^2_{(N_L+1)})-i\varepsilon}^{-\overline{m}_{(N_L+1)}(\overline{m}^2_{(N_L+1)})+i\varepsilon} ds' \int\limits_0^\infty \frac{T(s)}{(s+s')^2}\, ds \tag{8}$$

with the spectral density defined as $T(s) = \overline{m}_{(N_L+1)}(s) \sum\limits_{n=0}^{4} t_n^M a_s^n(s)$. It can be obtained from the dispersion-type expression for the **Euclidean** series

$$F(Q^2) = \overline{m}_{(N_L+1)}(Q^2) \sum_{n=0}^{4} f_n^E(N_L) a_s^n(Q^2) = Q^2 \int\limits_0^\infty \frac{T(s)}{(s+Q^2)^2}\, ds, \tag{9}$$

where $\overline{m}_{(N_L+1)}(Q^2)$ and $a_s(Q^2)$ are the heavy-quark masses and the QCD coupling constant which are «running» in the **Euclidean** region. The application of Eq. (8) allows one to fix the relations between coefficients $f_n^E(N_L)$ and $t_n^M(N_L)$ of the perturbative series in the time-like and space-like regions as $f_0^E = t_0^M$, $f_1^E = t_1^M$, $f_2^E(N_L) = t_2^M(N_L) + e_2(N_L)$, $f_3^E(N_L) = t_3^M(N_L) + e_3(N_L)$, $f_4^E(N_L) = t_4^M(N_L) + e_4(N_L)$. The kinematic π^2 terms enter the derived in [5] explicit expressions for the $e_i(N_L)$ contributions, namely

$$e_2(N_L) = \frac{\pi^2}{6} t_0^M \gamma_0(\beta_0+\gamma_0) = 5.89435 - 0.274156 N_L, \tag{10}$$

$$e_3(N_L) = \frac{\pi^2}{3}\left\{ t_1^M(\beta_0+\gamma_0)\left(\beta_0 + \frac{\gamma_0}{2}\right) + t_0^M\left[\frac{\beta_1\gamma_0}{2} + \gamma_1(\beta_0+\gamma_0)\right]\right\} = \tag{11}$$

$$= 105.622 - 10.0448 N_L + 0.198001 N_L^2, \tag{12}$$

$$\begin{aligned} e_4(N_L) = \pi^2 \Bigg\{ & t_2^M\left(\beta_0 + \frac{\gamma_0}{2}\right) + t_1^M\left[\frac{\beta_1}{2}\left(\frac{5}{3}\beta_0+\gamma_0\right) + \frac{\gamma_1}{3}(2\beta_0+\gamma_0)\right] + \\ & + t_0^M\left[\frac{\beta_2\gamma_0}{6} + \frac{\gamma_1}{3}\left(\beta_1 + \frac{\gamma_1}{2}\right) + \gamma_2\left(\frac{\beta_0}{2}+\frac{\gamma_0}{3}\right)\right]\Bigg\} + \\ & + \frac{7\pi^4}{60} t_0^M \gamma_0(\beta_0+\gamma_0)\left(\beta_0+\frac{\gamma_0}{2}\right)\left(\beta_0+\frac{\gamma_0}{3}\right) = \\ & = 2272.02 - 403.951 N_L + 20.6768 N_L^2 - 0.315898 N_L^3. \end{aligned} \tag{13}$$

Their N_L dependence results from N_L dependence of the coefficients $\beta_i(N_L)$ with $i \geqslant 0$ in Eq. (3), $\gamma_i(N_L)$ with $i \geqslant 1$ in Eq. (2) and t_2^M in Eq. (4), which has the following numerical form [14]:

$$t_2^M = 13.44396 - 1.041367 N_L \tag{14}$$

and comes from the analytical expression of [15], confirmed by the independent calculations of [16]. Notice, that the results of [15, 16] contain the **explicit dependence** on $\zeta_2 = \pi^2/6$ terms. The discussions presented above clarify that the part of these π^2 terms, explicitly visible in the formulae of [15,16], appear from the analytical continuation effect of Eq. (10). This our claim can be generalized to the level of t_3^M corrections, evaluated analytically in [14] and semi-analytically in [17]. In this case, kinematic π^2 contributions are determined by Eq. (11). The coefficients of the relation between heavy-quark Euclidean masses, defined in the MOM on-shell, and $\overline{\mathrm{MS}}$-scheme masses, contain only remaining transcendental terms, typical of the on-shell scheme calculations.

3. ESTIMATES OF α_s^4 CORRECTIONS

We consider now two perturbative series, namely the one of Eq. (4) and the related to it relation

$$M_{(N_L+1)} = \overline{m}_{(N_L+1)}(M^2_{(N_L+1)}) \sum_{n=0}^{4} v_n^M(N_L) a_s^n(M^2_{(N_L+1)}). \tag{15}$$

Keeping in mind that for $0 \leqslant n \leqslant 3$ the values of the explicit dependence on N_L of the coefficients $t_n^M(N_L)$ and $v_n^M(N_L)$ are already known [1, 14], we will study the problem of estimates of the α_s^3 and α_s^4 coefficients, using the effective-charges (ECH) inspired approach, developed and used in [5, 6]*. It is known that the applications of this approach in the Euclidean region at the level of α_s^3 and α_s^4 corrections give correct in signs and in order of magnitude estimates of the perturbative contributions to the number of physical quantities (see, e.g., [5, 6, 22, 23]). As to the application of this procedure to the Minkowskian quantities, two ways are possible. The first prescribes to apply the procedure of estimates in the Euclidean region and **add explicitly calculable** kinematic π^2 terms afterwards. Within the second way one may use the procedure of estimates in the Minkowski region directly. It should be noted, that both ways are leading to reasonable predictions of signs and numerical values of perturbative series for physical quantities. Moreover, in the case of **direct** application of this approach in the Minkowski region, the order α_s^4 estimates are sometimes even closer to the results of the explicit calculations (see, e.g., [22]). However, in the latter case the estimates do not reproduce the known values of the analytical continuation effects, similar to the ones of Eq. (11) and Eq. (13). Note, that their precise knowledge is important for applying different approaches of resummations of these contributions (see, e.g., [8, 11, 24–27]). Following the two ways mentioned above we first estimate the values of $t_3^M(N_L)$ coefficients and compare them with the results for $t_3^{\mathrm{exact}}(N_L)$ obtained in [14, 17]. Satisfied by this comparison we are going one step further and estimate $t_4^M(N_L)$ coefficients, taking into account the numerical expressions for $t_3^{\mathrm{exact}}(N_L)$. The concrete numbers are presented in the Table. One can see, that the estimates obtained give correct in signs and in order of magnitude estimates for the values of $t_3^M(N_L)$ terms. Thus, one may

*The method of ECH was proposed and developed in [18, 19] and independently in [20] (see also [21]).

The estimates for $t_3^M(N_L)$, $t_4^M(N_L)$

N_L	$t_3^{\rm exact}$	$t_3^{\rm ECH}$	$t_3^{\rm ECH\,direct}$	$t_4^{\rm ECH}$	$t_4^{\rm ECH\,direct}$
5	73.6366	58.0645	48.4906	719.339	710.016
4	94.4175	100.74	78.243	986.097	1045.5
3	116.504	147.303	111.315	1281.05	1438.75

hope that the estimates for $t_4^M(N_L)$ are not far from reality. We present now concrete numbers for the coefficients of the series of Eq. (4), where for the α_s^4 coefficients we use the estimates $t_4^{\rm ECH}(N_L)$ from the Table:

$$M_c \approx \overline{m}_c(\overline{m}_c^2)\left[1+\frac{4}{3}a_s(\overline{m}_c^2)+10.3a_s^2(\overline{m}_c^2)+116.5a_s^3(\overline{m}_c^2)+1281a_s^4(\overline{m}_c^2)\right], \quad (16)$$

$$M_b \approx \overline{m}_b(\overline{m}_b^2)\left[1+\frac{4}{3}a_s(\overline{m}_b^2)+9.28a_s^2(\overline{m}_b^2)+94.4a_s^3(\overline{m}_b^2)+986a_s^4(\overline{m}_b^2)\right], \quad (17)$$

$$M_t \approx \overline{m}_t(\overline{m}_t^2)\left[1+\frac{4}{3}a_s(\overline{m}_t^2)+8.24a_s^2(\overline{m}_t^2)+73.6a_s^3(\overline{m}_t^2)+719a_s^4(\overline{m}_t^2)\right]. \quad (18)$$

The similar relations for Eq. (15) with on-shell normalizations of running parameters read

$$M_c \approx \overline{m}_c(M_c^2)\left[1+\frac{4}{3}a_s(M_c^2)+13a_s^2(M_c^2)+156a_s^3(M_c^2)+1853a_s^4(M_c^2)\right], \quad (19)$$

$$M_b \approx \overline{m}_b(M_b^2)\left[1+\frac{4}{3}a_s(M_b^2)+12a_s^2(M_b^2)+131a_s^3(M_b^2)+1460a_s^4(M_b^2)\right], \quad (20)$$

$$M_t \approx \overline{m}_t(M_t^2)\left[1+\frac{4}{3}a_s(M_t^2)+11a_s^2(M_t^2)+107a_s^3(M_t^2)+1101a_s^4(M_t^2)\right]. \quad (21)$$

The results presented in Eq. (20) give the following ratios of the squares of running and pole b-quark masses:

$$\frac{\overline{m}_b^2(M_b^2)}{M_b^2} = 1-\frac{8}{3}a_s(M_b^2)-18.5559a_s^2(M_b^2)-175.797a_s^3(M_b^2)-1684a_s^4(M_b^2), \quad (22)$$

where the last term is fixed by the result of application of the ECH-motivated approach with adding kinematic π^2 contributions at the final step. In the case when the Euclidean and kinematic π^2 corrections are summed up at the intermediate steps, the last coefficient in Eq. (22) should be changed from **–1684** to **–1835**. Note, that in the process of analyzing the uncertainties of QCD predictions for $\Gamma(H^0 \to b\overline{b})$, performed in the work of [28], we used slightly lower estimate, namely **–1892**. The difference is explained in part by smaller number of significant digits taken into account in the values of coefficients, which enter in the procedure of corresponding estimates. However, this difference between the values of estimated order $O(\alpha_s^4)$ contributions is not so numerically important. Other possible physical applications, like the comparison with the renormalon-based analysis of asymptotic behaviour of perturbative series in Eqs. (19)–(21) (for the related theoretical discussions one can see [29–31]) are beyond the scope of this study.

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