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ACTION-ANGLE VARIABLES AND NOVEL SUPERINTEGRABLE SYSTEMS

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In this paper we demonstrate the effectiveness of the action-angle variables in the study of superintegrable systems. As an example, we construct the spherical and pseudospherical generalizations of the two-dimensional superintegrable models introduced by Tremblay, Turbiner and Winternitz, and by Post and Winternitz.

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INTRODUCTION

Prominent in the theory of integrable systems is the Liouville theorem, which states that any $2N$-dimensional Hamiltonian system with N mutually commuting constants of motion is integrable. Besides, the theorem states that, if the level surface of these constants of motion are *compact and connected manifolds*, then they are diffeomorphic to N-dimensional tori. This enables one to introduce action-angle variables $(\mathbf{I}, \mathbf{\Phi})$, so that the Hamiltonian depends only on the action variables $\mathbf{I}$, which are constants of motion. The formulation of an integrable system in terms of these variables yields a comprehensive geometric description of its dynamics and is a useful tool for developing perturbation theory [1]. For these reasons, action-angle variables have been widely exploited in celestial mechanics since the 19th century and play a central role in the Bohr–Sommerfeld semiclassical quantization. However, after invention of the canonical quantization and completing the theory of quantum mechanics, the interest to the action-angle variables waned. To the moment, besides the standard textbook problems such as the harmonic oscillator or the Kepler potential, action-angle variables seem to be absent in the literature for the vast variety of known integrable models, invented over the last fifty years, such as integrable systems in a curved space, multi-particle systems of Calogero type [2] (except for rational Calogero models [3]), as well as a particle systems coupled to a monopole or instanton background. Therefore, we have recently begun to develop this issue by analyzing the (one-dimensional) dihedral systems related to the three-particle Calogero model [4] as well as two-dimensional oscillator-like systems in magnetic field, which are relevant to certain models of quantum rings and lenses [5]. Then we suggested the integrable deformations of the d-dimensional oscillator and Coulomb systems and of their generalizations to (pseudo)spheres [6, 7], based on the use of action angle-variables [8]. Finally, we used the action-angle variables for the analyses of the relativistic particle system moving near the horizon of extreme black holes [9, 10]. We found, to our surprize, that, in fact, in each considered example, the use of action-angle variables immediately provide us with some qualitatively new information, which was not observed in previous studies performed by common methods. Say, action-angle variable formulation of the (one-dimensional!) dihedral systems [4] immediately established the equivalence of A_2 and G_2 rational Calogero models, as well as allowed us to demonstrate the locality of the equivalence of the rational Calogero model and free particle, which has been intensively discussed in literature in the last decade [11]; while action-angle description of the relativistic particle near extreme Kerr throat visualized the existence of the critical point $|p_\phi| = mcR_{\rm Sch}$ (with m being the mass of the particle, c denoting the speed of light, $R_{\rm Sch} = 2\gamma M/c^2$ being the Schwarzschild radius of a black hole with mass M, and γ denoting the gravitational constant), where the trajectories close, and the system becomes exactly solvable [10]. Finally, we found that action-angle variables give transparent explanation of the superintegrability property of the recently suggested deformations of the two-dimensional oscillator system (Tremblay–Turbiner–Winternitz system) [12] and of two-dimensional Coulomb system (Post–Winternitz system) [13], and allow us to immediately construct their (pseudo)spherical generalizations [8]. The discussion of the last issue is the subject of this paper.

The paper is arranged as follows. In Sec. 1 we give the necessary information on action-angle variables and their supergeneralizations. In Sec. 2 we give the action-angle variable formulation of the certain dihedral systems on the circle and discuss their consequences concerning rational Calogero models. In Sec. 3 we construct the spherical and pseudospherical generalizations of the TTW and PW systems. We demonstrate the superintegrability of these systems and write down their hidden constants of motion.

1. ACTION-ANGLE VARIABLES

The well-known Liouville theorem gives the exact criterion of integrability of the N-dimensional mechanical system, that is, the existence of N mutually commuting constants of motion $F_1 = H, \ldots, F_n$: $\{F_i, F_j\} = 0$, $i, j = 1, \ldots N$. The theorem also states that if the level surface $M_f = ((p_i, q_i) : F_i = \text{const})$ is a *compact and connective manifold*, then it is diffeomorphic to the N-dimensional torus T^N. The natural angular coordinates $\mathbf{\Phi} = (\Phi_1, \ldots, \Phi_N)$ parameterizing that torus satisfy the motion equations of a free particle moving on a circle. These coordinates form, with their conjugate momenta $\mathbf{I} = (I_1, \dots, I_N)$, a full set of phase space variables called «action-angle» variables. One of the results of the theorem is that the momenta $\mathbf{I}$ depend on constants of motion only $\mathbf{I} = \mathbf{I}(\mathbf{F})$. So, there exists a canonical transformation to the new variables $(\mathbf{p}, \mathbf{q}) \mapsto (\mathbf{I}, \mathbf{\Phi})$, in which the Hamiltonian depends on the constants of motion $\mathbf{I}$ (which are called action variables) only. Consequently, the equations of motion read

$$\frac{d\mathbf{I}}{dt} = 0, \quad \frac{d\mathbf{\Phi}}{dt} = \frac{\partial H(I)}{\partial \mathbf{I}}, \quad \{I_i, \Phi_j\} = \delta_{ij}, \quad \Phi_i \in [0, 2\pi), \quad i, j = 1, \dots, N. \tag{1.1}$$

Besides the practical importance, the action-angle formulation has an academic interest as well. From the academic viewpoint, it gives a precise indication of the (non)equivalence of different Hamiltonian systems. Indeed, gauging the integrable system by action-angle variables, we preserve the freedom only in the functional dependence of the Hamiltonian from the action variables, $H = H(\mathbf{I})$, and in the range of validity of the action variables, $I_i \in [\beta_i^-, \beta_i^+]$. Hence formulating the systems in terms of action-angle variables, we can indicate the (non)equivalence of different integrable systems.

The general prescription for the construction of action-angle variables looks as follows [1]. In order to construct the action-angle variables, we should fix the level surface of the Hamiltonian $\mathbf{F} = \mathbf{c}$ and then introduce the generating function for the canonical transformation $(\mathbf{p}, \mathbf{q}) \mapsto (\mathbf{I}, \mathbf{\Phi})$, which is defined by the expression

$$S(\mathbf{c}, \mathbf{q}) = \int\limits_{\mathbf{F}=\mathbf{c}} \mathbf{p}\, d\mathbf{q}, \tag{1.2}$$

where $\mathbf{p}$ is expressed via $\mathbf{c}, \mathbf{q}$ by the use of the constants of motion. The action variables $\mathbf{I}$ can be obtained from the expression

$$I_i(c) = \frac{1}{2\pi} \oint\limits_{\gamma_i} \mathbf{p}\, d\mathbf{q}, \tag{1.3}$$

where γ_i is some loop of the level surface $\mathbf{F} = \mathbf{c}$. Then inverting these relations, we can get the expressions of $\mathbf{c}$ via action variables: $\mathbf{c} = \mathbf{c}(\mathbf{I})$. The angle variables $\mathbf{\Phi}$ can be found from the expression

$$\mathbf{\Phi} = \frac{\partial S(\mathbf{c}(\mathbf{I}), \mathbf{q})}{\partial \mathbf{I}}. \tag{1.4}$$

The supergeometric generalization of the Liouville theorem has been known for many years [14]. For our context of one-dimensional supersymmetric mechanics, we follow here the construction of action-angle (super)variables as presented in [15]. Let us have $\mathcal{N}$=$2M$ supersymmetric mechanics defined on a

$(2N|2M)$-dimensional phase superspace, coordinatized by $(p_\varphi, \varphi|\theta^\alpha, \overline{\theta}^\beta)$. The supersymmetry algebra reads

$$\begin{gathered}\{Q^\alpha, \overline{Q}^\beta\} = 2\delta^{\alpha\beta} H_s,\\ \{Q^\alpha, H_s\} = \{\overline{Q}^\beta, H_s\} = 0 = \{Q^\alpha, Q^\beta\} = \{\overline{Q}^\alpha, \overline{Q}^\beta\},\end{gathered} \tag{1.5}$$

where $\alpha, \beta = 1, \ldots, M$.

Fixing the level supersurface, $H_s = h_s$, $Q^\alpha = q^\alpha$ and $\overline{Q}^\alpha = \overline{q}^\alpha$, we arrive at a $(N|0)$-dimensional torus in the phase superspace. On this torus, one defines bosonic action-angle variables $(\mathbf{\Phi}_s, \mathbf{I}_s)$, analogous to the non-supersymmetric case, as well as fermionic ones, $\Theta^\alpha = Q^\alpha/\sqrt{2h_s}$, with the following non-zero Poisson brackets

$$\{\Phi^i_s, I^j_s\} = \delta^{ij} \quad \text{and} \quad \{\Theta^\alpha, \overline{\Theta}^\beta\} = \delta^{\alpha\beta}. \tag{1.6}$$

In these variables, the Hamiltonian does not depend on Θ^α or $\overline{\Theta}^\alpha$, hence $H_s(I_s)$ is just like previously.

Hence, for any integrable Hamiltonian system, formulated in action-angle variables, we can construct formal $\mathcal{N} = 2M$ supersymmetric extension, defined by supercharges $Q^\alpha = \sqrt{2h_s}\Theta^\alpha$. Nevertheless, the canonical transformation from the initial to the action-angle supervariables does mix bosonic and fermionic degrees of freedom. Moreover, this supersymmetrization procedure has no any sense until defining the supercanonical transformation from the initial phase superspace variables $(\mathbf{p},\ \mathbf{q},\ \theta^\alpha,\ \overline{\theta}^\alpha)$ to the action-angle ones, given by $(\mathbf{I}_s,\ \mathbf{\Phi}_s,\ \Theta^\alpha,\ \overline{\Theta}^\alpha)$.

Let us demonstrate this issue on the simple case of $\mathcal{N} = 2$ supersymmetric mechanics [4]. For this purpose, let us choose a more flexible form of the supercharges, namely

$$Q = \sqrt{2H_s}\,\mathrm{e}^{\imath\lambda(\mathbf{I},\mathbf{\Phi})}\theta = \sqrt{2H_s}\,\Theta \quad \text{and} \quad \overline{Q} = \sqrt{2H_s}\,\mathrm{e}^{-\imath\lambda(\tilde{I},\Phi)}\overline{\theta} = \sqrt{2H_s}\,\overline{\Theta}, \tag{1.7}$$

where $\lambda(\mathbf{I}, \mathbf{\Phi})$ is an arbitrary real function of the action-angle variables of the underlying bosonic system. By expressing $(\mathbf{I}, \mathbf{\Phi})$ through $(\mathbf{p}, \mathbf{q})$, the supercharges are functions of the initial phase superspace variables. These supercharges also generate the superalgebra (1.5) (with M=1) and produce the Hamiltonian

$$H := \frac{1}{2}\{Q, \overline{Q}\} = H_s + \imath\theta\overline{\theta}\{H_s, \lambda\}. \tag{1.8}$$

The freedom of an arbitrary real function $\lambda(\mathbf{I}, \mathbf{\Phi})$ leads to a well-known variety of $\mathcal{N} = 2$ supersymmetric extensions of a given bosonic system. For example, it was used in [16] for the construction of $\mathcal{N} = 2$ superconformal extension of the particle near Kerr–Neeman AdS–dS black hole throat.

Applying the (super-)Liouville theorem to the supersymmetric system given by (1.7) and (1.8), we obtain

$$\mathbf{I}_s = \mathbf{I} + \imath\theta\overline{\theta}\,\{\lambda(\mathbf{I},\mathbf{\Phi}),\mathbf{\Phi}\}, \quad \mathbf{\Phi}_s = \mathbf{\Phi} + \imath\theta\overline{\theta}\,\{\lambda(\mathbf{I},\mathbf{\Phi}),\mathbf{I}\}, \tag{1.9}$$

$$\Theta = \mathrm{e}^{\imath\lambda(\mathbf{I},\mathbf{\Phi})}\theta, \quad \overline{\Theta} = \mathrm{e}^{-\imath\lambda(\mathbf{I},\mathbf{\Phi})}\overline{\theta}. \tag{1.10}$$

As already said, the Hamiltonian in these variables is of the same form as the non-supersymmetric one, $H_s = H$.

Let us demonstrate the procedure for the simplest case of $\mathcal{N}{=}2$, given by the classical counterpart of Witten's model of supersymmetric mechanics. It is defined by

$$H_s = \frac{1}{2}\big(p^2 + W'^2(q)\big) + \imath\theta\overline{\theta}\,W''(q), \quad Q = \theta\,\big(p + \imath W'(q)\big), \quad \overline{Q} = \overline{\theta}\,\big(p - \imath W'(q)\big), \tag{1.11}$$

with a chosen superpotential function $W(q)$. These functions obey the superalgebra (1.5) with $M{=}1$, by virtue of

$$\{p,q\} = 1 \quad \text{and} \quad \{\theta,\overline{\theta}\} = 1. \tag{1.12}$$

Quantization replaces θ and $\overline{\theta}$ by the Pauli matrices $\boldsymbol{\sigma}_+ = (1/2)(\boldsymbol{\sigma}_1 + \imath\boldsymbol{\sigma}_2)$ and $\boldsymbol{\sigma}_- = (1/2)(\boldsymbol{\sigma}_1 - \imath\boldsymbol{\sigma}_2)$, respectively, and $\imath\theta\overline{\theta}$ goes to $\boldsymbol{\sigma}_3$. In this way, we arrive at one-dimensional $\mathcal{N}{=}2$ supersymmetric quantum mechanics of a spinning particle interacting with an external field. However, when passing to action-angle variables, it turns out that there is no spin interaction, and the supersymmetric extension is rather trivial. On the other hand, Witten's model is quite special: its supercharges allow no momentum dependence in the nilpotent part of the Hamiltonian.

To formulate the standard $\mathcal{N}{=}2$ supersymmetric mechanics construction (1.11) in action-angle variables, we must choose

$$\sqrt{2H}\,\mathrm{e}^{\imath\lambda(I(p,q),\Phi(p,q))} = p + \imath W'(q) \;\Rightarrow\; \tan\lambda = \frac{W'(q)}{p}. \tag{1.13}$$

Note that $\lambda = \lambda(I)$ yields trivial supersymmetry, with no spin interaction. Another interesting case is $\lambda = \Phi/I$, which produces a coordinate-independent spin-background interaction.

2. DIHEDRAL SYSTEMS

Let us construct the action-angle variables for the dihedral systems on a circle, which are defined by the Hamiltonian

$$\mathcal{I}(p_\varphi,\varphi|k) = \frac{1}{2}p_\varphi^2 + V_k(\varphi), \quad V_k(\varphi) = \sum_{\ell=0}^{k-1}\frac{1}{(\mathbf{a}_\ell\cdot\mathbf{n})^2}, \quad \text{where} \quad \mathbf{n} = \begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix} \tag{2.1}$$

and $\mathbf{a}_\ell$ are the positive roots of a two-dimensional Coxeter system $I_2(k)$ called dihedral system. The full set of roots forms a regular star shape with an angular separation of π/k. Since the symmetry relates the root lengths as $|\mathbf{a}_\ell|^2 = |\mathbf{a}_{\ell+2}|^2$, for odd k all roots have the same length, say α_0, while for even k we may put $|\mathbf{a}_{\rm even}| = \alpha_1$ and $|\mathbf{a}_{\rm odd}| = \alpha_2$. Clearly, we have to distinguish between k being even or odd. As $\mathbf{a}_\ell{\cdot}\mathbf{n}$ is proportional to $\cos(\phi - \ell\pi/k)$, it is a matter of simple algebra to perform the finite sums and obtain

$$V_k(\varphi) = \frac{k^2\alpha_0^2}{2\cos^2 k\varphi} \quad \text{for} \quad k = 2k'{+}1, \tag{2.2}$$

$$V_k(\varphi) = \frac{(k'\alpha_1)^2}{2\cos^2 k'\varphi} + \frac{(k'\alpha_2)^2}{2\sin^2 k'\varphi} \quad \text{for} \quad k = 2k', \tag{2.3}$$

with $k' \in \mathbb{N}$. Hence, the odd systems feature one coupling (α_0), while the even ones allow for two (α_1, α_2), all naturally positive. For $\alpha_1{=}\alpha_2$, the even potential attains the same form as the odd one. So, we get precisely the Pöschl–Teller system for the odd k, and modified Pöschl–Teller system for the even k [17]. Their action-angle variable looks as follows.

For the systems with odd k, one has

$$I = \frac{1}{k}\sqrt{2\mathcal{I}(p_\varphi,\varphi)} - \alpha_0, \quad \Phi = \arcsin\left\{\frac{\sqrt{2\mathcal{I}(p_\varphi,\varphi)}}{\sqrt{2\mathcal{I}(p_\varphi,\varphi) - k^2\alpha_0^2}}\sin k\varphi\right\}. \tag{2.4}$$

In these variables the Hamiltonian reads

$$\mathcal{I} = \frac{k^2}{2}(I + \alpha_0)^2. \tag{2.5}$$

For the case (2.3), i.e., for the systems with even k, $k = 2k'$, the action-angle variables read

$$I = \frac{1}{k'}\sqrt{2\mathcal{I}} - (\alpha_1 + \alpha_2), \quad \Phi = \frac{1}{2}\arcsin\left\{\frac{1}{a}[\cos 2k'\varphi + b]\right\}, \tag{2.6}$$

where a and b are defined by the expressions

$$a = \sqrt{1 - \frac{k'^2(\alpha_1^2 + \alpha_2^2)}{\mathcal{I}(p_\varphi,\varphi)} + \frac{k'^4(\alpha_1^2 - \alpha_2^2)^2}{4\mathcal{I}^2(p_\varphi,\varphi)}}, \quad b = \frac{k'^2(\alpha_2^2 - \alpha_1^2)}{2\mathcal{I}(p_\varphi,\varphi)}. \tag{2.7}$$

Respectively, the Hamiltonian looks as follows:

$$\mathcal{I} = \frac{k'^2}{2}(I + (\alpha_1 + \alpha_2))^2. \tag{2.8}$$

Comparing the results (2.5) and (2.8), obtained by a canonical transformation from (2.2) and (2.3), respectively, we conclude that they differ in the «mass» of the (locally equivalent) free particle as well as in the domain of the momentum (action) variable. Thus, in general, all systems can be distinguished globally. Interestingly, however, any odd system $(k_{\rm odd};\alpha_0)$ matches *globally* to a one-parameter family system of even systems $(k_{\rm even};\alpha_1,\alpha_2)$ by the equivalence

$$(k_{\rm odd};\alpha_0)\sim(2k_{\rm odd};\beta,\alpha_0-\beta)\quad \text{with}\quad 0<\beta<\alpha_0. \tag{2.9}$$

Particularly, choosing $k=2$, we establish the global equivalence of A_2 and G_2 rational Calogero models, and their local equivalence with a free particle on the circle.

3. SUPERINTEGRABLE SYSTEMS

The action-angle variables elegantly explain the superintegrability of the recently suggested deformation of the two-dimensional oscillator system introduced by Tremblay–Turbiner–Winternitz [12] and also of the Coulomb versions treated by Post–Winternitz [13]. They also allow us to construct analogous deformations of the spherical and pseudospherical generalizations of oscillator and Coulomb systems suggested in [8],

$$H=\frac{p_r^2}{2}+\frac{\mathcal{I}_{\rm PT}(p_\varphi,\varphi)}{r^2}+V(r),\quad \{p_r,r\}=1. \tag{3.1}$$

Here, we introduced a radial coordinate r and momentum p_r, and $\mathcal{I}_{\rm PT}$ is a generalized Pöschl–Teller system on the circle

$$\mathcal{I}(p_\varphi,\varphi|k)=\frac{1}{2}p_\varphi^2\;+\frac{(k\alpha_1)^2}{2\cos^2 k\varphi}+\frac{(k\alpha_2)^2}{2\sin^2 k\varphi},\quad \{p_\varphi,\varphi\}=1. \tag{3.2}$$

Choosing the oscillator potential

$$V(r)=\frac{1}{2}\omega^2r^2, \tag{3.3}$$

we will arrive at the Tremblay–Turbiner–Winternitz system.

With the choice of the Coulomb potential

$$V(r)=-\frac{\gamma}{r},\quad \gamma>0, \tag{3.4}$$

we will get Post–Winternitz system.

Their generalizations to the sphere and pseudosphere are obvious. Those are defined by the following redefinition:

$$S^N: \quad r = r_0 \sin\chi, \qquad p_r = r_0^{-1} p_\chi, \quad V(r) \to V(r_0 \tan\chi), \tag{3.5}$$

$$\mathbb{R}^N: \quad r = r_0 \chi, \qquad p_r = r_0^{-1} p_\chi, \quad V(r) \to V(r_0 \chi), \tag{3.6}$$

$$H^N: \quad r = r_0 \sinh\chi, \quad p_r = r_0^{-1} p_\chi, \quad V(r) \to V(r_0 \tanh\chi), \tag{3.7}$$

where r_0 is the radial scale and $\{p_\chi, \chi\} = 1$ is a dimensionless canonical pair.

Let us show that these systems are superintegrable ones for the rational k. Formulating these system in the action-angle variables, we get that the Hamiltonian of the Tremblay–Turbiner–Winternitz system looks as follows:

$$H_\omega = \begin{cases} \omega\left(2I_\chi + k(I + \alpha_1 + \alpha_2)\right) & \text{for} \quad \mathbb{R}^2, \\ \dfrac{1}{2}(2I_\chi + k(I + \alpha_1 + \alpha_2) + \omega)^2 - \dfrac{\omega^2}{2} & \text{for} \quad S^2, \\ -\dfrac{1}{2}(2I_\chi + k(I + \alpha_1 + \alpha_2) - \omega)^2 + \dfrac{\omega^2}{2} & \text{for} \quad H^2 \end{cases} \tag{3.8}$$

and depends only on the combination $2I_\chi + k(I + \alpha_1 + \alpha_2)$. Thus, the evolution of the angle variables is given by

$$\Phi_\chi(t) = 2\Omega t, \quad \Phi_\varphi(t) = k\Omega t, \quad \text{with} \quad \Omega = \frac{dH_\omega}{d(2I_\chi + k(I + \alpha_1 + \alpha_2))}. \tag{3.9}$$

For rational values of k the trajectories are closed. It then follows that the hidden constant of motion is*

$$I_{\text{hidden}} = \cos\left(m\Phi_\chi - 2n\Phi_\varphi\right) \quad \text{for} \quad k = m/n. \tag{3.10}$$

The construction of superintegrable deformations of the Coulomb system, i.e., the Post–Winternitz system and its generalization to the (pseudo)spherical environment, proceeds completely similarly. The Hamiltonians depend only on the combination $I_\chi + k(I + \alpha_1 + \alpha_2)$,

$$H_\gamma = \begin{cases} -\dfrac{\gamma^2}{2}(I_\chi + k(I + \alpha_1 + \alpha_2))^{-2} & \text{for} \quad \mathbb{R}^2, \\ -\dfrac{\gamma^2}{2}(I_\chi + k(I + \alpha_1 + \alpha_2))^{-2} + \dfrac{1}{2}(I_\chi + k(I + \alpha_1 + \alpha_2))^2 & \text{for} \quad S^2, \\ -\dfrac{\gamma^2}{2}(I_\chi + k(I + \alpha_1 + \alpha_2))^{-2} - \dfrac{1}{2}(I_\chi + k(I + \alpha_1 + \alpha_2))^2 & \text{for} \quad H^2. \end{cases} \tag{3.11}$$

*For the oscillator case, the hidden constants of motion have been constructed in [18].

Hence, for rational $k = m/n$ the trajectories are closed, while the hidden symmetry is defined by the expression

$$I_{\text{hidden}} = \cos(m\Phi_\chi - n\Phi_\varphi). \tag{3.12}$$

Thus, we get superintegrable (pseudo)spherical analogs of the Tremblay–Turbiner–Winternitz and of the Post–Winternitz models.

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