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ASPECTS OF POHLMEYER REDUCTION FOR SUPERSTRINGS IN $AdS_5 \times S^5$

A. A. Tseytlin

Blackett Laboratory, Imperial College, London
and Theoretical Physics Division, Lebedev Institute, Moscow

We review some recent work on Pohlmeyer reduced theory associated with superstring theory in $AdS_5 \times S^5$. We discuss the S-matrix of the reduced theory and also the computation of the 2-loop correction to the partition function in a nontrivial background.

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Below we shall review some aspects of our recent work [1,2] on Pohlmeyer reduced theory associated with superstring theory in $AdS_5 \times S^5$. Ref. [1] continued the investigation [3,4] of the S-matrix of the Pohlmeyer reduced version of superstring theory on $AdS_5 \times S^5$. One motivation is to shed light on an eventual first-principles solution of the $AdS_5 \times S^5$ superstring based on quantum integrability.

One may view the reduced $AdS_5 \times S^5$ theory as a member of a class of $AdS_n \times S^n$ ($n = 2, 3, 5$) theories which are Pohlmeyer reductions of Green–Schwarz superstring sigma models based on $AdS_n \times S^n$ supercosets. These reduced theories [5–7] are fermionic extensions of generalized sine-Gordon models. Various examples of such bosonic models (also called «symmetric space sine-Gordon models») are based on a G/H gauged WZW theory with an integrable potential term. Due to their relation via Pohlmeyer reduction [8] to classical GS superstring theory on $AdS_n \times S^n$ (and, more generally, their bosonic truncations to classical string theory on symmetric spaces), there has been recent interest in these models.

The fields of the reduced theory are all valued in certain subspaces of a particular representation of the superalgebra $\hat{\mathfrak{f}}$, whose corresponding supergroup $\hat{F}$ is the global symmetry of the original superstring sigma model. The latter is based on the supercoset $\hat{F}/G$, where G is a bosonic subgroup of $\hat{F}$ and is the gauge group of the superstring sigma model. For the $AdS_5 \times S^5$ superstring [9] the supercoset is

$$\frac{PSU(2,2|4)}{Sp(2,2) \times Sp(4)}. \tag{1}$$

The gauge group H of the reduced theory is a subgroup of G that appears upon solving the Virasoro constraints. The reduced theory action is a fermionic extension of the G/H gauged WZW theory with an integrable potential [5]*

$$\mathcal{S} = \frac{k}{4\pi}\,\mathrm{STr}\Bigg[\frac{1}{2}\int d^2x\, g^{-1}\partial_+ g g^{-1}\partial_- g - \frac{1}{3}\int d^3x\,\epsilon^{mnl} g^{-1}\partial_m g g^{-1}\partial_n g g^{-1}\partial_l g +$$
$$+\int d^2x\big(A_+\partial_- g g^{-1} - A_- g^{-1}\partial_+ g - g^{-1}A_+ g A_- + A_+A_- + \mu^2\,(g^{-1}TgT - T^2)\big) +$$
$$+\int d^2x\,\left(\Psi_{_L} T D_+ \Psi_{_L} + \Psi_{_R} T D_- \Psi_{_R} + \mu g^{-1}\Psi_{_L} g \Psi_{_R}\right)\Bigg]. \quad (2)$$

Here $g \in G$, $A_\pm \in \mathfrak{h} = \mathrm{alg}\,(\mathfrak{H})$ and the fermionic fields $\Psi_{_L}$, $\Psi_{_R}$ take values in fermionic subspaces of $\hat{\mathfrak{f}}$; k is a coupling constant (level); μ is a parameter defining the mass of perturbative excitations near $g = \mathbf{1}$. The constant matrix T defining the potential commutes with H.

In the case of the $AdS_5 \times S^5$ superstring, the Pohlmeyer reduced theory has certain unique features; in particular, it is UV-finite and its one-loop semiclassical partition function is equivalent to that of the original $AdS_5 \times S^5$ superstring [10]. This suggests [5] that it may be quantum-equivalent to the $AdS_5 \times S^5$ superstring. If this were the case, the Pohlmeyer reduced theory could be used as a starting point for a 2-d Lorentz covariant «first-principles» solution of the $AdS_5 \times S^5$ superstring. The Lorentz invariance of (2) is a desirable feature as lack of 2-d Lorentz symmetry in the light-cone gauge $AdS_5 \times S^5$ superstring S-matrix leads, e.g., to a complicated structure for the corresponding thermodynamic Bethe ansatz for the full quantum superstring spectrum.

The form of the light-cone gauge $AdS_5 \times S^5$ superstring S-matrix (corresponding to the spin-chain magnon S-matrix on the gauge theory side) is fixed, up to a phase, by the residual global $PSU(2|2) \times PSU(2|2)$ symmetry of the light-cone gauge Hamiltonian. This S-matrix is the starting point for the conjectured Bethe ansatz solution for the superstring energy spectrum based on its integrability (see [11]). Just as for the standard 2-d sigma models or other similar massive theories, the starting point for solving the Pohlmeyer reduced theory is to find its exact S-matrix. Any proposal for the exact quantum S-matrix should be, of course, consistent with the perturbative S-matrix computed from the path integral defined by the classical action.

*We choose Minkowski signature in 2 dimensions with $d^2x = dx^0dx^1$, $\partial_+ \equiv \partial_0 \pm \partial_1$. For algebras $[\mathfrak{a}]^2 = \mathfrak{a}\oplus\mathfrak{a}$, i.e., the direct sum. We also use the notation $\Subset$ for a semidirect sum of algebras and $\ltimes$ for a central extension. For example, for the semidirect sum $\mathfrak{a}\Subset\mathfrak{b}$ we have the commutation relations: $[\mathfrak{a},\mathfrak{a}] \subset \mathfrak{a}$, $[\mathfrak{b},\mathfrak{b}] \subset \mathfrak{b}$ and $[\mathfrak{a},\mathfrak{b}] \subset \mathfrak{b}$.

This motivates the study of the perturbative S-matrix of the Pohlmeyer reduced $AdS_n \times S^n$ superstring. In [3] we computed the tree-level two-particle S-matrix for the 8+8 massive excitations of the reduced $AdS_5 \times S^5$ theory employing the light-cone $A_+ = 0$ gauge. Remarkably, the resulting S-matrix factorises in the same way as the non-Lorentz invariant $[\mathfrak{psu}(2|2)]^2 \ltimes \mathbb{R}^3$ symmetric light-cone gauge S-matrix of the $AdS_5 \times S^5$ superstring. The factorized S-matrix has an intriguing similarity with a particular limit of the quantum-deformed $\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3$ invariant R-matrix of [12, 13].

In [4] the perturbative computation was extended to the one-loop level for the bosonic part of the $G/H = SO(N+1)/SO(N)$ theories defined by (2). It was argued that the Lagrangian describing the physical fields constructed from the gauged WZW model (2) should be supplemented by a particular one-loop counterterm coming from the path integral. For the $G/H = SU(2)/U(1)$ theory*, the one-loop counterterm contributions to the S-matrix were computed in three different ways, all giving the same result. These contributions were precisely those needed to restore the validity of the Yang–Baxter equation (YBE) at the one-loop level and to match the exact quantum soliton S-matrix proposed in [14].

In [1] we investigated the quantum S-matrices for the perturbative massive excitations of the models (2), which are the Pohlmeyer reductions of the $AdS_2 \times S^2$, $AdS_3 \times S^3$ and $AdS_5 \times S^5$ superstring models. Studying the three models together turns out to be useful as it reveals certain universal features of their symmetries and S-matrices, and thus helps to shed light on the structure of the most nontrivial case of the $AdS_5 \times S^5$ theory. The reduced $AdS_2 \times S^2$ theory is equivalent [5] to the $\mathcal{N} = 2$ supersymmetric sine-Gordon model. In the reduced $AdS_3 \times S^3$ theory, the one-loop S-matrix computed starting from the classical Lagrangian does not satisfy the Yang–Baxter equation, but we have shown that one can find a local counterterm that restores the YBE and thus integrability, much like in the bosonic complex sine-Gordon theory case discussed in [4]. The addition of the counterterm not only restores the validity of the YBE, but also ensures the group factorization property and leads to a novel quantum-deformed supersymmetry of the S-matrix. The existence of a hidden 2-d supersymmetry in the classical reduced $AdS_3 \times S^3$ and $AdS_5 \times S^5$ theories was conjectured, by analogy with the $AdS_2 \times S^2$ case, in [5] and was recently demonstrated in [15, 16].

Assuming that the group-factorization and the quantum-deformed supersymmetry are true symmetries of the theory, we conjectured [1] an exact 2-d Lorentz invariant quantum S-matrix for the perturbative excitations of the reduced $AdS_3 \times S^3$ theory. The phase factor is fixed by the unitarity and crossing

*This theory is classically equivalent to the complex sine-Gordon theory, as seen by fixing a gauge on the group field g and integrating out $A_\pm$.

constraints (and is similar to that in the reduced $AdS_2 \times S^2$ theory). We checked that the resulting exact S-matrix expanded in $1/k$ agrees with our one-loop computation.

In the $AdS_5 \times S^5$ case, we observed that the one-loop S-matrix group-factorizes in the same way as at the tree level in [3]. However, there is a tree-level anomaly in the YBE [3,4], which is a general feature of the models (2) with a non-Abelian gauge group H. As in the bosonic case [7], this anomaly cannot be cancelled by adding a local two-derivative counterterm without breaking the manifest non-Abelian symmetry, indicating some subtlety with a realization of integrability.

Motivated by the quantum-deformed supersymmetry, we discovered in the S-matrix of the reduced $AdS_3 \times S^3$ theory and the quantum-deformed non-Abelian symmetry expected in soliton S-matrices of the bosonic theories [17] we conjectured that the S-matrix for the perturbative massive excitations of the reduced $AdS_5 \times S^5$ theory may be related to a trigonometric relativistic limit of the quantum-deformed $\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3$ invariant R-matrix of [12, 13], which satisfies the YBE by construction. The phase factor can be again fixed by unitarity and crossing and is the same as in the reduced $AdS_2 \times S^2$ and $AdS_3 \times S^3$ theories. The one-loop expansion of the resulting R-matrix indeed has a similar structure to the one-loop S-matrix that we found by direct computation.

One possibility is that the S-matrix for the perturbative excitations in a gauge-fixed Lagrangian is given by a certain non-unitary rotation of the quantum-deformed R-matrix. The violation of the YBE by the S-matrix computed directly from the Lagrangian may be related to some tension between gauge-fixing of the non-Abelian H symmetry and the conservation of hidden charges. It is also possible that the physical excitations whose S-matrix is the quantum-deformed R-matrix may be some nontrivial gauge-invariant combinations of the Lagrangian fields*. At the moment it is still an open question as to what is the origin of the quantum deformation. An alternative approach based on semiclassical quantization of solitons [16, 17] may shed more light on this issue.

Let us now turn to the question of quantum partition function of reduced theory [2]. While the conformal-gauge $AdS_5 \times S^5$ string theory (ST) and the associated Pohlmeyer-reduced theory (PRT) are closely related at the classical level, PRT has important simplifying features being 2d Lorentz invariant and quadratic in fermions, which have standard 2d kinetic terms. When expanded near the respective vacua, the two theories are described by the equivalent sets of $8 + 8$ boson+fermion physical 2d fields. This raises a hope that PRT may be useful in an attempt to solve the $AdS_5 \times S^5$ ST from first principles.

*In the reduced $AdS_3 \times S^3$ and $AdS_5 \times S^5$ theories, the quantum deformation parameter is, respectively, $q = \mathrm{e}^{-2i\pi/k}$ and $\mathrm{e}^{-i\pi/k}$; i.e., it is the coupling constant k that controls the deformation.

The relation between the $AdS_5 \times S^5$ ST and PRT is established at the classical level and involves a transformation from coset currents to new fields in a way that solves the conformal gauge conditions algebraically (both theories originate from the same set of first-order equations for the currents). The classical solutions are thus in correspondence (though the values of the two actions on the associated solutions are, in general, different).

The relation between the $AdS_5 \times S^5$ ST and PRT at the quantum level is a priori unclear. Nevertheless, given their classical connection, and the integrability and UV finiteness of both theories, one may conjecture that the two quantum theories should also be closely related. The precise form of such a relation remains to be understood. An indication of a quantum relation is the equality of the one-loop partition functions of the two theories computed by expanding near «dual» solutions [10]

$$Z^{(1)}_{\rm ST} = Z^{(1)}_{\rm PRT}. \tag{3}$$

While one may be tempted to view this one-loop relation as a consequence of the classical equivalence of the two theories (suggesting that determinants of small fluctuation operators found by perturbing the classical solutions should match), it is still a nontrivial test* of the correspondence between the underlying physical degrees of freedom of the two theories.

The aim of [2] was to explore possible relations between the two quantum partition functions at the two-loop level. Since the two-loop computations in a nontrivial background are, in general, very complicated, here we will consider the simplest string solution — the infinite spin (scaling) limit of the folded spinning (S, J) string in $AdS_3 \times S^1$ subspace of $AdS_5 \times S^5$ [18] and the associated solution of the reduced theory. As a further simplification, we will consider the limit $J \to 0$ when the logarithm of the string theory worldsheet partition function is simply proportional to a function of the coupling constant (i.e., to the universal scaling function of string tension on the string theory side). The conclusion (in both $AdS_5 \times S^5$ and $AdS_3 \times S^3$ cases) is the following: while the nontrivial parts of the two two-loop partition functions (coming from the most complicated two-loop integrals) appear to be a direct correspondence, the reduced theory partition function contains an extra two-loop term proportional to the square of the one-loop coefficient. Thus, if the two quantum partition functions are indeed related, this relation may be effectively nonlinear. It is also possible that the matching of two partition functions may be restored by modifying the PRT action by a certain one-loop counterterm that may be required to maintain its quantum integrability, i.e., to preserve certain hidden (super)symmetries.

*One may, in principle, construct a pair of classically equivalent theories that have different one-loop partition functions.

Let us recall the known structure of the two-loop string partition in the long spinning string background [19, 20]. The (S, J) spinning string background in the large spin limit has the following form in terms of the $AdS_5 \times S^5$ embedding coordinates ($S = \sqrt{\lambda}\mathcal{S}$, $J = \sqrt{\lambda}\mathcal{J}$, where $\sqrt{\lambda}/2\pi$ is string tension):

$$\begin{gathered} Y_0 + iY_5 = \cosh(\ell\sigma)\,\mathrm{e}^{i\kappa\tau}, \quad Y_1 + iY_2 = \sinh(\ell\sigma)\,\mathrm{e}^{i\kappa\tau}, \quad Y_{3,4} = 0, \\ X_1 + iX_2 = \mathrm{e}^{i\mu\tau}, \quad X_{3,4,5,6} = 0, \end{gathered} \tag{4}$$

where the parameters $\kappa \gg 1$, $\ell \gg 1$ and μ are related by

$$\kappa^2 = \ell^2 + \mu^2, \quad \ell = \frac{1}{\pi}\ln\mathcal{S} \gg 1, \quad \mathcal{J} = \mu. \tag{5}$$

We will be interested in the limit $\mu \to 0$ when $\kappa \to \ell$ is the only scale in the problem. The logarithm of the resulting quantum partition function is given by (V_2 is the volume)

$$\Gamma_{\rm ST} = -\ln Z_{\rm ST} = \frac{1}{2\pi} f(\lambda) V_2, \tag{6}$$

$$f(\lambda) = a_1 + \frac{a_2}{\sqrt{\lambda}} + O\left(\frac{1}{(\sqrt{\lambda})^2}\right), \tag{7}$$

$$a_1 = -3\ln 2, \quad a_2 = a_{2B} + a_{2F} = K - 2K = -K. \tag{8}$$

Here a_1 is the one-loop and a_2 is the two-loop contribution (K is Catalan's constant). In a_2 we indicated separately the part coming from purely bosonic graphs and graphs involving fermions. The spectrum of the string fluctuation modes [18] includes one AdS_3 mode with $m^2 = 4$, two AdS_5 modes «transverse» to AdS_3 with $m^2 = 2$, five S^5 modes with $m^2 = 0$ and eight fermionic modes with $m^2 = 1$. Contributions proportional to K originate from two-loop «sunset» graphs with three propagators that are expressed in terms of the following momentum integrals:

$$I[m_i^2, m_j^2, m_k^2] \equiv \int \frac{d^2q_i\, d^2q_j\, d^2q_k}{(2\pi)^4} \frac{\delta^{(2)}(q_i + q_j + q_k)}{(q_i^2 + m_i^2)(q_j^2 + m_j^2)(q_k^2 + m_k^2)}, \tag{9}$$

$$I[4, 2, 2] = \frac{1}{(4\pi)^2} K, \quad I[2, 1, 1] = \frac{2}{(4\pi)^2} K. \tag{10}$$

Here both the bosonic $I[4, 2, 2]$ and the fermionic $I[2, 1, 1]$ contributions involve the «transverse» AdS_5 modes with $m^2 = 2$. Since such modes are absent in the case of the $AdS_3 \times S^3$ superstring theory, one expects to find there no Catalan constant contribution. Indeed, there [2]

$$AdS_3 \times S^3: \quad a_1 = -2\ln 2, \quad a_2 = 0. \tag{11}$$

Let us now summarize the results of the corresponding two-loop computation in the reduced theory [2]. The coupling constant k of the reduced theory is undetermined by the classical reduction procedure. If the quantum string theory and the quantum reduced theory are to be related at all, k should be related to the string tension or $\sqrt{\lambda}$. Observing that the μ-dependent terms in the reduced theory Lagrangian (2) are exactly equal to the superstring Lagrangian (with the components of the coset current replaced by its reduced theory values $J_+^{(2)} = \mu T$, $J_-^{(2)} = \mu g^{-1} T g$, etc.), one may conjecture that

$$k = 2\sqrt{\lambda}. \tag{12}$$

While the matching of the one-loop partition functions (3) is not sensitive to the values of the two coupling constants (as they do not enter the determinants of the quadratic fluctuation operators), the comparison of higher-loop quantum corrections crucially depends on a relation like (12).

The aim is to compute the two-loop correction to the partition function of the PRT expanded near a solution which is a counterpart of the long spinning string solution (4). In the reduced theory, the parameter μ of the solution in (4) becomes identified with the μ in the PRT action*. The logarithm of the quantum partition function in the reduced theory has a similar form as in string theory (cf. (6), (7)):

$$\Gamma_{\rm PRT} = -\ln Z_{\rm PRT} = \frac{1}{2\pi} f(k) V_2, \tag{13}$$

$$f(k) = a_1 + \frac{2a_2}{k} + O\left(\frac{1}{k^2}\right). \tag{14}$$

Explicit results for the coefficients a_n are [2] (cf. (8))

$$a_1 = -3\ln 2, \quad a_2 = \bar{a}_2 + \tilde{a}_2, \quad \bar{a}_2 = -K, \quad \tilde{a}_2 = -\frac{1}{4}(a_1)^2 = -\frac{9}{4}(\ln 2)^2. \tag{15}$$

The value of the one-loop coefficient a_1 matches the string theory one in (8), and in agreement with (3). The Catalan constant term in $\bar{a}_2$ has exactly the same coefficient as in the string partition function in (8), provided we assume the identification of couplings in (12). Moreover, the pattern of the bosonic and fermionic contributions (i.e., $+K - 2K = -K$) is exactly the same as in the string theory expression in (8).

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*While we keep μ nonzero at the intermediate stages, to be able to obtain the explicit two-loop result, we will take the $\mu \to 0$ limit in the final expression; i.e., we will do the two-loop quantum PRT computation for the counterpart of the long spinning string with $J = 0$.

While the mass spectra of the quadratic fluctuation Lagrangians are equivalent, the interaction vertices could, in principle, generate additional nontrivial contributions in PRT, e.g., proportional to $I[4,4,4]$ in (9), which are not related to Catalan's constant. However, all such extra nontrivial integrals happen not to appear in PRT. This and the matching of Catalan's constant is a strong indication that the two quantum theories are indeed closely connected.

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