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ANALYTICAL APPROACH TO 2D HOLOGRAPHIC SUPERCONDUCTOR

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We briefly discuss the phase transitions in 2D Holographic Superconductors in the probe limit, and in rotating and non-rotating Black Hole backgrounds.

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1. MOTIVATION

Superconductivity is one of the most fascinating phenomena in condensed matter physics, which gave a great impact on the development of science and technology. Discovered 100 years ago, it has a rich history of events, when the first half of the «Superconductivity century» was devoted to extending the number of what we are presently calling conventional superconductors, and to establishing their theory. Nonconventional, high-temperature, superconductors (HTSCs) are known since 1986, but the solid theoretical ground to describe them still remains uncovered.

Recall, conventional superconductors are described by BCSB [1] theory. The main ingredients of BCSB are: 1. (second order) phase transition (described by the Landau–Ginsburg phenomenological theory); 2. the electron Cooper pairs condensate formation; 3. forming an energy gap. The effective coupling of BCSB is the electron-phonon coupling, so BCSB is one of the best examples of a weak coupling constant theory. In contrast, the description of HTSCs can be achieved within a theory in the strong coupling constant regime [2]. So the question is: how to formulate such a theory? Below we will exploit the prescription, which comes from String theory: to describe Holographic Superconductivity [4] in the frameworks of the AdS/CFT correspondence.

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2. AdS/CFT ON A NUTSHELL AND BASIC INGREDIENTS FOR A HOLOGRAPHIC SUPERCONDUCTOR

The main principle of the AdS/CFT correspondence [3] is formulated as follows: Gauge theory on the AdS_{d+1} boundary, and at a strong coupling constant, is dual to the bulk gravity with matter, and at a weak coupling. A local CFT may be defined by the asymptotic expansion of a bulk field near the boundary of AdS_{d+1}

$$\Theta(z) = \mathcal{A} z^{\Delta_-} (1 + \dots) + \mathcal{B} z^{\Delta_+} (1 + \dots), \tag{1}$$

where Δ is the scaling dimension, depending on the spin and on the mass of the bulk field, $\mathcal{A}$ is the source to the dual boundary CFT operator $\mathcal{O}$, and $\mathcal{B}$ is its expectation value $\langle\mathcal{O}\rangle$. For a massless vector field with just one temporal component, $A_t(z \to 0) = \mathcal{A} z^{(d-2)}(1 + \dots) + \mathcal{B}(1 + \dots)$, z-dependent part has a fast falloff, hence it is not a background field in the dual theory. Therefore, $\mathcal{A}$ fixes the electric charge density of the state [4]. The finite part of $A_t(z \to 0)$ is a chemical potential for the electric charge density ρ. For $d = 2$, one has to use [5] $z^\epsilon = 1 + \epsilon \ln z + \dots$ with $\epsilon = (d-2)$, subsequently rescaling the chemical potential $\mu \to -\mu/\epsilon$ as well.

Let's focus on AdS_3/CFT_2. This case is technically more simple, less studied, and includes main ingredients of holographic superconductors (HSCs). It has a relation to real systems like superconducting nanowires, and exhibits the rich symmetry structure (pure 2d CFT with infinitely dimensional conf. symmetry, entropy relation to the algebra central charge, etc.). Though gravity is not dynamical in the bulk, there are AdS_3 Black Holes (BHs) [6]. In the probe limit, without backreaction on the metric, the studied system is described by the action [4,5]

$$S = -\int d^3x \sqrt{-g} \left(\frac{1}{4} F_{mn} F^{mn} + (\partial_m - iA_m)\Psi(\partial^m + iA^m)\Psi^* + m^2 \Psi\Psi^* \right) \tag{2}$$

in the background of neutral AdS BH [6]

$$ds^2 = [L^2/z^2] \left(-f(z)dt^2 + dx^2 + \frac{dz^2}{f(z)} \right), \quad f(z) = 1 - \frac{z^2}{z_H^2}. \tag{3}$$

Here L is a length of AdS, the boundary is located at $z = 0$, the BH horizon is at $z = z_H$. Metric (3) solves the Einstein equation $2R_{mn} = g_{mn}\left(R + 2/L^2\right)$, and the BH temperature is $T = 1/(2\pi z_H)$.

The dynamics of fields in the bulk is described by

$$\begin{aligned} D_m(\sqrt{-g} D^m \Psi) - \sqrt{-g} m^2 \Psi = 0, \\ \partial_n(\sqrt{-g}\, F^{nm}) + \sqrt{-g}\, i(\Psi \overline{D^m} \Psi^* - \Psi^* D^m \Psi) = 0, \end{aligned} \tag{4}$$

with $D\Psi \equiv (\partial_m - iA_m)\Psi$, $\overline{D^m}\Psi^* = (D\Psi)^*$, $F_{mn} = 2\partial_{[m}A_{n]}$. Following [4], we are going to solve (4) with the ansatz $\Psi = \psi(z)$, $A = \phi(z)dt$. On account of the latter and of the BH metric (3), equations (4) become

$$\begin{aligned} &\psi'' + \left(\frac{f'}{f} - \frac{1}{z}\right)\psi' + \left(\frac{\phi^2}{f^2} - \frac{m^2L^2}{z^2 f}\right)\psi = 0, \\ &\phi'' + \left(\frac{\phi'}{z} - \frac{2\psi^2\phi L^2}{z^2 f}\right) = 0. \end{aligned} \tag{5}$$

To solve (5), as analytically (see, e.g., [7]) as well as numerically [5,8], boundary conditions (BCs) should be specified. At $z = z_H$ and at the AdS boundary the BCs are

$$\begin{aligned} &\phi(z_H) = 0, \quad \psi'(z_H) = \frac{\psi(z_H)}{2z_H}, \quad \phi = \mu \ln z - \rho, \\ &\psi = \psi^{(2)} z \to \langle \mathcal{O} \rangle_\psi = \psi^{(2)}. \end{aligned} \tag{6}$$

3. PHASE TRANSITIONS IN BTZ BACKGROUNDS

With the setup in above, solving (4) analytically involves several steps [7]: expand the fields $\Phi = (\phi, \psi)$, $\Phi(z) = \Phi(Z) + \Phi'(Z)(z-Z) + \Phi''(Z)(z-Z)^2/2 + \ldots$ near the horizon ($Z = z_H$) and at the boundary ($Z = 0$); take the BCs into account; evaluate a few first coefficients in the fields series expansions near the boundary and near the horizon by the use of the BCs and equations of motion; and sew the so-obtained fields' series expansions, and their first derivatives, at an intermediate point z, that will lead to the result. Following the prescription, and setting the sew point at $z = 1/2z_H$, we get [9]

$$T_c = 2\frac{\mu}{\pi\sqrt{123}} \approx 0.057\mu \tag{7}$$

for the critical temperature of the phase transition, and ($L = 1$)

$$\langle \mathcal{O} \rangle_\psi \approx 12.7\sqrt{TT_c}\left(1 - \frac{T}{T_c}\right)^{1/2} \xrightarrow{T \to T_c} \langle \mathcal{O} \rangle_\psi \approx 12.7\, T_c \left(1 - \frac{T}{T_c}\right)^{1/2}. \tag{8}$$

The critical exponent and the temperature dependence of the scalar dual operator expectation value is typical for the second-order phase transitions occurred in superconductors. The numerical coefficient is in a good agreement with that obtained in the numerical studies [5, 8]. There one gets $\langle \mathcal{O} \rangle_\psi \approx 12.2\, T_c(1 - T/T_c)^{1/2}$, $T_c/\mu \approx 0.136$. The discrepancy in T_c is expected for this type of analytical calculations, and may be slightly improved by choosing the appropriate sewing point.

Let's try to extend the standard setup modifying fields in the bulk, or the background in which they propagate. A natural modification for AdS_3 Maxwell field consists in adding the Chern–Simons topological term, and to make the gauge field massive, i.e.,

$$S = S_0(A, \Psi, \partial A, D\Psi) + \theta/2 \int A \wedge dA, \tag{9}$$

However, such a modification is nontrivial once new magnetic degrees of freedom (DOF) appear in the ansatz for A, $A = \phi(z)dt + B(z)dz$ ($\Psi = \psi(z)$). It is possible in the probe limit, but it cannot be realized in the complete setup with the backreaction on the metric. The reason is all the magnetic BTZ-type solutions in EMS(CS) systems are horizonless [10].

But inclusion of external magnetic field can be realized in another way [11], with taking into account the Barnett–London effect of magnetization of uncharged, but rotated body, and the Lense–Thirring dragging force effect which guarantees the AdS boundary rotation in the background of rotating BTZ BH. Hence, we have to put our Maxwell-Scalar interacting system in the background of a rotating BTZ BH (with $f(z) = 1 - Mz^2/L^2 + J^2z^4/(4L^6)$)

$$ds^2 = [\frac{L^2}{z^2}] \left[-f(z)\, dt^2 + \frac{dz^2}{f(z)} + L^2 \left(d\varphi - \frac{Jz^2}{(2L^4)}\, dt \right)^2 \right]. \tag{10}$$

Now Eqs. (4) have to be solved with the following ansatz $\Psi = \Psi(z, \tilde{\varphi})$, $A = \phi(z)\, dt + \xi(z) d\tilde{\varphi}$, $\tilde{\varphi} = L\varphi$, and in the background (10). The system looks complicated, so we need a simplification. The small angular momentum approximation, in which $\Psi = \Psi(z, \tilde{\varphi})$, $A = \phi(z)\, dt + \xi(z)\, d\tilde{\varphi}$, $J \ll 1$, $\xi(z) \ll \phi(z)$, makes the problem more tractable (see [9] for details). The system of equations of motion transforms in the limit to [9]

$$\psi'' + \left(\frac{f'}{f} - \frac{1}{z} \right) \psi' + \left(\frac{\phi^2}{f^2} - \frac{m^2L^2}{z^2 f} - \frac{z}{Lf}\lambda \right) \psi \approx 0, \tag{11}$$

$$\phi'' + \frac{1}{z} \left(1 + \frac{Jz^3}{L^3} \right) \phi' - \frac{L^2}{z^2} \frac{2\psi^2 \mathrm{e}^{2i\alpha\tilde{\varphi}}}{f} \phi \approx 0 \tag{12}$$

with the separation constant λ, coming from the solution to the angular part of the scalar equation of motion: $\lambda \approx L\alpha^2/z + \alpha J z \phi(z)/(2L^2 f(z))$.

Doing the same machinery of analytical calculations as in the nonrotating case, we get the following T dependence of the CFT scalar operator expectation value

$$\langle \mathcal{O} \rangle_\psi \approx 16.68\, T \left(1 - \frac{\sqrt{123}}{4\mu} \left[2\pi T + \frac{17\alpha^2}{164\pi T} - \frac{\alpha \mathcal{J}_R}{8\pi^2 T^2 L^3} \right] \right)^{1/2}. \tag{13}$$

Here $\mathcal{J}_R$ is the renormalized angular momentum (see [9] for details), and we are within the approximation $\alpha\mathcal{J}_R \ll 1$, $\alpha \ll 1$.

What are we expecting to get when the BH becomes rotating? In the background of the rotating BTZ BH the radial part of the scalar equation is $\partial_m(\sqrt{-g}g^{mn}\partial_n\Psi) + f(A,\partial)\Psi - V(\Psi) = 0$, with some operator $f(A,\partial)$ and the effective potential $V(\Psi) = \sqrt{-g}[m^2 + A_t g^{tt} A_t + A_{\tilde{\varphi}} g^{\tilde{\varphi}\tilde{\varphi}} A_{\tilde{\varphi}}]$. If there are magnetic DOF in A, the condensation becomes hard in comparison with the pure electric case $A_{\tilde{\varphi}} = 0$, when $g^{tt} < 0$, and the effective mass decreases. In the electro-magnetic case, due to $g^{\tilde{\varphi}\tilde{\varphi}}$, the effective mass gets decreased smaller, making the condensation hard. Though we have not magnetic DOF in the ansatz, external magnetic field is modeled by the rotation. Hence we expect the critical temperature of the phase transition will become lower in comparison with the nonrotating case, see Fig. 1 in [9]. However, one may encounter the situation when the critical temperature of the phase transitions becomes higher than in the nonrotating HSC, see Fig. 2 in [9].

4. CONCLUSIONS

To summarize, we have applied the analytical methods to a 2D holographic superconductor in the probe limit. It was found a good agreement between the boundary scalar operator expectation value in analytical and numerical approaches, but the value of the critical temperature, estimated analytically, is about twice less than that reproduced in the numerical calculations. This discrepancy is typical within the approach we followed, as it comes from Table 1 in [12]. It can be slightly improved by the appropriate choice of the sewing point, however the coefficient in the scalar CFT operator will be changed. It turns out that, in dependence on the choice of free parameters of the model (α, J_R), one may encounter as «normal» the lowering of the critical temperature due to the BH rotating, as well as «abnormal» the T_c increasing. It would be interesting to reproduce this effect in numerical simulations of the complete problem with backreaction, and out of the small angular momentum approximation.

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