ФИЗИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ И АТОМНОГО ЯДРА
2013. Т. 44. ВЫП. 2

ADVECTION OF A PASSIVE VECTOR FIELD BY THE GAUSSIAN VELOCITY FIELD WITH FINITE CORRELATIONS IN TIME

A. V. Gladyshev [1, *], *E. Jurčišinová* [2, **], *M. Jurčišin* [2, ***], *R. Remecký* [1,2 ****]

[1]Joint Institute for Nuclear Research, Dubna

[2]Institute of Experimental Physics, Slovak Academy of Sciences, Košice, Slovakia

Using the field-theoretic renormalization-group technique, the model of a passive vector field advected by an incompressible turbulent flow is investigated up to the second order of the perturbation theory (two-loop approximation). The turbulent environment is given by statistical fluctuations of the velocity field that has a Gaussian distribution with zero mean and defined noise with finite correlations in time. Two-loop analysis of all possible scaling regimes in general d-dimensional space is done in the plane of exponents $\varepsilon - \eta$, where ε characterizes the energy spectrum of the velocity field in the inertial range $E \propto k^{1-2\varepsilon}$, and η is related to the correlation time at the wave number k which is scaled as $k^{-2+\eta}$. It is shown that the scaling regimes of the present model of vector advection have essentially different properties than the scaling regimes of the corresponding model of passively advected scalar quantity. The results demonstrate the fact that, within the present model of passively advected vector field, the internal tensor structure of the advected field can have nontrivial impact on the diffusion processes deep inside in the inertial interval of given turbulent flow.

PACS: 47.27.ef; 47.27.tb; 05.10.Cc

INTRODUCTION

One of the main problems in the theory of developed turbulence is to verify the validity of the basic principles of the classical phenomenological Kolmogorov–Obukhov (KO) theory [1–3] in the framework of a microscopic model and to identify and understand possible deviations from its predictions. According to the KO theory [1–6], the statistical properties of random fields deep inside in the inertial interval $l \ll r \ll L$ are independent of the integral scale L (a typical scale on which the energy is pumped into the system) as well as the viscous

*E-mail: gladysh@theor.jinr.ru
**E-mail: jurcisine@saske.sk
***E-mail: jurcisin@saske.sk
****E-mail: remecky@saske.sk

scale l (a typical scale on which the energy starts to dissipate). This behavior is usually formulated in the form of the well-known first and second Kolmogorov hypotheses. Then, by using a simple dimensional analysis, one obtains the scaling behavior of correlation functions of the model with definite exponents.

For example, consider experimentally measured single-time structure functions of the velocity field defined as follows:

$$S_N(r) = \langle [v_r(t, \mathbf{x}) - v_r(t, \mathbf{x}')]^N \rangle, \quad r = |\mathbf{x} - \mathbf{x}'|, \tag{1}$$

where v_r denotes the component of the velocity field directed along the vector $\mathbf{r} = \mathbf{x} - \mathbf{x}'$. The Kolmogorov hypotheses, together with dimensional analysis, lead to the following scale-invariant behavior of the structure functions (1):

$$S_N(r) = \mathrm{const} \times (\bar{\epsilon} r)^{N/3}, \tag{2}$$

where $\bar{\epsilon}$ is the mean dissipation rate.

However, both experimental and theoretical studies show the existence of deviations from the predictions of the KO theory. Namely, the dependence of the correlation functions on the integral scale L is detected in contradiction with the first Kolmogorov hypothesis [4, 6–8]. Such deviations, referred to as anomalous or nondimensional scaling, manifest themselves in a singular dependence of the correlation functions on the distances and the integral scale L and, as a consequence, the simple scaling representation given in Eq. (2) must be replaced by the following one:

$$S_N(r) = (\bar{\epsilon} r)^{N/3} R_N(r/L), \tag{3}$$

with some unknown scaling functions R_N. The assumption that they have powerlike asymptotic behavior in the region $r \ll L$ in the form

$$R_N(r/L) \sim (r/L)^{q_N}, \tag{4}$$

with singular dependence on L in the limit $L \to \infty$ and nonlinearity of the exponents q_N as functions of N, is called «anomalous scaling», and it is explained by the existence of strong developed fluctuations of the dissipative rate (intermittency) [4–8].

During the last two decades, the problem of anomalous scaling was intensively studied, especially in the framework of various models of passively advected scalar and vector fields advected by «synthetic» velocity fields with given Gaussian statistics. The reason for this is twofold. First, it was shown that the deviations from the classical phenomenological theory are even more strongly noticeable for passively advected fields than for the velocity field itself, see, e.g., [7–9], and second, the problem of passive advection is considerably easier for theoretical investigation. At the same time, many anomalous properties of genuine turbulent heat or mass transport observed in experiments are reproduced by these relatively simple models. Thus, the theoretical study of the models of a passive scalar or vector advection can be treated as the first step on the long way of the investigation of intermittency and anomalous scaling in fully developed turbulence.

During a long period, the crucial role in the theoretical investigations of anomalous scaling was played by the simple model of a passive scalar quantity advected by a random Gaussian velocity field, white in time and self-similar in space, the so-called Kraichnan rapid-change model [10]. Namely, in the framework of the rapid-change model, for the first time, the anomalous scaling was established on the basis of a microscopic model and corresponding anomalous exponents were calculated within controlled approximations in the framework of the so-called zero-mode approach (see, e.g., [8] and references cited therein).

An effective and powerful method for investigation of self-similar scaling behavior is the renormalization group (RG) technique [11–13]. By using the RG technique, a considerable progress was also achieved in the understanding of the anomalous scaling in turbulence. In this respect, in [14–16], the field-theoretic RG and the operator-product expansion (OPE) were used in the systematic investigation of the anomalous scaling in Kraichnan's rapid-change model. It was shown that, in the framework of the field-theoretic RG approach, the anomalous scaling is related to the existence in the model of *dangerous* composite operators with negative critical dimensions in the OPE (see, e.g., [13, 17] for details). Thereafter, the field-theoretic RG technique was widely used for investigation of the anomalous behavior of various descendants of the Kraichnan model, e.g., models with inclusion of small-scale anisotropy, compressibility, models with the finite correlation time of the velocity field, and spatial parity violation (helicity) (see, e.g., [9, 18–22] and references cited therein). Besides, advection of the passive vector field by the Gaussian self-similar velocity field (with and without large- and small-scale anisotropy, pressure, compressibility, and finite correlation time) has also been investigated, and all possible asymptotic scaling regimes and crossover among them have been classified and anomalous scaling was analyzed [23–31]. A general conclusion of all these investigations is that the anomalous scaling remains valid for all generalized models.

Let us briefly describe the general solution of the problem of anomalous scaling in the framework of the field-theoretic approach [13,17]. It can be divided into two main stages. In the first stage, the multiplicative renormalizability of the corresponding field-theoretic model is demonstrated and the differential RG equations for its correlation functions are obtained. The asymptotic behavior of the latter on their ultraviolet argument (r/l) for $r \gg l$ and any fixed (r/L) is given by infrared stable fixed points of those equations. It involves some «scaling functions» of the infrared argument (r/L), whose form is not determined by the RG equations. In the second stage, the behavior of scaling functions at $r \ll L$ is found from the OPE within the framework of the general solution of the RG equations. There, the crucial role is played by the critical dimensions of various composite operators, which give rise to an infinite family of independent aforementioned scaling exponents (and hence to multiscaling).

However, one specific model of passively advected vector field, namely, the so-called $A = 0$ model (see the next section for definition of the model), has an interesting property, namely, it formally resembles the Navier–Stokes turbulence (see, e.g., [24] and references cited therein). In the framework of this model, the leading anomalous exponents are determined by the composite operators built solely of the gradients of the vector field. It leads to the fact that the number and the form of the operators that are important for the investigation of the anomalous scaling of correlation functions of a given order N depend on N. It means that different correlation (or structure) functions should be studied separately. An analogous problem exists in the genuine theory of fully developed turbulence based on the stochastic Navier–Stokes equation. Thus, from this point of view, the investigation of $A = 0$ model can be considered as an important next step to the investigation of anomalous scaling of the structure functions of the velocity field in fully developed turbulence. In this respect, the field-theoretic RG analysis of the model in the leading order of approximation (one-loop approximation) was done in [24, 25]. However, it is evident that it is necessary to go beyond the leading-order approximation to confirm the stability of obtained behavior with respect to the perturbation corrections and to make the predictions more accurate, especially in the situation when one works in the theory with a strong coupling. However, unlike the investigations of the anomalous scaling of passive scalar admixture in the framework of the Kraichnan model, generalized Kraichnan model [9], as well as in the model with advection by the Navier–Stokes velocity field [32], which were done up to the second-order (two-loop) approximation (in the case of the Kraichnan model also three-loop analysis of the anomalous exponents has been done [15, 16]), the complete field-theoretic RG analysis of the passively advected vector field, even within the simplest model, the so-called Kazantsev–Kraichnan kinematic magnetohydrodynamics (MHD) turbulence, is known only to the first order of approximation. The only exception is recent studies [33, 34], where brief RG discussions of Kazantsev–Kraichnan model have been done in two-loop approximation.

In the present paper, we would like to start with the investigation of the aforementioned $A = 0$ model of passively advected vector field [24, 25] in the two-loop approximation. In the present paper, we shall concentrate on detailed analysis of all possible scaling regimes in a generalized model, where the velocity field is supposed to have a Gaussian statistics with finite correlations in time [25]. Thus, in the present paper, we shall consider only the first stage of the solution of the problem of anomalous scaling in the framework of the field-theoretic approach, i.e., we shall establish all possible scaling regimes. The next step will be to use the obtained results for the investigation of the properties of the scaling functions of the structure functions of the advected field in the OPE to determine the critical dimensions of the most important composite operators that lead to the anomalous scaling. However, the problem of anomalous scaling will be studied elsewhere.

The paper is organized as follows. In Sec. 1, the model of the passively advected vector field is introduced and its field-theoretic formulation is given. In Sec. 2, the RG analysis of the model is done, and the possible scaling regimes and their IR stability under the influence of helicity are given in Sec. 3. In Conclusion, the discussion of results is presented.

1. FIELD-THEORETIC FORMULATION OF THE PASSIVE VECTOR ADVECTION

1.1. The Model. Let us consider the advection of a transverse (solenoidal) passive vector field $\theta \equiv \theta(x)$ $(x \equiv (t, \mathbf{x}))$ by an incompressible velocity field $\mathbf{v} \equiv \mathbf{v}(x)$ described by the following advection-diffusion equation (the so-called $A = 0$ model):

$$\partial_t \theta = \nu_0 \triangle \theta - (\mathbf{v} \cdot \partial) \theta - \partial P + \mathbf{f}^\theta, \tag{5}$$

where $\partial_t \equiv \partial / \partial t$, $\partial_i \equiv \partial / \partial x_i$, $\triangle \equiv \partial^2$ is the Laplace operator, ν_0 represents the diffusivity (in what follows, a subscript 0 will denote bare parameters of the unrenormalized theory), and $P(x)$ is the pressure. Thus, both $\mathbf{v}$ and θ are divergence-free vector fields: $\partial \cdot \mathbf{v} = \partial \cdot \theta = 0$. Due to transversality conditions, it can be shown that the pressure can be rewritten as the solution of the following Poisson equation [24]:

$$\triangle P = -\partial_i v_j \, \partial_j \theta_i. \tag{6}$$

The transverse Gaussian random noise $\mathbf{f}^\theta = \mathbf{f}^\theta(x)$ with zero mean and the correlation function

$$D^b_{ij}(x; 0) \equiv \langle f^\theta_i(x) \, f^\theta_j(0) \rangle = \delta(t) \, C_{ij}(|\mathbf{x}|/L) \tag{7}$$

represents the source of the fluctuations of the vector field θ that maintains the steady state of the system. Here, L is an integral scale related to the corresponding stirring, and C_{ij} is a function finite in the limit $L \to \infty$. The detailed form of the function C_{ij} is unimportant here, the only condition which must be satisfied is that C_{ij} decreases rapidly for $|\mathbf{x}| \gg L$. If C_{ij} depends on the direction of the vector $\mathbf{x}$ and not only on its modulus $r = |\mathbf{x}|$, then it can be considered as a source of the large-scale anisotropy (see, e.g., [24, 25]).

Usually, it is supposed that the velocity field $\mathbf{v}(x)$ satisfies the stochastic Navier–Stokes equation. However, in what follows, we shall work with the statistics of the velocity field given in the form of Gaussian distribution with zero mean and correlation function [25, 35, 36]

$$\langle v_i(x)v_j(x')\rangle \equiv D^v_{ij}(x;x') = \int \frac{d\omega\, d^d\mathbf{k}}{(2\pi)^{d+1}} P_{ij}(\mathbf{k})\tilde{D}^v(\omega,k)\times \times \exp\left[-i\omega(t-t') + i\mathbf{k}(\mathbf{x}-\mathbf{x}')\right], \quad (8)$$

with

$$\tilde{D}^v(\omega,k) = \frac{g_0\nu_0^3 k^{4-d-2\varepsilon-\eta}}{(i\omega + u_0\nu_0 k^{2-\eta})(-i\omega + u_0\nu_0 k^{2-\eta})}, \quad (9)$$

where $k = |\mathbf{k}|$ is the wave number; ω is frequency; d is the dimensionality of the $\mathbf{x}$ space. The geometric properties of the velocity correlator are given by the form of the transverse (due to incompressibility of the fluid) projector $P_{ij}(\mathbf{k})$. In the full symmetric turbulent system studied here, it has the form of the standard transverse projector $P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j/k^2$.

The symmetry of the present vector model also allows one to add to the advection-diffusion equation (5) the term of the form $(\theta\cdot\partial)\mathbf{v}$. In this case, one comes to an extended Kazantsev–Kraichnan model of kinematic MHD (see, e.g., [25] and references cited therein). The present model without this «stretching term» can be considered as a linearized Navier–Stokes equation with the prescribed statistics of the background field $\mathbf{v}$. Besides, the model is formally also resembling with the Navier–Stokes turbulence [24]. As was discussed in Introduction, namely, this similarity with the Navier–Stokes turbulence that is demonstrated, e.g., in the analogous anomalous behavior of the structure functions of the vector field θ in the present model and the corresponding structure functions of the velocity field $\mathbf{v}$ in the Navier–Stokes turbulence, is usually the main reason for detailed investigation of the vector model (5) with prescribed statistics of the velocity field.

The correlator (9) is related to the energy spectrum via the frequency integral [35, 37–41]

$$E(k) \simeq k^{d-1}\int d\omega\, \tilde{D}^v(\omega,k) \simeq \frac{g_0\nu_0^2}{u_0} k^{1-2\varepsilon}. \quad (10)$$

Thus, the coupling constant g_0 and the exponent ε describe the equal-time velocity correlator or, equivalently, energy spectrum. Besides, the constant u_0 and the second exponent η are related to the frequency $\omega \simeq u_0\nu_0 k^{2-\eta}$ which characterizes the mode k [35, 37–43]. It means that, in our notation, the value $\varepsilon = 4/3$ corresponds to the well-known Kolmogorov «five-thirds law» for the spatial statistics of velocity field, and $\eta = 4/3$ corresponds to the Kolmogorov frequency. Simple dimensional analysis shows that the parameters (charges) g_0 and u_0 are related to the characteristic ultraviolet (UV) momentum scale Λ (of the order of inverse Kolmogorov length) by

$$g_0 \simeq \Lambda^{2\varepsilon+\eta}, \quad u_0 \simeq \Lambda^{\eta}. \quad (11)$$

The specific feature of the model given by the Gaussian statistics of the velocity field given in Eqs. (8) and (9) is that it is not Galilean invariant [36] and, as a consequence, it does not take into account the self-advection of turbulent eddies. As a result of these so-called «sweeping effects», the different time correlations of the Eulerian velocity are not self-similar and depend strongly on the integral scale; see, e.g., [44–47]. However, the results presented in [36] show that the Gaussian model gives reasonable description of the passive advection in the appropriate frame, where the mean velocity field vanishes. One more argument to justify the model (8), (9) is that, in what follows, we shall be interested in the equal-time, Galilean invariant quantities (structure or correlation functions), which are not affected by the sweeping, and, therefore, as we expect (see, e.g., [25, 35, 48–50]), their absence in the Gaussian model (8), (9) is not essential.

The model (8), (9) contains two special cases that are interesting themselves. First of them is the so-called rapid-change model limit (see, e.g., [24] and references cited therein). It is obtained for $u_0 \to \infty$ and $g_0' \equiv g_0/u_0^2 = \text{const}$,

$$\tilde{D}^v(\omega, k) \to g_0'\, \nu_0\, k^{-d-2\varepsilon+\eta}. \tag{12}$$

The second one is the so-called quenched (time-independent or frozen) velocity field limit which is defined by $u_0 \to 0$ and $g_0'' \equiv g_0/u_0 = \text{const}$,

$$\tilde{D}^v(\omega, k) \to g_0''\, \nu_0^2\, \pi\delta(\omega)\, k^{-d+2-2\varepsilon}, \tag{13}$$

which is mathematically similar to the well-known models of the random walks in random environment with long-range correlations; see, e.g., [51–56].

1.2. Field-Theoretic Formulation of the Model. By using the well-known Martin–Siggia–Rose formalism [57–60], the stochastic problem given in Eqs. (5)–(9) can be rewritten into the field-theoretic model of the double set of fields $\Phi \equiv \{\theta, \theta', \mathbf{v}\}$ with the following action functional:

$$\begin{aligned} S(\Phi) = -\frac{1}{2}\int dt_1\, d^d\mathbf{x_1}\, dt_2\, d^d\mathbf{x_2}\, v_i(t_1, \mathbf{x_1})[D^v_{ij}(t_1, \mathbf{x_1}; t_2, \mathbf{x_2})]^{-1} \times \\ \times v_j(t_2, \mathbf{x_2}) + \frac{1}{2}\int dt_1\, d^d\mathbf{x_1}\, dt_2\, d^d\mathbf{x_2} \times \\ \times \theta_i'(t_1, \mathbf{x_1}) D^\theta_{ij}(t_1, \mathbf{x_1}; t_2, \mathbf{x_2}) \theta_j'(t_2, \mathbf{x_2}) + \\ + \int dt\, d^d\mathbf{x}\, \theta_i'[-\partial_t\theta_i + \nu_0\triangle\theta_i - v_j\partial_j\theta_i], \end{aligned} \tag{14}$$

where θ' is an auxiliary vector field with the same properties as the field θ, and D^θ_{ij} and D^v_{ij} are correlators (7) and (9), respectively. In the action (14), all the required integrations over $x = (t, \mathbf{x})$ and summations over the vector indices are understood. The second and the third integral in Eq. (14) represent the Dominicis–Jansen-type action for the stochastic problem (5), (7) at fixed $\mathbf{v}$, and the first integral represents the Gaussian averaging over $\mathbf{v}$.

The pressure term is omitted in the action functional (14) as a result of the transversality of the auxiliary field θ' that gives

$$\int dx\, \theta'_i\, \partial_i P = - \int dx\, P\, \partial_i \theta'_i = 0. \tag{15}$$

The model (14) corresponds to a standard Feynman diagrammatic perturbation theory with bare propagators (in frequency-momentum representation):

$$\langle \theta_i(\omega, \mathbf{k}) \theta'_j(-\omega, -\mathbf{k}) \rangle_0 = \frac{P_{ij}(\mathbf{k})}{-i\omega + \nu_0 k^2}, \tag{16}$$

$$\langle \theta'_i(\omega, \mathbf{k}) \theta_j(-\omega, -\mathbf{k}) \rangle_0 = \langle \theta_i(\omega, \mathbf{k}) \theta'_j(-\omega, -\mathbf{k}) \rangle_0^*, \tag{17}$$

$$\langle \theta_i(\omega, \mathbf{k}) \theta_j(-\omega, -\mathbf{k}) \rangle_0 = \frac{C_{ij}(\mathbf{k})}{|-i\omega + \nu_0 k^2|^2}, \tag{18}$$

$$\langle \theta'_i(\omega, \mathbf{k}) \theta'_j(-\omega, -\mathbf{k}) \rangle_0 = 0. \tag{19}$$

On the other hand, the bare propagator $\langle v_i v_j \rangle_0$ for the velocity field is given directly in Eqs. (8) and (9). $C_{ij}(\mathbf{k})$ in Eq. (18) is the Fourier transform of the function $C_{ij}(|\mathbf{x}|/L)$ from Eq. (7). The graphical representation of nonzero propagators is presented in Fig. 1 (the end with a slash in the propagator $\langle \theta_i \theta'_j \rangle_0$ corresponds to the field θ' and the end without a slash corresponds to the field θ). The triple (interaction) vertex

$$-\theta'_i v_j \partial_j \theta_i = \theta'_i V_{ijl} \theta_j v_l, \tag{20}$$

with the vertex factor (in frequency-momentum representation)

$$V_{ijl} = i k_l \delta_{ij}, \tag{21}$$

is shown in Fig. 2, where the momentum $\mathbf{k}$ is flowing into the vertex via the auxiliary field θ'.

Fig. 1. Graphical representation of the propagators of the model

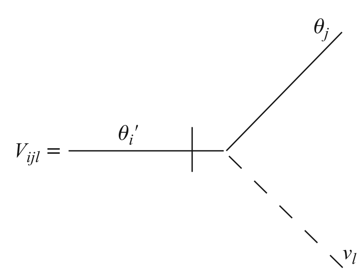

Fig. 2. The interaction vertex of the model

The formulation of the problem through the action functional (14) replaces the statistical averages of random quantities in the stochastic problem defined by Eqs. (5)–(9) with equivalent functional averages with weight $\exp S(\Phi)$. The generating functionals of the total Green's functions $G(A)$ and the connected Green's functions $W(A)$ are then defined by the functional integral [13]

$$G(A) = \mathrm{e}^{W(A)} = \int \mathcal{D}\Phi \, \mathrm{e}^{S(\Phi)+A\Phi}, \tag{22}$$

where $A(x) = \{\mathbf{A}^{\theta}, \mathbf{A}^{\theta'}, \mathbf{A}^{\mathbf{v}}\}$ represents a set of arbitrary sources for the set of fields Φ, $\mathcal{D}\Phi \equiv \mathcal{D}\theta\, \mathcal{D}\theta'\, \mathcal{D}\, \mathbf{v}$ denotes the measure of functional integration, and the linear form $A\Phi$ is standardly defined as

$$A\Phi = \int d\, x[A_i^{\theta}(x)\theta_i(x) + A_i^{\theta'}(x)\theta_i'(x) + A_i^{v}(x)v_i(x)], \tag{23}$$

where $dx \equiv dt\, d\mathbf{x}$.

2. RENORMALIZATION-GROUP ANALYSIS OF THE MODEL

To make the RG analysis of the present model, first of all, it is necessary to obtain the information about possible UV divergences which can be found by the standard analysis of canonical dimensions [11–13]. The field-theoretic model defined by the action functional (14) belongs among the so-called two-scale models [13, 17] for which the total canonical dimension d_Q of some quantity Q (which plays the same role in the renormalization theory of our dynamical model as the simple momentum dimension does in static models) is defined by two numbers, namely, the momentum dimension d_Q^k and the frequency dimension d_Q^{ω} with the standard normalization conditions $d_k^k = -d_x^k = 1$, $d_{\omega}^{\omega} = -d_t^{\omega} = 1$, $d_k^{\omega} = d_x^{\omega} = d_{\omega}^k = d_t^k = 0$. In the present model, the total canonical dimension is given as $d_Q = d_Q^k + 2d_Q^{\omega}$.

The canonical dimensions of the model under consideration are presented in table, where also the canonical dimensions of the renormalized parameters

Canonical dimensions of the fields and parameters of the model under consideration

Q	$\mathbf{v}$	θ	θ'	m, Λ, μ	ν_0, ν	g_0	u_0	g, u
d_Q^k	-1	0	d	1	-2	$2\varepsilon + \eta$	η	0
d_Q^{ω}	1	0	0	0	1	0	0	0
d_Q	1	0	d	1	0	$2\varepsilon + \eta$	η	0

are shown. The model is logarithmic for $\varepsilon = \eta = 0$ (the coupling constants g_0 and u_0 are dimensionless); therefore, the UV divergences in the correlation functions have the form of the poles in ε, η, and their linear combinations.

Detailed analysis of the possible divergences of the present model (see, e.g., [24, 25]) shows that the only superficially divergent function of the model is the one-irreducible Green's function $\langle \theta_i' \theta_j \rangle_{1-\mathrm{ir}}$, and the corresponding UV divergence can be removed multiplicatively by the only counterterm $\theta_i' \triangle \theta_j$ and it can be explicitly expressed in the multiplicative renormalization of the bare parameters g_0, u_0, and ν_0 in the following form:

$$\nu_0 = \nu Z_\nu, \quad g_0 = g\mu^{2\varepsilon+\eta} Z_g, \quad u_0 = u\mu^\eta Z_u. \tag{24}$$

Here, the dimensionless parameters g, u, and ν are the renormalized counterparts of the corresponding bare ones; μ is the renormalization mass (a scale setting parameter) in the minimal subtraction (MS) scheme; and $Z_i = Z_i(g, u)$ are renormalization constants.

At the same time, the corresponding renormalized action functional has the following form:

$$S^R(\Phi) = -\frac{1}{2} \int dt_1\, d^d\mathbf{x_1}\, dt_2\, d^d\mathbf{x_2}\, v_i(t_1, \mathbf{x_1})[D_{ij}^v(t_1, \mathbf{x_1}; t_2, \mathbf{x_2})]^{-1} \times$$
$$\times\, v_j(t_2, \mathbf{x_2}) + \frac{1}{2} \int dt_1\, d^d\mathbf{x_1}\, dt_2\, d^d\mathbf{x_2} \theta_i'(t_1, \mathbf{x_1}) D_{ij}^\theta(t_1, \mathbf{x_1}; t_2, \mathbf{x_2}) \times$$
$$\times\, \theta_j'(t_2, \mathbf{x_2}) + \int dt\, d^d\mathbf{x}\ \theta_i' \left[-\partial_t \theta_i + \nu Z_1 \triangle \theta_i - v_j \partial_j \theta_i\right], \tag{25}$$

where Z_1 is the only independent renormalization constant which is related to the renormalization constants defined in Eq. (24) as follows (the terms with correlators D_{ij}^v and D_{ij}^b, as well as the fields, are not renormalized):

$$Z_\nu = Z_1, \quad Z_g = Z_\nu^{-3}, \quad Z_u = Z_\nu^{-1}. \tag{26}$$

Here, the second and third relations are direct consequence of the absence of the renormalization of the term with D^v in renormalized action (25), i.e.,

$$g_0 \nu_0^3 = g\nu^3 \mu^{2\epsilon+\eta}, \quad u_0 \nu_0 = u\nu\mu^\eta. \tag{27}$$

The renormalization constant Z_1, in general, contains poles of linear combinations of ε and η, i.e., $Z_1 = Z_1(g, u, d; \varepsilon, \eta)$. However, as detailed analysis shows, to obtain all important quantities as the γ functions, β functions, coordinates of fixed points, and the critical dimensions, the knowledge of the renormalization

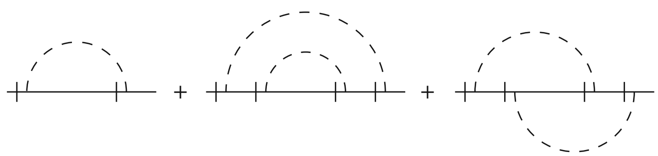

Fig. 3. The one- and two-loop contributions to the self-energy operator $\Sigma^{\theta'\theta}$

constants for the special choice $\eta = 0$ is sufficient up to two-loop approximation (see details in [35]). It is important here that the parameter ε alone provides the UV regularization for the theory; hence, the renormalization constant Z_1 remains finite at $\eta = 0$.

The renormalization constant Z_1 can be determined by the requirement that the one-irreducible Green's function $\langle\theta'_i\theta_j\rangle_{1-\mathrm{ir}}$ must be UV-finite when is written in the renormalized variables; i.e., it has no singularities in the limit $\varepsilon \to 0$ (as was briefly discussed above, one can put $\eta = 0$ in calculations). Using this condition, the renormalization constant Z_1 is determined up to a UV-finite contribution which is fixed by the choice of the renormalization scheme.

On the other hand, one-irreducible Green's function $\langle\theta'_i\theta_j\rangle_{1-\mathrm{ir}}$ is related (through the Dyson equation [12, 13]) to the self-energy operator $\Sigma^{\theta'\theta}$, which is expressed via the corresponding set of Feynman diagrams shown in Fig. 3.

The explicit integral expressions for the singular parts of diagrams in Fig. 3 are the following:

$$A = -\frac{S_d}{(2\pi)^d}\frac{g\nu p^2\delta_{ij}}{4u(1+u)}\frac{d^2-3}{d(d+2)}\left(\frac{\mu}{m}\right)^{2\varepsilon}\frac{1}{\varepsilon}, \tag{28}$$

$$B_1 = \frac{S_d^2}{(2\pi)^{2d}}\frac{g^2\nu p^2\delta_{ij}}{16u^2(1+u)^3}\frac{(d^2-3)}{d(d-1)(d+2)}\left(\frac{\mu}{m}\right)^{4\varepsilon}\times$$
$$\times\frac{1}{\varepsilon}\left[\frac{(d^2-3)(d-1)}{2d(d+2)\varepsilon}+\frac{S_{d-1}}{S_d}\int_0^1 dx\,(1-x^2)^{\frac{d-1}{2}}\int_1^\infty dz\,\frac{Y_1}{Z}\right], \tag{29}$$

$$B_2 = \frac{S_dS_{d-1}}{(2\pi)^{2d}}\frac{g^2\nu p^2\delta_{ij}}{16u^2(1+u)^2}\frac{(d-2)}{d(d-1)(d+2)}\left(\frac{\mu}{m}\right)^{4\varepsilon}\frac{1}{\varepsilon}\times$$
$$\times\int_0^1 dx\,(1-x^2)^{\frac{d-1}{2}}\int_1^\infty dz\,\frac{Y_2}{Z}, \tag{30}$$

where

$$\begin{aligned} Y_1 = 4z\{&-8x^6z^2 + (1+z^2)^2 + x^4[6 - 8(d-3)z^2 + 6z^4] + \\ &+ u^2(1 - 3x^2 + 2x^4)(1+z^2)^2 + \\ &+ x^2[-9 - 14z^2 - 9z^4 + 2d(1+z^2)^2] + \\ &+ 2u[(1+z^2)^2 - 2x^2(2 + 3z^2 + 2z^4) + x^4(3 + 4z^2 + 3z^4)]\}, \end{aligned} \quad (31)$$

$$\begin{aligned} Y_2 = 4(1+z^2)\{&(x^2-4)(z^4+1) + 2z^2(7x^2-4) - \\ &- d(x^2-1)[-(1+z^2)^2 - u(1 - 2(x^2-1)z^2 + z^4)] + \\ &+ u[-2x^4z^2 - 4(1+z^2)^2 + x^2(1 + 16z^2 + z^4)] + \\ &+ d^2(1+u)[1 + 2(1-2x^2)z^2 + z^4]\}, \end{aligned} \quad (32)$$

and

$$\begin{aligned} Z = [(1+u)(1+z^2) - 2xz][(1+u)(1+z^2) + 2xz]\times \\ \times (1 - 2xz + z^2)(1 + 2xz + z^2). \end{aligned} \quad (33)$$

Here, A given in Eq. (28) corresponds to the one-loop contribution (the first diagram in Fig. 3), B_1 in Eq. (29) is related to the second diagram in Fig. 3, and B_2 in Eq. (30) is the result for the third diagram. In Eqs. (28)–(30), $S_d = 2\pi^{d/2}/\Gamma(d/2)$ denotes the d-dimensional sphere.

In the end, one comes to the following expression for the renormalization constant $Z_1 = Z_\nu$:

$$\begin{aligned} Z_\nu = 1 - \frac{\bar g}{\varepsilon}\frac{d^2-3}{d(d+2)}\frac{1}{4u(1+u)} + \frac{\bar g^2}{\varepsilon^2}\frac{(d^2-3)^2}{d^2(d+2)^2}\frac{1}{32u^2(1+u)^3} + \\ + \frac{\bar g^2}{\varepsilon}\frac{S_{d-1}}{S_d}\frac{1}{d(d-1)(d+2)}\frac{1}{16u^2(1+u)^2}\left[\frac{d^2-3}{1+u}I_1 + (d-2)I_2\right], \end{aligned} \quad (34)$$

where $\bar g = gS_d/(2\pi)^d$ and

$$I_1 = \int_0^1 dx\,(1-x^2)^{\frac{d-1}{2}} \int_1^\infty dz \frac{Y_1}{Z}, \quad (35)$$

$$I_2 = \int_0^1 dx\,(1-x^2)^{\frac{d-1}{2}} \int_1^\infty dz \frac{Y_2}{Z}. \quad (36)$$

The basic RG differential equation, for example, for the renormalized connected correlation functions $W^R = \langle\Phi\cdots\Phi\rangle^R$ (the counterparts of the bare connected correlation functions $W = \langle\Phi\cdots\Phi\rangle)$ are obtained from the relation $S(\Phi, e_0) = S^R(\Phi, e, \mu)$, where e_0 stands for the complete set of bare parameters and e stands for the renormalized one, together with the fact that fields $\mathbf{v}, \mathbf{b}$, and $\mathbf{b}'$ are not renormalized. It leads to the relation

$$W^R(e, \mu, \ldots) = W(e_0, \ldots), \tag{37}$$

where the dots stand for other arguments which are untouched by renormalization, e.g., coordinates and times. Further, using the fact that unrenormalized correlation functions are independent of the scale-setting parameter μ, one can apply the differential operator $\mu\partial_\mu$ at fixed unrenormalized parameters on both sides of Eq. (37), which leads to the basic RG equation

$$\mathcal{D}_{\mathrm{RG}} W^R(A, e, \mu) = 0, \tag{38}$$

where operator $\mathcal{D}_{\mathrm{RG}}$ has the following explicit form:

$$\mathcal{D}_{\mathrm{RG}} = \mu\partial_\mu + \beta_g(g, u)\partial_g + \beta_u(g, u)\partial_u - \gamma_\nu(g, u)\mathcal{D}_\nu, \tag{39}$$

where we denote $\mathcal{D}_\nu \equiv \nu\partial_\nu$ and the RG functions (the β and γ functions) are given by well-known definitions, and in our case, using relations (26) for renormalization constants, they have the following form:

$$\gamma_\nu \equiv \mu\partial_\mu \ln Z_\nu, \tag{40}$$
$$\beta_g \equiv \mu\partial_\mu g = g(-2\varepsilon - \eta + 3\gamma_\nu), \tag{41}$$
$$\beta_u \equiv \mu\partial_\mu u = u(-\eta + \gamma_\nu). \tag{42}$$

Using the definition of the anomalous dimension γ_ν in Eq. (40) and the explicit expression for $Z_1 = Z_\nu$ as given in Eq. (34), one obtains

$$\gamma_\nu = -2(\bar{g}\mathcal{A} + 2\bar{g}^2\mathcal{B}), \tag{43}$$

where

$$\mathcal{A} = -\frac{d^2-3}{d(d+2)}\frac{1}{4u(1+u)} \tag{44}$$

is the one-loop contribution to anomalous dimension γ_ν and the two-loop one is

$$\mathcal{B} = \frac{S_{d-1}}{S_d}\frac{1}{d(d-1)(d+2)}\frac{1}{16u^2(1+u)^2}\left[\frac{d^2-3}{1+u}I_1 + (d-2)I_2\right], \tag{45}$$

where integrals I_1 and I_2 are defined in Eqs. (35) and (36), and functions Y_1, Y_2, and Z are given in Eqs. (31)–(33).

The possible asymptotic scaling regimes of the model deep inside in the inertial interval are given by the IR stable fixed points of the RG equations.

At the same time, the coordinates of all possible fixed points g_* and u_* are determined by the requirement of vanishing of the β functions (41) and (42), namely,

$$\beta_g(g_*, u_*) \equiv g_*(-2\varepsilon - \eta + 3\gamma_\nu^*) = 0, \tag{46}$$

$$\beta_u(g_*, u_*) \equiv u_*(-\eta + \gamma_\nu^*) = 0, \tag{47}$$

where γ_ν^* denotes the function (43) taken at the fixed points g_*, u_*.

The main aim of the present paper is to analyze the structure of the fixed points of the model. All possible fixed points and the corresponding scaling regimes will be classified and regions of their IR stability will be studied in the next section.

On the other hand, the existence of a stable IR fixed point means that the correlation functions of the model exhibit scaling behavior with given critical dimensions in the IR range. The issue of special interest is multiplicatively renormalizable equal-time two-point quantities $G(r)$ (see below). The IR scaling behavior of a function $G(r)$ (for $r/l \gg 1$ and any fixed r/L), namely,

$$G(r) \simeq \nu_0^{d_G^\omega} l^{-d_G} \left(\frac{r}{l}\right)^{-\Delta_G} R\left(\frac{r}{L}\right), \tag{48}$$

is related to the existence of IR stable fixed point of the RG equations (38). In Eq. (48), d_G^ω and d_G are the corresponding canonical dimensions of the function G (the canonical dimensions of the model are given in table), $l = 1/\Lambda$, $L = 1/m$, $R(r/L)$ is a scaling function, which cannot be determined by the RG equations (see, e.g., [13]), and Δ_G is the critical dimension defined as

$$\Delta_G = d_G^k + \Delta_\omega d_G^\omega + \gamma_G^*. \tag{49}$$

Here, γ_G^* is the fixed point value of the anomalous dimension $\gamma_G \equiv \mu\partial_\mu \ln Z_G$, where Z_G is the renormalization constant of the multiplicatively renormalizable quantity G, i.e., $G = Z_G G^R$ [13], and $\Delta_\omega = 2 - \gamma_\nu^*$ is the critical dimension of the frequency with γ_ν^*, which is defined in (43), taken at the corresponding fixed point. However, from Eqs. (46) and (47), one can immediately find the exact values of the γ_ν^* for the corresponding scaling regimes. They are exact one-loop results; i.e., no higher-loop corrections to the γ_ν^* exist. It also means that the critical dimension of frequency Δ_ω for the corresponding scaling regime is also known exactly, as well as the critical dimensions of the fields.

An example of the equal-time quantities built of the vector field θ are the equal-time two-point structure functions

$$S_N(r) = \langle[\theta_r(t, \mathbf{x}) - \theta_r(t, \mathbf{x}')]^N\rangle, \quad r = |\mathbf{x} - \mathbf{x}'|, \tag{50}$$

studied deep inside in the inertial range $l \ll r \ll L$, where θ_r denotes the component of the vector field θ directed along the vector $\mathbf{r} = \mathbf{x} - \mathbf{x}'$ [24, 25].

As was already mentioned, in the present paper, we shall concentrate only on the analysis of the possible asymptotic scaling regimes of the present model, and the analysis of the so-called anomalous scaling of the correlation (structure) functions of the model will be given elsewhere.

3. IR STABLE FIXED POINTS AND THE SCALING REGIMES OF THE MODEL

As was already mentioned in the previous section, possible scaling regimes of a renormalized model are directly given by the IR stable fixed points of the corresponding system of the RG equations [12, 13]. The fixed point of the RG equations is defined by β functions, namely, by requirement of their vanishing. In the present model, the coordinates g_*, u_* of the fixed points are found from the system of two equations, namely,

$$\beta_g(g_*, u_*) = \beta_u(g_*, u_*) = 0 \tag{51}$$

that are explicitly shown in Eqs. (46) and (47). To investigate the IR stability of a fixed point, it is enough to analyze the eigenvalues of the matrix Ω of first derivatives:

$$\Omega_{ij} = \begin{pmatrix} \dfrac{\partial \beta_g}{\partial g} & \dfrac{\partial \beta_g}{\partial u} \\ \dfrac{\partial \beta_u}{\partial g} & \dfrac{\partial \beta_u}{\partial u} \end{pmatrix}. \tag{52}$$

Possible IR asymptotic behaviors are governed by the IR stable fixed points, i.e., those for which both eigenvalues are positive.

The possible scaling regimes of the model in the framework of the one-loop approximation were investigated in [25]. The aim of the present paper is to analyze the problem in the two-loop approximation and to compare the results to the corresponding problem of passive scalar advection studied in [20–22].

Let us start with the so-called rapid-change limit: $u \to \infty$. Here, it is convenient to make the transformation to new variables, namely, $w \equiv 1/u$ and $g' \equiv g/u^2$ [25, 35], with the corresponding changes in the β functions:

$$\beta_{g'} = g'(\eta - 2\varepsilon + \gamma_\nu), \tag{53}$$

$$\beta_w = w(\eta - \gamma_\nu), \tag{54}$$

and the anomalous dimension γ_ν acquires the following form:

$$\gamma_\nu = -2(\bar{g}'\mathcal{A}' + 2\bar{g}'^2\mathcal{B}'), \tag{55}$$

where $\bar{g}' = g' S_d/(2\pi)^d$. After this transformation, the one-loop contribution $\mathcal{A}'$ acquires the form

$$\mathcal{A}' = -\frac{d^2-3}{d(d+2)}\frac{1}{4(1+w)} \tag{56}$$

and the two-loop correction $\mathcal{B}'$ is

$$\mathcal{B}' = \frac{S_{d-1}}{S_d}\frac{1}{d(d-1)(d+2)}\frac{1}{16(1+w)^2}\left[\frac{w(d^2-3)}{1+w}I_1' + (d-2)I_2'\right], \tag{57}$$

where now

$$I_1' = \int_0^1 dx\,(1-x^2)^{\frac{d-1}{2}}\int_1^\infty dz\frac{Y_1'}{Z'}, \tag{58}$$

$$I_2' = \int_0^1 dx\,(1-x^2)^{\frac{d-1}{2}}\int_1^\infty dz\frac{Y_2'}{Z'}, \tag{59}$$

with

$$\begin{aligned} Y_1' = 4z\{&(1-3x^2+2x^4)(1+z^2)^2 + 2w[(1+z^2)^2 - \\ &- 2x^2(2+3z^2+2z^4) + x^4(3+4z^2+3z^4)] + w^2\times \\ &\times[-8x^6z^2 + (1+z^2)^2 + x^4(6-8(d-3)z^2+6z^4) + \\ &+ x^2(-9-14z^2-9z^4+2d(1+z^2)^2)]\}, \end{aligned} \tag{60}$$

$$\begin{aligned} Y_2' = 4w(1+z^2)\{&-4+x^2-8z^2+16x^2z^2-2x^4z^2- \\ &-4z^4+x^2z^4+d^2(1+w)[1+2(1-2x^2)z^2+z^4]+ \\ &+d(x^2-1)[1-2(x^2-1)z^2+z^4+w(1+z^2)^2]+ \\ &+w[-4(1+z^2)^2+x^2(1+14z^2+z^4)]\}, \end{aligned} \tag{61}$$

and

$$\begin{aligned} Z' = &[1+z^2+w(1-2xz+z^2)](1-2xz+z^2)\times \\ &\times[1+z^2+w(1+2xz+z^2)](1+2xz+z^2). \end{aligned} \tag{62}$$

However, in the rapid-change limit $w \to 0$ ($u \to \infty$), one comes to the known result that the two-loop contribution $\mathcal{B}'$ is equal to zero. It is related to the fact that in the rapid-change limit [24] no higher-loop corrections to the self-energy operator exist and the anomalous dimension γ_ν is determined exactly at the one-loop level of approximation and has the following form:

$$\gamma_\nu = \lim_{w\to 0} \frac{(d^2-3)\bar{g}'}{2d(d+2)(1+w)} = \frac{(d^2-3)\bar{g}'}{2d(d+2)}. \tag{63}$$

In the rapid-change limit, one has two different fixed points. Let us denote them as FPI and FPII. The first fixed point is trivial, namely,

$$\text{FPI:} \quad w_* = g'_* = 0, \tag{64}$$

with $\gamma_\nu^* = 0$, and diagonal matrix Ω with eigenvalues (diagonal elements)

$$\lambda_1 = \eta, \quad \lambda_2 = \eta - 2\varepsilon. \tag{65}$$

Thus, this fixed point is IR stable for $\eta > 0$ and $\eta > 2\varepsilon$, as shown in Fig. 4. The second point is defined by the coordinates

$$\text{FPII:} \quad w_* = 0, \quad \bar{g}'_* = \frac{2d(d+2)}{d^2-3}(2\varepsilon - \eta), \tag{66}$$

with $\gamma_\nu^* = 2\varepsilon - \eta$. This is an exact one-loop expression as a result of non-existence of the higher-loop corrections. The corresponding matrix of first derivatives is triangular with diagonal elements (eigenvalues):

$$\lambda_1 = 2(\eta - \varepsilon), \quad \lambda_2 = 2\varepsilon - \eta, \tag{67}$$

i.e., this fixed point is IR stable if conditions $\eta > \varepsilon$ and $\eta < 2\varepsilon$ are fulfilled simultaneously. The region of stability of this fixed point is shown explicitly in Fig. 4.

The second limit of the present model corresponds to the so-called «frozen regime» with frozen velocity field with $u \to 0$. To study this transition, it is appropriate to change the variable g to the new variable $g'' \equiv g/u$ [25, 35]. In this case, the β functions are transformed to the following ones:

$$\beta_{g''} = g''(-2\varepsilon + 2\gamma_\nu), \tag{68}$$

$$\beta_u = u(-\eta + \gamma_\nu). \tag{69}$$

In this notation, the anomalous dimension γ_ν has the form

$$\gamma_\nu = -2(\bar{g}''\mathcal{A}'' + 2\bar{g}''^2\mathcal{B}''), \tag{70}$$

where again $\bar{g}'' = g''S_d/(2\pi)^d$. The one-loop contribution $\mathcal{A}''$ is now

$$\mathcal{A}'' = -\frac{d^2-3}{d(d+2)}\frac{1}{4(1+u)}, \tag{71}$$

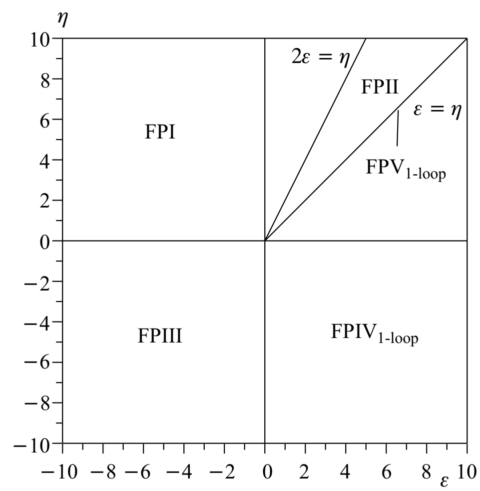

Fig. 4. Regions of the stability for the fixed points in one-loop approximation. The regions of stability for fixed points FPI, FPII, and FPIII are exact, i.e., are not influenced by loop corrections. The fixed point FPIV is shown in one-loop approximation. The d-dependence of the FPIV in two-loop approximation is shown in Fig. 5

and the two-loop one, $\mathcal{B}''$, is given as

$$\mathcal{B}'' = \frac{S_{d-1}}{S_d} \frac{1}{d(d-1)(d+2)} \frac{1}{16(1+u)^2} \left[\frac{d^2-3}{1+u} I_1 + (d-2) I_2 \right], \tag{72}$$

where integrals I_1 and I_2 are defined in Eqs. (35) and (36), and functions Y_1, Y_2, and Z are given in Eqs. (31)–(33). In the limit $u \to 0$, the functions $\mathcal{A}''$ and $\mathcal{B}''$ acquire the following form:

$$\mathcal{A}_0'' = -\frac{d^2-3}{4d(d+2)} \tag{73}$$

and

$$\mathcal{B}_0'' = \frac{S_{d-1}}{S_d} \frac{(d^2-3) I_1'' + (d-2) I_2''}{16 d(d-1)(d+2)}, \tag{74}$$

where

$$I_1'' = \int_0^1 dx\, (1-x^2)^{\frac{d-1}{2}} \int_1^\infty dz \frac{Y_1''}{Z''}, \tag{75}$$

$$I_2'' = \int_0^1 dx\, (1-x^2)^{\frac{d-1}{2}} \int_1^\infty dz \frac{Y_2''}{Z''}, \tag{76}$$

with

$$Y_1'' = 44z\{-8x^6z^2 + (1+z^2)^2 + x^4[6 - 8(d-3)z^2 + 6z^4] + \\ + x^2[-9 - 14z^2 - 9z^4 + 2d(1+z^2)^2]\}, \quad (77)$$

$$Y_2'' = 4(1+z^2)\{-4 + x^2 - 8z^2 + 14x^2z^2 - 4z^4 + x^2z^4 + \\ + d(x^2-1)(1+z^2)^2 + d^2[1 + 2(1-2x^2)z^2 + z^4]\}, \quad (78)$$

and

$$Z'' = [(1 - 2xz + z^2)(1 + 2xz + z^2)]^2. \quad (79)$$

The system of β functions (68) and (69) exhibits two fixed points that are related to the corresponding two scaling regimes. Let us denote them as FPIII and FPIV. One of them is again trivial, namely,

$$\mathrm{FPIII:} \quad u_* = g_*'' = 0, \quad (80)$$

with $\gamma_\nu^* = 0$. The eigenvalues of the corresponding matrix Ω, which is diagonal, are

$$\lambda_1 = -2\varepsilon, \quad \lambda_2 = -\eta. \quad (81)$$

Thus, this regime is IR stable only if both parameters ε and η are negative simultaneously, as can be seen in Fig. 4. The second, nontrivial, point is

$$\mathrm{FPIV:} \quad u_* = 0, \quad \bar{g}_*'' = -\frac{\varepsilon}{2\mathcal{A}_0''} - \frac{\mathcal{B}_0''}{2\mathcal{A}_0''^3}\varepsilon^2, \quad (82)$$

where $\mathcal{A}_0''$ and $\mathcal{B}_0''$ are defined in Eqs. (73) and (74), respectively, and the anomalous dimension γ_ν taken in the fixed point is $\gamma_\nu^* = \varepsilon$.

The eigenvalues of the matrix Ω (taken at the fixed point) are

$$\lambda_1 = 2\varepsilon\left(1 - \frac{\mathcal{B}_0''}{\mathcal{A}_0''^2}\varepsilon\right), \quad \lambda_2 = \varepsilon - \eta. \quad (83)$$

The conditions $\bar{g}_*'' > 0$, $\lambda_1 > 0$, and $\lambda_2 > 0$ for the IR stable fixed point lead to the following restrictions on the values of the parameters ε and η:

$$\varepsilon > 0, \quad \varepsilon > \eta, \quad \varepsilon\mathcal{B}_0'' < \mathcal{A}_0''^2. \quad (84)$$

The region of stability in the plane d–ε is shown in Fig. 5. It is evident that allowed values of the parameter ε are essentially restricted. For example, for the most interesting case $d = 3$, one obtains that for the IR stable fixed point, it is necessary to have $\varepsilon < 0.85$. These two-loop results for the frozen limit of

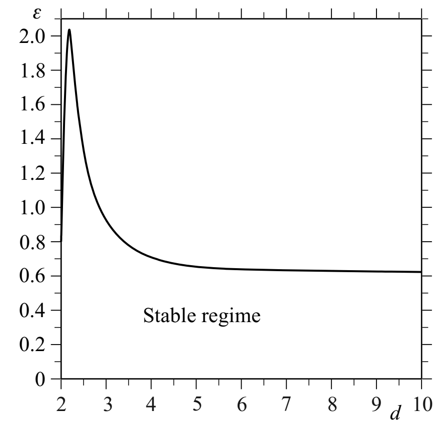

Fig. 5. Region of the stability of the fixed point FPIV (frozen limit) in two-loop approximation. The IR fixed point is stable in the region given in figure. At the same time, inequalities $\varepsilon > 0$, $\varepsilon > \eta$ must be fulfilled

the present vector model are essentially different from the corresponding results obtained for the model of passively advected scalar field, where the two-loop calculations give much weaker restriction on parameter ε (see [20–22]).

However, the most interesting scaling regime of the present model is the regime with finite value of the fixed point for the variable u. In this case, it is evident that the system of equations (see also [25, 35])

$$\beta_g = g(-2\varepsilon - \eta + 3\gamma_\nu) = 0, \tag{85}$$

$$\beta_u = u(-\eta + \gamma_\nu) = 0 \tag{86}$$

can be fulfilled simultaneously for finite values of g, u only when the parameter ε is equal to η: $\varepsilon = \eta$. Thus, in this case, the function β_g is proportional to function β_u. As a result, we have not one fixed point of this type but a curve of fixed points in the $g{-}u$ plane. The value of the fixed point for variable g in the two-loop approximation is given as follows (we denote this fixed point as FPV):

$$\mathrm{FPV:}\quad \bar{g}_* = -\frac{1}{2\mathcal{A}_*}\,\varepsilon - \frac{1}{2}\frac{\mathcal{B}_*}{\mathcal{A}_*^3}\,\varepsilon^2, \tag{87}$$

with exact one-loop result for $\gamma_\nu^* = \varepsilon = \eta$ (this is already directly given by Eq. (86)). Here, $\mathcal{A}_*$ and $\mathcal{B}_*$ are expressions $\mathcal{A}$ and $\mathcal{B}$ from Eqs. (44) and (45) which are taken in the fixed point value u_* of the variable u. The possible values of the fixed point for variable u can be restricted as we shall discuss below. The matrix of the first derivatives Ω has the following eigenvalues:

$$\lambda_1 = 0, \quad \lambda_2 = 3\bar{g}^*\left(\frac{\partial\gamma_\nu}{\partial g}\right)_* + u^*\left(\frac{\partial\gamma_\nu}{\partial u}\right)_*, \tag{88}$$

where γ_ν is given in Eq. (43). The vanishing of the λ_1 is related to the degeneracy of the system of Eqs. (85) and (86).

Explicit form of the coordinate $\bar{g}_*$ of the fixed point as function of the spatial dimension d and arbitrary fixed point value of parameter u_* is given as follows:

$$\bar{g}_* = \frac{2d(2+d)u_*(1+u_*)}{d^2-3}\varepsilon + \frac{S_{d-1}}{S_d}\frac{2d^2(d+2)^2 u_*(1+u_*)}{(d-1)(d^2-3)^3}\times \\ \times\left[\frac{d^2-3}{1+u_*}I_1^* + (d-2)I_2^*\right]\varepsilon^2, \quad (89)$$

where I_1^* and I_2^* are the integrals defined in Eqs. (35) and (36) taken at the fixed point value u_*. To have positive value of the fixed point for variables g and u, one finds restriction on parameter ε: $\varepsilon > 0$. Possible restrictions on the IR fixed point value of the variable u can be found from the condition $\lambda_2 > 0$. The explicit form of λ_2 is the following:

$$\lambda_2 = \frac{2+u_*}{1+u_*}\varepsilon - \frac{S_{d-1}}{S_d}\frac{d(d+2)\varepsilon^2}{(d-1)(d^2-3)^2(1+u_*)^2}\times \\ \times\{u_*(1+u_*)[(d^2-3)I_3^* + (d-2)(1+u_*)I_4^*]+ \\ + [d^2(2+u_*) - 2(3+2u_*)]I_1^* + (d-2)(1+u_*)(2+u_*)I_2^*\}, \quad (90)$$

where again I_1^* and I_2^* are the integrals defined in Eqs. (35) and (36) taken at the fixed point value u_*, and I_3^* and I_4^* are defined as follows:

$$I_3^* = \left(\frac{\partial I_1}{\partial u}\right)_*, \quad I_4^* = \left(\frac{\partial I_2}{\partial u}\right)_*. \quad (91)$$

In Fig. 6, the regions of stability for the fixed point FPV in the $u{-}\varepsilon$ plane for different space dimension d are shown. Thus, one can see again that in the two-loop approximation a nontrivial d-dependence of IR stability of the fixed point appears in contrast to the one-loop approximation [25]. It is evident that the strongest restriction on the region of IR stability of this scaling regime is near the frozen limit. When the fixed point value of the parameter u increases, the region of stability of the scaling regime enlarges too. Besides, again one can conclude that the region of IR stability, as well as properties of this fixed point, is essentially different for the present vector model in comparison with the corresponding fixed point obtained in the model of passively advected scalar field.

Thus, having the results of the present section, one can conclude that the first stage of the two-loop scaling analysis of the present model is completed; i.e., all possible scaling regimes are found and their IR stability is investigated to the second order of the corresponding perturbation theory. The next step is to analyze the scaling properties of the scaling functions $R(r/L)$ of important correlation

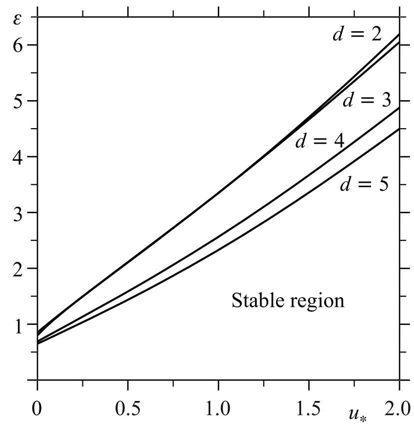

Fig. 6. Regions of stability for the fixed point FPV in the two-loop approximation. The d-dependence of the IR stability is shown

functions of the advected vector field (the single-time two-point structure functions defined in Eq. (50) are an example) by using the OPE. However, this question is beyond the scope of the present paper and will be studied elsewhere.

CONCLUSION

In the present paper, we have studied the model of advection of a vector field by a turbulent environment, where the turbulent flow is given by the Gaussian statistics of the velocity field with finite correlations in time. The advection of the vector field is described by the so-called $A = 0$ model, where the corresponding stochastic advection-diffusion equation does not contain the «stretching term» of the form $(\theta \cdot \partial)\mathbf{v}$. The complete two-loop analysis of all possible scaling regimes is done, and the IR stability of the corresponding fixed points of the RG equations is analyzed in detail. It is shown that, although the present model of passively advected vector field exhibits the same scaling regimes as the corresponding model of passively advected scalar field (see, e.g., [20–22]), the properties of the scaling regimes (the coordinates of the fixed points) as well as the regions of their IR stability are essentially different.

It is shown that the two-loop corrections lead to the rather strong restriction on the space of parameters, where some of the scaling regimes exhibit IR stability. This restriction is especially essential for the so-called frozen limit of the model $u_* = 0$, where the velocity correlations are independent in time. Here, only relatively small values of the parameter ε are allowed for which the scaling regime is IR stable. On the other hand, when the fixed point value of the parameter u increases; i.e., when the correlations of the velocity become finite in time, the region of IR stability of the scaling regime enlarges. In the limit $u_* \to \infty$, i.e., in the rapid-change limit, the restrictions completely disappear. Thus, we can conclude that the structure of the scaling regimes of the present model of passively advected vector quantity is now completely known.

However, from the point of view of the general analysis of the scaling behavior of the correlation functions of the model, in the present paper only the first stage of the solution of the problem is given; i.e., we have established all possible scaling regimes of the model and discussed their IR stability. The second stage of the analysis, namely, the investigation of the behavior of the scaling functions of the important correlation functions of the model deep inside in the inertial interval by using the OPE technique within two-loop approximation, is still open. Thus, the next step will be to use the OPE and the results obtained in the present paper in the two-loop analysis of the scaling functions of the correlation functions of the advected vector field to determine the critical dimensions of the most important composite operators that lead to the anomalous scaling. This open problem will be studied elsewhere.

Acknowledgements. E. J., M. J., and R. R. gratefully acknowledge the hospitality of the Bogoliubov Laboratory of Theoretical Physics of the Joint Institute for Nuclear Research, Dubna. The work was supported by the realization of the project ITMS No. 26220120029, based on the supporting operational Research and Development Program financed from the European Regional Development Fund.

REFERENCES

1. *Kolmogorov A. N.* // Dokl. Akad. Nauk SSSR. 1941. V. 30. P. 301 (reprinted in Proc. Roy. Soc. London. A. 1991. V. 434. P. 9).
2. *Kolmogorov A. N.* // Dokl. Akad. Nauk SSSR. 1941. V. 31. P. 538.
3. *Kolmogorov A. N.* // Ibid. V. 32. P. 16 (reprinted in Proc. Roy. Soc. London. A. 1991. V. 434. P. 15).
4. *Monin A. S., Yaglom A. M.* Statistical Fluid Mechanics. V. 2. Cambridge: MIT Press, 1975.
5. *McComb W. D.* The Physics of Fluid Turbulence. Oxford: Clarendon, 1990.
6. *Frisch U.* Turbulence: The Legacy of A. N. Kolmogorov. Cambridge Univ. Press, 1995.
7. *Sreenivasan K. R., Antonia R. A.* // Ann. Rev. Fluid Mech. 1997. V. 29. P. 435.
8. *Falkovich G., Gawedzki K., Vergassola M.* // Rev. Mod. Phys. 2001. V. 73. P. 913.
9. *Antonov N. V.* // J. Phys. A. 2006. V. 39. P. 7825.
10. *Kraichnan R. H.* // Phys. Fluids. 1968. V. 11. P. 945.

11. *Amit D. J.* Field Theory, Renormalization Group, and Critical Phenomena. McGraw-Hill, 1978.

12. *Zinn-Justin J.* Quantum Field Theory and Critical Phenomena. Oxford: Clarendon, 1989.

13. *Vasil'ev A. N.* Quantum-Field Renormalization Group in the Theory of Critical Phenomena and Stochastic Dynamics. Boca Raton: Chapman & Hall/CRC, 2004.

14. *Adzhemyan L. Ts., Antonov N. V.* // Phys. Rev. E. 1998. V. 58. P. 1823.

15. *Adzhemyan L. Ts. et al.* // Phys. Rev. E. 2001. V. 63. P. 025303(R).

16. *Adzhemyan L. Ts. et al.* // Ibid. V. 64. 056306.

17. *Adzhemyan L. Ts., Antonov N. V., Vasil'ev A. N.* The Field Theoretic Renormalization Group in Fully Developed Turbulence. London: Gordon & Breach, 1999.

18. *Jurčišinová E. et al.* // Intern. J. Mod. Phys. B. 2008. V. 22. P. 3589.

19. *Jurčišinová E., Jurčišin M.* // Phys. Rev. E. 2008. V. 77. P. 016306.

20. *Chkhetiani O. G. et al.* // Czech. J. Phys. 2006. V. 56. P. 827.

21. *Chkhetiani O. G. et al.* // J. Phys. A: Math. Gen. 2006. V. 39. P. 7913.

22. *Chkhetiani O. G. et al.* // Phys. Rev. E. 2006. V. 74. P. 036310.

23. *Antonov N. V., Lanotte A., Mazzino A.* // Phys. Rev. E. 2000. V. 61. P. 6586.

24. *Adzhemyan L. Ts., Antonov N. V., Runov A. V.* // Phys. Rev. E. 2001. V. 64. P. 046310.

25. *Antonov N. V. et al.* // Phys. Rev. E. 2003. V. 68. P. 046306.

26. *Antonov N. V. et al.* // Phys. Rev. E. 2000. V. 62. P. R5891.

27. *Adzhemyan L. Ts., Antonov N. V., Runov A. V.* // Phys. Rev. E. 2001. V. 64. P. 046310.

28. *Hnatich M. et al.* // Phys. Rev. E. 2005. V. 71. P. 066312.

29. *Hnatich M. et al.* // Acta Phys. Slov. 2002. V. 52. P. 559.

30. *Novikov S. V.* // J. Phys. A: Math. Gen. 2006. V. 39. P. 8133.

31. *Jurčišinová E. et al.* // Phys. Part. Nucl. Lett. 2008. V. 5. P. 219.

32. *Adzhemyan L. Ts. et al.* // Phys. Rev. E. 2005. V. 71. P. 016303.

33. *Jurčišinová E., Jurčišin M., Remecky R.* // Springer Proc. in Physics. 2009. V. 132. P. 961.

34. *Antonov N. V., Gulitskiy N. M.* // Lecture Notes in Comp. Science. 2012. V. 7125. P. 128.

35. *Antonov N. V.* // Phys. Rev. E. 1999. V. 60. P. 6691.

36. *Holzer M., Siggia E. D.* // Phys. Fluids. 1994. V. 6. P. 1820.

37. *Avellaneda M., Majda A.* // Commun. Math. Phys. 1990. V. 131. P. 381.

38. *Avellaneda M., Majda A.* // Commun. Math. Phys. 1992. V. 146. P. 139.

39. *Majda A.* // J. Stat. Phys. 1993. V. 73. P. 515.

40. *Horntrop D., Majda A.* // J. Math. Sci. Univ. Tokyo. 1994. V. 1. P. 23.

41. *Zhang Q., Glimm J.* // Commun. Math. Phys. 1992. V. 146. P. 217.
42. *Chertkov M., Falkovich G., Lebedev V.* // Phys. Rev. Lett. 1996. V. 76. P. 3707.
43. *Eyink G.* // Phys. Rev. E. 1996. V. 54. P. 1497.
44. *Kraichnan R. H.* // Phys. Fluids. 1964. V. 7. P. 1723.
45. *Kraichnan R. H.* // Phys. Fluids. 1965. V. 8. P. 575.
46. *Chen S., Kraichnan R. H.* // Phys. Fluids A. 1989. V. 1. P. 2019.
47. *L'vov V. S.* // Phys. Rep. 1991. V. 207. P. 1.
48. *Antonov N. V.* // Physica D. 2000. V. 144. P. 370.
49. *Antonov N. V.* // Zap. Nauchn. Semin. POMI. 2000. V. 269. P. 79.
50. *Adzhemyan L. Ts., Antonov N. V., Honkonen J.* // Phys. Rev. E. 2002. V. 66. P. 036313.
51. *Bouchaud J. P. et al.* // J. Phys. (Paris). 1987. V. 48. P. 1445.
52. *Bouchaud J. P. et al.* // J. Phys. (Paris). 1988. V. 49. P. 369.
53. *Bouchaud J. P., Georges A.* // Phys. Rep. 1990. V. 195. P. 127.
54. *Honkonen J., Karjalainen E.* // J. Phys. A. 1988. V. 21. P. 4217.
55. *Honkonen J., Pis'mak Yu. M., Vasil'ev A. N.* // J. Phys. A. 1989. V. 21. P. L835.
56. *Honkonen J., Pis'mak Yu. M.* // J. Phys. A. 1989. V. 22. P. L899.
57. *Martin P. C., Siggia E. D., Rose H. A.* // Phys. Rev. A. 1973. V. 8. P. 423.
58. *De Dominicis C.* // J. Phys. (Paris). Colloq. 1976. V. 37. P. C1–C247.
59. *Janssen H. K.* // Z. Phys. B. 1976. V. 23. P. 377.
60. *Bausch R., Janssen H. K., Wagner H.* // Ibid. V. 24. P. 131.