ФИЗИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ И АТОМНОГО ЯДРА
2013. Т. 44. ВЫП. 3

ANALYTICAL RESULTS FOR THE FOUR-LOOP RG FUNCTIONS IN THE 2D NONLINEAR $O(n)$ σ-MODEL ON THE LATTICE

O. Veretin *

II. Institut für Theoretische Physik, Universität Hamburg, Hamburg, Germany

We recalculate four-loop renormalization group functions in 2-dimensional nonlinear $O(n)$ σ-model using coordinate-space method. The high accuracy of calculation allows us to find the analytical form of β and γ functions (anomalous dimension).

PACS: 11.15.Ha; 11.10.Gh; 12.20.Ds

INTRODUCTION

Nonlinear σ-models have been the objects of the intensive studies for many years. The particular case of these models, considered in this paper, is the 2-dimensional nonlinear $O(n)$ σ-model. This model is known to be asymptotically free and can be applied, e.g., to the study of ferromagnetic systems. It can also serve as a toy model for the strong interactions in particle physics.

In calculations of physically interesting characteristics it is important to know the β function and anomalous dimension γ. The know of them allows one, in particular, to predict the correlation length ξ and the spin susceptibility χ. In the regime of weak coupling, β and γ functions can be evaluated as perturbative series in the coupling constant. In order to study the whole range of the coupling constant, one has to appeal to the lattice simulations. Due to the asymptotical freedom, this model is especially suitable for such a study. For the precise comparison of Monte Carlo data with perturbative expansions, higher loop calculations within the lattice regularization are required. Such a calculation to two loops has been done analytically in [1] and then pushed forward to four loops in [2] numerically and checked in [3]. At the same time, analogous results at the four-loop order in the continuum limit are known analytically [4].

*E-mail: veretin@mail.desy.de

The goal of this work is to find the analytical expressions for the renormalization group (RG) coefficients to the four-loop order in the lattice perturbation theory. In order to do this, we use the methods proposed in the continuum field theory for the evaluation of the multiloop integrals. Diagrams on the lattice, as well as in the continuum limit, are related to each other algebraically. Such relations arise due to the integration by parts method [5], which leads, in general, to the reduction of the number of independent integrals. However, the realization of this algorithm in the lattice already at the three-loop level is quite difficult task.

In Sec. 1 we give the definitions and discuss the method. In Sec. 2 our results are presented, and in Appendix A we give all integrals from [2] separately.

1. DEFINITIONS

The action of the nonlinear $O(n)$ σ-model is usually written in the form

$$S = \frac{1}{2f_0}\int d^2x(\partial_\mu q(x)\,\partial_\mu q(x)), \tag{1}$$

where $q_i(x)$ is an n-component real vector field of unit length and f_0 is the bare coupling constant. In the lattice formulation the derivatives are, as usual, understood as finite differences.

The perturbative expansions of the $\hat{\beta}$ and $\hat{\gamma}$ functions can be written as follows*:

$$\hat{\beta}(f) = -a\frac{d}{da}f_0 = -2\pi(n-2)\sum_{L=1}\hat{b}^{(L)}\left(\frac{f_0}{2\pi}\right)^{L+1}, \tag{2}$$

$$\hat{\gamma}(f) = a\frac{d}{da}\ln Z = 2\pi(n-1)\sum_{L=1}\hat{c}^{(L)}\left(\frac{f_0}{2\pi}\right)^{L+1}, \tag{3}$$

where a is the lattice spacing and Z is the renormalization constant of the field. Prefactors $(n-2)$ and $(n-1)$ in the above formulae always factorize and we take them in front of the expressions.

Coefficients $\hat{b}^{(L)}$ and $\hat{c}^{(L)}$ can be computed using the technique of Feynman diagrams. Generally, Feynman diagrams on the lattice are more difficult to evaluate than the ones in the continuum field theory. Therefore, the analytical results in the lattice are known only to two loops [1], while analogous quantities in the continuum theory are known to four loops [4]. The RG coefficients were computed on the lattice numerically to four loops [2], where they were expressed

*Our coefficients $\hat{b}^{(L)}$ and $\hat{c}^{(L)}$ are defined slightly differently than those in [2].

in terms of 12 different integrals. The evaluation of these integrals has been repeated in [3] to somewhat better accuracy (about $\sim 10^{-9}$) and the wrong notation of [2] was clarified in [6].

It is known that between different Feynman diagrams there are many algebraic relations, which can be obtained by partial integration [5]. This explains the fact that a big number of different integrals could be expressed as linear combinations of few constants (irrationalities) with rational coefficients. Moreover, there were proposed some rules how to predict the constants that occur in higher loop calculations [7, 8]. The interesting question arises: which constants appear in the lattice diagrams calculation? We make a conjencture that they are the same as in the continuum case, proposed in [8]. To test this conjencture, the so-called PSLQ test [9] has been used.

Let us briefly describe this approach. Suppose that we have some irrational numbers $\eta_1, \ldots, \eta_n$ given to some a certain precision with d decimal digits. We say that they obey an integer relation with norm bound N if $\eta_1, \ldots, \eta_n$ are linear-dependent with integer coefficients. Precisely, there exist *integer* numbers $c_1, \ldots, c_n$ such that

$$|c_1\eta_1 + \ldots + c_n\eta_n| < \epsilon, \quad \text{provided that} \quad \max|c_i| < N, \tag{4}$$

where $\epsilon > 0$ is some small number of the order 10^{-d} and N is norm bound.

Given accuracy d, «detection threshold» ϵ and norm bound N, the PSLQ test allows one to find out whether relation (4) exists or not (for detail, see [9]). This approach has been applied in several calculations (see, e.g., [10]).

The crucial point is the knowledge of the basis elements η_j. We suppose, naturally, that the basis for lattice integrals under consideration is the same as for those in continuum field theory for a single-scale diagrams. The reason for that is that the finite part of diagrams contains the same class of functions, regardless of which kind of regularization has been used. It was suggested in [7] that the basis elements form an algebra; i.e., if η_1 and η_2 belong to the basis, then the product $\eta_1\eta_2$ does either. Thus, some «higher» elements (but not all of them) are constructed from «lower» ones by forming all possible products of the latters. In addition, the integral and the basic elements can be ordered by their «weights» (for detail, see [7, 8]), which are determined by the number of loops but not by the topology of a diagram (for several single-scale diagrams it has been tested in [12]).

Thus, we come to the following basis elements:

$$\begin{aligned}
&\pi,\ \log 2,\\
&\pi^2,\ \pi\log 2,\ \log^2 2,\ G,\\
&\pi^3,\ \pi^2\log 2,\ \pi\log^2 2,\ \log^3 2,\ G\pi,\ G\log 2,\ \zeta_3,\ \mathrm{Ls}_3(\pi/2)
\end{aligned} \tag{5}$$

and

$$\frac{\pi}{\sqrt{3}},\ \log 3,$$
$$\pi^2,\ \frac{\pi}{\sqrt{3}}\log 3,\ \log^2 3,\ \frac{\mathrm{Ls}_2(\pi/3)}{\sqrt{3}}, \tag{6}$$
$$\frac{\pi^3}{\sqrt{3}},\ \pi^2\log 3,\ \frac{\pi}{\sqrt{3}}\log^2 3,\ \log^3 3,\ \pi\mathrm{Ls}_2(\pi/3),\ \log 3\frac{\mathrm{Ls}_2(\pi/3)}{\sqrt{3}},\ \zeta_3,\ \frac{\mathrm{Ls}_3(2\pi/3)}{\sqrt{3}},$$

where $\zeta_k = \zeta(k)$ is Riemann ζ function, $G = 0.915965594177219015\ldots$ is the Catalan constant and the constant $\mathrm{Ls}_2(\pi/3) = 1.014941606409653625\ldots$ is defined through the so-called log-sine integral [13]

$$\mathrm{Ls}_k(\theta) = \int_0^\theta \log^{k-1}\left(2\sin\frac{\theta'}{2}\right)d\theta'.$$

In Eqs. (5) and (6) the first, second and third lines correspond to weights $1, 2$ and 3, respectively. The elements of higher weights would correspond to higher loop integrals and do not appear here.

2. RESULTS AND DISCUSSION

We applied the ideas explained above to the lattice integrals presented in [2]. The integrals were computed to accuracy better than 10^{-40} using the coordinate-space method proposed in [14]. The most problematic integrals V_3 and V_6 were computed even to higher accuracy. The analysis established that these integrals can be expressed within bases (5) and (6) plus one more constant, introduced below. From 28 elements of (5) and (6) only five do contribute. Namely, we were able to express all integrals evaluated numerically in [2, 3] in terms of the following six irrational constants:

$$\pi,\quad \pi^2,\quad \zeta_3,\quad G,\quad \frac{\mathrm{Ls}_2(\pi/3)}{\sqrt{3}},\quad \text{and}\quad (2\pi)^3K, \tag{7}$$

where integral K is the same three-loop bubble as in [2, 3].

Among these integrals only for K we did not find a relation to the bases (5) and (6). Therefore, we include it as an independent constant. However, it is not excluded that $(2\pi)^3K$ can be rewritten as a linear combination of elements (5) and (6) and the possible reason for our misfinding is the lack of the accuracy for the numerical value of this integral.

For the last constant K, we give numerical result accurate to 10^{-37}:

$$(2\pi)^3K = 23.7849506237378578142256363314563137344(1). \tag{8}$$

Coefficients $\hat{b}^{(L)}$ of beta function (2) now read

$$\hat{b}^{(1)} = 1, \tag{9}$$

$$\hat{b}^{(2)} = 1, \tag{10}$$

$$\hat{b}^{(3)} = \frac{n-7}{24}\pi^2 + \frac{1}{2}\pi - \frac{n-4}{2}, \tag{11}$$

$$\begin{aligned}\hat{b}^{(4)} = &-\frac{28n^2-66n-38}{12}\zeta_3 - \frac{(n-2)(n+1)}{8}(2\pi)^3 K + \frac{3n-1}{12}\pi^3 - \\ &- 10(n-2)\pi\frac{\mathrm{Ls}_2(\pi/3)}{\sqrt{3}} + 20(n-2)\pi G + \frac{6n^2-26n-1}{12}\pi^2 - \\ &- 2(n-2)(n+20)\frac{\mathrm{Ls}_2(\pi/3)}{\sqrt{3}} - 4(n-2)G - \frac{5n-12}{2}\pi + \frac{2n^2-3n-1}{2}.\end{aligned} \tag{12}$$

For the anomalous dimension (3) we have

$$\hat{c}^{(1)} = 1, \tag{13}$$

$$\hat{c}^{(2)} = \frac{1}{2}\pi, \tag{14}$$

$$\hat{c}^{(3)} = \frac{n+9}{24}\pi^2 - \frac{n-2}{2}, \tag{15}$$

$$\begin{aligned}\hat{c}^{(4)} = &\frac{(n-2)(127n-121)}{24}\zeta_3 + \frac{(n-2)(n+1)}{16}(2\pi)^3 K - \frac{4n-11}{24}\pi^3 + \\ &+ 5(n-2)\pi\frac{\mathrm{Ls}_2(\pi/3)}{\sqrt{3}} - 10(n-2)\pi G - \frac{3n^3-11n+2}{6}\pi^2 + \\ &+ (n-2)(7n+8)\frac{\mathrm{Ls}_2(\pi/3)}{\sqrt{3}} + (2n-4)G + \frac{13(n-2)}{4}\pi - \\ &- \frac{(n-2)(10n-21)}{2}.\end{aligned} \tag{16}$$

In conclusion, we expressed RG functions within the lattice regularization in terms of six irrational constants given by (7). The algebraic structure of the above results suggests that there should exist a method of algebraic reduction of diagrams to a set of a few master intergrals. As is mentioned at the beginning of the paper, such a method exists in continuum field theory and is based on the integration by parts [5] in the momentum space. On the lattice, however, reduction algorithms are not so obvious. In the simplest case of vacuum one-loop bubble diagrams algebraic method was discussed in [15]. In more complicated cases, only few investigations have been done in this direction (see, e.g., [16]). The development of algebraic methods is desirable and they could be very useful tools for higher loop computations on the lattice.

Acknowledgements. This work was supported in part by the German Federal Ministry for Education and Research BMBF, Grant No. 05H12GUE, and by the Helmholtz Association HGF, Grant No. Ha 101.

Appendix
INTEGRALS

In this appendix we present separately our analytical results for the integrals that enter RG functions. They are given in [3] and [14] numerically. So our results for these integrals read

$$(2\pi)^2 G_1 = \frac{1}{2}\zeta_2 + 1, \tag{17}$$

$$(2\pi)^2 R = \frac{\mathrm{Ls}_2(\pi/3)}{\sqrt{3}}, \tag{18}$$

$$(2\pi)^3 J = -24\zeta_2\pi + 96\zeta_2, \tag{19}$$

$$(2\pi)^3 L_1 = -\frac{7}{2}\zeta_3 + 3\zeta_2, \tag{20}$$

$$(2\pi)^3 V_1 = \frac{7}{2}\zeta_3, \tag{21}$$

$$(2\pi)^3 V_2 = \frac{14}{3}\zeta_3 - 4\zeta_2 + 8\frac{\mathrm{Ls}_2(\pi/3)}{\sqrt{3}} - 4, \tag{22}$$

$$(2\pi)^3 V_3 = \frac{56}{3}\zeta_3 - 16\zeta_2 + 24\frac{\mathrm{Ls}_2(\pi/3)}{\sqrt{3}} - 16 + (2\pi)^3 K, \tag{23}$$

$$(2\pi)^3 V_4 = -\frac{13}{24}\zeta_3, \tag{24}$$

$$(2\pi)^3 V_5 = \frac{19}{2}\zeta_3 - 3\pi\zeta_2 + 4\zeta_2, \tag{25}$$

$$(2\pi)^3 V_6 = \frac{14}{3}\zeta_3 - 8\zeta_2 + \frac{1}{2}(2\pi)^3 K, \tag{26}$$

$$(2\pi)^2 W_1 = -\frac{1}{2}\frac{\mathrm{Ls}_2(\pi/3)}{\sqrt{3}}, \tag{27}$$

$$(2\pi)^3 \hat{W}_2 = \frac{1}{2}\zeta_3 + \frac{3}{2}\pi\frac{\mathrm{Ls}_2(\pi/3)}{\sqrt{3}} - \frac{5}{2}\pi G + \frac{1}{2}\zeta_2 + \frac{11}{2}\frac{\mathrm{Ls}_2(\pi/3)}{\sqrt{3}} + \frac{1}{2}G - \frac{1}{2}. \tag{28}$$

And according to [6]

$$W_2 = \hat{W}_2 + \frac{85}{2304\pi^3}\zeta_3. \tag{29}$$

REFERENCES

1. *Falcioni M., Treves A.* // Nucl. Phys. B. 1986. V. 265. P. 671;
   Caracciolo S., Pelissetto A. // Nucl. Phys. B. 1994. V. 420. P. 141.
2. *Caracciolo S., Pelissetto A.* // Nucl. Phys. B. 1995. V. 455. P. 619.
3. *Shin D.* // Nucl. Phys. B. 1999. V. 546. P. 669.
4. *Hikami S., Brezin E.* // J. Phys. A. 1978. V. 11. P. 1141;
   Hikami S. // Phys. Lett. B. 1981. V. 98. P. 208; Nucl. Phys. B. 1983. V. 215. P. 555;
   Bernreuther W., Wegner F. J. // Phys. Rev. Lett. 1986. V. 57. P. 1383;
   Wegner F. // Nucl. Phys. B. 1989. V. 316. P. 663.
5. *Chetyrkin K. G., Tkachov F. V.* // Nucl. Phys. B. 1981. V. 192. P. 159.
6. *Alles B. et al.* // Nucl. Phys. B. 1999. V. 562. P. 581.
7. *Broadhurst D. J.* // Eur. Phys. J. C. 1999. V. 8. P. 311.
8. *Fleischer J., Kalmykov M. Yu.* // Phys. Lett. B. 1999. V. 470. P. 168.
9. *Ferguson H. R. P., Bailey D. H.* RNR Technical Report RNR-91-032;
   Ferguson H. R. P., Bailey D. H., Arno S. NASA Technical Report NAS-96-005.
10. *Kniehl B. A. et al.* // Phys. Rev. Lett. 2006. V. 97. P. 042001;
    Kniehl B. A., Kotikov A. V., Veretin O. L. // Phys. Rev. Lett. 2008. V. 101. P. 193401;
    Kniehl B. A., Kotikov A. V., Veretin O. L. arXiv:0909.1431 [hep-ph].
11. *Davydychev A. I., Kalmykov M. Yu.* // Nucl. Phys. B. 2001. V. 605. P. 266.
12. *Kalmykov M. Yu., Veretin O. L.* // Phys. Lett. B. 2000. V. 483. P. 315;
    Fleischer J., Kalmykov M. Yu. // Comp. Phys. Commun. 2000. V. 128. P. 531.
13. *Lewin L.* Polylogarithms and Associated Functions. Amsterdam: North-Holland, 1981.
14. *Shin D.* // Nucl. Phys. B. 1998. V. 525. P. 457.
15. *Caracciolo S., Menotti P., Pelissetto A.* // Nucl. Phys. B. 1992. V. 375. P. 195.
16. *Becher T., Melnikov K.* // Phys. Rev. D. 2002. V. 66. P. 074508.