ФИЗИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ И АТОМНОГО ЯДРА
2013. Т. 44. ВЫП. 5

THE MATHEMATICAL AND GEOMETRICAL STRUCTURE OF THE SPACE–TIME AND THE CONCEPT OF UNIFICATION OF MATTER AND ENERGY

D. J. Cirilo-Lombardo *

International Institute of Physics,
Federal University of Rio Grande do Norte, Natal-RN, Brazil and
Joint Institute for Nuclear Research, Dubna

*E-mail: diego777jcl@gmail.com

ФИЗИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ И АТОМНОГО ЯДРА
2013. Т. 44. ВЫП. 5

THE MATHEMATICAL AND GEOMETRICAL STRUCTURE OF THE SPACE–TIME AND THE CONCEPT OF UNIFICATION OF MATTER AND ENERGY

D. J. Cirilo-Lombardo *

International Institute of Physics,
Federal University of Rio Grande do Norte, Natal-RN, Brazil and
Joint Institute for Nuclear Research, Dubna

Following the guidelines of previous works of the author, the geometrical analysis of a new type of Unified Field Theoretical models (UFT) is presented. These new unified theoretical models are characterized by an underlying hypercomplex structure, zero nonmetricity; and the geometrical action is determined fundamentally by the curvature provenient of the breaking of symmetry of a group manifold in higher dimensions. This mechanism of the Cartan–MacDowell–Mansouri type permits us to construct geometrical actions of determinantal type leading to a nontopological physical Lagrangian due to the splitting of a reductive geometry. Our goal is to take advantage of the geometrical and topological properties of this theory in order to determine the minimal group structure of the resultant space–time manifold able to support a fermionic structure. From this fact, the relation between antisymmetric torsion and Dirac structure of the space–time is determined, and the existence of an important contribution of the torsion to the gyromagnetic factor of the fermions is shown. Also we resume and analyze previous cosmological solutions in this new UFT, where, as in our work [3] for the non-Abelian Born–Infeld model, the Hosoya and Ogura ansatz is introduced for the important cases of tratorial, totally antisymmetric and general torsion fields. In the case of space–time with torsion, the real meaning of the spin-frame alignment is found and the question of the minimal coupling is discussed.

Представлен геометрический анализ нового типа теоретических моделей единой теории поля. Эти новые модели характеризуются наличием гиперкомплексной структуры, нулевой неметричностью, и геометрическое действие определяется фундаментально кривизной, возникающей из-за нарушения симметрии группового множества в пространстве больших измерений. Механизм Картана–Макдауэлла–Мансури позволяет строить геометрические действия детерминантного типа, приводящие к нетопологическому физическому лагранжиану благодаря редуктивной геометрии. Наша цель — получить преимущество геометрических и топологических свойств этой теории, чтобы определить минимальную групповую структуру результирующего пространственно-временного множества, способного поддерживать фермионную структуру. Это обстоятельство определяет антисимметричную торсионную и дираковскую структуру пространства-времени и

*E-mail: diego777jcl@gmail.com

наличие важного вклада торсиона в гиромагнитный фактор для фермионов. Также заново пересматривается анализ предыдущих космологических решений в предложенной новой единой теории поля, где, как и в нашей работе [3] для неабелевой модели Борна–Инфельда, вводится анзац Хосоя и Огура для важных случаев полностью асимметричного и обобщенного торсионного полей. В случае пространства-времени с торсионом устанавливается реальное значение выстраивания спиновой структуры и обсуждается проблема минимальной связи.

PACS: 04.20.Cv; 04.20.Jb; 04.20.Gz

1. MOTIVATION AND SUMMARY OF THE RESULTS

For a long time in the history of the modern theoretical physics, the possibility of the unification of all fundamental forces has been treated from the mathematical and theoretical point of view. Several models, formulations and sophisticated mathematical tools were used in order to solve the intricate puzzle of conciliating the gravity with the other fundamental forces of the nature: electromagnetic, weak and strong. Although many attempts were made, this issue is still without concrete solution: the string theory is a typical case. In the string theory, the claiming is common on the consistent solution of the unification trouble; but, beside particular formulations, the theoretical and conceptual environment joined with an obscure mathematical basis put certainly in doubt the affirmative acceptation of such a claim.

As was pointed out by us in the later works [1, 2], the cornerstone of the problem is where to start to conceptually reformulate the theoretical arena where the fundamental unified theory will be placed, and where the geometry is the unifying essence. According to Mach, space–time does not exist without matter. Then, two basic ideas immediately arise how to fulfill the observation given by Mach: the concept of dualistic or nondualistic theories. In the first one, the simplest and economical description can be formulated in terms of the gravitational field without torsion plus the energy momentum tensor that, however, is added «by hand» in order to cover the lack of knowledge of a fundamental structure of the space–time giving the matter plus energy distribution. In the second one, there are not prescriptions for the interaction of gravity with the «matter» fields because they are arising from the same fundamental geometrical structure.

In our previous works, we presented a new model of a nondualistic unified theory. The idea that we introduced first in our preliminary model in [1] is absolutely consistent from the mathematical and geometrical point of view and is based on a manifold equipped with an *underlying hypercomplex structure and zero nonmetricity*. It leads to the important fact that the torsion of the space–time structure turns out to be totally antisymmetric. As is well known, in the particular case of totally antisymmetric torsion tensor, the affine geometrical framework has the geodesic and the minimal length equations that are equivalent, and the most important is that it is the only case when the equivalence principle is fulfilled as was shown in [9, 10] and we demonstrate it here also.

The other goal that we introduce as the main ingredient in [1, 2] and here, is that the specific form of our action is determined by *the curvature from the breaking of symmetry of a group manifold in higher dimensions* via the Cartan–MacDowell–Mansouri mechanism [1, 2]. This mechanism permits us to construct geometrical actions of determinantal type which due to the splitting of a reductive geometry (as is the case of the group manifold treated here) via the breaking of a higher dimensional group (i.e., as is the typical case $SO(1,4) \to SO(1,3) \oplus \mathbb{M}_{1,3}$), leads to a nontopological physical Lagrangian.

Following the guidelines of our last works [1–3], in this paper we complete the previous analysis considering the same fundamental model of UFT. The organization of the paper with the corresponding results is as follows: in Sec. 2, the geometrical framework is introduced, and the theoretical basis of the model, based on a geometrical action that takes physical meaning through a breaking of symmetry, is described. In Sec. 3, the dynamic equations are analyzed, and the geometrical and physical meaning is elucidated.

In Sec. 4, we resume and analyze previous cosmological solutions in the new UFT: as in our work [3] for the non-Abelian Born–Infeld model, the Hosoya and Ogura ansatz is introduced for the important cases of tratorial and totally antisymmetric torsion. The real meaning of the spin-frame alignment in the case with torsion is found. Also, we explicitly show that, contrary to the case of the Poincare theory of gravitation (see [4]), the possibility in our theory of the co-existence of both types of torsion in cosmological space–times certainly exists.

Section 5 is the most important in the sense that the fermionic structure of the space–time is described, and the possibility of geometrical unification is realized: a unified theory of QED and GR can be derived from $P(G, M)$, the Principal fiber bundle of frames over the 4D space–time manifold with G as its structure group. In the subsections, the action of the UFT is analyzed from the group-theoretical point of view considering the G-symmetry of the model. In Sec. 6, the derivation of the Dirac equation from the G-manifold, the relation between the electromagnetic field/fermionic structure of the space–time, and the contribution of the torsion to the gyromagnetic factor are explicitly shown. However, the physical consequences are explained. Finally, Sec. 7 is devoted to discussion of the cohomological interplay between the fields involved in the space–time structure, and in Sec. 8 the concluding remarks are given.

2. THE SPACE–TIME MANIFOLD AND THE GEOMETRICAL ACTION

The starting point is a hypercomplex construction of the (metric compatible) space–time manifold [1]

$$M, g_{\mu\nu} \equiv e_\mu \cdot e_\nu, \tag{1}$$

where for each point $\in M$ there exists a local space affine A. The connection over A, $\widetilde{\Gamma}$ defines a generalized affine connection Γ on M specified by (∇, K), where K is an invertible $(1,1)$ tensor over M. We will demand that the connection is compatible and rectilinear

$$\nabla K = KT, \quad \nabla g = 0, \tag{2}$$

where T is the torsion, and g (the space–time metric, used to raise and to lower indices and determines the geodesics) is preserved under parallel transport. This generalized compatibility condition ensures that the affine generalized connection Γ maps autoparallels of Γ on M in straight lines over the affine space A (locally). The first equation is equal to the condition determining the connection in terms of the fundamental field in the nonsymmetric UFT. For instance, K can be identified with the fundamental tensor in the nonsymmetric fundamental theory. This fact gives us the possibility of restricting the connection with an (anti)Hermitian theory.

The covariant derivative of a vector with respect to the generalized affine connection is given by

$$\begin{aligned} A^{\mu}_{;\nu} &\equiv A^{\mu}{}_{,\nu} + \Gamma^{\mu}_{\alpha\nu} A^{\alpha}, \\ A_{\mu;\nu} &\equiv A_{\mu}{}_{,\nu} - \Gamma^{\alpha}_{\mu\nu} A_{\alpha}. \end{aligned} \tag{3}$$

The generalized compatibility condition (2) determines the 64 components of the connection by the 64 equations as follows:

$$K_{\mu\nu;\alpha} = K_{\mu\rho} T^{\rho}_{\nu\alpha}, \quad \text{where} \quad T^{\rho}_{\nu\alpha} \equiv 2\Gamma^{\rho}_{[\alpha\nu]}. \tag{4}$$

Notice that by contraction of indices ν and α in the first equation of (4), an additional condition for this hypothetic fundamental (nonsymmetric) tensor K is obtained

$$K_{\mu\alpha;\alpha} = 0,$$

that, geometrically speaking, is

$$d^{*}K = 0,$$

this is a current-free condition for the tensor K that can be exemplified nicely with the prototype of nonsymmetric fundamental tensor $K_{\mu\nu} = g_{\mu\nu} + f_{\mu\nu}$:

$$d^{*}K = d^{*}g + d^{*}f \Rightarrow d^{*}f = 0 \quad \text{(current-free e.o.m.)},$$

with, however, $g_{\mu\nu}$ playing the role of space–time metric; and $f_{\mu\nu}$, the role of electromagnetic field.

The metric is uniquely determined by the metricity condition that puts 40 restrictions on the partial derivatives of the metric

$$g_{\mu\nu,\rho} = 2\Gamma_{(\mu\nu)\rho}. \tag{5}$$

The space–time curvature tensor, that is defined in the usual way, has two possible contractions: the Ricci tensor $R^{\lambda}_{\mu\lambda\nu} = R_{\mu\nu}$ and the second contraction $R^{\lambda}_{\lambda\mu\nu} = 2\Gamma^{\lambda}_{\lambda[\nu,\mu]}$, which is identically zero due to the metricity condition (2). In order to find a symmetry of the torsion tensor, if we denote the inverse of K by $\widehat{K}$, $\widehat{K}$ is uniquely specified by $\widehat{K}^{\alpha\rho}\ K_{\alpha\sigma} = K^{\alpha\rho}\ \widehat{K}_{\alpha\sigma} = \delta^{\rho}_{\sigma}$. As was pointed out in [1], inserting explicitly the torsion tensor as the antisymmetric part of the connection in (4) and multiplying by $\widehat{K}^{\alpha\nu}/2$, result after straighforward computations in

$$(\ln\sqrt{-K})_{,\mu} - \Gamma^{\nu}_{(\mu\nu)} = 0, \tag{6}$$

where $K = \det(K_{\mu\rho})$. Notice that from expression (6) we arrive at the following condition between the determinants K and g: $K/g = \text{const}$. Now we can write

$$\Gamma^{\nu}_{\alpha\nu,\beta} - \Gamma^{\nu}_{\beta\nu,\alpha} = \Gamma^{\nu}_{\nu\beta,\alpha} - \Gamma^{\nu}_{\nu\alpha,\beta}, \tag{7}$$

due to the fact that the first term of (7) is the derivative of a scalar. Then, the torsion tensor has the symmetry

$$T^{\nu}_{\nu[\beta,\alpha]} = T^{\nu}_{\nu[\alpha,\beta]} = 0. \tag{8}$$

That means that the trace of the torsion tensor defined as $T^{\nu}_{\nu\alpha}$, is the gradient of a scalar

$$T_{\alpha} = \nabla_{\alpha}\phi.$$

The second important point is the following: let us consider [1] the extended curvature [8]

$$\mathcal{R}^{ab}_{\mu\nu} = R^{ab}_{\mu\nu} + \Sigma^{ab}_{\mu\nu} \tag{9}$$

with

$$\begin{aligned} R^{ab}_{\mu\nu} &= \partial_{\mu}\omega^{ab}_{\nu} - \partial_{\nu}\omega^{ab}_{\mu} + \omega^{ac}_{\mu}\omega^{b}_{\nu c} - \omega^{ac}_{\nu}\omega^{b}_{\mu c}, \\ \Sigma^{ab}_{\mu\nu} &= -(e^{a}_{\mu}e^{b}_{\nu} - e^{a}_{\nu}e^{b}_{\mu}). \end{aligned} \tag{10}$$

We assume that ω^{ab}_{ν} is the $SO(d-1,1)$ connection, and e^{a}_{μ} is the vierbein field. Equations (9) and (10) can be obtained, for example, using the formulation that was pioneering introduced in seminal works by E. Cartan long time ago [1]. It is well known that in such a formalism the gravitational field is represented as a connection 1-form associated with some group which contains the Lorentz group as subgroup. The typical example is provided by the $SO(d,1)$ de Sitter gauge theory of gravity. In this specific case, the $SO(d,1)$ gravitational gauge field $\omega^{AB}_{\mu} = -\omega^{BA}_{\mu}$ is broken into the $SO(d-1,1)$ connection ω^{ab}_{μ} and the $\omega^{da}_{\mu} = e^{a}_{\mu}$ vierbein field. Then, the de Sitter (anti-de Sitter) curvature

$$\mathcal{R}^{AB}_{\mu\nu} = \partial_{\mu}\omega^{AB}_{\nu} - \partial_{\nu}\omega^{AB}_{\mu} + \omega^{AC}_{\mu}\omega^{B}_{\nu C} - \omega^{AC}_{\nu}\omega^{B}_{\mu C} \tag{11}$$

splits in the curvature (9). At this point, our goal is to enlarge the group structure of the space–time manifold in such a manner that the curvature (11), obviously after the breaking of symmetry, permits us to define the geometrical Lagrangian of the theory as

$$L_g = \sqrt{\det \mathcal{R}^a_\mu \mathcal{R}_{a\nu}} = \sqrt{\det G_{\mu\nu}}, \tag{12a}$$

where we have defined the following geometrical object:

$$\mathcal{R}^a_\mu = \lambda(e^a_\mu + f^a_\mu) + R^a_\mu \quad (M^a_\mu \equiv e^{a\nu} M_{\nu\mu}), \tag{12b}$$

where f^a_μ (in sharp contrast to e^a_μ) carries the following symmetry:

$$e_{a\mu} f^a_\nu = f_{\mu\nu} = -f_{\nu\mu}.$$

The action will contain, as usual, $\mathcal{R} = \det(\mathcal{R}^a_\mu)$ as the geometrical object that defines the dynamics of the theory. The particularly convenient definition of $\mathcal{R}^a_\mu$ makes easy to establish the equivalent expression in the spirit of the unified theories developed long ago by Eddington, Einstein and Born, and Infeld, for example:

$$\sqrt{\det \mathcal{R}^a_\mu \mathcal{R}_{a\nu}} = \sqrt{\det\left[\lambda^2(g_{\mu\nu} + f^a_\mu f_{a\nu}) + 2\lambda R_{(\mu\nu)} + 2\lambda f^a_\mu R_{[a\nu]} + R^a_\mu R_{a\nu}\right]}, \tag{13}$$

where $R_{\mu\nu} = R_{(\mu\nu)} + R_{[\mu\nu]}$.

The important point to be considered in this simple Cartan inspired model is that, although a cosmological constant λ is required, the expansion of the action in four dimensions leads automatically to the Hilbert–Einstein part when $f^a_\mu = 0$. Explicitly $(R = g^{\alpha\beta} R_{\alpha\beta})$

$$\begin{aligned} S = \int d^4x(e+f)\Big\{ &\lambda^4 + \lambda^3(R + f^a_\mu R^\mu_a) + \\ &+ \frac{\lambda^2}{2!}\left[R^2 - R^{\mu\nu}R_{\mu\nu} + (f^a_\mu R^\mu_a)^2 - f^{\mu\nu} f^{\rho\sigma} R_{\mu\rho} R_{\nu\sigma}\right] + \\ &+ \frac{\lambda}{3!}\left[R^3 - 3RR^{\mu\nu}R_{\mu\nu} + 2R^{\mu\alpha}R_{\alpha\beta}R^\beta_\mu + (f^a_\mu R^\mu_a)^3 - \right. \\ &\left. -3(f^a_\mu R^\mu_a) f^{\mu\nu} f^{\rho\sigma} R_{\mu\rho} R_{\nu\sigma} + 2 f^{\mu\nu} R^\alpha_\mu R_{\alpha\beta} R^\beta_\nu\right] + \det(R_{\mu\nu})\Big\}. \end{aligned} \tag{14}$$

Notice that the tetrad property was used here. In the remaining part of the work, this property will be used or not, wherever the case.

3. THE DYNAMICAL EQUATIONS

In this case, the variation with respect to the metric remains the same as in the previous works (see [1], Eq. (9)), e.g.,

$$\delta_g \sqrt{G} = \frac{\sqrt{G}}{2} (G^{-1})^{\mu\nu} \delta_g G = 0.$$

The variation with respect to the connection gives immediately

$$\frac{\delta\sqrt{G}}{\delta\Gamma^{\omega}_{\mu\nu}} = \{-\nabla_\sigma[\sqrt{G}(G^{-1})^{\alpha\nu}\mathcal{R}^{\sigma}_{\alpha}]\delta^{\mu}_{\omega} + \\ + \nabla_\omega[\sqrt{G}(G^{-1})^{\alpha\nu}\mathcal{R}^{\mu}_{\alpha}] + \sqrt{G}(G^{-1})^{\alpha\nu}\mathcal{R}^{\sigma}_{\alpha}\Gamma^{\mu}_{[\sigma\omega]}\}, \quad (15)$$

where the general form of Palatini's identity has been used and

$$G_{\mu\nu} \equiv \mathcal{R}^{a}_{\mu}\mathcal{R}_{a\nu},$$

with the $\mathcal{R}^{a}_{\mu}$ from Eq. (12b). Defining $\Sigma^{\nu\sigma} \equiv \sqrt{G}(G^{-1})^{\alpha\nu}\mathcal{R}^{\sigma}_{\alpha}$, the above equation can be written in a more suggestive form but due to the variation with respect to the metric it is identically zero (due to the lack of energy momentum tensor) and the only information, till known in our disposal is through the antisymmetric part of the variation with respect to the metric (see (12) of [1])

$$R_{\mu\nu} = -\lambda(g_{\mu\nu} + f_{\mu\nu}) \Rightarrow R_{[\mu\nu]} = (\nabla_\alpha + 2T_\alpha)(T^{\alpha}_{\mu\nu} + T_\nu\delta^{\alpha}_{\mu} - T_\mu\delta^{\alpha}_{\nu}) = -2\lambda f_{\mu\nu}, \quad (16)$$

with T_α being the trace of the torsion tensor. Now we have to explore the role played by $f_{\mu\nu}$:

i) If $f_{\mu\nu}$ plays the role of the electromagnetic field, then we have a one-form vector potential $f_{\mu\nu}$ which is derived. Notice the important fact that such an existence not necessarily can follow «a priori» from the definition of $f_{\rho\tau}$. This fact leads to the usual Euler–Lagrange equations, where the variation is made with respect to the electromagnetic potential a_τ

$$\frac{\delta\sqrt{G}}{\delta a_\tau} = \nabla_\rho\left(\frac{\partial\sqrt{G}}{\partial f_{\rho\tau}}\right) \equiv \nabla_\rho \mathbb{F}^{\rho\tau} = 0. \quad (17)$$

Explicitly

$$\nabla_\rho\left[\frac{\lambda^2 N^{\mu\nu}(\delta^{\sigma}_{\mu} f^{\rho}_{\nu} + \delta^{\sigma}_{\nu} f^{\rho}_{\mu})}{2\mathbb{R}}\right] = 0, \quad (18)$$

where $N^{\mu\nu}$ is given by expression (32) of [1]. The set of equations to solve for this particular case is

$$R_{(\mu\nu)} = \overset{\circ}{R}_{\mu\nu} - T^{\alpha}_{\mu\rho} T^{\rho}_{\alpha\nu} = -\lambda g_{\mu\nu}, \tag{19a}$$

$$R_{[\mu\nu]} = (\nabla_\alpha + 2T_\alpha)(T^{\alpha}_{\mu\nu} + T_\nu \delta^{\alpha}_{\mu} - T_\mu \delta^{\alpha}_{\nu}) = -\lambda f_{\mu\nu}, \tag{19b}$$

$$\nabla_\rho \left[\frac{\lambda^2 N^{\mu\nu} (\delta^{\sigma}_{\mu} f^{\rho}_{\nu} + \delta^{\sigma}_{\nu} f^{\rho}_{\mu})}{2\mathbb{R}} \right] = 0, \tag{19c}$$

where the quantities with a little circle «○» are defined from the Christoffel connection (as in general relativity). From Eqs. (19), the link between T and f will be determined.

ii) The $f_{\mu\nu}$ has only the role to be the antisymmetric part of a fundamental (nonsymmetric) tensor K, i.e., $f_{\mu\nu}$ closed but not necessarily exact. Then, the variation of the geometrical Lagrangian $\delta_f \sqrt{G}$ gives the same information that $\delta_g \sqrt{G}$. That means that the remaining equations are

$$R_{(\mu\nu)} = \overset{\circ}{R}_{\mu\nu} - T^{\alpha}_{\mu\rho} T^{\rho}_{\alpha\nu} = -\lambda g_{\mu\nu}, \tag{20a}$$

$$R_{[\mu\nu]} = (\nabla_\alpha + 2T_\alpha)(T^{\alpha}_{\mu\nu} + T_\nu \delta^{\alpha}_{\mu} - T_\mu \delta^{\alpha}_{\nu}) = -\lambda f_{\mu\nu}. \tag{20b}$$

3.1. Analysis and Reduction of the Dynamical Equations. One important equation, that appears in the two sets recently described (independently of the specific role of the antisymmetric tensor $f_{\mu\nu}$) brings us a lot of information about the link between T and f (Eqs. (19b) and (20b)). Precisely, this equation $R_{[\mu\nu]} = -\lambda f_{\mu\nu}$ plus the condition $\nabla_\alpha T^{\alpha}_{\mu\nu} = 0$ lead immediately to

$$\nabla_\mu T_\nu - \nabla_\nu T_\mu = -(\lambda f_{\mu\nu} + 2T_\alpha T^{\alpha}_{\mu\nu}), \tag{21}$$

then, the quantity that *naturally appears* in the r.h.s. is the «definition» in the current literature of the *minimal coupling* electromagnetic tensor $\mathcal{F}_{\mu\nu}$ in a space–time with torsion. Notice the important fact that $\nabla_\alpha T^{\alpha}_{\mu\nu} = 0$ is equivalent to

$$d^* T = 0,$$

the torsion is current free. Two cases naturally arise:

i) If we assume the existence of the potential vector, we have

$$\nabla_\mu T_\nu - \nabla_\nu T_\mu \equiv \mathcal{F}_{\mu\nu} = -\lambda \Bigl(\overbrace{\partial_\mu a_\nu - \partial_\nu a_\mu}^{f_{\mu\nu}} \Bigr) - 2T_\alpha T^{\alpha}_{\mu\nu}, \tag{22}$$

a link between a_ν and T_ν clearly appears: $T_\nu = -\lambda a_\nu$. The important fact to remark here is that, although in [11] the link between the trace of the torsion and the vector potential of the electromagnetic field was proposed, but in the theory presented in this paper this relation is derived automatically from its geometrical basis. Beside this point, note that $\mathcal{F}_{\mu\nu} = F_{\mu\nu} + B_{\mu\nu}$, with $B_{\mu\nu}$ being such a type of «background» field generated by the space–time torsion.

ii) If $f_{\mu\nu}$ has only the role to be the antisymmetric part of a fundamental (nonsymmetric) tensor K, it acquires a potential automatically, being in this manner *an exact form*, where T_ν takes the role of potential vector. Clearly, now f cannot be a potential for the torsion from this point of view (in a nontrivial topology, it can be, of course).

From above statements over the «trace» of the torsion, it is clearly seen that two ansatz appear as candidates for the torsion tensor structure: the «tratorial» structure $T^{\alpha}_{\mu\nu} \sim (\delta^{\alpha}_{\mu} a_\nu - \delta^{\alpha}_{\nu} a_\mu)$ and the «product» structure $T^{\alpha}_{\mu\nu} = k^\alpha f_{\mu\nu}$, where the vector k^α is the eigenvector of the antisymmetric tensor $f_{\mu\nu}$, in general (notice that torsion tensor with this «product structure» also has the possibility to be fully antisymmetric).

The other possibility is to take $\nabla_\alpha T^{\alpha}_{\mu\nu} = -\lambda f_{\mu\nu}$, then $\nabla_\mu T_\nu - \nabla_\nu T_\mu = -2T_\alpha T^{\alpha}_{\mu\nu}$, but their interpretation is not so clean as before. Even more, it brings us to a «product structure», with the torsion tensor being not fully antisymmetric.

3.2. A Potential for the Torsion. As was shown in [1], if we impose the restriction $T_{\alpha\beta\gamma} = T_{[\alpha\beta\gamma]}$ (e.g., totally antisymmetric torsion tensor), from Eq. (2), for example, we note that only the antisymmetric part of the fundamental tensor $K_{\alpha\beta}$ determines fully the torsion tensor. Then, due to the assumption of a torsion tensor to be completely antisymmetric, the potential torsion $f_{\mu\nu}$ exists and arises in a natural form (the ∇ for the covariant derivative with respect the full connection Γ). This potential torsion has the following properties:

$$\begin{aligned} f_{\mu\nu} = \overline{f}_{\mu\nu} &= -f_{\nu\mu} \in \mathbb{HC}, \\ \nabla_{[\rho} f_{\mu\nu]} &= T_{\mu\nu\rho}, \\ &= \varepsilon_{\mu\nu\rho\sigma} h^\sigma, \end{aligned} \tag{23}$$

with the last equality coming from the full antisymmetry of the torsion field. Immediately we can see, as a consequence of the above statements, the following:

i) The torsion is the dual of an axial vector h^σ.

ii) From i), the existence in the space–time of a completely antisymmetric tensor is covariantly constant $\varepsilon_{\mu\nu\rho\sigma} (\nabla\varepsilon = 0)$.

Notice the choice for the real nature of the metric and the pure hypercomplex potential tensor coming from the Hermitian nature of the theory, as was clearly explained in [1].

For expression (13) of [1], we have a highly nonlinear dynamical (propagating) equation for the torsion field, where the variation was performed with respect to their potential $f_{\mu\nu}$ and has a nonlinear term proportional to $f_{\mu\nu}$ playing the role of current for the $\mathbb{T}^{\rho\sigma\tau}$. Then, 2-form potential is associated nonlinearly to the torsion field in a similar manner as the electromagnetic field and the spin in particle physics.

For the expression (12) of [1], firstly, it is useful to split the equation into the symmetric and the antisymmetric parts using $R_{\mu\nu}$ explicitly as before

$$R_{(\mu\nu)} = \overset{\circ}{R}_{\mu\nu} - T^{\alpha}_{\mu\rho}T^{\rho}_{\alpha\nu} = -2\lambda g_{\mu\nu}, \tag{24}$$

$$R_{[\mu\nu]} = \overset{\circ}{\nabla}_{\alpha}T^{\alpha}_{\mu\nu} = -2\lambda f_{\mu\nu}, \tag{25}$$

$$= \nabla_{\alpha}T^{\alpha}_{\mu\nu}$$

(the last equality coming from the total antisymmetry of the torsion).

Notice the important fact that $-2\lambda f_{\mu\nu}$ is the «current» for the torsion field, as the terms proportional to the 1-form potential vector a_{μ} act as current of the electromagnetic field $f_{\mu\nu}$ in the equation of motion for the electromagnetic field in the standard theory: $\nabla_{\alpha}f^{\alpha}_{\mu} = J_{\mu}$ (constants absorbed into the J_{μ}).

The symmetric part (24) can be written in a «GR» suggestive fashion

$$\overset{\circ}{R}_{\mu\nu} = -2\lambda g_{\mu\nu} + T^{\alpha}_{\mu\rho}T^{\rho}_{\alpha\nu}; \tag{26}$$

we can advertise that the equation has the aspect of the Einstein equations with the cosmological term modified by the torsion symmetric term $T^{\alpha}_{\mu\rho}T^{\rho}_{\alpha\nu}$. This can be interpreted, as was shown in [1], by the energy of the gravitational field itself.

The second antisymmetric part (25) is more involved. In order to understand it, it will be necessary to use the language of differential forms to rewrite them, that, beside their symbolic and conceptual simplicity, permits us to check consistency and covariance step by step

$$\begin{aligned} \nabla_{\alpha}T^{\alpha}_{\mu\nu} &= -2\lambda f_{\mu\nu}, \\ d^{*}T &= -2\lambda^{*}f. \end{aligned} \tag{27}$$

Now, using $T = {}^{*}h$

$$dh = -2\lambda^{*}f \Rightarrow {}^{*}f = -\frac{1}{2\lambda}dh \tag{28}$$

in more familiar form

$$\nabla_{\mu}h_{\nu} - \nabla_{\nu}h_{\mu} = -2\lambda^{*}f_{\mu\nu}, \tag{29}$$

it follows, using again $T = df = {}^{*}h$ and Eq. (27), that

$$d^{*}f = 0, \tag{30}$$

and fundamentally

$$df = -\frac{1}{2\lambda}d^{*}dh = T = {}^{*}h, \tag{31}$$

$$d^{*}dh = -2\lambda^{*}h. \tag{32}$$

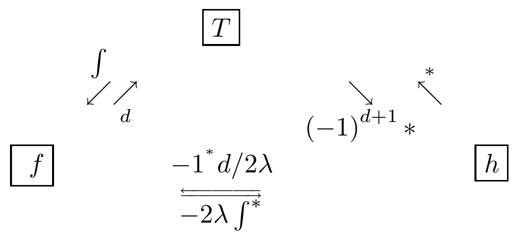
We can recognize the Laplace–de Rham operator that helps us to write the wave covariant equation

$$[(d\delta + \delta d) + 2\lambda]^* h = 0, \quad (\Delta + 2\lambda)^* h = 0. \tag{33}$$

Starting with the potential, it is not difficult to see that equivalent equation can be found

$$(\Delta + 2\lambda)^* f = 0. \tag{34}$$

Notice that equation (33) comes from (28) and is a consequence of the Tfh-relation ($T = df = {}^*h$), but (34) comes directly from (27). The geometric interplay is the following*:



(35)

4. EXACT SOLUTIONS IN THE NEW UFT THEORY

The main motivation in this Section is clear: we must equip our «theoretical arena» by studying wormhole solutions beyond the Einstein equations coupled to possible matter fields. Then, let us construct wormhole solutions from the viewpoint of the UFT model introduced here. The action in four dimensions is given by

$$S = -\frac{1}{16\pi G}\int d^4x \sqrt{\det |G_{\mu\nu}|}, \tag{36}$$

$$\mathbb{R} \equiv \sqrt{\gamma^4 - \frac{\gamma^2}{2}\overline{G}^2 - \frac{\gamma}{3}\overline{G}^3 + \frac{1}{8}\left(\overline{G}^2\right)^2 - \frac{1}{4}\overline{G}^4}. \tag{37}$$

4.1. Totally Antisymmetric Torsion. Scalar curvature R and the torsion 2-form field $T^a_{\mu\nu}$ with a $SU(2)$ Yang–Mills structure are defined in terms of the affine connection $\Gamma^\lambda_{\mu\nu}$ and the $SU(2)$ potential torsion f^a_μ by

$$\begin{gathered} R = g^{\mu\nu} R_{\mu\nu}, \quad R_{\mu\nu} = R^\lambda_{\mu\lambda\nu}, \quad R^\lambda_{\mu\lambda\nu} = \partial_\nu \Gamma^\lambda_{\mu\rho} - \partial_\rho \Gamma^\lambda_{\mu\nu} + \ldots, \\ T^a_{\mu\nu} = \partial_\mu f^a_\nu - \partial_\nu f^a_\mu + \varepsilon^a_{bc} f^b_\mu f^c_\nu, \end{gathered} \tag{38}$$

*In order to be consistent with the action of the Hodge operator ($*$), in this subsection, we assume an even number of dimensions [2].

G and Λ are the Newton gravitational constant and the cosmological constant, respectively. Notice the important fact that from the last equation for the torsion 2-form, the potential f^a_μ must be proportional to the antisymmetric part of the affine connection $\Gamma^\lambda_{\mu\nu}$ as in the Strauss–Einstein UFT. As in the case of Einstein–Yang–Mills systems, for our new UFT model, it can be interpreted as a prototype of gauge theories interacting with gravity (e.g., QCD, GUTs, etc.). Upon varying the action, we obtain the gravitational «Einstein–Eddington-like» equation

$$R_{\mu\nu} = -2\lambda(g_{\mu\nu} + f_{\mu\nu}) \tag{39}$$

and the field equation for the torsion 2-form in differential form

$$d^*\mathbb{T}^a + \frac{1}{2}\varepsilon^{abc}(f_b \wedge^* \mathbb{T}_c - {}^*\mathbb{T}_b \wedge f_c) = \mathbb{F}^a, \tag{40}$$

where we define as usual

$$\mathbb{T}^a_{bc} \equiv \frac{\partial L_G}{\partial T^{bc}_a}, \quad \mathbb{F}^a_{bc} \equiv \frac{\partial L_G}{\partial F_a}.$$

We are going to seek for a classical solution of Eqs. (39) and (40) with the following spherically symmetric ansatz for the metric and gauge connection:

$$ds^2 = d\tau^2 + a^2(\tau)\sigma^i \otimes \sigma^i \equiv d\tau^2 + e^i \otimes e^i. \tag{41}$$

Here τ is the Euclidean time and the dreibein is defined by $e^i \equiv a(\tau)\sigma^i$. The gauge connection is

$$f^a \equiv f^a_\mu dx^\mu = h\sigma^a \tag{42}$$

for $a = 1, 2, 3$, and for $a = 0$ it is

$$f^0 \equiv f^0_\mu dx^\mu = s\sigma^0. \tag{43}$$

This choice for the potential torsion is the most general and consistent from the physical and mathematical point of view due to the symmetries involved in the problem, as we will show soon.

The σ^i 1-form satisfies the $SU(2)$ Maurer–Cartan structure equation

$$d\sigma^a + \varepsilon^a_{bc}\sigma^b \wedge \sigma^c = 0. \tag{44}$$

Notice that in the ansatz, the frame and isospin indexes are identified as for the case with the NBI Lagrangian of [3]. The torsion 2-form

$$T^\gamma = \frac{1}{2}T^\gamma_{\mu\nu}dx^\mu \wedge dx^\nu \tag{45}$$

becomes

$$\begin{aligned} T^a &= df^a + \frac{1}{2}\varepsilon^a_{bc} f^b \wedge f^c, \\ &= \left(-h + \frac{1}{2}h^2\right)\varepsilon^a_{bc}\sigma^b \wedge \sigma^c. \end{aligned} \tag{46}$$

Notice that f^0 plays no role here because we take simply $ds = 0$ (the $U(1)$ component of $SU(2)$, in principle, does not form a part of the space spherical symmetry), and the expression for the torsion is analogous to the non-Abelian 2-form strength field of [3]. It is important to note that, when we go from the Lorentzian to Euclidean gravitational regime, then $it \to \tau$, and the torsion passes from the field of the *Hypercomplex* to the *Complex* numbers. Geometrically, multiplication of hypercomplex numbers preserves the (square) Minkowski norm $(x^2 - y^2)$ in the same way that multiplication of complex numbers preserves the (square) Euclidean norm $(x^2 + y^2)$. Inserting T^a from Eq. (46) into the dynamical equation (40) we obtain

$$\begin{aligned} d^*\mathbb{T}^a + \frac{1}{2}\varepsilon^{abc}(f_b \wedge^* \mathbb{T}_c - {}^*\mathbb{T}_b \wedge f_c) &= {}^*\mathbb{F}^a, \\ (-2h + h^2)(1-h)\, d\tau \wedge e^b \wedge e^c &= -2\lambda d\tau \wedge e^b \wedge e^c, \end{aligned} \tag{47}$$

where

$${}^*\mathbb{T}^a \equiv \frac{\lambda\sqrt{|g|}}{\sqrt{3}} hA(-2h + h^2)\, d\tau \wedge \frac{e^a}{a^2}, \tag{48}$$

$${}^*\mathbb{F}^a = -\frac{2\lambda^2\sqrt{|g|}}{\sqrt{3}} hA \frac{d\tau \wedge e^b \wedge e^c}{a^3}, \tag{49}$$

$$A \equiv \lambda^4 \left[(1+\alpha)^2 + \frac{\alpha}{2}\right], \tag{50}$$

and

$$\alpha = \frac{1}{2}(s^2 + 3h^2); \tag{51}$$

from expression (47) we have an algebraic cubic equation for h

$$(-2h + h^2)(1-h) + 2\lambda = 0. \tag{52}$$

We can see that, in contrast with our previous work with a dualistic theory [3], where the energy-momentum tensor of Born–Infeld was considered, for h there exist three nontrivial solutions depending on the cosmological constant λ. But, at this preliminary analysis of the problem, only the values of h that make the quantity $(-h + (1/2)h^2) \in \mathbb{R}$ are relevant for our proposals: due to the pure imaginary character of T in the Euclidean framework and mainly to comparison with the NABI wormhole solution of our previous work (the question of the $h \in \mathbb{C}$ will be the focus of a further paper [5]). As the value of $h \in \mathbb{R}$ is -1 and in 4 space–time dimensions $\lambda = |1 - d| = 3$, then

$$T^a_{bc}|_{h_1} = \frac{3}{2}\frac{\varepsilon^a_{bc}}{a^2}, \quad T^a_{0c} = 0. \tag{53}$$

Namely, only the magnetic field is nonvanishing while the electric field vanishes. An analogous feature can be seen in the solution of Giddings and Strominger and in our previous paper [3]. Substituting the expression for the torsion 2-form (53) into the symmetric part of the variational equation, namely*,

$$R_{(\mu\nu)} = \overset{\circ}{R}_{\mu\nu} - T^\alpha_{\mu\rho} T^\rho_{\alpha\nu} = -2\lambda g_{\mu\nu}, \tag{54}$$

we reduce equation (24) to an ordinary differential equation for the scale factor a,

$$\left[\left(\frac{\dot{a}}{a}\right)^2 - \frac{1}{a^2}\right] = \frac{2\lambda}{3} - \frac{9}{2a^4}, \tag{55}$$

$$\frac{\ln\left[1 + 4a^2 + 2\sqrt{-9 + 2a^2 + 4a^4}\right]}{2\sqrt{2}} = \tau - \tau_0, \tag{56}$$

$$\begin{aligned} T^\alpha_{\mu\rho} T^\rho_{\alpha\nu} &= \frac{\left(-h + \frac{1}{2}h^2\right)^2}{a^4} 2\delta_{\mu\nu}, \\ &= \frac{9}{2a^4}\delta_{\mu\nu}. \end{aligned} \tag{57}$$

There are two values for the scale factor a: max. and min., respectively, namely,

$$a = \mp \frac{\mathrm{e}^{-\sqrt{2}(\tau-\tau_0)}\sqrt{37 - 2\,\mathrm{e}^{2\sqrt{2}(\tau-\tau_0)} + \mathrm{e}^{4\sqrt{2}(\tau-\tau_0)}}}{2\sqrt{2}}. \tag{58}$$

Expression (58) for the scale factor a is described in Fig. 1 for the real value of h.

As is easily seen from (58), the scale factor has an exponentially growing behavior, in sharp contrast to the wormhole solution from our previous work with the «dualistic» non-Abelian BI theory (Fig. 4). Also, for this particular value of the torsion, the wormhole tunneling interpretation (in the sense of Coleman's

*In the tetrad: $\overset{\circ}{R}_{00} = -3\frac{\ddot{a}}{a}$, $\overset{\circ}{R}_{ab} = -\left[\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 - \frac{2}{a^2}\right]$.

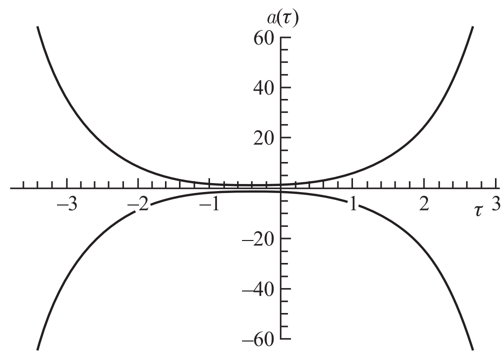


Fig. 1. Shape of the wormhole solution for values of the Euclidean time and torsion $\tau_0 = 1$ and $T^a_{bc} = (3/2)\varepsilon^a_{bc}$, respectively

mechanism) is fulfilled. Now we need to see what happens with equation (27) in this particular case under consideration: equation (27) takes the following form:

$$d^*T^a + \frac{1}{2}\varepsilon^{abc}(f_b \wedge^* T_c -^* T_b \wedge f_c) = -2\lambda^* f^a,$$
$$(-2h + h^2)(1 - h)\, d\tau \wedge e^b \wedge e^c = -2\lambda\, d\tau \wedge e^b \wedge e^c, \tag{59}$$

$$^*T^a \equiv h(-2h + h^2)\, d\tau \wedge \frac{e^a}{a^2}, \tag{60}$$

$$^*f^a = -h\frac{d\tau \wedge e^b \wedge e^c}{a^3}. \tag{61}$$

Then we arrive to the same equation for λ as it was given in (52), corroborating the self-consistency of the procedure.

4.2. «Tratorial» Torsion. To begin with, let us consider the problem involving the set of Eq. (19) with the usual definition for the $SU(2)$ electromagnetic field strength

$$f^\gamma = \frac{1}{2}f^\gamma_{\mu\nu}\, dx^\mu \wedge dx^\nu, \tag{62}$$

and, as before, we are going to seek for a classical solution of Eqs. (19) with the following spherically symmetric ansatz for the metric and gauge connection:

$$ds^2 = d\tau^2 + a^2(\tau)\sigma^i \otimes \sigma^i \equiv d\tau^2 + e^i \otimes e^i, \tag{63}$$

here τ is the Euclidean time and the dreibein is defined by $e^i \equiv a(\tau)\sigma^i$. However, in the case of the set (19), we assume that the 2-form f^γ comes from the 1-form potential A where, as in the non-Abelian Born–Infeld model of [3], it is defined as $A^a \equiv A^a_\mu dx^\mu = h\sigma^a$.

The extremely important fact in this case is that we know that σ^i 1-form satisfies the $SU(2)$ Maurer–Cartan structure equation, as fundamental geometrical structure of the non-Abelian electromagnetic field

$$d_{su(2)}\sigma^a + \varepsilon^a_{bc}\sigma^b \wedge \sigma^c = 0, \tag{64}$$

but now due to the identification assumed in (63):

$$e^i \equiv a(\tau)\sigma^i, \tag{65}$$

$$\Rightarrow de^a = T^a - e^a_b \wedge \sigma^b. \tag{66}$$

Here we make the difference between the exterior derivatives in the space–time with torsion and in the $SU(2)$ group manifold. It is clearly seen that a question of compatibility involving the identification of the gauge group with the geometrical structure of the space–time with torsion certainly exists. From (64)–(66), we see that

$$\partial_\tau a\, d\tau \wedge \sigma^a - a\varepsilon^a{}_{bc}\sigma^b \wedge \sigma^c = T^a - e^a_b \wedge \sigma^b. \tag{67}$$

If

$$e^a_b = -\varepsilon^a_{bc}\sigma^c \tag{68}$$

and

$$T^a = \delta^a_b(\partial_\tau a)\, d\tau \wedge \sigma^b \tag{69}$$

the space–time and gauge group are fully compatible, then

$$d\sigma^a + \varepsilon^a_{bc}\sigma^b \wedge \sigma^c = 0 \tag{70}$$

is restored. Hence, the general form assumed for the torsion field, due to the symmetry conditions prescribed above, is

$$T^\alpha_{\beta\gamma} = \xi(\delta^\alpha_\beta u_\gamma - \delta^\alpha_\gamma u_\beta) + \varsigma h_\delta \varepsilon^{\delta\alpha}_{\beta\gamma} \quad (\xi, \varsigma : \mathrm{const}). \tag{71}$$

Notice that the condition of compatibility, that imposes such a type of «trator» form for the torsion tensor in order to restore the behaviour of the volume form of the space–time with respect to the covariant derivative, here appears in a natural manner without introducing any extra scalar field (dilaton) or passing to other frame (i.e., Jordan, Einstein, etc.). Moreover, if we continue without making the correspondences to (68), (69), the equations of motion for the electromagnetic field itself bring automatically these conditions.

Notice that in the HO ansatz, the frame and isospin indexes are identified as for the case with the NBI Lagrangian of [3]. The electromagnetic field 2-form is

$$\begin{aligned} f^a &= dA^a + \frac{1}{2}\varepsilon^a_{bc} A^b \wedge A^c, \\ &= h\delta^a_b(\partial_\tau \ln a)\, d\tau \wedge \sigma^b + h\frac{T^a}{a} - \left(-h + \frac{1}{2}h^2\right)\varepsilon^a_{bc}\sigma^b \wedge \sigma^c, \\ &= \left(-h + \frac{1}{2}h^2\right)\varepsilon^a_{bc}\sigma^b \wedge \sigma^c, \end{aligned} \tag{72}$$

where in the last equality conditions (68), (69) have been assumed. The dynamical equations are

$$\begin{gathered} \mathbb{F}^a_{bc} \equiv \frac{\partial L_G}{\partial F_a} \Rightarrow \\ {}^*\mathbb{F}^a \equiv \frac{\lambda\sqrt{|g|}}{\sqrt{3}} h\mathbb{A}(-2h + h^2)\, d\tau \wedge \frac{e^a}{a^2} \equiv Mh(-2h + h^2)\, d\tau \wedge \frac{e^a}{a^2}. \end{gathered} \tag{73}$$

Inserting it in the Yang–Mills-type field equation (19c) we obtain

$$\begin{gathered} d^*\mathbb{F}^a + \frac{1}{2}\varepsilon^{abc}(A_b \wedge^* \mathbb{F}_c -^* \mathbb{F}_b \wedge A_c) = 0, \\ = Mh\, d\tau \wedge \sigma^b \wedge \sigma^c(-2h + h^2)(h-1), \\ \mathbb{A} \equiv \lambda^4[(1+\alpha)^2 + \alpha/2]. \end{gathered} \tag{74}$$

Then, there exists a nontrivial solution: $h = 1$ (with $s = 0$ in $\mathbb{A}$ as before in [1]). The electromagnetic field is immediately determined. It is the same form as in the non-Abelian Born–Infeld model considered in [3], namely,

$$f^a_{bc} = -\frac{\varepsilon^a_{bc}}{a^2}, \quad f^a_{0c} = 0, \tag{75}$$

we have only magnetic field.

Now considering only a «trator» form for the torsion, Eq. (16b) is identically null due to the magnetic character of f^a and the particular form of the symmetric coefficients of the connection. Inserting the torsion Eq. (69) into Eq. (19a), as in the previous section, we obtain

$$\left[\left(\frac{\dot{a}}{a}\right)^2 - \frac{1}{a^2}\right] = \frac{\lambda}{3}. \tag{76}$$

Integration of this last expression immediately leads to

$$a(\tau) = \left(\frac{\lambda}{3}\right)^{-1/2} \sinh\left[\left(\frac{\lambda}{3}\right)^{1/2}(\tau - \tau_0)\right]. \tag{77}$$

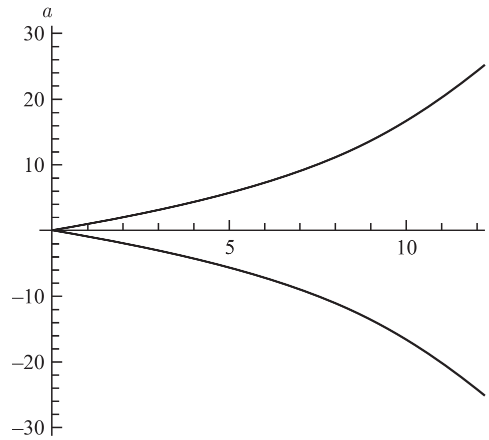


Fig. 2. Shape of the cosmological solution for values of $|\lambda| = 3$, $d = 4$, and the Euclidean time $\tau > 0$

Then it is quite evident that this particular case does not lead to wormhole configurations because there exists only eternal expansion with $a(\tau_0) = 0$ (the origin of the Euclidean time, Fig. 2).

Now considering only the product form for the torsion, Eq. (19c) does not change but Eq. (19b) takes the form of a wave equation for the scale factor

$$[\Box a + (\partial_0 a)(\partial^0 a)] = \lambda$$

due to $T^{\alpha}_{\beta\gamma} = \varsigma k^{\alpha}\varepsilon_{\beta\gamma} \to \varepsilon_{ab}(\partial^0 a)$. It is not difficult to see that the $su(2)$ structure of the electromagnetic tensor is in some manner transferred to the structure of the torsion. But here we enter in conflict because the system of Eqs. (19) turns out to be *overdetermined*: probably we need more freedom in the ansatz for f^a_{bc} ($s \neq 0$, or $h = h(\tau)$). This fact will be studied in the near future [5].

4.3. General Case. Let us assume the full form (71) for T^a

$$^*\mathbb{F}^a \equiv Mh\Big\{h\delta^a_d(\partial_\tau \ln a)\varepsilon^{d0}_{ed}\sigma^e \wedge \sigma^d + \frac{h}{a}[\xi(\delta^a_i u_j - \delta^a_j u_i) + \\ + \varsigma h_\delta \varepsilon^{\delta a}_{ji}]\varepsilon^{ij}_{kl}\omega^k \wedge \omega^l + (-2h + h^2)\varepsilon^a_{bc}\varepsilon^{bc}_{0d}\, d\tau \wedge \sigma^a\Big\}. \quad (78)$$

Here, in order to avoid the cumbersome expression in the second term due to the standard orthonormal splitting, $ij = 0, a, b, c$ and the ω^k are the corresponding

1-forms $(d\tau, \sigma^a \ldots)$ wherever the case. The YM-type equation can be written as

$$\begin{aligned} d^*\mathbb{F}^a + \frac{1}{2}\varepsilon^{abc}(A_b \wedge^* \mathbb{F}_c -^* \mathbb{F}_b \wedge A_c) = Mh\Bigg\{\Bigg[h\delta^a_b(\partial_\tau\partial_\tau \ln a) + \\ + \partial_\tau\left(\frac{h}{a}(\xi(\delta^a_b u_0 - \delta^a_0 u_b) + \varsigma h_c \varepsilon^{ca}_{b0})\right)\Bigg]\varepsilon^{b0}_{ed} d\,\tau \wedge \sigma^e \wedge \sigma^d + \\ + \left[h\delta^a_b(\partial_\tau \ln a) + \frac{h}{a}(\xi(\delta^a_b u_0 - \delta^a_0 u_b) + \varsigma h_c \varepsilon^{ca}_{b0})\right] 2d(\sigma^e \wedge \sigma^d)\Bigg\} + \\ + M\left[\frac{h}{a}(\xi(\delta^a_b u_0 - \delta^a_0 u_b) + \varsigma h_c \varepsilon^{ca}_{b0}) + (-2h + h^2)\right](h-1)\, d\tau \wedge \sigma^b \wedge \sigma^c = 0. \end{aligned} \tag{79}$$

From the above equation we obtain information about the determination of the f field and of the torsion field as in the previous cases: the first term

$$\left[h\delta^a_b(\partial_\tau\partial_\tau \ln a) + \partial_\tau\left(\frac{h}{a}(\xi(\delta^a_b u_0 - \delta^a_0 u_b) + \varsigma h_c \varepsilon^{ca}_{b0})\right)\right] = 0 \tag{80}$$

leads immediately to

$$\begin{aligned} [\eta_{ab}\partial_0 a + (\xi(\eta_{ab}u_0 - \eta_{a0}u_b) + \varsigma h_c \varepsilon^{ca}_{b0})] = \Xi^A_{ab0} + \Xi^s_{ab0} \Rightarrow \\ \Rightarrow \varsigma h_c \varepsilon^c_{ab0} \equiv \Xi^A_{ab0} \Rightarrow \eta_{ab}\partial_0 a + \xi(\eta_{ab}u_0 - \eta_{a0}u_b) = \Xi^S_{ab0}, \end{aligned} \tag{81}$$

where the tensor

$$\Xi_{ab0} = \Xi^A_{ab0} + \Xi^S_{ab0}$$

is independent of the time, and the superscripts A and S indicate the totally antisymmetric part of the other nontotally antisymmetric one. Then, the second and third equalities above follow. It is not difficult to see, that contracting indices, tracing and considering the symmetries involved, we obtain explicitly

$$T^a_{b0} = \delta^a_{[b}\partial_{0]}a - a\widetilde{\Xi}^{Sa}_{b0} + \Xi^A_{ab0}, \tag{82}$$

$$T^a_{bc} = -a\widetilde{\Xi}^{Sa}_{bc} + \varsigma h_0 \varepsilon^{0a}_{bc}, \tag{83}$$

$$T^0_{bc} = -a\widetilde{\Xi}^{S0}_{bc} + \varsigma h_c \varepsilon^{c0}_{bc}, \tag{84}$$

where the integration tensors (independent of time) are related with u_i and $\widetilde{\Xi}^{Sj}_{kl}\,(ij \ldots = 0, a, b, c)$ as follows:

$$u_c = -\frac{a\Xi^S_c}{2\xi}, \quad u_0 = -\frac{1}{2\xi}(3\partial_0 a + a\Xi^S_c), \quad \Xi^S_c \equiv \Xi^{Sj}_{cj}, \quad \Xi^S_0 \equiv \Xi^{Sj}_{0j}$$

$$\text{and} \quad \widetilde{\Xi}^{Sj}_{kl} \equiv \frac{-1}{2}(\delta^j_k \Xi^S_l - \delta^j_l \Xi^S_k).$$

The last term, however, indicates that there exists the simplest solution with $h = 1$, as in the previous case for the non-Abelian f. Then

$$f^a_{bc} = -\frac{\varepsilon^a_{bc}}{a^2}, \quad f^a_{b0} = 0$$

again, and the second equation is identically zero due to the symmetry of the torsion 2-form with respect to the tetrad defined by (63). Now the question is whether the system of equations is overdetermined or not. To this end, we put expressions (82)–(84) to Eq. (19b). Now, again, from the equations

$$\nabla_i T^i_{ab} + 2T_i T^i_{ab} = -\lambda f^c_{ab} e_c, \tag{85}$$

$$\nabla_i T^i_{ao} + 2T_i T^i_{a0} = 0 \tag{86}$$

we fix the torsion tensor components as

$$T^a_{b0} = \delta^a_{[b}\partial_{0]} a, \tag{87}$$

$$T^a_{bc} = -a\widetilde{\Xi}^{Sa}_{bc} + \varsigma h_0 \varepsilon^{0a}_{bc}, \tag{88}$$

$$T^0_{bc} = 0. \tag{89}$$

Expression (86) turns to a null identity, and from (85) we get

$$\begin{aligned} 4T_i T^i_{ab} = 4a\widetilde{\Xi}^S_c \overbrace{(-a\widetilde{\Xi}^{Sa}_{bc} + \varsigma h_0 \varepsilon^{0a}_{bc})}^{T^a_{bc}} = -\lambda f^c_{ab} e_c \Rightarrow \\ \Rightarrow 4(a^2 \widetilde{\Xi}^S_c \widetilde{\Xi}^{Sc}_{ab} - a\widetilde{\Xi}^S_c \varsigma h_0 \varepsilon^{0c} \varepsilon_{ab}) = -\lambda f^c_{ab} e_c, \\ a\widetilde{\Xi}^S_c \varsigma h_0 \varepsilon^{0c} \varepsilon_{ab} = -\lambda f^c_{ab} e_c = \frac{\lambda \varepsilon^c_{ab}}{a^2} e_c, \\ \widetilde{\Xi}^S_c \varsigma h_0 \varepsilon^{0c} \varepsilon_{ab} = \frac{\lambda \varepsilon^c_{ab}}{a^2} \sigma_c, \end{aligned} \tag{90}$$

where in the last line we use the property $\widetilde{\Xi}^S_c \widetilde{\Xi}^{Sc}_{ab} = \Xi^S_c(\delta^c_a \Xi^S_b - \delta^c_b \Xi^S_a) \equiv 0$ (see definitions above).

It is easily seen, that by squaring both sides of (90) and from (89), we obtain

$$h_0 = \frac{\lambda \sigma_0}{a^2 |\widetilde{\Xi}^S_c| 2\varsigma}, \quad h_c = \frac{\lambda |\widetilde{\Xi}^S_c| a \sigma_c}{2\varsigma},$$

and analogously to the previous cases, from Eqs. (19a) the equation to integrate takes the form

$$\frac{da}{d\tau} = \pm \frac{4}{5}\left[1 + \frac{\lambda}{3}a^2 + \frac{2}{3}a^4 |\widetilde{\Xi}^S_c|^2 + \frac{3}{8}\left(\frac{\lambda}{|\widetilde{\Xi}^S_c| a}\right)^2\right]^{1/2}.$$

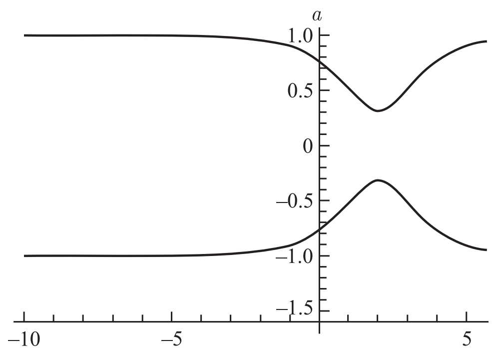


Fig. 3. Shape of the cosmological solution for values of $|\lambda| = 3$, $d = 4$

One interesting case when the above equation can be integrated exactly is precisely when $d = 4$. This condition, besides improving the integrability condition of the equation, fixes $|\widetilde{\Xi}_c^S|^2 > 3/2$. The scale factor $a(\tau)$ takes the following form:

$$a(\tau) = \sqrt{B + (A - B)\tanh^2\left[\frac{(\tau - \tau_0)\sqrt{(A - B)}}{2}\right]},$$

where A and B are nonlinear functions of the norm square $|\widetilde{\Xi}_c^S|^2$. The explicit form of these functions is not crucial: only the bound for $|\widetilde{\Xi}_c^S|^2 > 3/2$ needs to be preserved (also through the normalization of A and B into the graphic representation, see Figs. 3 and 4). Notice that the space–time is asymptotically Minkowskian with a wormhole $a(\tau_0) = \sqrt{B}$ (however the values of the constants have been selected according to the previous remarks). Other possibilities not enumerated here, lead space–times with cyclic singularities due to transcendental

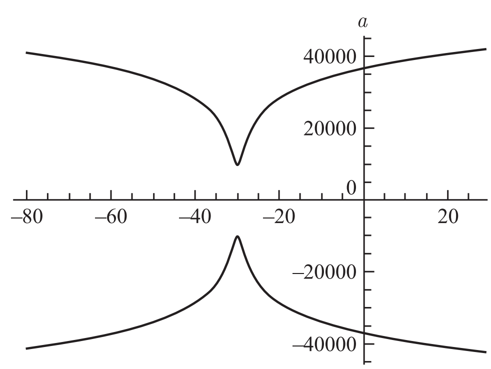


Fig. 4. Shape of the instanton–wormhole solution for $r_0 = 70$, $a = 40$

functions into the denominator of the expression for the scale factor $a(\tau)$. This issue is a focus of a future discussion somewhere [5].

4.4. Coexistence of Both Types of Torsion in Cosmological Space–Times. It is interesting to note that in [4] the field equations of vacuum Quadratic Poincare Gauge Field Theory (QPGFT) were solved for purely null tratorial torsion. The author there expressed the contortion tensor for such a case as

$$K_{\lambda\mu\nu} = -2(g_{\lambda\mu}a_\nu - g_{\lambda\nu}a_\mu).$$

However, the important thing is that the author has discussed the relationship between this class (tratorial) and a similar class of solutions with null axial vector torsion, arriving to the conclusion that cosmological solutions with different types of torsion are forbidden. The main reason of this situation can have two origins: the specific theory and action (QPGFT), or the Newman–Penrose method used in the computations that works, as is well known, with null geometric quantities. Here we show that this problem does not arise in our theory.

5. THE UNDERLYING DIRAC STRUCTURE OF THE SPACE–TIME MANIFOLD

The real structure of the Dirac equation in the form

$$(\gamma_0 p_0 - i\gamma \cdot \mathbf{p})\mathbf{u} = m\mathbf{v}, \tag{91}$$

$$(\gamma_0 p_0 + i\gamma \cdot \mathbf{p})\mathbf{v} = m\mathbf{u}, \tag{92}$$

with

$$\gamma_0 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & \sigma_0 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0 & -\sigma \\ \sigma & 0 \end{pmatrix}, \tag{93}$$

where σ are the Pauli matrices and $p = (\widehat{p}_1, \widehat{p}_2, \widehat{p}_3)$, determines a 4D real vector space with G as its automorphism, such that $G \subset L(4)$. This real vector space can coincide with the tangent space to the space–time manifold M, this being the idea. The principal fiber bundle (PFB) $P(G, M)$ with the structural group G determines the (Dirac) geometry of the space–time. We suppose now G with the general form

$$G = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}, \quad G^{+}G = I_4, \tag{94}$$

where A and B are 2×2 matrices. Also there exists a fundamental tensor $J^\lambda_\mu J^\nu_\lambda = \delta^\nu_\mu$ invariant under G with structure

$$J = \begin{pmatrix} 0 & \sigma_0 \\ -\sigma_0 & 0 \end{pmatrix}, \tag{95}$$

where, however, the Lorentz metric $g_{\lambda\mu}$ is also invariant under G due to its general form (94). Finally, the third fundamental tensor $\sigma_{\lambda\mu}$ is also invariant under G, where the following relations between the fundamental tensors are

$$J_\lambda^\nu = \sigma_{\lambda\mu} g^{\lambda\nu}, \quad g_{\mu\nu} = \sigma_{\lambda\mu} J_\nu^\lambda, \quad \sigma_{\lambda\mu} = J_\lambda^\nu g_{\mu\nu}, \tag{96}$$

where

$$g^{\lambda\nu} = \frac{\partial g}{\partial g_{\lambda\nu}} \quad (g \equiv \det(g_{\mu\nu})). \tag{97}$$

Then, the necessary fundamental structure is given by

$$G = L(4) \cap \mathrm{Sp}\,(4) \cap K(4), \tag{98}$$

which leaves concurrently invariant the three fundamental forms

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu, \tag{99}$$

$$\sigma = \sigma_{\lambda\mu}\, dx^\lambda \wedge dx^\mu, \tag{100}$$

$$\phi = J_\nu^\lambda w^\nu v_\lambda, \tag{101}$$

where w^ν are components of a vector $w^\nu \in V^*$: — the dual vector space. In expression (98), $L(4)$ is the Lorentz group in 4D, $\mathrm{Sp}\,(4)$ is the symplectic group in 4D real vector space, and $K(4)$ denotes the almost complex group that leaves ϕ invariant [6].

For instance, G leaves the geometric product invariant [7]

$$\gamma_\mu\gamma_\nu = \frac{1}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu) + \frac{1}{2}(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu) = \gamma_\mu\gamma_\nu - \gamma_\mu \wedge \gamma_\nu = g_{\mu\nu} + \sigma_{\mu\nu}, \tag{102}$$

where they are now regarded as a set of orthonormal basis vector in such a manner that any vector can be represented as $\mathbf{v} = v^\lambda \gamma_\lambda$ and

$$\varepsilon_{\alpha\beta\gamma\delta} \equiv \gamma_\alpha \wedge \gamma_\beta \wedge \gamma_\gamma \wedge \gamma_\delta. \tag{103}$$

In resume, the fundamental structure of the space–time is then represented by $P(G, M)$, where G is given by (98), which leaves invariant the fundamental forms (99)–(101), implying that

$$\nabla_\lambda g_{\mu\nu} = 0, \tag{104}$$

$$\nabla_\nu \sigma_{\lambda\mu} = 0, \tag{105}$$

$$\nabla_\lambda J_\nu^\lambda = 0, \tag{106}$$

where ∇_λ denotes the covariant derivative of the G connection. It is interesting to note that it is only necessary to consider two of above three equations: the third follows automatically. Then, we will consider (104), (105) because, in some sense, they represent the boson and fermion symmetry, respectively.

5.1. Field Equations and Group Structure. It is necessary to introduce now other antisymmetric tensor $\sigma'_{\mu\nu}$ which is not helical, that means that it differs from $\sigma_{\mu\nu}$ of (102) but also is invariant with respect to the generalized connection G: $\nabla_\nu \sigma_{\lambda\mu} = 0$. For instance, we can construct also the antisymmetric tensor $\vartheta_{\mu\nu} \equiv \sigma'_{\mu\nu} - \sigma_{\lambda\mu} \neq 0$, that obeys $\nabla_\nu \vartheta_{\mu\nu} = 0$ and obviously $(1/6)(\partial_\mu \vartheta_{\nu\lambda} + \partial_\nu \vartheta_{\lambda\mu} + \partial_\lambda \vartheta_{\mu\nu}) = T^\rho_{\nu\mu} \vartheta_{\rho\lambda}$ due to the completely antisymmetric nature of T.

5.2. Antisymmetric Torsion and Fermionic Structure of the Space–Time. We know that [8]

$$\Gamma^\rho_{\mu\lambda} = \left\{ {\rho \atop \mu\lambda} \right\} + g^{\rho\nu}(T_{\mu\lambda\nu} + T_{\lambda\nu\mu} + T_{\nu\mu\lambda}), \tag{107}$$

where $\Gamma^\rho_{\mu\lambda}$ are the coefficients of the G-connection and $\left\{ {\rho \atop \mu\lambda} \right\}$ denotes the coefficients of the Levi-Civita connection whose covariant derivative is denoted by $\overset{\circ}{\nabla}_\lambda$. From (105), we make the link between the fermionic structure of the fundamental geometry of the manifold and the torsion tensor

$$\nabla_{[\nu} \sigma_{\lambda\mu]} = 0 \Rightarrow \tag{108}$$

$$\Rightarrow \frac{1}{2} \partial_{[\nu} \sigma_{\lambda\mu]} = T^\rho_{[\nu\mu} \sigma_{\rho\lambda]}. \tag{109}$$

The most simple solution for T arises when the torsion tensor is totally antisymmetric [9]

$$T_{\mu\lambda\nu} = T_{[\mu\lambda\nu]} \tag{110}$$

in order that the equivalence principle be obeyed [5, 9, 10]. In this case, as we have shown already in [1, 2, 9], we have

$$T_{\mu\lambda\nu} = \varepsilon_{\mu\lambda\nu\rho} h^\rho, \tag{111}$$

where the axial vector h^ρ is still to be determined. As will be clear soon, it is useful to put for d dimensions [9]

$$h^\rho = \frac{1}{\sqrt{w}} J^\rho_\lambda P^\lambda, \tag{112}$$

where P^λ is the *generalized momentum vector*; if $d = 4$, $w = 6$.

Expression (109) can be simplified taking account of the symmetries of $T_{\mu\lambda\nu}$ and the contraction with the fundamental tensor J^λ_τ

$$T_{\lambda\mu\nu} = \frac{1}{w} J^\rho_\lambda \partial_{[\nu} \sigma_{\rho\mu]}. \tag{113}$$

5.3. About the Equivalence Principle (EP) and the Antisymmetry of the Torsion Tensor: A Theorem. As is well known, in order the experimental evidence to form the foundation of the theory, the PE has to be imposed as well as the foregoing symmetry principles.

Because the G-connection contains a torsion tensor by specific requirements, it is currently suspected that due to this fact, the EP can be violated. Then, a good question naturally arises: what is the implication of PE as defined (or better described in this context) by the G-geometry? Let us analyze specifically the question:

i) The PE implies that the tangent space M_p is to be a Minkowski space, then at M_p we have

$$(g_{\mu\nu})_p = \eta_{\mu\nu} \quad \text{and} \quad (\partial_\rho g_{\mu\nu})_p = 0, \tag{114}$$

where $\eta_{\mu\nu}$ is the Minkowski metric.

ii) The coefficients of the affine general connection are given by (17) [8, p. 141]

$$\Gamma^\rho_{\mu\lambda} = \left\{ {\rho \atop \mu\lambda} \right\} + \overbrace{g^{\rho\nu}(T_{\mu\lambda\nu} + T_{\lambda\nu\mu} + T_{\nu\mu\lambda})}^{\equiv S^\rho_{\mu\lambda}}, \tag{115}$$

where $T_{\nu\mu\lambda}$ is the torsion tensor and $S^\rho_{\mu\lambda}$ is the contortion.

iii) From

$$\nabla g = 0$$

we have, however,

$$\nabla_\lambda g_{\alpha\beta} = \overset{\circ}{\nabla}_\lambda g_{\alpha\beta} - T^\rho_{\lambda\alpha} g_{\rho\beta} - T^\rho_{\lambda\beta} g_{\alpha\rho} = 0, \tag{116}$$

which is valid at p also.

iv) From (114) and (116) we obtain

$$[T_{\beta\lambda\alpha} + T_{\alpha\lambda\beta}]_p = 0 \tag{117}$$

since (114) said

$$\left[\overset{\circ}{\nabla}_\lambda g_{\alpha\beta}\right]_p = 0. \tag{118}$$

v) The above relations have *tensorial character*, for instance, they are valid in all coordinate systems (and in all points p), then

$$T_{\beta\lambda\alpha} = -T_{\alpha\lambda\beta} \tag{119}$$

and

$$\overset{\circ}{\nabla}_\lambda g_{\alpha\beta} = 0. \tag{120}$$

These equations show geometrically that the imposition of the PE implies the following equivalence:

$$[\nabla_\lambda g_{\alpha\beta} = 0 \text{ and PE}] \Longleftrightarrow (\text{Eqs. (119) and (120)}). \qquad (121)$$

vi) But, from (119) and (120) we have that the torsion tensor has the full antisymmetric property

$$T_{\alpha\lambda\beta} = T_{[\alpha\lambda\beta]}. \qquad (122)$$

With this Proof we conclude that: *the full antisymmetry for the torsion tensor is the result of imposition of the Equivalence Principle (EP) on the space–time structure. It is not the result of a priori assumptions concerning the hypothetic or possible physical meaning of the torsion tensor*.

5.4. The G-Invariance of the Action. As is well known, the Palatini principle has a double role that is the determining of the connection required for the space–time symmetry as the field equations. By means of this principle, we were able to construct the action integral S. This action S necessarily needs to yield the G-invariant conditions (104)–(106) without prior assumption; and, the Einstein, Dirac and Maxwell equations need to arise from S as a causally connected closed system. This equations will be generalized inevitably, so that causal connections between them can be established. Our action fulfills the above requirements, having account that the role of $f_{\mu\nu}$, that enters symmetrically with $g_{\mu\nu}$ in S, is linked with the fundamental tensor $\vartheta_{\mu\nu}$ of the previous Subsec. 5.3 denoting the dual of $\vartheta_{\mu\nu}$ by

$$f_{\mu\nu} \equiv \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\vartheta^{\rho\sigma} = *\vartheta_{\mu\nu}$$

(where $\vartheta^{\mu\nu}$ is the inverse tensor to $\vartheta_{\mu\nu}$).

The usual Euler–Lagrange equations from the action with the explicit computation of the determinant in ($d = 4$) of expression (8), that will help us to compare the unitarian model introduced here (in the sense of Eddington (see [1, 2]) with the dualistic non-Abelian Born–Infeld model of [3], take the familiar form [1–3]

$$S = \frac{b^2}{4\pi}\int\sqrt{-g}\,dx^4\left\{\overbrace{\sqrt{\gamma^4 - \frac{\gamma^2}{2}\overline{G}^2 - \frac{\gamma}{3}\overline{G}^3 + \frac{1}{8}(\overline{G}^2)^2 - \frac{1}{4}\overline{G}^4}}^{\equiv\mathbb{R}}\right\}, \qquad (123)$$

$$G_{\mu\nu} \equiv [\lambda^2(g_{\mu\nu} + f^a_\mu f_{a\nu}) + 2\lambda R_{(\mu\nu)} + 2\lambda f^a_\mu R_{[a\nu]} + R^a_\mu R_{a\nu}], \qquad (124)$$

$$G^\nu_\nu \equiv [\lambda^2(d + f_{\mu\nu}f^{\mu\nu}) + 2\lambda(R_S + R_A) + (R^2_S + R^2_A)] \qquad (125)$$

with (the upper bar on the tensorial quantities indicates traceless condition)

$$R_S \equiv g^{\mu\nu} R_{(\mu\nu)}, \quad R_A \equiv f^{\mu\nu} R_{[\mu\nu]}, \quad \gamma \equiv \frac{G^{\nu}_{\nu}}{d},$$
$$\overline{G}_{\mu\nu} \equiv G_{\mu\nu} - \frac{g_{\mu\nu}}{4} G^{\nu}_{\nu}, \quad \overline{G}^{\nu}_{\rho}\overline{G}^{\rho}_{\nu} \equiv \overline{G}^{2}, \quad \overline{G}^{\nu}_{\lambda}\overline{G}^{\lambda}_{\rho}\overline{G}^{\rho}_{\nu} \equiv \overline{G}^{3}, \tag{126}$$
$$(\overline{G}^{\nu}_{\rho}\overline{G}^{\rho}_{\nu})^2 \equiv (\overline{G}^{2})^2, \quad \overline{G}^{\nu}_{\mu}\overline{G}^{\mu}_{\lambda}\overline{G}^{\lambda}_{\rho}\overline{G}^{\rho}_{\nu} \equiv \overline{G}^{4},$$

where the variation was made with respect to the electromagnetic potential a_τ as follows:

$$\frac{\delta\sqrt{G}}{\delta a_\tau} = \nabla_\rho \left(\frac{\partial\sqrt{G}}{\partial f_{\rho\tau}} \right) \equiv \nabla_\rho \mathbb{F}^{\rho\tau} = 0. \tag{127}$$

Explicitly

$$\nabla_\rho \left[\frac{\lambda^2 N^{\mu\nu}(\delta^{\sigma}_{\mu} f^{\rho}_{\nu} + \delta^{\sigma}_{\nu} f^{\rho}_{\mu})}{2\mathbb{R}} \right] = 0, \tag{128}$$

where $N^{\mu\nu}$ is given by

$$N^{\mu\nu} = g \left[-\gamma^2 G^{\mu\nu} - \gamma (G^2)^{\mu\nu} + \frac{(G^2)^{\mu}_{\mu} G^{\mu\nu}}{2} - (G^3)^{\mu\nu} + \right.$$
$$\left. + \frac{4\gamma^3 g^{\mu\nu}}{d} - \frac{\gamma (G^2)^{\mu}_{\mu} g^{\mu\nu}}{d} - \frac{(G^3)^{\mu}_{\mu} g^{\mu\nu}}{3d} \right]. \tag{129}$$

The set of equations to solve from the action (13) in this particular case is

$$R_{(\mu\nu)} = \overset{\circ}{R}_{\mu\nu} - T^{\alpha}_{\mu\rho} T^{\rho}_{\alpha\nu} = -\lambda g_{\mu\nu}, \tag{19a}$$
$$R_{[\mu\nu]} = \nabla_\alpha T^{\alpha}_{\mu\nu} = -\lambda f_{\mu\nu}, \tag{19b}$$
$$\nabla_\rho \left[\frac{\lambda^2 N^{\mu\nu}(\delta^{\sigma}_{\mu} f^{\rho}_{\nu} + \delta^{\sigma}_{\nu} f^{\rho}_{\mu})}{2\mathbb{R}} \right] = 0, \tag{19c}$$

from this set, the link between T and f will be determined (f is not a priori potential for the torsion T).

The key point now is Eq. (112)

$$\overset{\circ}{R}_{\mu\nu} = -\lambda g_{\mu\nu} + T^{\alpha}_{\mu\rho} T^{\rho}_{\alpha\nu}, \tag{130}$$
$$= -\lambda g_{\mu\nu} + w h_\mu h_\nu = -\lambda g_{\mu\nu} + P_\mu P_\nu, \tag{131}$$

then we can obtain, as in the mass shell condition

$$P^2 = m^2 \Rightarrow m = \pm\sqrt{\overset{\circ}{R} + \lambda d}. \tag{132}$$

Notice that there exists a link between the dimension of the space–time and the scalar «Einstenian» curvature $\overset{\circ}{R}$. Moreover, the curvature is constrained to take definite values $\in \mathbb{N}$ — the natural number characteristic of the dimension. On the other hand, knowing that $|\lambda| = d - 1$ and accepting that the parameter $m \in \mathbb{R}$, the limiting condition on the physical values for the mass is $\overset{\circ}{R} \geqslant (1-d)d$.

Introducing the geometric product in the above equation (e.g., $\gamma^\mu \gamma^\nu P_\mu P_\nu = m^2$) plus the quantum condition: $P_\mu \to \widehat{P}_\mu - e\widehat{A}_\mu$, we have

$$\left[\gamma^\mu \gamma^\nu \left(\widehat{P}_\mu - e\widehat{A}_\mu\right)\left(\widehat{P}_\nu - e\widehat{A}_\nu\right) - m^2\right] \Psi = 0, \tag{133}$$

where $\Psi = \mathbf{u} + i\mathbf{v}$ are given in (91), (92). That is

$$\left[\gamma^\mu \left(\widehat{P}_\mu - e\widehat{A}_\mu\right) + m\right]\left[\gamma^\nu \left(\widehat{P}_\nu - e\widehat{A}_\nu\right) - m\right] u^\lambda = 0 \tag{134}$$

which leads to the Dirac equation

$$\left[\gamma^\mu \left(\widehat{P}_\mu - e\widehat{A}_\mu\right) + m\right] u^\lambda = 0 \tag{135}$$

with m given by (132). Notice that this condition, in the Dirac case, not only passes from classical variables to quantum operators, but in the case that the action does not contain explicitly $\widehat{A}_\mu, h_\mu$ remains without specification due to the gauge freedom in the momentum. Applying the geometric product to (133), it is not difficult to see that

$$\left[\left(\widehat{P}_\mu - e\widehat{A}_\mu\right)^2 - m^2 - \frac{1}{2} e\sigma^{\mu\nu} F_{\mu\nu}\right] u^\lambda + \frac{1}{2}\sigma^{\mu\nu} R^\lambda_{\rho[\mu\nu]} u^\rho - \\ - \frac{1}{2} e\sigma^{\mu\nu}\left(\widehat{A}_\mu \widehat{P}_\nu - \widehat{A}_\nu \widehat{P}_\mu\right) u^\lambda = 0. \tag{136}$$

It is interesting to see that:

i) The above formula is absolutely general for the type of geometrical Lagrangians involved containing the generalized Ricci tensor inside.

ii) For instance, the variation of the action will carry the symmetric contraction of components of the torsion tensor (i.e., Eq. (130)) and then the arising of terms as $h_\mu h_\nu$.

iii) The only thing that changes is the mass (132) and the explicit form of the tensors involved as $R^\lambda_{\rho[\mu\nu]}$, $F_{\mu\nu}$, etc., without variation of the Dirac general structure of the equation under consideration,

iv) Equation (136) differs from that obtained by Landau and Lifshitz by the appearance of the last two terms: the term involving the curvature tensor is due to the spin interaction with the gravitational field (due to torsion term in $R^\lambda_{\rho[\mu\nu]}$) and the last term is the spin interaction with the electromagnetic and mechanical momenta.

v) Expression (136) is valid for another vector v^λ, then it is valid for a bispinor of the form $\Psi = \mathbf{u} + i\mathbf{v}$.

vi) The meaning for a quantum measurement of the space–time curvature is mainly due to the term in (136) involving explicitly the curvature tensor.

The important point here is that the spin-gravity interaction term is so easily derived as the spinors are represented as space–time vectors whose covariant derivatives are defined in terms of the G-(affine) connection. In their original form the Dirac equations would have, in curved space–time, their momentum operators replaced by covariant derivatives in terms of «spin-connection» whose relation is not immediately apparent.

6. DIRAC STRUCTURE, ELECTROMAGNETIC FIELD AND ANOMALOUS GYROMAGNETIC FACTOR

The interesting point now is based on the observation that if we introduce expression (19b) in (136), then

$$\left[\left(\widehat{P}_\mu - e\widehat{A}_\mu\right)^2 - m^2 - \frac{1}{2}e\sigma^{\mu\nu}F_{\mu\nu}\right]u^\lambda - \frac{\lambda}{d}\frac{1}{2}\sigma^{\mu\nu}f_{[\mu\nu]}u^\lambda - \\ - \frac{1}{2}e\sigma^{\mu\nu}\left(\widehat{A}_\mu\widehat{P}_\nu - \widehat{A}_\nu\widehat{P}_\mu\right)u^\lambda = 0, \quad (137)$$

$$\left[\left(\widehat{P}_\mu - e\widehat{A}_\mu\right)^2 - m^2 - \frac{1}{2}\sigma^{\mu\nu}\left(eF_{\mu\nu} + \frac{\lambda}{d}f_{\mu\nu}\right)\right]u^\lambda - \\ - \frac{e}{2}\sigma^{\mu\nu}\left(\widehat{A}_\mu\widehat{P}_\nu - \widehat{A}_\nu\widehat{P}_\mu\right)u^\lambda = 0, \quad (138)$$

we can see clearly that if $\widehat{A}_\mu = ja_\mu$ (with j arbitrary constant), $F_{\mu\nu} = jf_{\mu\nu}$, the last expression takes the suggestive form

$$\left[\left(\widehat{P}_\mu - e\widehat{A}_\mu\right)^2 - m^2 - \frac{1}{2}\left(ej + \frac{\lambda}{d}\right)\sigma^{\mu\nu}f_{\mu\nu}\right]u^\lambda - \\ - \frac{e}{2}\sigma^{\mu\nu}\left(\widehat{A}_\mu\widehat{P}_\nu - \widehat{A}_\nu\widehat{P}_\mu\right)u^\lambda = 0 \quad (139)$$

with the result that the gyromagnetic factor has been modified to $2/(j + \lambda/ed)$. Notice that in a unified theory with the characteristics introduced here, it is reasonable the identification introduced in the previous step ($F \leftrightarrows f$) in order that the fields arise from the same geometrical structure.

The concrete implications about this important contribution of the torsion to the gyromagnetic factor will be given elsewhere with more details on the dynamical property of the torsion field. We remark only the following:

i) There exists an important contribution of the torsion to the gyromagnetic factor that can have implications to the problem of the anomalous momentum of the fermionic particles.

ii) This contribution appears (taking the second equality of expression (19b) as a modification on the vertex of interaction, almost from the effective point of view.

iii) It is quite evident that this contribution will justify probably the appearance of the torsion at great scale, because we can bound the torsion due to the other well-known contributions to the anomalous momenta of the elementary particles (QED, weak, hadronic contribution, etc.).

iv) The form of the spin-geometric structure coupling coming from the first principles, such as the Dirac equation.

v) Then, from iii) the work of the covariant derivative in presence of torsion is determined by the G structure of the space–time.

vi) The Dirac equation (137) (where the second part of the equivalence (19b) coming from the equation of motion was introduced), shows that the vertex was modified without a dynamical function of propagation. Then, other way to see the problem treated in this paragraph is to introduce the propagator for the torsion corresponding to the first part of the equivalence (19b). This important possibility will be studied elsewhere [5].

7. SPACE–TIME AND STRUCTURAL COHOMOLOGIES

As is well know from the physical and mathematical point of view, the cohomological interplay between the fields involved in any well-possessed geometrical and unified theory is crucial. The importance of this fact arises as a consequence of the logical (and causal) structure of the physical fields (sources, fields, conserved quantities) and not only as a mathematical toy. In the theory presented here, there exist two cohomological structures: *space–time cohomology* and *structural cohomology*.

The difference between them is that in the *space–time cohomology* the Dirac (fermionic) structure of the space–time is not involved directly in the relations between the fields involved. The main equations necessary for the construction are

$$\nabla_\alpha T^\alpha_{\mu\nu} = -\lambda f_{\mu\nu}, \quad d^*T = -\lambda^* f = dh, \tag{140}$$

the interplay being schematically as

$$\begin{array}{ccccc} & & \boxed{T} & & \\ & {}^{A_-}\swarrow\;\nearrow_{A_+} & & \searrow_{B_-}\;\nwarrow^{B_+} & \\ \boxed{f} & & \overset{C_+}{\underset{C_-}{\leftrightarrows}} & & \boxed{h}\,, \end{array} \tag{141}$$

where the operators are

$$\begin{gathered}
A_{-} \equiv (-1)^{d+1}(-\lambda) * \textstyle\int *, \quad A_{+} \equiv (-\lambda)^{-1} * d*, \\
B_{-} \equiv (-1)^{d+1} *, \quad B_{+} \equiv *, \\
C_{-} \equiv -\lambda \textstyle\int^{*}, \quad C_{+} \equiv [(-1)^{d+1}(-\lambda)]^{-1} * d, \\
D_{-} \equiv (-1)^{d+1} * d, \quad D_{+} \equiv (-1)^{d+1} * \displaystyle\int, \\
E_{-} \equiv d, \quad E_{+} \equiv \displaystyle\int, \\
G_{-} \equiv [(-1)^{d+1}(-\lambda)]^{-1} *, \quad G_{+} \equiv -\lambda * .
\end{gathered} \tag{142}$$

The structural cohomology, in contrast, involves directly the fermionic structure of the space–time due to that in the basic formulas $\vartheta_{\mu\nu}$ enters directly into the cohomological game, as is easily seen below

$$\begin{array}{ccccc}
 & & \boxed{a} & & \\
 & {}^{E_-}\swarrow\nearrow_{E_+} & & {}_{D_+}\nwarrow\searrow^{D_-} & \\
\boxed{f} & & \underset{B_+}{\overset{B_-}{\rightleftarrows}} & & \boxed{\vartheta} \\
 & {}^{C_+}\nwarrow\searrow_{C_-} & & {}_{A_-}\swarrow\nearrow^{A_+} & \\
 & & \boxed{h} & & \\
 & & G_- \uparrow\downarrow G_+ & & \\
 & & \boxed{a} & &
\end{array} . \tag{143}$$

Notice the important thing: in this case it is clear that the degree of the relations between the quantities involved is more fundamental than in the previous case (jerarquical sense).

8. CONCLUDING REMARKS

In this paper we make an exhaustive analysis of the model based on the theory developed in early papers of the author. The simplest structure of the space–time described by this new theory makes, beside the connection between curvature and matter, the link between the torsion and the spin.

As was well explained through all this paper, the mechanism of rupture of symmetry is responsible for that the geometrical Lagrangian can be written in a suggestive Eddington–Born–Infeld-like form. Three cases were treated from the point of view of the solutions, depending on the form of torsion used: totally antisymmetric (with torsion potential), not totally antisymmetric («tratorial» type), and with a torsion tensor with both characteristics. In all the cases, they were compared from the point of view of the obtained solutions with the nondualistic model of reference [3], namely, the non-Abelian Born–Infeld model.

In all these cases, the (nondualistic) unified model proposed here differs deeply from the dualistic non-Abelian Born–Infeld model of our early reference [3].

The first obvious difference comes from a conceptual framework: the geometrical action will provide, besides the space–time structure, the matter-energy spin distribution. This fact is the same basis of the unification: all the (apparently disconnected) theories and interactions of the natural world appear naturally as a consequence of the intrinsic space–time geometry.

For the case of totally antisymmetric tensor torsion with torsion potential, several points were answered and elucidated:

i) As to the Hosoya and Ogura ansatz, the natural question arising was: Why does the identification of the isospin structure of the Yang–Mills field with the space frame lead to a similar physical situation as with a nondualistic unified theory with torsion? The answer is: because at once such identification is implemented, a potential torsion is introduced and the solution of the set of equations is the consistency between the definition of the torsion tensor from the potential and the Cartan structure equations [1, 2].

ii) As to the obtained solutions for the scale factor, the difference with our previous work is precisely the particular form of the energy-momentum tensor in the NABI case (in the UFT model presented here, there is not energy-momentum tensor, of course): both solutions describe a wormhole-instanton but the final form of the differential equations for the scale factor is different: then, the scale factor here has an exponentially growing behavior, in sharp contrast to the wormhole solution from our previous work with the «dualistic» non-Abelian BI theory. Also, for this particular value of the torsion, the wormhole tunneling interpretation (in the sense of Coleman's mechanism) is fulfilled.

The contact point between the compared models, however, are the dynamical equations that are very similar although the existence of a «current term» in the UFT model (cf. (45)) that does not appear in the NABI case. This fact was pointed out in a slightly different context by N. Chernikov.

For the case of nontotally antisymmetric torison (tratorial type), the space–time structure was analyzed from the point of view of the interacting fields arising from the same geometry of the space–time and relaxing now the condition of a totally antisymmetric torsion. Then, one also neglects the condition of the prior existence of an antisymmetric 2-form potential for it. The precise results can be easily enumerated as:

(i) From its $SL(2C)$ underlying structure: the notion of minimal coupling has been elucidated and has come naturally from the compatibility condition between the gauge field structure of the antisymmetric part of the fundamental tensor and the $SL(2, C)$ structure of the base manifold.

(ii) Through exact cosmological solutions from this model, where the geometry is Euclidean $R \otimes O(3) \sim R \otimes SU(2)$, the relation between the space–time geometry and the structure of the gauge group was explicitly shown.

(iii) This relation is directly connected with the relation of the spin and torsion fields.

From the point of view of the obtained solutions, a solution of this model was explicitly compared with our previous ones and we find that:

(i) The torsion is not identified directly with the Yang–Mills type strength field.

(ii) There exists a compatibility condition connected with the identification of the gauge group with the geometric structure of the space–time: this fact leads to the identification between derivatives of the scale factor a with the components of the torsion in order to allow the Hosoya–Ogura ansatz (namely, the alignment of the isospin with the frame geometry of the space–time).

(iii) This compatibility condition precisely marks the fact that local gauge covariance, coordinate independence and arbitrary space–time geometries are harmonious concepts and

(iv) of two possible structures of the torsion, the «tratorial» form forbids wormhole configurations, leading only to cosmological instanton space–time in eternal expansion.

For the general case, i.e., with torsion with totally antisymmetric and tratorial parts, the full analysis was given in a clear manner in Sec. 4. Here we point out that the Hosoya and Ogura ansatz can be implemented as in the previous cases, and, the most important is the fact that wormhole solutions can be obtained for some particular cases. The solutions are asymptotically flat, where appear vector and tensor integration constants that are constrained in norm to bring physical consistency to the solution.

As to the problem of the possibility of coexistence of the trace of the torsion due to the tratorial part and the axial vector from the totally antisymmetric part of the torsion, we saw here that there is no problem in the new theory: there are tratorial and antisymmetric torsion fields without contradictions.

The fact that in [4] the field equations of vacuum quadratic Poincare gauge field theory (QPGFT) were solved for purely null tratorial torsion, will permit one to express the contortion tensor for such a case as (tratorial form, with notation of [4])

$$K_{\lambda\mu\nu} = -2(g_{\lambda\mu}a_\nu - g_{\lambda\nu}a_\mu),$$

then it does not permit the coexistence with an axial torsion vector, as was clearly shown by Singh in the beautiful paper [4]. The two points that lead to such a discrepancy are

i) because the described theories are fundamentally different, one is unitarian and the other [4] is dualistic

ii) and the fact that the Newman–Penrose formulation that was used in [4], works in a null tetrad.

8.1. On the Geometrical Structure. From the point of view of the concrete structure able to explain the content of the bosonic and fermionic matter of the Universe, the present paper is left open-ended as many physical consequences need to be explored. Some words concerning the realization and the choice of the correct group structure of the tangent space to M is that $G = L(4) \cap \mathrm{Sp}\,(4) \cap K(4)$ preserves the boson and fermion symmetry simultaneously without implying supersymmetry of the model. As we would like to show in a future work, the supergravitational extension of the model will be discussed in connection with the problem of its quantization, where the key point will be precisely the group structure of the tangent space to the space–time manifold M. Here we conclude enumerating the main results concerning the basic structure of the manifold supporting a unified field theoretical model:

i) The simplest geometrical structure able to support the fermionic fields was constructed based on a tangent space with a group structure $G = L(4) \cap \mathrm{Sp}\,(4) \cap K(4)$.

ii) Then, the explicit link of the fermionic structure with the torsion field was realized and the Dirac-type equation was obtained from the same space–time manifold.

iii) Notice that the matter was not included in the geometrical Lagrangian of the unified theory presented here: only symmetry arguments (that will lead to the correct dynamical equations for the material fields arising from the same manifold) are needed to allow the appearance of matter and this fact is not the essence of the unification, of course (several references trying to include matter into the Eddington-«type» theories by hand without physical and symmetry principles).

8.2. On the Energy Concept. 1) About the equation

$$\overset{\circ}{R}_{\mu\nu} = -\lambda g_{\mu\nu} + T^{\alpha}_{\mu\rho} T^{\rho}_{\alpha\nu}$$

notice that the concept, here, of the terms that arise as «energy-momentum» part coming from the symmetric contraction of the torsion components is different in essence to the concept coming from the inclusion of the energy-momentum tensor in the Einstein theory. The conceptual framework that «matter and energy curve the space–time» implicitly carries the idea of some «embedding-like» situation where the matter and energy are put on some Minkowskian flexible carpet and you see how it is curved under the «weight» of the «ball» (matter + energy). Here, in the theory presented, the situation is that the torsion terms (contributing as «energy momentum» in above equation) arise from the same geometry, then we have the picture as a unique entity: the interplay fields-space–time; the idea is the same as the solitonic vortex in the water.

This fact can be also interpreted as that the concept of force is introduced due to the torsion in the unified model, thing that is lost in the Einstein theory [10], where the concept is that there is no force, but curvature only.

2) Some remarks on the general Hodge–de Rham decomposition of $h = h_\alpha dx^\alpha$.

Theorem 1. *If $h = h_\alpha dx^\alpha \notin F'(M)$ is a 1-form on M, then there exist a zero-form Ω, a 2-form $\alpha = A_{[\mu\nu]}\, dx^\mu \wedge dx^\nu$ and a harmonic 1-form $q = q_\alpha\, dx^\alpha$ on M that*

$$h = d\Omega + \delta\alpha + q \rightarrow h_\alpha = \nabla_\alpha \Omega + \varepsilon_\alpha^{\beta\gamma\delta} \nabla_\beta A_{\gamma\delta} + q_\alpha.$$

Notice that if h is not harmonic and assuming that q_α is a polar vector, then an axial vector can be added and above expression takes the form

$$h_\alpha = \nabla_\alpha \Omega + \varepsilon_\alpha^{\beta\gamma\delta} \nabla_\beta A_{\gamma\delta} + \varepsilon_\alpha^{\beta\gamma\delta} M_{\beta\gamma\delta} + q_\alpha,$$

where $M_{\beta\gamma\delta}$ is a completely antisymmetric tensor.

3) Notice the important fact that when the torsion is totally antisymmetric tensor field, $-2\lambda f_{\mu\nu}$ takes the role of «current» for the torsion field, as usual, the terms proportional to the 1-form potential vector a_μ act as the current of the electromagnetic field $f_{\mu\nu}$ in the equation of motion for the electromagnetic field in the standard theory: $\nabla_\alpha f_\mu^\alpha = J_\mu$ (constants absorbed into the J_μ). The interpretation and implications of this question will be analyzed concretely in [5].

Acknowledgements. I am very grateful to Prof. Yu Xin who introduced me into the subject of the unified theories based only on geometrical concepts and the Mach principle. Many thanks are given to Profs. Yu. P. Stepanovsky and A. E. Dorokhov for their interest and discussions. This work is partially supported by Brazilian Ministry of Science and Technology (MCT).

REFERENCES

1. *Cirilo-Lombardo D. J.* // Intern. J. Theor. Phys. 2010. V. 49. P. 1288–1301 and references therein.
2. *Cirilo-Lombardo D. J.* // Intern. J. Theor. Phys. DOI 10.1007/s10773-011-0678-1 and references therein.
3. *Cirilo-Lombardo D. J.* // Class. Quant. Grav. 2005. V. 22. P. 4987–5004 and references therein.
4. *Singh P.* // Class. Quant. Grav. 1990. V. 7. P. 2125.
5. *Cirilo-Lombardo D. J.* To be published.
6. *Kobayashi S., Nomizu K.* Foundations of Differential Geometry. V. 1 and 2. Wiley-Interscience, 1996.

7. *Hestenes D.* Space–Time Algebra. Gordon & Breach, 1966.

8. *Schouten J.* Ricci-Calculus. Heidelberg: Springer, 1954.

9. *Xin Yu.* General Relativity on Spinor-Tensor Manifold // Quantum Gravity: Intern. School on Cosmology & Gravitation, XIV Course / Eds. P. G. Bergman, V. de. Sabbata & H. J. Treder. World Scientific, 1996. P. 382–411.

10. *Xin Yu.* Private communication.

11. *Borchsenius K.* // Gen. Rel. Grav. 1976. V. 7. P. 527;
Hammond R. T. // Class. Quant. Grav. 1989. V. 6. P. L195;
Poplawski N. J. // Intern. J. Theor. Phys. 2010. V. 49. P. 1481.

12. *Kibble T. W. B.* // J. Math. Phys. 1961. V. 2. P. 212;
Sciama D. W. // Rev. Mod. Phys. 1964. V. 36. P. 463; 1103;
Hehl F. W. // Gen. Rel. Grav. 1973. V. 4. P. 333; 1974. V. 5. P. 491;
Hehl F. W. et al. // Rev. Mod. Phys. 1976. V. 48. P. 393.