

FACTORIZATION OF WAVE FUNCTIONS OF THE QUANTUM INTEGRABLE PARTICLE SYSTEMS

*V. I. Inozemtsev*¹

Joint Institute for Nuclear Research, Dubna

The relation between the characteristics of the equilibrium configurations of the classical Calogero–Moser integrable systems and properties of the ground state of their quantum analogs is found. It is shown that, under the condition of factorization of the wave function of these systems, the coordinates of classical particles at equilibrium are zeroes of the polynomial solutions of the second-order linear differential equation. It turns out that, under these conditions, the dependence of classical and quantum minimal energies on the parameters of the interaction potential is the same.

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The most known example of the correspondence between classical and quantum dynamical systems is the Bohr–Sommerfeld rule for one-dimensional quasi-classical motion. It allows one to find the characteristics of the states with large quantum numbers without complicated procedure of investigations of the solutions of the Schrödinger equation. But in general case there is no analogy between the properties of the ground state of the quantum mechanical systems and their classical counterparts.

However, for integrable particle systems there might be a much deeper connection between the classical and quantum dynamics due to the common symmetry of the Hamiltonians having the same group-theoretical grounds. The aim of the present work is demonstration of the existence of such a connection for integrable cases of the motion of the Calogero–Moser particle systems [1, 2] in the external field discovered by mine [3, 4]. These systems of arbitrary number of particles N with the mutual two-particle interaction given by the potential $V(q)$ are supposed to move in the external field with the potential $W(q)$ and are defined by the quantum Hamiltonian

$$H = \sum_{j=1}^N \left[- \left(\frac{\partial}{\partial q_j} \right)^2 + W(q_j) \right] + \sum_{j>k}^N V(q_j - q_k), \quad (1)$$

where

$$V(q) = g^2(\sinh q)^{-2}, \quad W(q) = 8g^2(2A^2 \cosh 4q + B \cosh 2q + C \sinh 2q). \quad (2)$$

¹E-mail: inozv@thsun1.jinr.ru

The parameters A, B, C and coupling constant g are absolutely arbitrary real numbers. The case of $A = B = C = 0$ corresponds to the usual Calogero–Moser hyperbolic systems [1, 2].

It was shown in the paper [4] that, under the condition

$$B = -Ag^{-1}(1 + \alpha(N - 1)), \quad \alpha = \frac{1}{2} + \sqrt{g^2 + \frac{1}{4}}, \quad (3)$$

the ground-state wave function of the quantum systems defined by (1) is factorized,

$$\psi(q_1, \dots, q_N) = \prod_{j>k}^N |\sinh(q_j - q_k)|^\alpha \prod_{j=1}^N \exp \left[-g \left(4A \cosh 2q_j + \frac{C}{A} q_j \right) \right]. \quad (4)$$

Later on, it was shown in [5] that, under conditions similar to (3), there are several solutions which differ from (3) by polynomials in $\{\cosh 2q_j\}$ and the model belongs to the class of quasi-exactly solvable ones, i.e., it is possible to determine analytically some finite set of the eigenvalues of the Hamiltonian (1). It was also found that the eigenvalue which corresponds to the wave function (4), i.e., the ground-state energy, is given by the formula

$$E_q = -\frac{N}{2} \left[\frac{\alpha^2}{3}(N^2 - 1) + \frac{C^2}{A^2}g^2 - 32A^2g^2 \right]. \quad (5)$$

We use the system of units in which $\hbar = 1$; the parameters of the potentials (2) slightly differ from those used in [4] for convenience. If the condition (3) does not take place, the factorization (4) does not hold and the energy of the ground state depends on the parameters A, B, C by much more complicated form. The solution of the corresponding quantum problem is not known till now.

The natural question arises: which properties of the corresponding *classical* systems do correspond to (3)–(5)? It is natural to suppose that they reveal themselves for the states with minimal energy, i.e., classical equilibrium points. The description of these points might be much easier under the condition similar to (3). It will be shown later that such a correspondence indeed takes place. Note also that at $A = B = C = 0$ there are no equilibrium points at all.

It is well known that there is the relation between the coordinates of particles of some, more simple than (1), (2), classical systems of particles at equilibrium, and zeroes of classical orthogonal polynomials. It was discovered first by Stiltjes in the 19th century. It was used for getting various «sum rules» for zeroes of Hermit, Jacobi, Laquerre polynomials and zeroes of Bessel functions [6, 7]. As for the systems (1), (2) in some limit, when $W(q)$ coincides with the Morse potential $8g^2A^2(\exp(4q) - \exp(2q))$, this relation was established in the paper [8]. Let us now show that the simplification of the equilibrium equations takes place even in the general case of potential (2). These equations can be written as

$$-\sum_{k \neq j}^N \frac{\cosh(q_j - q_k)}{\sinh^3(q_j - q_k)} + 8[4A^2 \sinh(4q_j) + B \sinh(2q_j) + C \cosh(2q_j)] = 0. \quad (6)$$

With the use of variables $z_j = \exp(2q_j)$, the system of equations (6) can be cast in the rational form

$$-\sum_{k \neq j} \frac{z_k(z_j + z_k)}{(z_j - z_k)^3} + 4A^2(z_j - z_j^{-3}) + B + C - (B - C)z_j^{-2} = 0. \quad (7)$$

Let us now use the following trick [7] for transformation of the system (7) to more simple form. Let $\{z_j\}$ be the solution of (7). Let us construct the polynomial

$$P_N(z) = \prod_{j=1}^N (z - z_j)$$

and consider the integral on closed contour removed enough from the origin and surrounding all $\{z_j\}$, for the function

$$F_j(z) = \frac{z(z + z_j)P'_N(z)}{(z - z_j)^3 P_N(z)}.$$

Since $F_j(z) \sim z^{-2}$ as $z \rightarrow \infty$, this integral and, correspondingly, the sum of the residues at the poles of $F_j(z)$, equal zero. The residues at the poles of the first order at $z_k, k \neq j$, equal

$$\text{res } F_j(z) = \frac{z_k(z_k + z_j)}{(z_k - z_j)^3},$$

and the system (7) can be written as

$$4A^2(z_j - z_j^{-3}) + B + C - (B - C)z_j^{-2} = -\text{res } F_j(z)|_{z=z_j}. \quad (8)$$

The pole of $F_j(z)$ at the point $z = z_j$ is of the fourth order, and the right-hand side of (8) is calculated with the use of the formula

$$\begin{aligned} -\text{res } F_j(z)|_{z=z_j} &= -\frac{1}{6} \frac{d^3}{d\xi^3} \left[\xi(z_j + \xi)(2z_j + \xi) \frac{P'_N(z_j + \xi)}{P_N(z_j + \xi)} \right] \Big|_{\xi=0} = \\ &= -\frac{1}{4} [2a + z_j(4b - 3a^2) + z_j^2(a^3 - 2ab + c)], \quad (9) \end{aligned}$$

where

$$a = \frac{P''_N(z_j)}{P'_N(z_j)}, \quad b = \frac{P_N^{(3)}(z_j)}{P'_N(z_j)}, \quad c = \frac{P_N^{(4)}(z_j)}{P'_N(z_j)}. \quad (10)$$

If we suppose that $P_N(z)$ is a solution of the second-order differential equation

$$\mu(z)P_N''(z) + \rho(z)P_N'(z) + \lambda(z)P_N(z) = 0 \quad (11)$$

(the functions μ, ρ, λ can depend on N), then the value (10) and the residue (9) can be expressed through μ, ρ, λ and their derivatives. Our purpose is to choose such kind of μ, ρ, λ which guarantees the transformation of equations (8) to identities and also allows one to prove the existence of the polynomial solutions to (11). One can show that all these requirements can be satisfied if and only if

$$\mu(z) = z^2.$$

Then the residue (9) can be relatively simply expressed through ρ and λ ,

$$\text{res } F_j(z)|_{z=z_j} = \frac{1}{4} \left(-\frac{\rho^2}{z_j^3} + \frac{\rho\rho'}{z_j^2} - \rho'' - 2\lambda' \right). \quad (12)$$

The validity of equations (8) happens if

$$\rho(z) = pz^2 + qz + r, \quad \lambda(z) = uz + v. \tag{13}$$

The constants p, q, r, u are determined from (8),

$$p = -r = -4A, \quad pq - 2(p + u) = 4(B + C), \quad qr = 4(B - C). \tag{14}$$

The polynomial of the degree N can be a solution of (11) in accordance with (13) if and only if the condition $pN + u = 0$ is satisfied. It results finally in

$$B = -A(N - 1), \tag{15}$$

$$q = -\frac{C}{A} - N + 1. \tag{16}$$

It is clear now that the solutions to the system (7) are determined by the roots of the polynomial $P_N(z)$ satisfying the second-order differential equation (11) if and only the equality (14) takes place. The coefficient B cannot be arbitrary. Equation (11) with the coefficients (13) does not fall into the hypergeometric class since it has irregular singular points at zero and infinity. This is the characteristic feature of the Hill equation. The parameter v in (13) must be determined by the condition of compatibility of the system of the recurrence relations for the coefficients of the polynomial

$$P_N(z) = \sum_{l=0}^{N-1} d_{N-l} z^l + z^N, \tag{17}$$

which appear under the substitution of (17) into (11). This condition can be represented in the form of the algebraic equation of $(N + 1)$ th order. Its solution for v should be chosen so that all the roots of $P_N(z)$ must be real and positive. I cannot point out the explicit dependence of v on A and C . However, it turns out that it is not necessary for the calculation of the energy of the ground state.

This calculation can be performed as follows. Let us write the expression for the ground-state energy in the form

$$E_{cl} = 4g^2(S + 2A^2(S_2 + S_{-2}) + (B + C)S_1 + (B - C)S_1), \tag{18}$$

where

$$S = \sum_{k < j} \frac{z_j z_k}{(z_j - z_k)^2}, \quad S_\alpha = \sum_{j=1}^N z_j^\alpha.$$

The sum of the residues of the function

$$\psi(z) = \sum_{j=1}^N \frac{z z_j}{(z - z_j)^2} \frac{P'_N(z)}{P_N(z)}$$

in all its poles must equal zero since $\psi(z) \sim z^{-2}$ as $z \rightarrow \infty$. The calculation of these residues with the use of equalities (9), (10) and equation (11) with the coefficients (13)–(16) allows one to express the double sum S through S_α ,

$$S = -\frac{p^2}{24}(S_2 + S_{-2}) - \frac{S_1}{12}(pq - 3p - 2u) - \frac{S_{-1}}{12}r(q + 1) - \frac{N}{12}(2pr + (q - 1)^2 - 4v - 1). \tag{19}$$

All the sums $\{S_\alpha\}$ are expressed through p, q, r, u, v with the use of (11) and (17). It follows from (17) that

$$S_1 = -d_1, \quad S_1^2 - S_2 = 2d_2, \quad S_{-1} = -\frac{d_{N-1}}{d_N}, \quad S_1^2 - S_{-2} = 2\frac{d_{N-2}}{d_N}.$$

The substitution of (17) into (11) leads to simple expressions for all values in the right-hand sides of all these equations. The subsequent substitution of them into (18) and (19) results, after very long but not too tedious calculations, in the very simple formula

$$E_{\text{cl}} = -\frac{Ng^2}{2} \left[\frac{N^2 - 1}{3} + \frac{C^2}{A^2} - 32A^2 \right]. \quad (20)$$

All nonzero degrees of the parameter v which appear on intermediate stages of the calculation cancel, and getting the final answer (20) does not need the explicit form of v .

Let us also show that Eq.(11) with the coefficients (13) can be considered as one-particle Schrödinger equation. Indeed, after the change of the variable $z = \exp(2q)$ and the substitution

$$P_N(e^{2q}) = \psi(q) \exp \left[4A \cosh 2q + \left(\frac{C}{A} + N \right) q \right],$$

one obtains from (11) the equation

$$\frac{-\psi''}{2} + \tilde{W}(q)\psi = \varepsilon\psi, \quad (21)$$

where

$$\begin{aligned} \tilde{W}(q) &= 8(2A^2 \cosh 4q - A(N+1) \cosh 2q + C \sinh 2q), \\ \varepsilon &= 16A^2 + 2v - \frac{1}{2} \left(\frac{C}{A} + N \right)^2. \end{aligned}$$

To summarize, it is established that, under the condition (15), the coordinates of particles of the classical systems (2) at equilibrium coincide with zeroes of the wave function of N th level of one-particle wave function, the solution to the one-dimensional Schrödinger equation (21) with the potential $\tilde{W}(z)$. The condition (15) is completely analogous to the condition of the factorization of the wave function of the ground state for the *quantum* Calogero–Moser problem (1), (2) and exactly coincides with it in the limit of large coupling constants g (i.e., at $\hbar \rightarrow 0$). In this limit, the minimal classical and quantum energies (5), (20) do coincide; the transition to the usual systems of units corresponds to the changes as in (3)–(5)

$$g \rightarrow \frac{g}{\hbar}, \quad \varepsilon_q(g) \rightarrow \hbar^2 \varepsilon_q \left(\frac{g}{\hbar} \right).$$

The dependence of the energies (5) and (20) on the parameters of the external field are identical. It seems that this fact is not casual. Nevertheless, I do not know the theoretical group interpretation of such a coincidence. It is quite possible that it might be found if one finds a way of inclusion of (1), (2) into the Kirillov–Kostant [9] scheme of geometrical quantization. However, till now it was used only for the investigation of much more simple dynamical systems.

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