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TOWARDS A SUPERSTATISTICAL $SU(2)$ YANG–MILLS EoS

T. S. Biró [a], *Z. Schram* [b]

[a] MTA KFKI Research Institute for Particle and Nuclear Physics, Budapest

[b] Department of Theoretical Physics, University of Debrecen, Debrecen, Hungary

We study an $SU(2)$ lattice gauge field system on a 10^4 symmretric and on a $10^3 \times 2$ asymmetric lattice in order to learn about the equation of state.

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INTRODUCTION

Lattice field theory, the most successful nonperturbative, regularized treatment of continuum field theory, has accomplished in the last three decades a lot: equilibrium calculations of ground-state (vacuum) properties, spectra of low-excited states (hadrons) and phase transitions at finite temperature are most widely known achievements in understanding of the strong and electroweak interaction. Recent developments in computational techniques (fat link actions, complex pole techniques, fast numerical inversions including pseudoinverse methods, chiral symmetric fermion actions and reweighting of configurations) demonstrate that this field is still developing, aimed towards physical situations which were not feasible for numerical studies in the past. Equilibrium calculations at finite baryochemical potential, refined equations of state (pressure and energy density) information, realistically low quark masses, decay properties of hadronic resonances, an increasing knowledge on the nature of deconfinement phase transition and close-ups on topologically nontrivial field configurations circumline a picture of very rich physics due to the numerical study of quantum fields [1–11].

Unfortunately, the space of all possible field configurations constituting a path integral included in the calculation of any quantum expectation value is so enormous, exponentially growing with the number of considered space-time points, that it is hopeless to sum all contributions with the proper complex weight factors, $\exp(iS)$ with a given action S. For equilibrium situations the canonical ensemble technique formally treats the time as pure imaginary, resulting in exponentially suppressing factors, $\exp(-S_E)$, for most of the configurations. Corresponding algorithms, summing the path integral with Monte Carlo methods, can then be utilized for calculations. Out of equilibrium this fortunate circumstance is no more established.

There can be, however, another way to surpass the traditional canonical paradigm. Some, even very drastically nonequilibrium dynamics may lead to a stationary repetition (or cascading) of self-similar physical situations which leave their imprint on experimental observations as there were a non-Gibbs-like distribution of different energy states in the background. On the other hand, the equilibrium itself is not necessarily an element of a canonical Gibbs ensemble. As was recently pointed out [12], a single degree of freedom under the influence of a stochastic force and a stochastic damping constant has the Tsallis distribution as the stationary solution. This distribution goes over into a power law at high energy. In experiments, pion and proton transverse momentum distributions were observed in elementary-particle and heavy-ion collisions. They show nonexponential, power-law-like distribution tails.

Nonconventional distributions are based on a nonconventional entropy formula (which replaces the Boltzmann entropy), or in general on nonadditive composition rules for thermodynamical systems [16]. Such a formula is the Tsallis entropy [13], discussed vividly in recent years (although the same formula has been suggested earlier in the context of information measure theory [15]). It leads to a power law at high energy as canonical distribution, coinciding with the stationary solution of the above-mentioned stochastic problem; but the entropy is not extensive. A monotonic function of it, locating the maximum for the same canonical distribution, is proven to be extensive (i.e., additive when the probabilities factorize). It is easy to see that this formula is a monotnic function of the Rényi entropy [14].

Nonextensive thermodynamics is announced to be an effective theory for nonequilibrium and long-range order phenomena [Tsallis]. Particularly interesting are connections to anomalous (fractal) diffusion and Levy distributions. Its canonical distribution is a power law, which occurs in particle and heavy-ion physics experiments [20–28]. It is particularly interesting to produce the Tsallis distribution with the help of a Gamma distribution for the inverse temperature, which may have consequences on the EoS of quark matter [18]. A possible source of such fluctuations is a multiplicative noise in the heat conduction [19].

In this paper we implement the canonical Tsallis distribution for lattice field theory. The fluctuation of the inverse temperature is simulated by a Gamma-distributed anisotropy parameter, $t = a_t/a_s$. The mean value is 1, the finite width, $1/\sqrt{c}$, is a parameter in this approach. Here we present numerical results for a test system with $SU(2)$ gauge fields.

1. THE TSALLIS DISTRIBUTION

Tsallis has suggested a nonextensive thermodynamics, derived on the basis of a generalization of the Boltzmann entropy formula[1]. In the canonical case the different states of a system are weighted by

$$w_i = \frac{1}{Z_{\mathrm{Ts}}}\left(1+\frac{\beta E_i}{c}\right)^{-c}, \tag{1}$$

in the $c \to \infty$ limit leading back to the Gibbs factor:

$$\lim_{c\to\infty} w_i = \frac{1}{Z_G}\exp\left(-\beta E_i\right). \tag{2}$$

The quantity $q = 1 + 1/c$ is the Tsallis index. The average energy (and the average number in the grand canonical approach) is given by a sum with slightly different weight factors due to the derivation of a power,

$$\langle E\rangle = \frac{1}{Z}\frac{\partial}{\partial\beta}Z = \frac{1}{Z}\sum_i E_i\left(1+\frac{\beta E_i}{c}\right)^{-(c+1)}. \tag{3}$$

[1] It seems there are earlier publications of this formula in the information theory [15], but without claiming to be a universal framework.

This factor can be obtained as the stationary distribution of a Langevin equation for a particle with momentum p, $\dot{p} + \gamma p = \xi$, with stochastic γ and ξ factors. For $\langle\gamma\rangle = G$, $\langle\xi\rangle = 0$ and white noise correlations with the corresponding strengths $2C$ and $2D$, one arrives at the following correspondence: $c = 1 + 2G/C$ and $\beta = mG/D$ with the energy $E = p^2/2m$ [12]. The Tsallis distribution weight factor, w_i, on the other hand, can be obtained as an integral of Gibbs factors over the Gamma distribution,

$$w_i = \frac{1}{Z_{\mathrm{Ts}}} \int_0^\infty dt\, w_c(t) \exp\left(-t\beta E_i\right), \tag{4}$$

with

$$w_c(t) = \frac{c^c}{\Gamma(c)} t^{c-1}\, \mathrm{e}^{-ct}. \tag{5}$$

$\Gamma(c) = (c-1)!$ for integer c is Euler's Gamma function. By its definition, $w_c(t)$ is normalized to one.

Based on this, any canonical Gibbs expectation value, if known as a function of β, can be converted into the corresponding expectation values with the canonical Tsallis distribution. The respective partition functions, Z_G and Z_{Ts} ensure the normalization of the w_i probabilities, $\sum\limits_i w_i = 1$. They are related to each other:

$$Z_{\mathrm{Ts}}(\beta) = \sum_i \int_0^\infty dt\, w_c(t) \exp\left(-t\beta E_i\right) = \int_0^\infty dt\, w_c(t) Z_G(t\beta). \tag{6}$$

The above formula can be interpreted as averaging over different β valued Gibbs simulations, as an instance of the superstatistical approach [17].

2. APPLICATION TO LATTICE GAUGE FIELD THEORY

The question arises, which strategy is the best to follow in order to perform lattice field theory simulations with Tsallis statistics instead of the Gibbs one. In the following we factorize the arguments of path integrals to an observable which does not explode exponentially with the lattice size, and to a weight factor which scales like $\exp(-N)$ with the total lattice size $N = N_t N_s^3$. We model the Tsallis distribution through a Gamma distributed inverse temperature in the physical system. The lattice simulation incorporates the physical temperature by the period length in the Euclidean time direction: $\beta = N_t a_t$. Due to the restriction to a few integer values of N_t, we simulate the Gamma distribution of the physical $\beta = 1/T$ by a Gamma distribution of the timelike link lengths, a_t. We assume that its mean value is equal to the spacelike lattice spacing, a_s. Then the ratio $t = a_t/a_s$ follows a normalized Gamma distribution with the mean value 1 and a width of $1/\sqrt{c}$. (In the view of ZEUS e^+e^- data $c \approx 5.8 \pm 0.5$, the width is about 40%.)

For calculating expectation values in field theory a generating functional based on the Legendre transform of Z is used. Our starting assumption is formula (6) with

$$Z_G\left[t\beta\right] = \int \mathcal{D}U\, \mathrm{e}^{-S[U,t]} \tag{7}$$

in a shorthand notation of path integrals. Since we simulate the canonical power-law distribution by a lattice with fluctuating asymmetry ratio, there are two limiting strategies to execute the Legendre transformation: i) in the *annealing* scenario the lattice fluctuates slowly and one considers first summations over field configurations, in the ii) *quenched* scenario on the contrary, the lattice fluctuations are fast, form an effective action (virtually reweighting the occurrence probability of a field configuration), and the summation over possible field configuration is the slower process performing the second (i.e., the path-) integral. In the first case the coupling of the external source current is instantaneous and therefore in the $a_t a_s^3$ factor inherent in $J \cdot U = \int d^4x\, J(x)U(x)$ the fluctuating ratio t is present, so we consider

$$Z_{\rm Ts}\,[J] = \int dt\, w_c(t) \int \mathcal{D}U\, \mathrm{e}^{-S[U,t]} \mathrm{e}^{tJ\cdot U}. \tag{8}$$

For the simplest expectation value, for the field itself, we obtain in this case

$$\langle U \rangle = \frac{1}{Z_{\rm Ts}} \frac{\delta}{\delta J} Z_{\rm Ts}\bigg|_{J=0} = \frac{\int dt\, w_c(t) \int \mathcal{D}U \mathrm{e}^{-S}\, tU}{\int dt\, w_c(t) \int \mathcal{D}\, U\, \mathrm{e}^{-S}} \tag{9}$$

which can be rewritten with the help of an effective action defined by

$$\mathrm{e}^{-S_{\rm eff}[U;v]} = \int dt\, w_c(t)\, t^v\, \mathrm{e}^{-S[U,t]}, \tag{10}$$

in the form

$$\langle U \rangle = \frac{\int \mathcal{D}U\, \mathrm{e}^{-S_{\rm eff}[U;\,1]} U}{\int \mathcal{D}U\, \mathrm{e}^{-S_{\rm eff}[U;\,0]}}. \tag{11}$$

The extra t factor resembles the occurrence of the power $-(c+1)$ in place of $-c$ by calculating the average energy with the help of the Tsallis distribution.

In the quenched case the field evolution and therefore the coupling to an external source feels the mean ratio only, which is one. In this case

$$Z_{\rm Ts}\,[J] = \int \mathcal{D}U \mathrm{e}^{J\cdot U} \int dt\, w_c(t)\, \mathrm{e}^{-S[U,t]}, \tag{12}$$

and the field expectation value does not contain an extra power of t. We arrive at

$$\langle U \rangle_{\rm quench} = \frac{\int \mathcal{D}U\, \mathrm{e}^{-S_{\rm eff}[U;\,0]} U}{\int \mathcal{D}U\, \mathrm{e}^{-S_{\rm eff}[U;\,0]}}. \tag{13}$$

Now one generic effective action governs all expectation values, but since this picture does not follow the behavior obtained by considering multiplicative noise, which coincides with the rules derived in Tsallis thermodynamics, we consider in the following the annealing option.

In any case the effect of t fluctuation is an effective weight for field configurations, which may depend on a scaling power according to the time (or energy) dimension of the operator under study. In general, we consider the Tsallis expectation value of an observable $\hat{A}[U]$ over lattice field configurations U. $\hat{A}$ may include the timelike link length, say with the power v: $\hat{A} = t^{\,v} A$. The Tsallis expectation value then is an average over all possible a_t link lengths according to a Gamma distribution of a_t/a_s. We obtain

$$\langle A \rangle_{\rm Ts} = \frac{1}{Z_{\rm Ts}} \frac{c^c}{\Gamma(c)} \int dt\, t^{c-1}\, \mathrm{e}^{-c\,t} \int \mathcal{D}U A\,[U]\, t^v\, \mathrm{e}^{-S[t,U]}, \tag{14}$$

with

$$Z_{\mathrm{Ts}} = \frac{c^c}{\Gamma(c)} \int dt\, t^{c-1}\, \mathrm{e}^{-c\,t} \int \mathcal{D}U\, \mathrm{e}^{-S[t,U]}. \tag{15}$$

The t dependence of the lattice gauge action is known for long: due to the time derivatives of vector potential in the expression of electric fields, the «kinetic» part scales like $a_t a_s^3/(a_t^2 a_s^2) = a_s/a_t$, and the magnetic («potential») part like $a_t a_s^3/(a_s^2 a_s^2) = a_t/a_s$ [1]. This leads to the following expression for the general lattice action:

$$S\,[t, U] = a\,t + b/t, \tag{16}$$

where $a = S_{\mathrm{ss}}[U]$ contains space–space-oriented plaquettes and $b = S_{\mathrm{ts}}[U]$ contains time–space-oriented plaquettes. The simulation runs in lattice units anyway, so actually the U configurations are selected according to weights containing a and b. In the $c \to \infty$ limit the scaled Gamma distribution approximates $\delta(t-1)$ (its width narrows extremely, while its integral is normalized to one), and one gets back the traditional lattice action $S = a + b$ and the traditional averages. For finite c, one can exchange the t integration and the configuration sum (path integral) and obtains exactly the power-law-weighted expression. The expectation value (14) becomes

$$\langle A \rangle_{\mathrm{Ts}} = \frac{\int \mathcal{D}U\, W_{v,\,c}\,[U]}{\int \mathcal{D}U\, W_{0,\,c}\,[U]}, \tag{17}$$

with the Gamma fluctuating time-link averaged general weight factor,

$$W_{v,c} = \frac{c^c}{\Gamma(c)} \int dt\, t^{v+c-1}\, \mathrm{e}^{-c\,t}\, \mathrm{e}^{-S[t,U]}. \tag{18}$$

The t integration can be carried out analytically using the replacement $t = \mathrm{e}^t \sqrt{b/(a+c)}$ and Eq. (16). The result contains the K-Bessel function:

$$W_{v,c} = \frac{c^c}{\Gamma(c)} \left(\frac{b}{a+c}\right)^{\frac{c+v}{2}} 2\, K_{v+c}\left(2\sqrt{b(a+c)}\right). \tag{19}$$

The K-Bessel function has an exponentially decreasing asymptotics, so we are in principle able to utilize known Monte Carlo techniques in order to calculate Tsallis expectation values. On the other hand, we cannot simply use old data produced according to the weight $\mathrm{e}^{-(a+b)}$, because the argument of the K-Bessel function is not $a + b$. This makes it necessary to redo lattice calculations — but only with a slightly increased effort.

In the $c \to \infty$ limit the conventional Gibbs thermodynamics is restored; here we consider the Gamma-distributed integral of a Taylor-expandable function of the asymmetry parameter, t:

$$I = \frac{c^c}{\Gamma(c)} \int_0^\infty dt\, t^{c-1}\, \mathrm{e}^{-ct}\, f(t), \tag{20}$$

with

$$f(t) = \sum_{n=0}^{\infty} f_n t^n. \tag{21}$$

[1]This generalizes to all lattice field actions: kinetic and mass terms scale like $1/t$, potential terms like t.

The integral for a given t power can be done analytically and we arrive at

$$I = \sum_{n=0}^{\infty} f_n \frac{\Gamma(c+n)}{c^n \Gamma(c)}. \tag{22}$$

The expression involving the Gamma functions can be expanded as a product with n factors, each being one plus something in the order of $1/c$. In the large c limit this starts by one, and continues as the sum of integers from zero to n starting with $n = 2$. So we get the following expansion:

$$I = f(1) + \frac{1}{2c} f''(1) + \mathcal{O}(1/c^2). \tag{23}$$

The leading term is the original function at $t = 1$ (this is the mean value of the Gamma distribution), the subleading term is proportional to the second derivative of the function at $t = 1$. For $1/c = 0$ one gets $f(1)$, proving that the t distribution approaches $\delta(t-1)$. The subleading term corresponds to a Gaussian approximation for the fluctuations.

3. EQUATION OF STATE

The calculation of the equation of state in lattice field theory is based on asymmetric lattices, therefore we extend our treatment now to lattices, where the timelike lattice constant not only fluctuates, but also its mean value is not necessarily equal to its spacelike pendant. The partition function is given by the average over the Gamma distribution,

$$Z = \int dt\, w_c(t) \int \mathcal{D}U\, \mathrm{e}^{-S(U;ta_t,a_s)}, \tag{24}$$

where the action is built from an asymmetric summation of space–space- and time–space-like plaquette contributions, P_s and P_t, given by the generic form of $P = 1 - \frac{1}{N}\mathrm{Re}\,\mathrm{Tr}\,(U_{\mathrm{plaq}})$ for the gauge group $SU(N)$:

$$S = \frac{N}{g^2} \sum \left(\frac{a_t}{a_s} t P_s + \frac{a_s}{a_t} \frac{P_t}{t} \right). \tag{25}$$

The pressure and energy density will be obtained from the partition function Z according to standard thermodynamical rules with $\beta = 1/T$ inverse temperature [2,4]:

$$\begin{aligned} p &= \frac{1}{\beta} \frac{\partial}{\partial V} \ln Z = \frac{1}{N_t N_s^3} \frac{1}{3 a_t a_s^2} \frac{\partial}{\partial a_s} \ln Z, \\ e &= -\frac{1}{V} \frac{\partial}{\partial \beta} \ln Z = -\frac{1}{N_t N_s^3} \frac{1}{a_s^3} \frac{\partial}{\partial a_t} \ln Z. \end{aligned} \tag{26}$$

For an ideal gas $Z \sim VT^3$, so $\ln Z \sim 3(\ln a_s - \ln a_t)$. In this case $p = e/3$, as it is well known. Therefore, it is particularly interesting to consider the interaction measure,

$$\Delta = \frac{e - 3p}{T^4}. \tag{27}$$

Another trivial extreme case is that of the constant field configuration. In this case $\ln Z \sim a_t a_s^3 \sim \beta V$ leading to $e + p = 0$. Such a relation is typical for the bag constant.

In general, the lattice gauge field theory describes a strongly interacting system. Besides the trivial a_s and a_t factors due to operator dimensions, a dependence of the coupling strength due to renormalization is taken into account. The «line of constant physics» requires in the one-loop approximation

$$a\frac{\partial}{\partial a}\left(\frac{N}{g^2}\right) = -2N\mathcal{B} \tag{28}$$

for both the a_t- and a_s-dependences ($\mathcal{B} = -11/(24\pi^2)$ for $SU(N)$). Restoring the symmetry of the lattice spacings at the end (but keeping $N_t \neq N_s$), we arrive at the following pressure and energy density in lattice units:

$$\begin{aligned} a^4 p &= 2N\mathcal{B}(\overline{tP_s} + \overline{P_t/t}) + \frac{N}{g^2}(\overline{tP_s} - \overline{P_t/t}), \\ a^4 e &= -2N\mathcal{B}(\overline{tP_s} + \overline{P_t/t}) + \frac{N}{g^2}(\overline{tP_s} - \overline{P_t/t}). \end{aligned} \tag{29}$$

This formula is the extension of the standard expressions used so far in lattice gauge theory in Gibbs–Boltzmann thermodynamics [29]. Finally, the comparison of the interaction measure, Δ on symmetric and asymmetric lattices should approach the physical value, which then can be obtained from the $1/N_t^4$-like scaling of the plaquette expectation values:

$$\Delta|_{N_t=N_s}^{N_t\ll N_s} = 6N_t^4\left(1-\frac{1}{c}\right)^4 2N\mathcal{B}\left(2P_0 - \overline{tP_s} - \overline{P_t/t}\right), \tag{30}$$

with

$$P_0 = \frac{1}{2}(\overline{tP_s} + \overline{P_t/t}) \tag{31}$$

denoting the average plaquette contribution on the symmetric lattice. (According to our previous discussion in the large $N_t = N_s$ limit the t distribution approaches a Dirac delta, so there is practically no fluctuation of the formal temperature.)

4. NUMERICAL RESULTS

We apply a Metropolis algorithm for updating links. The update criterion is built by an action where the electric (space–time-oriented plaquette) contributions are divided by t and the magnetic (space–space-oriented plaquette) contributions are multiplied by a factor t, randomly chosen as a deviate from the Gamma distribution at each new sweep over the whole lattice. This way we take into account interactions inside the lattice generated by the $SU(2)$ Yang–Mills action, but do not let the system relax to a Gibbs distribution. Instead the convolution of the exponential and the Gamma distribution — eventually a Tsallis distribution — is the attractor in the Monte Carlo process.

We have studied an $SU(2)$ lattice gauge field system on a 10^4 symmretric and on a $10^3 \times 2$ asymmetric lattice in order to learn about the equation of state.

In the followings we discuss our results presented in the figures. In Fig. 1 histograms of the Euler Gamma distributions are plotted. This distribution of the asymmetry factor t

has been applied on the top of the lattice Monte Carlo code, renewed after each sweep over the whole lattice. For comparison a hypothetical distribution with 2 million draws (spiky line in the background) is shown. In Fig. 2 the average of the time–space- and space–space-like plaquettes multiplied with proper asymmetry factors during the Monte Carlo process is shown. It leads to an equipartition, but with different magnitude of fluctuations dependning on the width parameter, c. In Fig. 3 the action difference, the basis of obtaining e/T^4, as a function of the effective coupling, $b = 4/g^2$, monotonically depending on the temperature ratio T/T_c is shown. For different values of nonextensivity (parameter c), see the legend. In Fig. 4 the action sum, the basis of obtaining $(e-3p)/T^4$, as a function of the effective

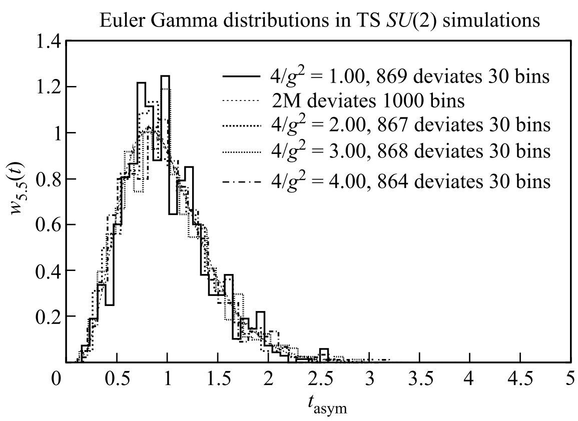

Fig. 1. Histograms of the Euler Gamma distributions applied on the top of the lattice Monte Carlo code for the asymmetry factor, t. For comparison a hypothetical distribution with 2 million draws (spiky line in the background) is shown

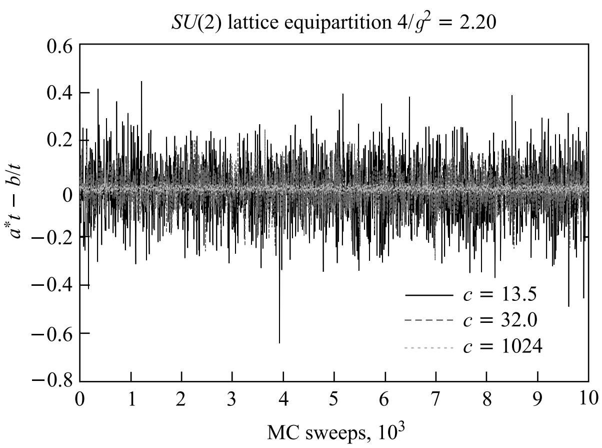

Fig. 2. The equipartition of the time–space- and space–space-like plaquettes multiplied with proper asymmetry factors during the Monte Carlo process leads to the average values above. Different symbols belong to different values of the width parameter, c, according to the legend

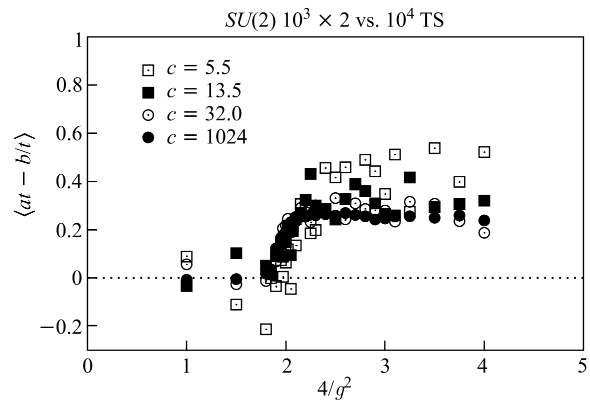

Fig. 3. The action difference, the basis of obtaining e/T^4, as a function of the effective coupling, $b = 4/g^2$, monotonically depending on the temperature ratio T/T_c. For different values of nonextensivity (parameter c), see the legend

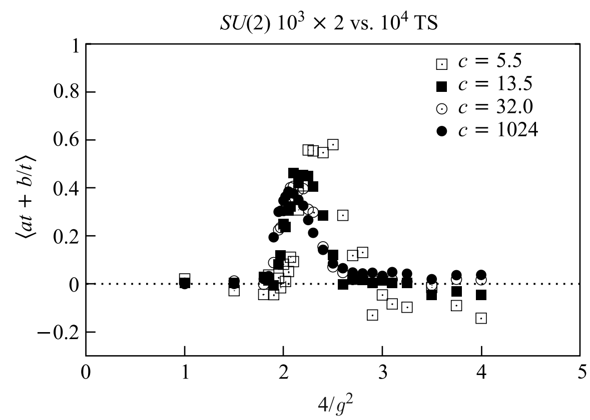

Fig. 4. The action sum, the basis of obtaining $(e - 3p)/T^4$, as a function of the effective coupling, $b = 4/g^2$, monotonically depending on the temperature ratio T/T_c. For different values of nonextensivity (parameter c), see the legend

coupling, $b = 4/g^2$, monotonically depending on the temperature ratio T/T_c is plotted, for different values of nonextensivity parameter c (see the legend).

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