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ФИЗИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ И АТОМНОГО ЯДРА. ТЕОРИЯ

QUANTIZATION OF UNIVERSAL TEICHMÜLLER SPACE

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Universal Teichmüller space $\mathcal{T}$ is the quotient of the group QS(S^1) of quasi-symmetric homeomorphisms of S^1 modulo Möbius transformations. The quantization problem for $\mathcal{T}$ arises in the theory of nonsmooth closed bosonic strings. Because of nonsmoothness of strings, the natural QS(S^1)-action on $\mathcal{T}$ is also not smooth, so there is no classical Lie algebra, associated to QS(S^1). However, using the methods of noncommutative geometry, we can define a quantum Lie algebra of observables Der^q(QS), yielding the quantization of $\mathcal{T}$.

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The universal Teichmüller space $\mathcal{T}$ is the quotient of the group QS(S^1) of quasi-symmetric homeomorphisms of the unit circle S^1 (i.e., homeomorphisms of S^1, extending to quasi-conformal homeomorphisms of the unit disc) modulo Möbius transformations. In particular, this space contains the quotient $\mathcal{S}$ of the group Diff$_+(S^1)$ of diffeomorphisms of S^1, preserving orientation, modulo Möbius transformations. Both groups act naturally on the Sobolev space $V := H_0^{1/2}(S^1, \mathbb{R})$ of half-differentiable functions on the circle by reparameterization.

The spaces $\mathcal{T}$ and $\mathcal{S}$ arise in the closed bosonic string theory as the phase spaces of this theory. The main difference between them is that in the case of $\mathcal{S}$, we restrict to smooth strings with the reparameterization group Diff$_+(S^1)$, while in the case of $\mathcal{T}$ we consider the maximal possible phase space, consisting of half-differentiable strings. In this case the role of the reparameterization group is played by QS(S^1).

Accordingly, in the case of $\mathcal{S}$, the algebra of classical observables coincides with the Lie algebra Vect(S^1) of the group Diff$_+(S^1)$. For the quantization space in this case, we can take any of the Fock spaces $F(V^{\mathbb{C}}, J)$ over the complexified Sobolev space $V^{\mathbb{C}} = H_0^{1/2}(S^1, \mathbb{C})$, associated with a complex structure J on V. The infinitesimal version of the projective Diff$_+(S^1)$-action on the union of Fock spaces $F(V^{\mathbb{C}}, J)$ over the space of Hilbert–Schmidt complex structures J generates an irreducible representation of Vect(S^1) in any of these Fock spaces, yielding the quantization of $\mathcal{S}$.

For the universal Teichmüller space $\mathcal{T}$, the situation is more complicated since the QS(S^1)-action on $\mathcal{T}$ is not smooth. By this reason, we cannot associate with $\mathcal{T}$ any Lie algebra of classical observables, associated with the group QS(S^1). However, we can define a Lie algebra of quantum observables Der^q(QS), generated by the quantum differentials, acting on

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$F(V^{\mathbb{C}}, J)$. These differentials originate from the integral operators $d^q h$ on V with kernels, given essentially by the finite-difference derivatives of $h \in \mathrm{QS}(S^1)$.

Briefly on the content of the paper. In Sec. 1, we define the universal Teichmüller space $\mathcal{T}$ and describe its properties. The space $\mathcal{S}$ of diffeomorphisms of S^1 may be considered as a regular part of $\mathcal{T}$.

Quasi-symmetric homeomorphisms of S^1, acting on the Sobolev space V of half-differentiable functions on S^1 by reparameterization, generate linear bounded symplectic operators on V. Respectively, diffeomorphisms of S^1 are realized as Hilbert–Schmidt symplectic operators on V. This action of quasi-symmetric homeomorphisms of S^1 on the Sobolev space V yields a Grassmann realization of $\mathcal{T}$ and $\mathcal{S}$ in terms of the Hilbert–Schmidt Grassmannian.

In Sec. 2, we explain how to quantize the regular part $\mathcal{S}$ of $\mathcal{T}$, using the embedding of $\mathcal{S}$ into the Hilbert–Schmidt Grassmannian. According to the Shale–Berezin theorem, there exists a holomorphic Fock bundle over the space of Hilbert–Schmidt complex structures which can be provided with a projective action of the Hilbert–Schmidt symplectic group, covering the natural action of this group on the base. The infinitesimal version of this action yields the Dirac quantization of $\mathcal{S}$.

However, this quantization method does not extend to the whole universal Teichmüller space $\mathcal{T}$ because of the nonsmoothness of the $\mathrm{QS}(S^1)$-action on $\mathcal{T}$. For the quantization of $\mathcal{T}$, considered in Sec. 3, we use another approach, based on the ideas of noncommutative geometry of Connes. Namely, we introduce a quantized infinitesimal version of the $\mathrm{QS}(S^1)$-action on $\mathcal{T}$ in terms of which the Connes quantization of $\mathcal{T}$ is defined. The last Section is based on the author's paper [4].

1. UNIVERSAL TEICHMÜLLER SPACE

1.1. Definition. The universal Teichmüller space is defined in terms of quasi-symmetric homeomorphisms. A homeomorphism $f : S^1 \to S^1$, preserving the orientation, is called *quasi-symmetric* if it extends to a quasi-conformal homeomorphism of the unit disc Δ.

We recall that a homeomorphism $w : \Delta \to \Delta$ with locally integrable derivatives is called *quasi-conformal* if there exists a function $\mu \in L^\infty(\Delta)$ with the norm $\|\mu\|_\infty =: k < 1$ such that the following *Beltrami equation*

$$w_{\bar{z}} = \mu w_z$$

holds almost everywhere in Δ. In this case μ is called the *Beltrami differential*.

For $\mu \equiv 0$, the Beltrami equation converts into the Cauchy–Riemann equation so in this case its solutions are given by conformal maps. We shall consider such solutions as trivial ones and factorize them out in the sequel.

Recall that the differential $dw(z)$ of a conformal map sends circles with center in z to circles centered at $w(z)$. If w is a smooth quasi-conformal map, then its differential sends circles with center in z to the ellipses, centered at $w(z)$, with the eccentricities being uniformly bounded in $z \in \mathcal{D}$ by a constant K, related to the above constant k by the formula

$$K = \frac{1+k}{1-k} \geqslant 1.$$

Quasi-conformal maps $w : \Delta \to \Delta$ form a group, i.e., the composition of quasi-conformal maps is again quasi-conformal as well as the inverse of a quasi-conformal map. It implies that quasi-symmetric homeomorphisms of S^1 also form a group with respect to the composition, denoted by $\mathrm{QS}(S^1)$.

Since any diffeomorphism of S^1, preserving the orientation, can be extended to a diffeomorphism of the closed unit disc $\overline{\Delta}$, which is evidently quasi-conformal, the group $\mathrm{Diff}_+(S^1)$ of orientation-preserving diffeomorphisms of S^1 is contained in the group $\mathrm{QS}(S^1)$. So we have the following chain of embeddings:

$$\mathrm{M\ddot{o}b}(S^1) \subset \mathrm{Diff}_+(S^1) \subset \mathrm{QS}(S^1) \subset \mathrm{Homeo}_+(S^1),$$

where $\mathrm{M\ddot{o}b}(S^1)$ is the group of fractional-linear automorphisms of the unit disc Δ, restricted to S^1, and $\mathrm{Homeo}_+(S^1)$ is the group of orientation-preserving homeomorphisms of S^1.

Definition 1. The *universal Teichmüller space* is the quotient

$$\mathcal{T} = \mathrm{QS}(S^1)/\mathrm{M\ddot{o}b}(S^1).$$

It can be identified with the space of *normalized* quasi-symmetric homeomorphisms of S^1, fixing the points $\pm 1, i \in S^1$.

It is a complex Banach manifold (cf. [4, 5]) which contains, according to the above remark, the space

$$\mathcal{S} := \mathrm{Diff}_+(S^1)/\mathrm{M\ddot{o}b}(S^1)$$

of normalized diffeomorphisms of S^1.

1.2. Sobolev Space of Half-Differentiable Functions

Definition 2. *The Sobolev space of half-differentiable functions* on the circle is the Hilbert space

$$V = H_0^{1/2}(S^1, \mathbb{R}),$$

consisting of functions $f \in L_0^2(S^1, \mathbb{R})$ with Fourier decompositions

$$f(z) = \sum_{k \neq 0} f_k z^k, \quad z = \mathrm{e}^{i\theta},$$

with $\bar{f}_k = f_{-k}$ and finite Sobolev norm

$$\sum_{k \neq 0} |k||f_k|^2 = 2 \sum_{k>0} k|f_k|^2 < \infty.$$

A *symplectic structure* ω on V can be defined by the formula

$$\omega(\xi, \eta) = 2\,\mathrm{Im} \sum_{k>0} k \xi_k \bar{\eta}_k \quad \text{for } \xi, \eta \in V.$$

We can also introduce a *complex structure* J^0 on V, given by

$$J^0 \xi = -i \sum_{k>0} \xi_k z^k + i \sum_{k<0} \xi_k z^k \quad \text{for } \xi \in V.$$

The introduced structures ω and J^0 are compatible with each other in the sense that they generate together a *Riemannian metric* g^0 on V, given by

$$g^0(\xi,\eta) := \omega(\xi, J^0\eta) = 2\,\mathrm{Re}\sum_{k>0} k\xi_k\bar{\eta}_k \quad \text{for } \xi,\eta \in V.$$

The complex structure J^0, being extended to the *complexified Sobolev space*

$$V^{\mathbb{C}} = H_0^{1/2}(S^1,\mathbb{C}),$$

generates a *polarization* of $V^{\mathbb{C}}$, i.e., decomposition of

$$V^{\mathbb{C}} = W_+ \oplus W_-$$

into the direct orthogonal sum of $(\mp i)$-eigenspaces of the complex structure operator J^0.

1.3. QS-Action on the Sobolev Space V. Associate with a homeomorphism $h : S^1 \to S^1$, preserving the orientation, the «change-of-variable» operator T_h, defined by

$$T_h\xi = \xi \circ h - \frac{1}{2\pi}\int_0^{2\pi} \xi(h(\theta))\,d\theta.$$

This operator is correctly defined on the Sobolev space V and has the following remarkable properties.

Theorem 1 (Nag–Sullivan theorem).
1. The operator $T_h : V \to V$ if and only if $h \in \mathrm{QS}(S^1)$.
2. For $h \in \mathrm{QS}(S^1)$ the operator T_h acts symplectically on V, i.e., it preserves the form ω.
3. The complex-linear extension of T_h to $V^{\mathbb{C}}$ preserves $W_\pm$ if and only if $h \in \mathrm{Möb}(S^1)$. In this case T_h acts on $W_\pm$ as a unitary operator.

The Nag–Sullivan theorem immediately implies that there is an embedding

$$\mathcal{T} = \mathrm{QS}(S^1)/\mathrm{Möb}(S^1) \longrightarrow \mathcal{J}(V) := \mathrm{Sp}(V)/\mathrm{U}(W_+),$$

where $\mathrm{Sp}(V)$ is the symplectic group of V, and $\mathrm{U}(W_+)$ is the unitary group of W_+, embedded diagonally into $\mathrm{Sp}(V)$.

The space $\mathcal{J}(V)$ on the right can be identified with the space of complex structures on V, compatible with symplectic form ω. Indeed, acting on the reference complex structure J^0 by operators from the group $\mathrm{Sp}(V)$, we shall obtain all complex structures on V, compatible with ω, and it remains to factor out the transforms which do not change the complex structure J^0. The group of such transforms coincides precisely with the diagonal subgroup $\mathrm{U}(W_+)$.

The restriction of the constructed embedding to the subspace $\mathcal{S} \subset \mathcal{T}$ yields an embedding

$$\mathcal{S} = \mathrm{Diff}_+(S^1)/\mathrm{Möb}(S^1) \longrightarrow \mathcal{J}_{\mathrm{HS}}(V) := \mathrm{Sp}_{\mathrm{HS}}(V)/\mathrm{U}(W_+),$$

where the *symplectic Hilbert–Schmidt group* is defined as

$$\mathrm{Sp}_{\mathrm{HS}}(V) = \{A \in \mathrm{Sp}(V) :\ \mathrm{pr}_+ \circ A \circ \mathrm{pr}_- \ \text{ is a Hilbert–Schmidt operator}\},$$

and $\mathrm{pr}_\pm$ denotes the orthogonal projection $\mathrm{pr}_\pm : V^{\mathbb{C}} \to W_\pm$.

2. QUANTIZATION OF $\mathcal{S}$

2.1. Dirac Quantization. Define first the classical systems to be quantized. A *classical system* is a pair $(M, \mathcal{A})$, consisting of the phase space M and algebra of classical observables $\mathcal{A}$. The *phase space* of the system is a symplectic manifold (M, ω), while the *algebra of observables* $\mathcal{A}$ is a subalgebra of the Lie algebra $C^\infty(M, \mathbb{R})$ of smooth real-valued functions on M, provided with the Poisson bracket, determined by ω.

A standard way to produce the algebras of classical observables is to consider a Lie subgroup Γ of symplectic diffeomorphisms of (M, ω) and take for $\mathcal{A}$ the Lie algebra Lie(Γ), consisting of Hamiltonian vector fields X_f, generated by functions (Hamiltonians) $f \in C^\infty(M, \mathbb{R})$. Such functions f form a Lie algebra of observables, associated with Γ.

The *quantum system*, corresponding to the classical system $(M, \mathcal{A})$, is determined by an irreducible linear representation r of the algebra $\mathcal{A}$, associating with every observable f from $\mathcal{A}$ a self-adjoint operator $r(f)$, acting in a complex Hilbert space H, called the *quantization space*. It is required that this representation should map

$$r : \{f, g\} \longmapsto \frac{1}{i}[r(f), r(g)] = \frac{1}{i}(r(f)r(g) - r(g)r(f)) \tag{1}$$

and satisfy the normalization condition: $r(1) = I$.

Sometimes it is more convenient to deal with the complexified algebras of observables $\mathcal{A}^{\mathbb{C}}$ (or, more generally, with complex Lie algebras, provided with an involution). The Dirac quantization of such a system $(M, \mathcal{A}^{\mathbb{C}})$ is given by an irreducible linear representation r of the observables $f \in \mathcal{A}^{\mathbb{C}}$ by closed linear operators $r(f)$ in H, satisfying, apart from (1) and normalization condition, also the conjugation rule: $r(\bar{f}) = r(f)^*$.

We are going to apply this definition to the quantization of infinite-dimensional systems in which both the phase space and the algebra of observables are infinite-dimensional. For the infinite-dimensional algebras of observables it is more natural to look for the projective representations. If we succeed in finding such a representation of the algebra of observables $\mathcal{A}$, it will mean that we have quantized an extended system $(M, \tilde{\mathcal{A}})$, where $\tilde{\mathcal{A}}$ is a suitable central extension of $\mathcal{A}$.

In our case the role of the classical system is played by the pair

$$(\mathcal{S}, \mathrm{Vect}(S^1)),$$

where $\mathrm{Vect}(S^1)$ is the Lie algebra of the Lie group $\mathrm{Diff}_+(S^1)$, coinciding with the Lie algebra of smooth tangent vector fields on S^1.

To quantize this system, we first extend it to a larger classical system, using the embedding

$$\mathcal{S} \longrightarrow \mathcal{J}_{\mathrm{HS}}(V) := \mathrm{Sp}_{\mathrm{HS}}(V)/\mathrm{U}(W_+),$$

constructed above.

The *extended classical system* is given by

$$(\mathcal{J}_{\mathrm{HS}}(V), \mathrm{sp}_{\mathrm{HS}}(V)),$$

where $\mathrm{sp}_{\mathrm{HS}}(V)$ is the Lie algebra of the Hilbert–Schmidt symplectic group $\mathrm{Sp}_{\mathrm{HS}}(V)$.

2.2. Quantization of the Extended System. We have to introduce first the quantization space for our extended classical system. It coincides with the Fock space of the Sobolev space V which is defined as follows.

Fix a complex structure $J \in \mathcal{J}(V)$, generating a decomposition

$$V^{\mathbb{C}} = W \oplus \overline{W}$$

into the direct sum of $(\mp i)$-eigenspaces of operator J and provide $V^{\mathbb{C}}$ with the associated inner product

$$\langle z, w \rangle_J := \omega(z, Jw).$$

Denote by $\mathfrak{S}(W)$ the algebra of symmetric polynomials in variables $z \in W$ and provide it with the inner product, generated by $\langle \cdot, \cdot \rangle_J$. On monomials of the same degree it is given by the formula

$$\langle z_1 \otimes \ldots \otimes z_n, z_1' \otimes \ldots \otimes z_n' \rangle_J := \sum_{(i_1, \ldots, i_n)} \langle z_1, z_{i_1}' \rangle_J \cdots \langle z_n, z_{i_n}' \rangle_J,$$

where the summation is taken over all permutations $\{i_1, \ldots, i_n\}$ of the set $\{1, \ldots, n\}$. On monomials of different degrees it is set to zero and extended to the whole algebra $\mathfrak{S}(W)$ by linearity.

The *Fock space*

$$F_J = F(V^{\mathbb{C}}, J)$$

is the closure of the algebra $\mathfrak{S}(W)$ with respect to the norm, determined by the introduced inner product.

If $\{w_n\}_{n=1}^{\infty}$ is an orthonormal base of the subspace W, then one can take for the orthonormal base of the Fock space F_J the system of homogeneous polynomials

$$P_K(z) = \frac{1}{\sqrt{K!}} \langle z, w_1 \rangle_J^{k_1} \cdots \langle z, w_n \rangle_J^{k_n},$$

where K runs through all finite index sets $K = (k_1, \ldots, k_n, 0, \ldots)$ and $K! = k_1! \cdots k_n!$.

We unify the different Fock spaces F_J with $J \in \mathcal{J}_{\mathrm{HS}}(V)$ into the *Fock bundle*

$$\mathcal{F} := \bigcup_{J \in \mathcal{J}_{\mathrm{HS}}(V)} F_J \longrightarrow \mathcal{J}_{\mathrm{HS}}(V) = \mathrm{Sp}_{\mathrm{HS}}(V)/\mathrm{U}(W_+).$$

Theorem 2 (Shale–Berezin theorem). *The Fock bundle $\mathcal{F} \to \mathcal{J}_{\mathrm{HS}}(V)$ is a holomorphic Hermitian Hilbert bundle. There is a projective unitary action of the symplectic Hilbert–Schmidt group $\mathrm{Sp}_{\mathrm{HS}}(V)$ on this bundle, covering its natural action on the base $\mathcal{J}_{\mathrm{HS}}(V)$ by left translations.*

The infinitesimal version of this action yields a projective representation of the Lie algebra $\mathrm{sp}_{\mathrm{HS}}(V)$ in the Fock space $F_0 = F(V^{\mathbb{C}}, J^0)$, i.e., the quantization of the extended classical system

$$\left(\mathcal{J}_{\mathrm{HS}}(V), \widetilde{\mathrm{sp}_{\mathrm{HS}}(V)}\right),$$

where $\widetilde{\mathrm{sp}_{\mathrm{HS}}(V)}$ is a suitable central extension of the symplectic Hilbert–Schmidt Lie algebra $\mathrm{sp}_{\mathrm{HS}}(V)$.

The restriction of this construction to the subspace $\mathcal{S} \subset \mathcal{J}_{\mathrm{HS}}(V)$ yields a holomorphic Hermitian Hilbert bundle

$$\mathcal{F}_{\mathcal{S}} = \bigcup_{J \in \mathcal{S}} F_J \longrightarrow \mathcal{S} = \mathrm{Diff}_+(S^1)/\text{Möb}(S^1)$$

together with a projective action of the diffeomorphism group $\mathrm{Diff}_+(S^1)$ on $\mathcal{F}_{\mathcal{S}}$, covering its action on the base $\mathcal{S}$. The infinitesimal version of this action yields a projective representation of the Lie algebra $\mathrm{Vect}(S^1)$ in the Fock space F_0, i.e., the quantization of the classical system

$$(\mathcal{S}, \mathrm{vir}),$$

where vir is the *Virasoro algebra*, i.e., an (essentially unique) central extension of the Lie algebra $\mathrm{Vect}(S^1)$.

3. QUANTIZATION OF $\mathcal{T}$

We can try to quantize the universal Teichmüller space $\mathcal{T}$ in the same way as we did it for $\mathcal{S}$. Namely, we can consider the Fock bundle of all complex structures from $\mathcal{J}(V)$ and try to pull back the $\mathrm{Sp}(V)$-action on $\mathcal{J}(V)$ to this Fock bundle. However, it is impossible, according to the Shale–Berezin theorem. So we use another method of quantization of $\mathcal{T}$, based on the noncommutative geometry of Connes.

In the following table we compare Connes and Dirac approaches to the quantization.

	Dirac approach	Connes approach
Classical system	$(M, \mathcal{A})$, where: M — phase space, $\mathcal{A}$ — involutive Lie algebra of observables	$(M, \mathfrak{A})$, where: M — phase space, $\mathfrak{A}$ — involutive associative algebra of observables with an exterior derivative d
Quantization	Irreducible representation $r : \mathcal{A} \to \mathrm{End}\, H$, sending $\{f, g\} \mapsto \frac{1}{i}[r(f), r(g)]$	Irreducible representation $\pi : \mathfrak{A} \to \mathrm{End}\, H$ sending $df \mapsto [S, \pi(f)]$, where $S = S^*$, $S^2 = I$ is a symmetry operator

We can also reformulate the Connes definition of quantization in terms of Lie algebras. For that, introduce the Lie algebra $\mathrm{Der}\,\mathfrak{A}$ of derivations of $\mathfrak{A}$, i.e., linear maps of $\mathfrak{A}$, satisfying the Leibniz rule. In these terms, the Connes quantization of $(M, \mathfrak{A})$ means an irreducible Lie algebra representation of $\mathrm{Der}\,\mathfrak{A}$ in the Lie algebra $\mathrm{End}\, H$ of linear closed operators in H, provided with the commutator as a Lie bracket.

If all observables from the algebra $\mathcal{A}$ are smooth (as we have assumed before), both definitions of the quantization are essentially equivalent. Indeed, the differential df of a smooth observable f is symplectically dual to the Hamiltonian vector field X_f, which establishes the relation between the associative algebra of observables $\mathfrak{A} \ni f$ and the Lie algebra of observables $\mathcal{A} \ni X_f$. The symmetry operator S is provided by the polarization $H = H_+ \oplus H_-$ of the quantization space H. It is related to the complex structure operator J, determined by the same polarization, by the evident formula $S = iJ$.

However, in the case when the algebra of classical observables contains nonsmooth functions, the Dirac approach is formally nonapplicable. In the Connes approach, the differential *df* of a nonsmooth observable f is also not defined but its quantum analogue

$$d^q f := [S, \pi(f)]$$

can have sense.

Consider the following example in which the role of the algebra of observables is played by

$$\mathfrak{A} := L^\infty(S^1, \mathbb{C}).$$

Any $f \in \mathfrak{A}$ determines the multiplication operator in the quantization space $H := L^2(S^1, \mathbb{C})$ by the formula

$$M_f : v \in H \longmapsto fv \in H.$$

The symmetry operator S in H is given by the *Hilbert transform*

$$(Sf)(\varphi) = \frac{1}{2\pi}\,\text{P.V.} \int_0^{2\pi} K(\varphi, \psi)\, f(\psi)\, d\psi,$$

where the integral is taken in the principal value sense and we identify $f(\varphi)$ with $f(\mathrm{e}^{i\varphi})$. The kernel of the above integral operator is given by

$$K(\varphi, \psi) = 1 + i \cot \frac{\varphi - \psi}{2}.$$

Note that for $\varphi \to \psi$ it behaves like $1 + \dfrac{2i}{\varphi - \psi}$.

The differential *df* of a general observable $f \in \mathfrak{A}$ is not defined but its quantum analogue

$$d^q f = [S, M_f]$$

is a correctly defined operator in H. Moreover, for $f \in V$ it coincides with the Hilbert–Schmidt operator, given by

$$d^q f(v)(\varphi) = \frac{1}{2\pi} \int_0^{2\pi} k(\varphi, \psi)\, v(\psi)\, d\psi,$$

where

$$k(\varphi, \psi) = K(\varphi, \psi)(f(\varphi) - f(\psi)).$$

For $\varphi \to \psi$, this kernel behaves like (up to a constant)

$$\frac{f(\varphi) - f(\psi)}{\varphi - \psi}.$$

It can be shown that the quasi-classical limit of this operator, obtained by performing the limit $\varphi \to \psi$, coincides with the multiplication operator $v \mapsto f'v$. So the quantization in this case reduces to the replacement of the derivative by its finite-difference analogue.

Turning to the quantization of $\mathcal{T}$, recall that we have defined earlier a natural action of the group $\mathrm{QS}(S^1)$ of quasi-symmetric homeomorphisms of the circle on the Sobolev space V. But this action is not smooth, so it does not admit the differentiation. In particular, there is no Lie algebra of classical observables, associated with $\mathcal{T}$, and no associated classical system. However, we shall construct directly a quantum algebra of observables, associated with $\mathcal{T}$.

For that we define first a *quantized infinitesimal version* of the $\mathrm{QS}(S^1)$-action on V, given by the integral operator $d^q f$. Next, we extend the operator $d^q f : V \to V$ to the Fock space $F_0 = F(V^{\mathbb{C}}, J^0)$ by defining it first on the basis polynomials $P_K(z)$ with the help of Leibniz rule and then extending by the linearity and closure to an operator $d^q f$ on F_0. The operators $d^q f$ with $f \in \mathrm{QS}(S^1)$ generate a Lie algebra $\mathrm{Der}^q(\mathrm{QS})$ which should be considered as a *quantum algebra of observables, associated with* $\mathcal{T}$. It can be also treated as the replacement of the (nonexisting) classical Lie algebra, associated with the group $\mathrm{QS}(S^1)$.

Let us compare now the Connes quantization of $\mathcal{T}$ with the Dirac quantization of the space $\mathcal{J}_{\mathrm{HS}}(V)$ of Hilbert–Schmidt complex structures on the Sobolev space V.

The *Dirac quantization of* $\mathcal{J}_{\mathrm{HS}}(V)$ involved the following steps:

(1) we started from the $\mathrm{Sp}_{\mathrm{HS}}(V)$-action on V;

(2) using Shale–Berezin theorem, we extended this action to a projective unitary action of $\mathrm{Sp}_{\mathrm{HS}}(V)$ on the Fock bundle;

(3) infinitesimal version of this action yielded a projective unitary representation of $\mathrm{sp}_{\mathrm{HS}}(V)$ in the Fock space F_0.

In the case of the *Connes quantization of* $\mathcal{T}$, we have proceeded as follows:

(1) we started from the $\mathrm{QS}(S^1)$-action on V;

(2) since the step (2) in the Dirac quantization of $\mathcal{J}_{\mathrm{HS}}$ is impossible in the case of $\mathcal{T}$ by Shale–Berezin theorem, we have defined instead the quantized infinitesimal action of $\mathrm{QS}(S^1)$ on V, given by quantum differentials $d^q f$;

(3) we extended the operators $d^q f : V \to V$ to F_0 and defined the quantum Lie algebra $\mathrm{Der}^q(\mathrm{QS})$, generated by operators $d^q f$ on F_0 with $f \in \mathrm{QS}(S^1)$.

So, the Connes quantization of the universal Teichmüller space $\mathcal{T}$ involves the following two steps:

(1) *first quantization*: construction of the quantized infinitesimal $\mathrm{QS}(S^1)$-action on V, given by the quantum differentials $d^q f$ with $f \in \mathrm{QS}(S^1)$.

(2) *second quantization*: extension of operators $d^q f : V \to V$ to operators $d^q f : F_0 \to F_0$, generating the quantum algebra of observables $\mathrm{Der}^q(\mathrm{QS})$, associated with $\mathcal{T}$.

The *correspondence principle* for the constructed Connes quantization of $\mathcal{T}$ means that this quantization, being restricted to $\mathcal{S}$, coincides with the Dirac quantization of $\mathcal{S}$.

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