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OPERATIONAL CALCULUS APPROACH TO EXPLICIT SOLVING OF INITIAL AND BOUNDARY VALUE PROBLEMS

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Short review of an approach to explicit solving of initial and boundary value problems (BVPs) for some partial differential equations (PDEs) is presented. A combination of two classical methods — the Fourier method and the Duhamel principle — is used in the framework of a two-dimensional operational calculus. It gives explicit solutions of some local and nonlocal BVPs for the classical equations of mathematical physics in rectangular domains.

Представлен краткий обзор одного подхода для решения задач с начальными и граничными условиями для некоторых уравнений с частными производными. Используется сочетание двух классических методов — метода Фурье и принципа Дюамеля — в рамках двумерного операционного исчисления. Это приводит к получению эксплицитных решений некоторых локальных и нелокальных граничных задач для классических уравнений математической физики в прямоугольных областях.

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INTRODUCTION

In a series of papers (see [1–4], for example), the operational calculus (OC) approach to solving initial and boundary value problems for some types of differential equations is considered. In this paper, the essence of this approach and its application for solving local and nonlocal boundary value problems for the heat equation, the wave equation and the equation of a free supported beam are briefly presented.

Oliver Heaviside (1850–1925) had developed his OC mainly for solving initial value problems for ordinary linear differential equations with constant coefficients. Polish mathematician Jan Mikusiński (1913–1987) developed a direct algebraic approach to the Heaviside OC, allowing it to be better accepted and used. His calculus is known as the Mikusiński's OC (see [5]).

If we are interested in application of the OC approach to PDEs, the Heaviside–Mikusiński's OC should be extended to multivariate functions. Such an extension for the Laplace transformation is proposed by Ditkin and Prudnikov in [6]. The principles of the applications of

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multivariate OC for solution of the Cauchy problems for linear PDEs with constant coefficients are developed by Gutterman in [7].

If we want to use convolutions devised especially for the given BVPs for the classical equations of the mathematical physics in finite domains, we need to make extension of the Mikusiński's approach in the way considered below (see [1] and [2]).

1. A TWO-VARIATE OPERATIONAL CALCULUS FOR BOUNDARY VALUE PROBLEMS

The following two types of convolutions, intended for one-variate OC, will be assembled to multivariate convolutions.

1.1. Convolutions for the Differentiation Operator. The basic BVP for the differentiation operator d/dt in the space $C[0,\infty)$ of the continuous functions $f(t)$, $0 \leqslant t < \infty$, is determined by an arbitrary linear functional χ on $C\,[0,\infty)$. It looks so

$$y' = f(t), \quad \chi(y) = 0.$$

In order the solution y to exist, it is necessary to assume $\chi\{1\} \neq 0$. For the simplicity sake, we take $\chi\{1\} = 1$. Then, the solution $y = lf(t)$ could be named a generalized integration operator. Evidently,

$$l\,f(t) = \int_0^t f(\tau)\,d\tau - \chi_\tau\left\{\int_0^t f(\tau)\,d\tau\right\}.$$

In [1], it is shown that the operation

$$(f * g)(t) = \chi_\tau\left\{\int_\tau^t f(t-\sigma+\tau)\,g(\sigma)\,d\sigma\right\} \tag{1}$$

is a bilinear, commutative and associative operation such that

$$lf = \{1\} * f.$$

1.2. Convolutions for the Square of the Differentiation Operator. Let us consider the space $C[0,a]$ of the continuous functions on $[0,a]$.

The simplest nonlocal BVP for d^2/dx^2 in $C[0,a]$ is given by

$$y'' = f(x), \quad y(0) = 0, \quad \Phi\{y\} = 0,$$

where Φ is a linear functional on $C^1[0,a]$. In order it to have a solution, it is necessary to assume $\Phi\{x\} \neq 0$. For the simplicity sake, we assume that $\Phi\{x\} = 1$. The solution $y = Lf(x)$ has the explicit form

$$Lf(x) = \int_0^x (x-\xi)\ f(\xi)\,d\xi - x\,\Phi_\xi\left\{\int_0^\xi (\xi-\eta)\ f(\eta)\,d\eta\right\}.$$

In [1], it is proved that the operation

$$(f * g)(x) = -\frac{1}{2}\,\Phi_\xi \left\{ \int_0^\xi h(x,\eta)\,d\eta \right\}, \tag{2}$$

where

$$h(x,\eta) = \int_x^\eta f(\eta + x - \zeta)\,g(\zeta)\,d\zeta - \int_{-x}^{\eta} f(|\eta - x - \zeta|)\,g(|\zeta|)\,\mathrm{sign}\,(\eta - x - \zeta)\,\zeta\,d\zeta,$$

is a bilinear, commutative and associative operation and $Lf(x) = \{x\} * f$.

If we consider the operations (1) and (2) simultaneously, we write:

$$(f \overset{(t)}{*} g)(t) = \chi_\tau \left\{ \int_\tau^t f(t + \tau - \sigma)\,g(\sigma)\,d\sigma \right\},$$

$$(f \overset{(x)}{*} g)(x) = -\frac{1}{2}\Phi_\xi \left\{ \int_0^\xi h(x,\eta)\,d\eta \right\}.$$

2. TWO-DIMENSIONAL CONVOLUTIONS. OPERATIONAL CALCULI FOR l AND L

The idea of a multivariate operational calculus is the following. Let $u = \{u(x,t)\}$ and $v = \{v(x,t)\}$ be arbitrary functions from the space $C = C([0,\infty) \times [0,a])$. We introduce a bilinear, commutative and associative operation $u * v$ in C such that the operators l and L are multipliers of the convolution algebra $(C, *)$ of the form $lu = \{1\} \overset{t}{*} u$ and $Lu = \{x\} \overset{x}{*} u$.

Theorem. *The operation*

$$\{u(x,t)\} * \{v(x,t)\} = -\frac{1}{2}\tilde{\Phi}_\xi \chi_\tau \{h(x,t;\xi,\tau)\} \tag{3}$$

with

$$h(x,t;\xi,\tau) = \int_\xi^x \int_\tau^t u(x + \xi - \eta, t + \tau - \sigma)\,v(\eta,\sigma)\,d\sigma d\eta -$$
$$- \int_{-\xi}^x \int_\tau^t u(|x - \xi - \eta|, t + \tau - \sigma)\,v(|\eta|,\sigma)\,\mathrm{sign}\,[(x - \xi - \eta)\eta]\;d\sigma\,d\eta$$

and with the functional $\tilde{\Phi}_\xi = \Phi \circ \int_0^\xi$, *is a convolution of the operators* L *and* l *in* $C(\Delta)$ *(where* $\Delta = (0,a] \times [0,\infty)$*), for which* $L\,l\,u = \{x\} * u$. *The operators* $lu = \{1\} \overset{(t)}{*} u(x,t)$ *and* $Lu = \{x\} \overset{(x)}{*} u(x,t)$ *are multipliers of this operation.*

This theorem gives us an operation $(u * v)(x,t)$ in $C(\Delta)$, which is a convolution of each of both operators l and L.

The basic formulae of the operational calculus for l and L are

$$\frac{\partial u}{\partial t} = su - [\chi_\tau\{u(x,\tau)\}]_t \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = Su - [\Phi_\xi\{u(\xi,t)\}]_x \,, \tag{4}$$

where the indices t and x mean that the corresponding functions of t and x are considered as "partial" numerical operators.

3. THE DUHAMEL-TYPE REPRESENTATIONS OF SOLUTIONS OF BVPs

Using multivariate operational calculi, algebraization of each BVP can be made. An explicit Duhamel-type representation of the solution is obtained, using one special solution satisfying simple boundary value conditions. The general solution is obtained as a multivariate convolution product of this solution with the right-hand side function or with a given boundary value function (see [2] and [3]). This representation can be used successfully for numerical computation of the solution. Representation formulae for the solutions of local and nonlocal BVPs for the heat equation, the wave equation and the equation of a supported beam are considered in [2] and [3].

One of the problems, solved by our approach, is the following nonlocal BVP for the wave equation:

$$u_{tt} = u_{xx}, \quad 0 < x < 1, \quad 0 < t < \infty,$$
$$u(0,t) = 0, \quad \int_0^1 u(\xi,t)\, d\xi = 0,$$
$$u(x,0) = f(x), \quad u_t(x,0) = g(x).$$

We obtained the following representation of the solution (for $f(x) = 0$):

$$u(x,t) = -2\int_0^x \Omega_x(x-\xi,t)g'(\xi)\,d\xi - \\ - \int_x^1 \Omega_x(1+x-\xi,t)g'(\xi)\,d\xi + \int_{-x}^1 \Omega_x(1-x-\xi,t)g'(|\xi|)\,d\xi.$$

The special solution $\Omega(x,t)$ was obtained using the Fourier method.

This solution is applied for the study of vibration of a real system [4].

The algorithms for solving the considered local and nonlocal BVPs are implemented, using the computer algebra system *Mathematica*. Visualization of the solutions is provided.

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