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КОМПЬЮТЕРНЫЕ ТЕХНОЛОГИИ В ФИЗИКЕ

STRUCTURE OF TOPOLOGICAL SOLITONS IN NONLINEAR SPINOR MODEL

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The 16-spinor field is suggested to unify Skyrme and Faddeev models describing baryons and leptons as topological solitons.

Для объединения моделей Скирма и Фаддеева, описывающих соответственно барионы и лептоны как топологические солитоны, предлагается использовать 16-спинорное поле.

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1. INTRODUCTION. 16-SPINOR FIELD MODEL

In the popular Skyrme model (SM) [1] baryons are considered as topological solitons (TS), the baryon number $\mathbb{B}$ being identified with the degree type topological charge: $\mathbb{B} = \deg(S^3 \to S^3)$ serving as a generator of the homotopy group $\pi_3(S^3) = \mathbb{Z}$. Similarly, in the Faddeev model (FM) [2] leptons are considered as TS endowed with the Hopf invariant Q_H, the latter playing the role of the lepton number $\mathbb{L} = Q_H$ serving as a generator of the homotopy group $\pi_3(S^2) = \mathbb{Z}$. To unify these two approaches, we introduce 16-spinor field $\Psi = \Psi_1 \oplus \Psi_2$, Ψ_i, $i = 1, 2$, being 8-spinors, and the two kinds of internal Pauli matrices: $\boldsymbol{\lambda} = I_4 \otimes \boldsymbol{\sigma} \otimes I_2$, $\boldsymbol{\Lambda} = I_8 \otimes \boldsymbol{\sigma}$, where $\boldsymbol{\sigma}$ stands for usual Pauli matrices and I_n — for n-dimensional unit matrix. As was shown in [3], for each 8-spinor ψ the following Brioschi identity [4] holds:

$$j_\mu j^\mu - \tilde{j}_\mu \tilde{j}^\mu = s^2 + p^2 + \mathbf{v}^2 + \mathbf{a}^2, \tag{1}$$

where standard bilinear spinor quantities are introduced: $s = \bar{\psi}\psi$, $p = i\overline{\psi}\gamma_5\psi$, $\mathbf{v} = \overline{\psi}\,\boldsymbol{\lambda}\psi$, $\mathbf{a} = i\overline{\psi}\gamma_5\boldsymbol{\lambda}\psi$, $j_\mu = \overline{\psi}\gamma_\mu\psi$, $\tilde{j}_\mu = \overline{\psi}\gamma_\mu\gamma_5\psi$, $\overline{\psi} = \psi^+\gamma_0$, with γ_μ, $\mu = 0, 1, 2, 3$, being Dirac matrices. In view of the identity (1), one can use the special structure of the Higgs potential V implying the spontaneous symmetry breaking:

$$V = \frac{\sigma^2}{8}\left(j_\mu j^\mu - \varkappa_0^2\right)^2, \tag{2}$$

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with σ and $\varkappa_0$ being parameters of the model and the natural boundary condition at space infinity being imposed:

$$\lim_{r\to\infty} j_\mu j^\mu = \varkappa_0^2, \quad r = |\mathbf{x}|. \tag{3}$$

As follows from (1) and (3), the two kinds of TS are possible [3] due to the choice of S^2- or S^3-manifolds as phase spaces, with the vacuum fixed point being determined by (3). In fact, S^2-manifold is determined by the $O(3)$-invariant $\mathbf{v}^2$ and corresponds to FM or lepton sector (LS). Similarly, S^3-manifold is determined by the chiral invariant $s^2 + \mathbf{a}^2$ and corresponds to SM or baryon sector (BS).

However, the unification of SM and FM supposes that the vacuum state Ψ_0 should be universal, that can be realized within the scope of 16-spinor field model, if one supposes that $\Lambda\Psi_0 = 0$, $\Lambda = (1 - \Lambda_3)/2$, and S^2-manifold is determined by the invariant $\mathbf{V}^2$, $\mathbf{V} = \overline{\Psi}\boldsymbol{\Lambda}\Psi$, S^3-manifold being unchanged. To choose LS, one remarks that condition $\mathbf{a} = \imath(\overline{\Psi}\gamma_5\boldsymbol{\lambda}\Psi) = 0$ implies $\mathbb{B} = 0$ and can be satisfied if one supposes the mirror symmetry (MS) of LS:

$$\Psi \to \gamma_0\Psi. \tag{4}$$

Introducing the 16-spinor of the form

$$\Psi = \oplus_{j=1}^2(\varphi_j \oplus \chi_j \oplus \xi_j \oplus \zeta_j), \tag{5}$$

where φ_j, χ_j, ξ_j, ζ_j stand for 2-spinors, one finds from invariance condition (4) that $\varphi_j = \chi_j$, $\xi_j = \zeta_j$ for the Weyl representation of γ-matrices; i.e., one gets an effective 8-spinor. In this case $\mathbf{V} = \overline{\Psi}\boldsymbol{\Lambda}\Psi \neq 0$ and, in accordance with the Brioschi identity,

$$\frac{j_\mu j^\mu - \mathbf{V}^2}{16} = \left(|\varphi_2|^2 + |\zeta_2|^2\right)\left(|\varphi_1|^2 + |\zeta_1|^2\right) - |\varphi_1^+\varphi_2 + \zeta_1^+\zeta_2|^2 \geqslant 0$$

due to the Schwarz inequality.

Now let us choose BS, taking into account the charge independence of strong interactions, or MS of the isotopic space:

$$\Psi \to \gamma_0\gamma_5\lambda_2\Psi^*. \tag{6}$$

Inserting (5) into the invariance condition (6) one finds for BS $\xi_j = \imath\sigma_2\varphi*_j$, $\zeta_j = \imath\sigma_2\chi*_j$; i.e., one gets an effective 8-spinor again. In this case $p = \mathbf{v} = V_2 = 0$, i.e., $\mathbb{L} = 0$, but $s \neq 0$, $\mathbf{a} \neq 0$ and, in accordance with the Brioschi identity,

$$\frac{s^2 + \mathbf{a}^2}{16} = |\varphi_1^+\chi_1 + \varphi_2^+\chi_2|^2 + |\varphi_1^T\sigma_2\chi_1 + \varphi_2^T\sigma_2\chi_2|^2 \leqslant \frac{j_\mu j^\mu}{16}.$$

Thus, the structure (2) of the Higgs potential is consistent with the topology constraints of LS or BS.

In view of these arguments, let us consider the following Lagrangian density [3]:

$$\mathcal{L}_{\rm spin} = \frac{1}{2\lambda^2}\overline{\partial_\mu\Psi}\gamma^\nu j_\nu\partial^\mu\Psi + \frac{\epsilon^2}{4}f_{\mu\nu}f^{\mu\nu} - V, \tag{7}$$

where $f_{\mu\nu} = (\overline{\Psi}\gamma^\alpha\partial_{[\mu}\Psi)(\overline{\partial_{\nu]}\Psi}\gamma_\alpha\Psi)$ is the antisymmetric tensor of Skyrme–Faddeev type and λ, ϵ are constant parameters. The first term in (5) is similar to the sigma-model one and includes the projector $P = \gamma^0\gamma^\nu j_\nu$ on the positive energy states.

2. INTERACTION WITH PHYSICAL VECTOR FIELDS

Due to the general gauge invariance principle, the group of phase transformations of Ψ with some charge generator Γ_e and left, right isotopic rotations with generators $P_{L,R}\lambda^a\Lambda/2$, $a = 1, 2, 3$, $P_{L,R} = (1 + \gamma_5)/2$, give rise to the interactions with the electromagnetic field A_μ and left, right Yang–Mills fields $A_\mu^{aL,R}$, respectively, through the extention of the derivative $\partial_\mu\Psi \to D_\mu\Psi = \partial_\mu\Psi - \imath e_0\Gamma_e A_\mu\Psi + (A_\mu^L + A_\mu^R)\Psi$, where we adopt $\Gamma_e = \lambda_3\Lambda$, $A_\mu^{L,R} = P_{L,R}e_{1L,R}A_\mu^{aL,R}\lambda^a\Lambda/(2\imath)$. Here e_0, e_{1L}, e_{1R} stand for corresponding coupling constants. However, the Yang–Mills fields being responsible for strong interactions should not give any contribution to LS. In view of MS (4), this is possible only if the sum $A_\mu^{aL} + A_\mu^{aR}$ is proportional to γ_5, or the following constraint holds:

$$e_{1L}A_\mu^{aL} + e_{1R}A_\mu^{aR} = 0. \tag{8}$$

In view of (8), the extended derivative reads

$$D_\mu\Psi = \partial_\mu\Psi - \imath e_0\lambda_3\Lambda A_\mu\Psi + g_0\gamma_5\mathbb{A}_\mu\Psi, \tag{9}$$

where $g_0 = e_{1L} = e_{1R}$, $\mathbb{A}_\mu = \lambda^a\Lambda A_\mu^{aL}/(2\imath)$, with the standard Yang–Mills Lagrangian taking the form

$$\mathcal{L}_{\rm YM} = \frac{1}{32\pi}\mathrm{Sp}\left\{(\partial_\mu\mathbb{A}_\nu - \partial_\nu\mathbb{A}_\mu)(\partial^\mu\mathbb{A}^\nu - \partial^\mu\mathbb{A}^\nu) + g_0^2[\mathbb{A}_\mu, \mathbb{A}_\nu][\mathbb{A}^\mu, \mathbb{A}^\nu]\right\}. \tag{10}$$

It is worth-while to stress that our spinor model admits a new possibility to derive usually adopted hypothesis of the electric charge Q_e quantization, i. e., $Q_e = ne_0$, $n \in \mathbb{Z}$. To this end, let us introduce the modified Maxwell Lagrangian density:

$$\mathcal{L}_{\rm em} = -\frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu}f(u), \tag{11}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $f(u)$ is some unknown function of the special Lorentz and dilatation invariant

$$u = (n_\mu A^\mu)^2(E_\nu E^\nu)^{-1/2}. \tag{12}$$

Here $E_\nu = n^\mu F_{\mu\nu}$, the unit time-like vector n^μ being identified with the normalized Dirac current: $n^\mu = j^\mu/j$, $j = (j_\nu j^\nu)^{1/2}$. In fact, for the static soliton with the charge q one finds the equation for the scalar potential $\Phi = A^0$:

$$F\ \mathrm{div}\,[\mathbf{E}(2f - uf'(u))] = 2uf'(u)E^2, \quad \mathbf{E} = -\bigtriangledown F. \tag{13}$$

At large distances $r \to \infty$, one gets $\Phi = (q/r)[1 + \mathcal{O}(1/r)]$, $u = q[1 + \mathcal{O}(1/r^2)]$ and the left-hand side of (13) behaves as $\mathcal{O}(1/r^5)$, or $f'(q) = 0$. The latter equation being considered as the quantization condition for Q_e, one can suggest the simplest choice of the function $f(u) = 1 + \mu_0\sin^2[\pi u/(2e_0)]$, where μ_0 stands for some small constant.

Finally, let us consider small excitations of our soliton configuration near the vacuum: $\Psi = \Psi_0 + \xi$ as $|\mathbf{x}| \to \infty$. Then the linearized equation for ξ after the substitution $\xi = k\Psi_0$ takes the form

$$\partial_\mu\partial^\mu k + \frac{1}{2}M_0^2(k^* + k) = 0, \quad M_0 \equiv 2\sigma\lambda\varkappa_0. \tag{14}$$

Thus, from (14) one derives the equations for real and imaginary parts of $k = k_1 + \imath k_2$:

$$\partial_\mu \partial^\mu k_1 + M_0^2 k_1 = 0, \quad \partial_\mu \partial^\mu k_2 = 0. \tag{15}$$

According to (15), our model admits two types of vacuum excitations: massive and massless ones. For massive soliton the excitation k_1 could be interpreted as the wave function if the parameter M_0 were replaced by the real mass M of the soliton. To satisfy this condition, we first introduce the interaction with the gravitational field via the new extended derivative $\bigtriangledown_\mu \Psi = (D_\mu - \Gamma_\mu)\Psi$, where $\Gamma\mu$ stands for spinor connection. Then we generalize the Higgs potential (15) by replacing σ^2 with the new invariant:

$$\sigma^2 = -\frac{2(8/7)^3 D^3}{\lambda^2 G^2 K^2 \varkappa_0^2}, \tag{16}$$

where G is the Newton gravitational constant, $D = \overline{\bigtriangledown_\mu \Psi} \gamma^\alpha j_\alpha \bigtriangledown^\mu \Psi$, and K is the Kraichnan invariant constructed with the help of the Riemannian curvature tensor: $K = R_{\mu\nu\sigma\lambda} R^{\mu\nu\sigma\lambda}/48$. The validity of this choice of the invariants in the Higgs potential becomes evident at large distances, since in view of the Schwarzschild metric $D = -7r_g^2 \varkappa_0^2/(16r^4)$, $K = r_g^2/r^6$, where $r_g = GM$ and natural units $\hbar = c = 1$ are used.

CONCLUSIONS

The resulting Lagrangian of our 16-spinor model reads: $\mathcal{L} = \mathcal{L}_{\rm spin} + \mathcal{L}_{\rm em} + \mathcal{L}_{\rm YM} + \mathcal{L}_g$, with $\mathcal{L}_g$ being the Einstein gravitational Lagrangian. It is worth-while to stress that the gravitational field plays an important role in our model, since the wave-particle duality principle of quantum mechanics has the gravitational origin and the vacuum excitation ξ can be considered as the wave function for the soliton center in the special stochastic representation [5, 6].

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