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AUXILIARY REPRESENTATIONS OF LIE ALGEBRAS AND THE **BRST** CONSTRUCTIONS

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1   Introduction

The BRST quantization procedure for a system of the first class constraints [1] – [2] is straightforward. By the definition, the first class constraints form a closed algebra with respect to the commutators (the Poisson brackets). For simplicity we consider only linear algebras – Lie algebras of constraints.

More general systems include the second class constraints as well, whose commutators contain terms which are nonzero on mass shell (on the subspace where all constraints vanish). In the simplest cases these terms are   numbers or central charges, but sometimes, they are operators which act nontrivially on the space of the physical states. Moreover, the commutators between these operators and the constraints can be nontrivial. In some cases the total system of the constraints and the operators mentioned above form a Lie algebra.

So, in such cases we have a system of operators which form a Lie algebra, but the physical meaning of different operators is different. Some of them play the role of constraints and annihilate the physical states, others are nonzero and simply transform the physical states into other ones. It means, that in the BRST approach for the description of the corresponding physical system we cannot use the standard BRST charge for the given Lie algebra. Instead, we have to construct the nilpotent BRST charge in a manner, that some of the operators play the role of the first class constraints, others are second class constraints and the others do not imply any conditions on the physical space of the system.

In this letter we demonstrate the possibility of   different BRST constructions for the given system of generators, which form a given Lie algebra and have different physical meaning.

In the Section 2 we discuss the general method of the BRST quantization, when some of the Cartan generators are excluded from the total system of constraints. The method is based on the introduction of an auxiliary representations of the algebra under consideration, in the way that these representations effectively lead to the desired properties of initial generators: some of them are of the first class, some - of the second class and rest of the generators (only the Cartan ones) do not imply any equations on the space of physical states. The auxiliary representations of the algebra are constructed by means of some additional degrees of freedom, which are usually exploited for the quantization of the systems with the second class constraints [3] – [5].

In the Section 3 we describe the construction of auxiliary representations of the algebra by means of the method of induced representations. The resulting operators act in a new Fock space, generated by a set of additional creation and annihilation operators. The construction automatically destroys the hermiticity properties of the generators. The consequence of this fact is that the BRST charge $\mathcal{Q}$ becomes non – hermitian: $\mathcal{Q}^+$ is not equal to $\mathcal{Q}$.

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As an example of this general construction the case of the $so(3,2)$ algebra is considered in detail.

In the Section 4 we show, how these hermiticity properties are restored by introduction of new scalar product with some kernel K in this auxiliary Fock space. As a consequence, the new operator KQ becomes hermitian and can be used for the construction of the lagrangians, which are gauge invariant due to the nilpotency of (non-hermitian) BRST charge Q.

2   The general method

In this section, we describe the method of the BRST construction, which leads to the desirable division of the generators of a given Lie algebra into the first and second class constraints. Let $\hat{H}^i$, $(i = 1, ..., k)$ and $\hat{E}^\alpha$ be the Cartan generators and root vectors of the algebra with the following commutation relations

$$\left[\hat{H}^i, \hat{E}^\alpha\right] = \alpha(i)\hat{E}^\alpha, \tag{2.1}$$

$$\left[\hat{E}^\alpha, \hat{E}^{-\alpha}\right] = \alpha^i \hat{H}^i, \tag{2.2}$$

$$\left[\hat{E}^\alpha, \hat{E}^\beta\right] = N^{\alpha\beta} \hat{E}^{\alpha+\beta}. \tag{2.3}$$

Roots $\alpha(i)$ and parameters α^i, $N^{\alpha\beta}$ are structure constants of the algebra in the Cartan – Weyl basis. Our goal is to construct nilpotent BRST charge, which after quantization leads to the following conditions: all positive root vectors $\hat{E}^\alpha$ $(\alpha > 0)$ of the algebra annihilate the physical states. Contrary, the operators $\hat{H}^i$ which form the Cartan subalgebra may or may not be constraints, depending on the physical nature of these operators.

The simplest case, when all Cartan generators annihilate the physical states, is well known. We introduce the set of anticommuting variables η_i, η_α, $\eta_{-\alpha} = \eta_\alpha^+$, having ghost number one and corresponding momenta $\mathcal{P}_i, \mathcal{P}_{-\alpha} = \mathcal{P}_\alpha^+, \mathcal{P}_\alpha$, with the commutation relations:

$$\{\eta_i, \mathcal{P}_k\} = \delta_{ik}, \ \{\eta_\alpha, \mathcal{P}_{-\beta}\} = \{\eta_{-\alpha}, \mathcal{P}_\beta\} = \delta_{\alpha\beta} \tag{2.4}$$

we define the "ghost vacuum" as

$$\eta_\alpha|0\rangle = \mathcal{P}_\alpha|0\rangle = \mathcal{P}_i|0\rangle = 0 \tag{2.5}$$

for positive roots α. The BRST charge for the Cartan – Weyl decomposition of the algebra has a standard form

$$Q = \sum_i \eta_i \hat{H}^i + \sum_{\alpha>0} \left(\eta_\alpha \hat{E}^{-\alpha} + \eta_{-\alpha} \hat{E}^\alpha\right) - \frac{1}{2} \sum_{\alpha\beta} N^{\alpha\beta} \eta_{-\alpha} \eta_{-\beta} \mathcal{P}_{\alpha+\beta} +$$

$$\sum_{\alpha>0,i} \left\{\alpha(i) \left(\eta_i \eta_\alpha \mathcal{P}_{-\alpha} - \eta_i \eta_{-\alpha} \mathcal{P}_\alpha\right) + \alpha^i \eta_\alpha \eta_{-\alpha} \mathcal{P}_i\right\}. \tag{2.6}$$

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The physical states are then the cohomology classes of the BRST operator.

The quantization in this case is similar to the quantization à la Gupta – Bleuler, because physical states satisfy equations

$$(\hat{H}^i + \sum_{\alpha>0} \alpha(i))|Phys\rangle = 0, \quad \hat{E}^\alpha|Phys\rangle = 0$$

only for positive values of α. The appearance of the $\sum \alpha(i)$ in the quantization conditions does not cause problems since these terms can be absorbed after the redefinition of $\hat{H}^i$ as we shall see below.

The situation becomes different if some of the Cartan operators $\hat{H}^i$, say $\hat{H}^{i_l}$, $l = 1, 2, ...N$ are nonvanishing from the physical reasons. In this case the following method can be used.

First of all we construct some auxiliary representation for the generators $\hat{H}^i$, $\hat{E}^\alpha$ of the algebra in terms of additional creation and annihilation operators. The only condition for this representation is that it depends on some parameters h^n. The total number of these parameters is equal to the number of the Cartan generators, which are nonzero in the physical sector. In what follows, we consider the realizations of the algebra with a linear dependence of the Cartan generators on these parameters: $H^m(h) = \tilde{H}^m + c_n^m h^n$, where c_n^m are some constants. The h^n dependence of other generators can be arbitrary. In the next section we describe the general method of construction of such representations. Here we simply assume that they exist.

The next step is to consider the realization of the algebra as a sum of "old" and "new" generators

$$\mathcal{H}^i = \hat{H}^i + H^i(h), \quad \mathcal{E}^\alpha = \hat{E}^\alpha + E^\alpha(h). \tag{2.7}$$

The BRST charge for the total system has the same form as (2.6), with modified generators:

$$\begin{aligned} \mathcal{Q} &= \sum_i \eta_i \mathcal{H}^i + \sum_{\alpha>0} \left(\eta_\alpha \mathcal{E}^{-\alpha} + \eta_{-\alpha}\mathcal{E}^\alpha \right) - \frac{1}{2} \sum_{\alpha\beta} N^{\alpha\beta}\eta_{-\alpha}\eta_{-\beta}\mathcal{P}_{\alpha+\beta} + \\ &\quad \sum_{\alpha>0,i} \left\{ \alpha(i)\left(\eta_i\eta_\alpha\mathcal{P}_{-\alpha} - \eta_i\eta_{-\alpha}\mathcal{P}_\alpha \right) + \alpha^i\eta_\alpha\eta_{-\alpha}\mathcal{P}_i \right\}. \end{aligned} \tag{2.8}$$

The ghost variables η_{i_l}, correspond to the set of nonvanishing generators $\hat{H}^{i_l}$ and therefore one needs to remove the η_{i_l} dependence

$$\mathcal{Q}_{i_l} = \eta_{i_l}\{\hat{H}^{i_l} + \tilde{H}^{i_l} + c_n^{i_l}h^n + \sum_{\beta>0} \beta(i_l)\left(\eta_\beta\mathcal{P}_{-\beta} - \eta_{-\beta}\mathcal{P}_\beta \right)\} \tag{2.9}$$

from the BRST charge. For this purpose consider an auxiliary N - dimensional space with coordinates x_{i_l} and conjugated momenta p^{i_l}, where $c_n^{i_l}h^n = p^{i_l}$:

$$\left[x_{i_l}, p^{i_n} \right] = i\delta_{i_l}^{i_n}. \tag{2.10}$$

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After the similarity transformation, which corresponds to the dimensional reduction [6]

$$\tilde{Q} = e^{i\pi^{i_l}x_{i_l}} Q e^{-i\pi^{i_l}x_{i_l}}, \tag{2.11}$$

where

$$\pi^{i_l} = \hat{H}^{i_l} + \tilde{H}^{i_l} + \sum_{\beta>0} \beta(i_l)\left(\eta_\beta \mathcal{P}_{-\beta} - \eta_{-\beta}\mathcal{P}_\beta\right) \tag{2.12}$$

the transformed BRST charge $\tilde{Q}$ does not depend on the ghost variables η_{i_l}. All parameters p^{i_l} in the BRST charge are replaced by the corresponding operators $-\pi^{i_l}$. The transformation (2.11) does not change the nilpotency property of the BRST charge. It means that the $\mathcal{P}_{i_l}$ independent part $\tilde{Q}_0$ of the total charge $\tilde{Q}$ is nilpotent as well. Moreover, as a consequence of the nilpotency of $\tilde{Q}$ all coefficients at the corresponding antighost operators $\mathcal{P}_{i_l}$ commute with $\tilde{Q}_0$. One can show that the quantization with the help of the BRST operator $\tilde{Q}_0$ will lead to the desirable reduced system of constraints on the physical states.

3 Construction of auxiliary representations of the algebra

Consider the highest weight representation of the algebra under consideration with the highest weight vector $|\Phi\rangle_V$, annihilated by the positive roots

$$E^\alpha |\Phi\rangle_V = 0 \tag{3.1}$$

and being the proper vector of the Cartan generators

$$H^i |\Phi\rangle_V = h^i |\Phi\rangle_V. \tag{3.2}$$

As it was shown in [7] the representations of this algebra can be (in principle) constructed by means of the so-called Gelfand – Tsetlin schemes [8]. However difficulties in such construction arise, if one considers algebras, different from the simplest ones of rank 1.

In this section we describe another method, based on the construction given in [9]. The representation which is given by (3.1) and (3.2) in the mathematical literature is called the Verma module [10]. Following the Poincare – Birkhoff – Witt theorem, the basis space of this representation is given by vectors

$$|n_1, n_2, \ldots, n_r\rangle_V = (E^{-\alpha_1})^{n_1}(E^{-\alpha_2})^{n_2}\ldots(E^{-\alpha_r})^{n_r}|\Phi\rangle_V \tag{3.3}$$

where $\alpha_1, \alpha_2, \ldots \alpha_r$ is some ordering of positive roots and $n_i \in N$.

Using the commutation relations of the algebra and the formula

$$AB^n = \sum_{k-1}^{n} \binom{n}{k} B^{n-k} [[\ldots [A, B], B] \ldots]$$

one can calculate the explicit form of the Verma module. In [9] it was shown that, making use of the map

$$|n_1, n_2, \ldots, n_r\rangle_V \longleftrightarrow |n_1, n_2, \ldots, n_r\rangle, \qquad (3.4)$$

where $|n_1, n_2, \ldots, n_r\rangle$ are base vectors of the Fock space

$$|n_1, n_2, \ldots, n_r\rangle = (b_1^+)^{n_1} (b_2^+)^{n_2} \ldots (b_r^+)^{n_r} |0\rangle. \qquad (3.5)$$

generated by creation and annihilation operators b_i^+, b_i $i = 1, 2, \ldots, r$ with the standard commutation relations

$$\left[b_i, b_j^+ \right] = \delta_{ij}, \qquad (3.6)$$

the Verma module can be rewritten as polynomials in creation operators on the Fock space.

As an explicit example of the construction given above let us consider the representations of $so(3, 2)$ algebra, which can be used for the description of the higher spin fields. In this case commutation relations between the corresponding generators $L_1^+, L_{12}^+, L_2^+, T^+, L_1, L_{12}, L_2, T, H_1, H_2$ are given by

$$\begin{aligned}
&[L_1, L_1^+] = H_1, \quad [L_2, L_2^+] = H_1, \quad [L_{12}, L_{12}^+] = H_1 + H_2, \\
&[T^+, L_1^+] = -L_{12}^+, \quad [T^+, L_{12}^+] = -2L_2^+, \quad [T, L_1^+] = 0, \\
&[T, L_1] = L_{12}, \quad [T, L_{12}] = -2L_2, \quad [T, T^+] = H_1 - H_2, \\
&[L_2, L_{12}^+] = -T, \quad [L_2, T^+] = -L_{12}, \quad [T, L_2^+] = -L_{12}^+, \qquad (3.7) \\
&[L_1, L_{12}^+] = -T^+, \quad [L_{12}, L_1^+] = -T, \quad [L_{12}, L_2^+] = -T^+, \\
&[T, L_{12}] = -2L_1^+, \quad [T^+, L_{12}] = 2L_1.
\end{aligned}$$

In this case we take the representation space the following space of vectors

$$|n\rangle_V = |n_1, n_2, n_3, n_4\rangle_V = (L_1^+)^{n_1} (L_{12}^+)^{n_2} (L_2^+)^{n_3} (T^+)^{n_4} |\Phi\rangle_V$$

and after the simple calculations one obtains

$$\begin{aligned}
&L_1^+ |n\rangle_V = |n_1 + 1, n_2, n_3, n_4\rangle_V, \quad L_{12}^+ |n\rangle_V = |n_1, n_2 + 1, n_3, n_4\rangle_V, \\
&L_2^+ |n\rangle_V = |n_1, n_2, n_3 + 1, n_4\rangle_V, \\
&T^+ |n\rangle_V = |n_1, n_2, n_3, n_4 + 1\rangle_V - n_1 |n_1 - 1, n_2 + 1, n_3, n_4\rangle_V \\
&\quad - 2n_2 |n_1, n_2 - 1, n_3 + 1, n_4\rangle_V, \\
&H_1 |n\rangle_V = (2n_1 + n_2 - n_4 + h_1)|n\rangle_V,
\end{aligned}$$

$$H_2|n\rangle_V = (n_2 + 2n_3 + n_4 + h_2)|n\rangle_V, \qquad (3.8)$$
$$L_1|n\rangle_V = (n_1 + n_2 - n_4 + h_1 - 1)n_1|n_1 - 1, n_2, n_3, n_4\rangle_V$$
$$-n_2|n_1, n_2 - 1, n_3, n_4 + 1\rangle_V + n_2(n_2 - 1)|n_1, n_2 - 2, n_3 + 1, n_4\rangle_V,$$
$$L_{12}|n\rangle_V = (2n_1 + n_2 + 2n_3 + h_1 + h_2 - 1)n_2|n_1, n_2 - 1, n_3, n_4\rangle_V$$
$$-n_3|n_1, n_2, n_3 - 1, n_4 + 1\rangle_V + n_3 n_1|n_1 - 1, n_2 + 1, n_3 - 1, n_4\rangle_V$$
$$+n_4(n_4 - 1)n_1|n_1 - 1, n_2, n_3, n_4 - 1\rangle_V$$
$$+(h_2 - h_1)n_4 n_1|n_1 - 1, n_2, n_3, n_4 - 1\rangle_V,$$
$$L_2|n\rangle_V = (n_2 + n_3 + n_4 + h_2 - 1)n_3|n_1, n_2, n_3 - 1, n_4\rangle_V$$
$$+n_2(n_2 - 1)|n_1 + 1, n_2 - 2n_3, n_4\rangle_V$$
$$+(h_2 - h_1 + n_4 - 1)n_4 n_2|n_1, n_2 - 1, n_3, n_4 - 1\rangle_V,$$
$$T|n\rangle_V = (h_1 - h_2 - n_4 + 1)n_4|n_1, n_2, n_3, n_4 - 1\rangle_V$$
$$-2n_2|n_1 + 1, n_2 - 1, n_3, n_4\rangle_V - n_3|n_1, n_2 + 1, n_3 - 1, n_4\rangle_V.$$

It can be seen that the action of the following operators in the Fock space

$$L_1^+ = b_1^+, \quad L_{12}^+ = b_2^+, \quad L_2^+ = b_3^+,$$
$$T^+ = b_4^+ - b_2^+ b_1 - 2b_3^+ b_2,$$
$$T = (h_1 - h_2 - b_4^+ b_4)b_4 - 2b_1^+ b_2 - b_2^+ b_3,$$
$$H_1 = 2b_1^+ b_1 + b_2^+ b_2 - b_4^+ b_4 + h_1,$$
$$H_2 = b_2^+ b_2 + 2b_3^+ b_3 + b_4^+ b_4 + h_2, \qquad (3.9)$$
$$L_1 = (b_1^+ b_1 + b_2^+ b_2 - b_4^+ b_4 + h_1)b_1 - b_4^+ b_2 + b_3^+ b_2 b_2,$$
$$L_{12} = (2b_1^+ b_1 + b_2^+ b_2 + 2b_3^+ b_3 + h_1 + h_2)b_2 + b_4^+ b_4 b_4 b_1$$
$$+b_2^+ b_3 b_1 - b_4^+ b_3 + (h_2 - h_1)b_4 b_1,$$
$$L_2 = (b_2^+ b_2 + b_3^+ b_3 + b_4^+ b_4 + h_2)b_3$$
$$+(h_2 - h_1)b_4 b_2 + b_1^+ b_2 b_2 + b_4^+ b_4 b_4 b_2$$

is identical to the expressions (3.8) for the Verma module.

As it can be concluded from the realization of the $so(3,2)$ algebra, the generators, corresponding to opposite roots are not hermitian conjugated to each other. Indeed, following the usual rules of hermitian conjugation for creation and annihilation operators

$$(b_i)^+ = b_i^+, \quad (b_i^+)^+ = b_i \qquad (3.10)$$

the operator conjugated to L_1^+ is not equal to the operator L_1 since $(L_1^+)^+ = (b_1^+)^+ = b_1 \neq L_1$. The same statement is true for all other pairs of root generators. It means that the BRST charge $\mathcal{Q}$, constructed with the help of these operators, though being nilpotent, is not hermitian. This causes the serious problems, because the BRST gauge invariance has been lost and consequently the lagrangians of the form $L \sim \langle \Psi | \mathcal{Q} | \Psi \rangle$ [6], [11] – [13] are no longer gauge invariant.

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4 The restoration of the hermiticity properties

The situation, when the generators corresponding to the opposite roots are not mutually hermitian conjugated holds for any algebra under consideration and is a consequence of the method used for the construction of the corresponding representations.

The reason is that if we consider the usual scalar product in the Fock space with basis (3.5), we find that these vectors form the orthogonal (not orthonormal) basis. At the same time the corresponding vectors (3.3) in Verma module are not orthogonal. For example, the scalar product of two vectors

$$|\Phi_1\rangle_V = L_1^+ T^+ |\Phi\rangle_V, \quad |\Phi_2\rangle_V = L_{12}^+ |\Phi\rangle_V \tag{4.1}$$

is different from zero

$$_V\langle\Phi_1|\Phi_2\rangle_V = {}_V\langle\Phi|TL_1 L_{12}^+|\Phi\rangle_V = (h_2 - h_1), \tag{4.2}$$

where we assumed that $_V\langle\Phi|\Phi\rangle_V = 1$. Therefore the correspondence between these two spaces is not complete because of the difference in the scalar products of pairs of corresponding vectors.

The idea how to improve the situation lies on the modification of the scalar product in the auxiliary Fock space. The standard scalar product of two vectors of type (3.5), namely $|\Phi_1\rangle$ and $|\Phi_2\rangle$ is defined as

$$(\Phi_1|\Phi_2)_{st} = \langle\Phi_1|\Phi_2\rangle \tag{4.3}$$

and is calculated by transition of the annihilation operators b_i to the right by means of the commutation relations (3.6) with the subsequent use of the property $b_i|0\rangle = 0$.

Let us introduce the new scalar product in the Fock space

$$(\Phi_1|\Phi_2)_{new} = \langle\Phi_1|K|\Phi_2\rangle, \tag{4.4}$$

with a kernel K, which depends on the creation and annihilation operators b_i and b_i^+. The only condition on this new scalar product in the Fock space is that it has to coincide with the scalar product in the Verma module. Therefore taking two arbitrary vectors in the Verma module $|\Phi_1\rangle_V, |\Phi_2\rangle_V$ and corresponding vectors in the Fock space $|\Phi_1\rangle, |\Phi_2\rangle$ one has the following defining relation:

$$\langle\Phi_1|K|\Phi_2\rangle = {}_V\langle\Phi_1|\Phi_2\rangle_V. \tag{4.5}$$

According to this relation the hermiticity properties of the root generators are restored in the following sense. Let us consider the scalar product of the states $|\Phi_1\rangle_V$ and $E^\alpha|\Phi_2\rangle_V$: $_V\langle\Phi_1|E^\alpha|\Phi_2\rangle_V$. Due to hermitian properties of

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the root generators in the Verma module it coincides with the scalar product of the states $E^{-\alpha}|\Phi_1\rangle_V$ and $|\Phi_2\rangle_V$ since

$$(E^{-\alpha}|\Phi_1\rangle_V)^+ = {}_V\langle\Phi_1|E^\alpha. \tag{4.6}$$

In the Fock space the relation (4.6) looks as

$$(E^{-\alpha}|\Phi_1\rangle_V)^+ = {}_V\langle\Phi_1|(E^{-\alpha})^+. \tag{4.7}$$

So, taking the new scalar product of the pairs of corresponding vectors in the Fock space $|\Phi_1\rangle$, $E^\alpha|\Phi_2\rangle$ and $E^{-\alpha}|\Phi_1\rangle$, $|\Phi_2\rangle$ one has the following relations

$$\langle\Phi_1|KE^\alpha|\Phi_2\rangle = \langle\Phi_1|(E^{-\alpha})^+K|\Phi_2\rangle. \tag{4.8}$$

Therefore all of the root generators of the algebra under consideration satisfy the relations

$$KE^\alpha = (E^{-\alpha})^+K, \tag{4.9}$$

which play the role of hermiticity relations.

Now, consider the part of the BRST charge in the Fock space dependent on the root generators

$$\mathcal{Q}_{nonh} = \sum_{\alpha>0}\left(\eta_\alpha E^{-\alpha} + \eta_{-\alpha}E^\alpha\right) \tag{4.10}$$

being the only non – hermitian part of the BRST charge

$$\mathcal{Q}_{nonh}^+ = \sum_{\alpha>0}\left(\eta_{-\alpha}(E^{-\alpha})^+ + \eta_\alpha(E^\alpha)^+\right) \neq \mathcal{Q}_{nonh}. \tag{4.11}$$

It can be easily shown, that the following relations take place

$$\mathcal{Q}_{nonh}^+K = K\mathcal{Q}_{nonh}, \quad \mathcal{Q}_{nonh}K = K\mathcal{Q}_{nonh}^+ .$$

So one can conclude, that the total BRST charge of the form (2.8) constructed with the help of the generators (2.7) satisfies the modified hermiticity relation

$$\mathcal{Q}^+K = K\mathcal{Q}. \tag{4.12}$$

This gives the possibility to construct the lagrangians of the form $L \sim \langle\Psi|K\mathcal{Q}|\Psi\rangle$, which are gauge invariant with the following transformation rules for the field $|\Psi\rangle$

$$\delta|\Psi\rangle = \mathcal{Q}|\Psi\rangle, \quad \delta\langle\Psi| = \langle\Psi|\mathcal{Q}^+. \tag{4.13}$$

Obviously the gauge invariance is guaranteed by the nilpotency of $\mathcal{Q}$ and $\mathcal{Q}^+$ and by the relation (4.9).

Below we prove the existence of the hermitian kernel K and show, how it can be constructed. The central role in this construction will play the matrix of scalar products of basic elements of the Verma module

$$C^{m_1,\cdots,m_r}_{n_1,\cdots,n_r} \equiv {}_V\langle n_1,\cdots,n_r|m_1,\cdots,m_r\rangle_V. \qquad (4.14)$$

Let us introduce the notion of ancestor for the pair of multiindices $\{n_1,\cdots,n_r|m_1,\cdots,m_r\}$. It is defined in the following way. Consider a pair of indices $\{n_k|m_k\}$ standing on the k-th place of the given pair of multiindices. The ancestor is the pair of representations, which has on the k-th place the following pair:

$\{n_k - m_k|0\}$ if $n_k > m_k$;

$\{0|m_k - n_k\}$ if $n_k < m_k$;

$\{0|0\}$ if $n_k = m_k$;

It means that we reduce the pair on the maximal common number. We can illustrate graphically this procedure for the case of $SO(3,2)$ algebra, which has the rank 2. Its root diagram is shown on the following picture.

$$(4.15)$$
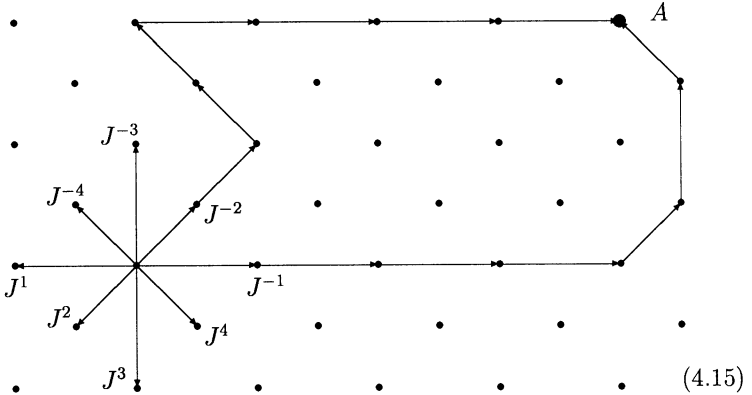
Here are drown two different vectors of the Verma module. Each vector corresponds to the line, which begins at the origin and ends at the point A. Different segments of these lines correspond to negative roots, which are present in the definition of the vector $|n_1,\cdots,n_r\rangle_V$. The first (lower) line corresponds to the vector $|4,1,1,1\rangle_V$, while the second one corresponds to the vector $|4,2,0,2\rangle_V$. The lines described above are not unique for the given vectors, since all lines with the same numbers of each negative roots represent the same vector. However this fact does not affect the result obtained with the help of this picture. The vectors with corresponding lines ended at the same point, say at the point A, are the only ones which can have nonzero scalar product. The pair of representations for the vectors drawn on the picture is $\{4,1,1,1|4,2,0,2\}$, while the corresponding ancestor

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is $\{0,0,1,0|0,1,0,1\}$. In general all pairs of representations are divided into the equivalence classes by their ancestors.

The following expression solves the problem of finding the kernel K.

$$K = \sum_{anc} (b_1^+)^{n_1} \cdots (b_r^+)^{n_r} \; C_{n_1,\cdots,n_r}^{m_1,\cdots,m_r} (b_1^+ b_1, b_2^+ b_2, \cdots, b_r^+ b_r) \; (b_1)^{m_1} \cdots (b_r)^{m_r},$$

(4.16)

where the summation goes over all possible ancestors and $C_{n_1,\cdots,n_r}^{m_1,\cdots,m_r}(b_1^+ b_1, b_2^+ b_2, \cdots, b_r^+ b_r)$ are the functions of the number operators $b_1^+ b_1, b_2^+ b_2, \cdots, b_r^+ b_r$ with the following properties:

$$C_{n_1,\cdots,n_r}^{m_1,\cdots,m_r}(0,0,\cdots,0) = C_{n_1,\cdots,n_r}^{m_1,\cdots,m_r},$$

(4.17)

$$C_{n_1,\cdots,n_r}^{m_1,\cdots,m_r}(l_1,\cdots,l_r) = C_{n_1+l_1,\cdots,n_r+l_r}^{m_1+l_1,\cdots,m_r+l_r}.$$

(4.18)

The hermiticity of the kernel K is a consequence of the relations

$$C_{n_1,\cdots,n_r}^{m_1,\cdots,m_r}(l_1,\cdots,l_r) = C_{m_1,\cdots,m_r}^{n_1,\cdots,n_r}(l_1,\cdots,l_r).$$

(4.19)

Therefore, the expression (4.16) solves the problem of finding the kernel for the scalar product for an arbitrary Lie algebra.

5 Conclusions

It might be interesting to apply the method described in the paper to construct the gauge invariant lagrangians for the particles with the higher spins [14] – [15]. Namely, the case of $so(3,2)$ algebra which we have considered in detail corresponds to the subset of constraints obtained after the quantization of the three – particle bound system (three – point discrete string) [16]. The description of the various irreducible representations of the Poincare group with the corresponding Young tableau having two rows can be achieved after elimination of the Cartan generators of $so(3,2)$ algebra from the total set of constraints. The application of the BRST approach given in this letter to the description of the above mentioned system will be given elsewhere.

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Вспомогательные представления алгебр Ли и БРСТ-конструкции

В свете БРСТ-подхода обсуждается построение вспомогательных представлений алгебр Ли. Соответствующий БРСТ-заряд оказывается неэрмитовым. Данная проблема решена с помощью включения дополнительного оператора ядра в определение скалярного произведения в пространстве Фока. Существование данного оператора доказано для произвольной алгебры Ли.

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Auxiliary Representations of Lie Algebras
and the BRST Constructions

The method of construction of auxiliary representations for a given Lie algebra is discussed in the framework of the BRST approach. The corresponding BRST charge turns out to be non-hermitian. This problem is solved by the introduction of the additional kernel operator in the definition of the scalar product in the Fock space. The existence of the kernel operator is proven for any Lie algebra.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

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