ОБЪЕДИНЕННЫЙ
ИНСТИТУТ
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ

Дубна

E2-2000-46

Dedicated to the memory
of Vladimir Lengyel

D.V.Shirkov

TOWARDS THE CORRELATED ANALYSIS OF PERTURBATIVE **QCD** OBSERVABLES

Submitted to «Nuclear Physics B»

2000

1 Introduction

The item of the low energy behavior of a strong interaction attracts more and more interest along with the further experimental data accumulation. In the perturbative quantum chromodynamics (pQCD) this behavior is spoiled by unphysical singularities associated with the scale parameter $\Lambda \simeq 300$ MeV. In the "small energy" and "small momentum transfer" regions ($\sqrt{s},\ Q \lesssim 3\Lambda$) these singularities violate the weak coupling regime and complicate theoretical interpretation of data. On the other hand, their existence contradicts some general statements of the local QFT.

Meanwhile, this issue has a rather elegant solution. As it has recently been shown [1, 2] (see, also recent review [3]), by combining spectral representations of Källen–Lehmann — and Jost–Lehmann–Dyson — type (which follows from general principles of local QFT like causality, unitarity, Poincaré invariance and spectrality) with renormalizability (that is with renormalization–group invariance), it is possible to formulate an *Invariant Analytic Approach* (IAA) for invariant coupling and observables of pQCD which obeys several remarkable properties:

— It enables one to get rid of unphysical singularities, poles and cuts, producing smooth expressions with the behavior correlated in spacelike and timelike domains.

— In particular, the IAA results in modified ghost-free expressions for invariant QCD coupling in spacelike $\alpha_{\rm an}(Q^2)$ and timelike $\tilde{\alpha}(s)$ regions which obey reduced higher–loops and renormalization–scheme sensitivity [2] - [8].

— Then, it yields changing the structure of perturbation expansion for observables — instead of common power series, as a result of its integral transformation, there appears asymptotic series [9] *à la Erdélyi* over the set of oscillating functions $\mathcal{A}_k(Q^2)$ and $\mathfrak{A}_k(s)$. These functions, at small and moderate argument values, diminish with the k growth much quicker than the corresponding powers $\alpha^k_{\rm an}(Q^2)$ and $\tilde{\alpha}^k(s)$, thus improving the expansion convergence.

1.1 Early attempts: pluses and minuses

It is worth noting that sporadic attempts to define the effective coupling $\alpha(s)$ in the timelike domain were made in late 70s. Omitting an early simple–minded trick with "mirror reflection"

$$\alpha_s(Q^2;f) \to \alpha(s;f) \equiv |\alpha_s(-s;f)|,$$

we mention here the practically simultaneous results of Radyushkin [10] and Krasnikov and Pivovarov [11]. In both the papers, the integral transformation $\tilde{\alpha}(s;f) = \mathbf{R}[\bar{\alpha}_s(Q^2;f)]$ reverse to "dipole representation" for the Adler function $\mathbf{R} = \mathbf{D}^{-1}$

$$D(Q^2) = \frac{Q^2}{\pi}\int_0^\infty \frac{ds}{(s+Q^2)^2}\,R(s) \;\equiv \mathbf{D}\left\{R(s)\right\} \tag{1}$$

in terms of an observable $R(s)$ in the timelike region has been used.

In [10, 11], as a starting point for observables in the spacelike domain $Q^2 > 0$, the perturbation series

$$D_{\rm pt}(Q^2) = 1 + \sum_{k\geq 1} d_k\,\bar{\alpha}_s^k(Q^2;f) \tag{2}$$

has been assumed. It contains powers of usual, RG summed, invariant coupling $\bar{\alpha}_s(Q^2;f)$ that obeys unphysical singularities in the infrared (IR) region around $Q^2 \simeq \Lambda_3^2$.

By using the reverse transformation

$$R(s) = \frac{i}{2\pi}\int_{s-i\varepsilon}^{s+i\varepsilon}\frac{dz}{z}\,D_{\rm pt}(-z) \equiv \mathbf{R}\left[D_{\rm pt}(Q^2)\right] \tag{3}$$

these authors arrived at the "**R**–transformed" expansion that, in our notation, reads

$$R_\pi(s) = 1 + \sum_{k\geq 1} d_k \mathfrak{A}_k(s;f)\,;\;\; \mathfrak{A}_k(s;f) = \mathbf{R}\left[\bar{\alpha}_s^k(Q^2;f)\right]\,. \tag{4}$$

For example, one has[1]

$$\mathbf{R}\left[\frac{1}{l_f}\right] = \frac{1}{2} - \frac{1}{\pi}\arctan\frac{L_f}{\pi}\,,\quad l_f = \ln\frac{Q^2}{\Lambda_f^2}\,;\quad L_f = \ln\left(\frac{s}{\Lambda_f^2}\right)\,;$$

[1]For the first of these expressions we use the form equivalent to that one first found in [12]. In old papers [10, 13] it was given in another form, nonadequate at $L_f \leq 0$. Quite recently, these expressions were rediscovered [14] without proper reference to the precursors.

$$\mathbf{R}\left[\frac{\ln l_f}{l_f^2}\right] = \frac{\ln\left[\sqrt{L_f^2+\pi^2}\right] + 1 - \frac{L_f}{\pi}\arctan\left(\frac{\pi}{L_f}\right)}{L_f^2+\pi^2}\,.$$

At the two–loop case with $\beta_0 = (33-2f)/12\pi$, $\beta_1 = (102-38f)/12\pi$,

$$\beta_{[f]}\bar{\alpha}_s^{(2)}(Q^2;f) = \frac{1}{l_f} - b\frac{\ln l_f}{l_f^2}\,,\quad \beta_{[f]} \equiv \beta_0\,;\ b = \frac{\beta_1}{\beta_0^2}\,,$$

by combining $\mathbf{R}\,[1/l_f] - b\mathbf{R}[\ln l_f/l_f^2]$ we obtain explicit expression for

$$\beta_{[f]}\tilde{\alpha}^{(2)}(s;f) = \beta_0\,\mathfrak{A}_1^{(2)}(s;f)\,.$$

Higher terms $\mathfrak{A}_k$ could be explicitely constucted in the analogous way. Here the iterative relation

$$k\mathbf{R}\left[1/l_f^{k+1}\right] = -\frac{d}{dL}\mathbf{R}\left[1/l_f^k\right]$$

turns out to be useful.

The positive feature of this construction was an automatic summation of the so–called "π^2 – terms" and observed [10] property

$$\left(\mathbf{R}\left[\bar{\alpha}_s^{k+1}\right]\right)^{1/(k+1)} < \left(\mathbf{R}\left[\bar{\alpha}_s^{k}\right]\right)^{1/k}$$

that improves the convergence of perturbation series.

However, there was one essential drawback. The dipole transformation (1), that is supposed to be reverse to $\mathbf{R}$, being applied to (4) does not return us to the input (2)

$$\mathbf{D}\left\{R_\pi(s)\right\} = \mathbf{D}\left\{\mathbf{R}\left[D_{\mathrm{pt}}\right]\right\} \neq D_{\mathrm{pt}}(Q^2) \quad \Rightarrow \quad \mathbf{D}\cdot\mathbf{R} \neq \mathbf{I}\,,$$

as far as the unphysical singularities of $\bar{\alpha}_s(Q^2;f)$ and of its powers are incompatible with analytic properties in the complex Q^2 plane of the integral in the r.h.s. of (1).

Resolution of this issue came 15 years later with the IAA. The "missing link" is the

1.2 Analyticization transformation

$$F(Q^2) \to F_{\mathrm{an}}(Q^2) = \mathbf{A}\cdot F(Q^2) \qquad (5)$$

first introduced [1] in terms of the Källén–Lehmann representation that follows from general statements of local QFT and reflects analytic properties of the Adler function contained in eq.(1).

Generally, this transformation is defined for a function F that should be analytic in the Q^2 complex plane with a cut along the negative part of the real axis. In our case, this function could be either invariant coupling $\bar{\alpha}_s$ itself [2] or its power, or some series in its powers.

Operation **A** consists of two elements

$$F_{\rm an}(Q^2) = \frac{1}{\pi}\int_0^\infty \frac{d\sigma}{\sigma + Q^2}\,\rho_{\rm pt}(\sigma) \quad \text{and} \quad \rho_{\rm pt}(\sigma) = \Im\, F(-\sigma)\,. \tag{6}$$

A couple of comments are in order.

- Operation **A**, being applied to the usual coupling[3] $F = \bar{\alpha}_s(Q^2; f)$, results in the analyticized coupling

$$\alpha_{\rm an}(Q^2; f) = \frac{1}{\pi}\int_0^\infty \frac{d\sigma}{\sigma + Q^2}\,\rho(\sigma; f)\ ; \quad \rho(\sigma; f) = \Im\bar{\alpha}_s(-\sigma; f) \tag{7}$$

  free of unphysical singularities, with a finite value at the origin

$$\alpha_{\rm an}(0; f) = 1/\beta_{[f]} \simeq 1.4$$

  which is remarkably independent of higher loop contributions.

  Here, ρ is defined as an imaginary part of the usual, RG invariant, effective coupling $\bar{\alpha}_s$ continued on the physical cut.

- Operation **A**, applied to power perturbation series (2) for an observable $D_{\rm pt}(Q^2)$ produces, a nonpower perturbation series

$$D_{\rm an}(Q^2; f) = 1 + \sum_{k\geq 1} d_k\,\mathcal{A}_k(Q^2; f)\ ; \quad \alpha_{\rm an}(Q^2; f) = \mathcal{A}_1(Q^2; f) \tag{8}$$

[2] As it has been explained in detail in the first papers [1, 2] on the IAA, the QCD invariant coupling, according to general properties of local QFT, should satisfy the Källén–Lehmann spectral representation.

[3] For the time being, we consider the massless case with a fixed number f of effective quark flavours in the $\overline{\rm MS}$ scheme. For the transition between the regions with different f values, see Section 2.3.

with

$$\mathcal{A}_k(x;f) = \frac{1}{\pi}\int\limits_0^\infty \frac{d\sigma}{\sigma + x}\,\rho_k(\sigma;f)\,; \quad \rho_k(\sigma;f) = \Im\left[\bar{\alpha}_s^k(-\sigma;f)\right]\,. \qquad (9)$$

Properties of the functions $\mathcal{A}_k$ and nonpower expansion (8) have been discussed in papers [9]. They are quite similar to those for $\mathfrak{A}_k$ and expansion (4) — see below.

1.3 Summary of the IAA

Here, we repeat in brief basic definitions of the Invariant Analytic Approach.

First, one has to transform the usual singular invariant coupling

$$\bar{\alpha}_s(Q^2;f) \to \alpha_{\rm an}(Q^2;f) = \mathbf{A}\cdot\bar{\alpha}_s(Q^2;f)$$

into the analyticized one, free of ghost singularities in the spacelike region.

Second, with the help of the operation $\mathbf{R}$, one defines[12] invariant coupling $\tilde{\alpha}(s;f)$ in the timelike domain

$$\alpha_{\rm an}(Q^2;f) \to \tilde{\alpha}(s;f) = \mathbf{R}\left[\alpha_{\rm an}\right] = \int\limits_s^\infty \frac{d\,\sigma}{\sigma}\rho(\sigma;f)\,.$$

Here, we have a possibility of reconstructing the Q^2–channel coupling $\alpha_{\rm an}(Q^2;f)$ from the s–channel one $\tilde{\alpha}(s;f)$ by the dipole representation

$$\alpha_{\rm an}(Q^2;f) = \frac{Q^2}{\pi}\int_0^\infty \frac{ds}{(s+Q^2)^2}\,\tilde{\alpha}(s;f) \equiv \mathbf{D}\left\{\tilde{\alpha}(s;f)\right\}\,. \qquad (10)$$

As it has been shown in [2, 3, 4], relations parallel to eqs.(7) and (10) are valid for powers of the pQCD invariant coupling. This can be resumed in the form of a self-consistent scheme.

2 Basic relations

2.1 Self-consistent scheme for observables.

First, one has to transform usual power perturbation series (2) of the Q^2 domain

$$\textbf{I.} \qquad D_{\rm pt}(Q^2) \to D_{\rm an}(Q^2) = \mathbf{A}\cdot D_{\rm pt}(Q^2)$$

into the nonpower one (8).

Second, with the help of the operation **R**,

$$\textbf{II.} \qquad D_{\rm an}(Q^2) \to R_\pi(s) = \mathbf{R}\left[D_{\rm an}(Q^2)\right]$$

one introduces the s–channel nonpower expansion $R_\pi(s)$ (4) and

$$\mathfrak{A}_k(s) = \int_s^\infty \frac{d\sigma}{\sigma}\rho_k(\sigma)\ ; \quad \rho_k(\sigma) = \Im\left[\alpha_s^k(-\sigma)\right] . \tag{11}$$

The third element is the closure of the scheme that is provided by the operation (10)

$$\textbf{III.} \qquad R_\pi(s) \to D_{\rm an}(Q^2) = \mathbf{D}\left\{R_\pi(s)\right\}$$

reverse to **II.**

In other words, to enjoy self-consistency $\mathbf{R}\cdot\mathbf{D} = \mathbf{D}\cdot\mathbf{R} = \mathbf{1}$, one has *to abandon completely* the usual power series $D_{\rm pt}$, eq.(2), applying operations $\mathbf{R}$ and $\mathbf{D} = \mathbf{R}^{-1}$ only to nonpower expansions $D_{\rm an}$ and R_π.

2.2 Expansion of observables over nonpower sets

Nonpower sets of the functions $\{\mathcal{A}\}$ and $\{\mathfrak{A}\}$. To realize the effect of transition from expansion over the "traditional" power set

$$\left\{\bar{\alpha}_s^k(Q^2)\right\} = \bar{\alpha}_s(Q^2),\, \bar{\alpha}_s^2,\, \ldots \bar{\alpha}_s^k \ldots$$

to expansions over non–power sets in the spacelike and timelike domains

$$\left\{\mathcal{A}_k(Q^2)\right\} = \alpha_{\rm an}(Q^2),\, \mathcal{A}_2(Q^2),\, \mathcal{A}_3 \ldots\ ;\ \left\{\mathfrak{A}_k(s)\right\} = \tilde{\alpha}(s),\, \mathfrak{A}_2(s),\, \mathfrak{A}_3 \ldots ,$$

it is instructive to learn properties of the latters.

In a sense, both nonpower sets are similar:

— They consist of functions that are free of unphysical singularities.

— The first functions, the new effective couplings, $\mathcal{A}_1 = \alpha_{\rm an}$ and $\mathfrak{A}_1 = \tilde{\alpha}$ are monotonously decreasing. They are finite and equal $\alpha_{\rm an}(0) = \tilde{\alpha}(0) = 1.4$ with the same infinite derivatives in the IR limit. Both have the same leading term $\sim 1/\ln x$ in the UV limit.

— All other functions ("effective coupling powers") of both the sets start from the zero IR values $\mathcal{A}_{k\geq 2}(0) = \mathfrak{A}_{k\geq 2}(0) = 0$ and obey the UV behavior $\sim 1/(\ln x)^k$ corresponding to $\bar{\alpha}_s^k(x)$. They are no longer monotonous. The second functions $\mathcal{A}_2$ and $\mathfrak{A}_2$ are positive with maximum around $s, Q^2 \sim \Lambda^2$. Higher functions $\mathcal{A}_{k\geq 3}$ and $\mathfrak{A}_{k\geq 3}$ oscillate in the region of low argument values and obey precisely $k-2$ zeroes.

Remarkably enough, the mechanism of liberation of ghost singularities is quite different. While in the spacelike domain it involves nonperturbative, power in Q^2, structures, in the timelike region, it is based only upon resummation of the "π^2 terms". Figuratively, (nonperturbative) *analyticization* in the Q^2–channel can be treated as a quantitavely distorted reflection (under $Q^2 \to s = -Q^2$) of (perturbative) *"pipization"* in the s–channel.

Nonpower expansions for observables. Summarise the main results essential for data analysis. Instead of the power perturbative series in the spacelike

$$D_{\rm pt}(Q^2) = 1 + d_{\rm pt}(Q^2)\,; \quad d_{\rm pt}(Q^2) = \sum_{k\geq 1} d_k\, \bar{\alpha}_s^k(Q^2; f) \qquad (2a)$$

and timelike regions

$$R_{\rm pt}(s) = 1 + r_{\rm pt}(s);\; r_{\rm pt}(s) = \sum_{k\geq 1} r_k\, \tilde{\alpha}^k(s; f);\; (r_{1,2} = d_{1,2}, r_3 = d_3 - \frac{\pi\beta^2_{[f]}}{3}),$$

one has to use asymptotic expansions

$$d_{\rm an}(Q^2) = \sum_{k\geq 1} d_k\, \mathcal{A}_k(Q^2)\,; \qquad r_\pi(s) = \sum_{k\geq 1} d_k\, \mathfrak{A}_k(s)$$

with the same coefficients d_k over nonpower sets of functions $\{\mathcal{A}\}$ and $\{\mathfrak{A}\}$.

2.3 Global formulation

To apply the new scheme for analysis of QCD processes, one has to formulate it "globally", in the whole experimental domain, i.e., for regions with different values of a number f of active quarks. For this goal, we revise the issue of the threshold crossing.

Threshold matching. In a real calculation, the procedure of the threshold matching is in use. One of the simplest is the matching condition in the massless $\overline{\rm MS}$ scheme

$$\bar{\alpha}_s(Q^2 = M_f^2; f-1) = \bar{\alpha}_s(Q^2 = M_f^2; f) \qquad (12)$$

related[4] to the mass squared M_f^2 of the f-th quark.

This condition allows one to define a function $\bar{\alpha}_s(Q^2)$ which consists of the smooth parts

$$\bar{\alpha}_s(Q^2) = \bar{\alpha}_s(Q^2; f) \quad \text{at} \quad M_{f-1}^2 \leq Q^2 \leq M_f^2 \qquad (13)$$

and is continuous in the whole spacelike interval of positive Q^2 values with discontinuity of derivatives at the matching points. We call it the *spline–continuous* function.

At first sight, any massless matching, yielding the spline–type function, violates the analyticity in the Q^2 variable, thus disturbing the relation between the $s-$ and Q^2–channels[5].

However, in the IAA, the original power perturbation series (2) with its unphysical singularities and possible threshold nonanalyticity has no direct relation to data, being a sort of a "raw material" for defining spectral density. Meanwhile, the discontinuous density is not dangerous. Indeed, expression of the form

$$\rho_k(\sigma) = \rho_k(\sigma; 3) + \sum_{f\geq 4} \theta(\sigma - M_f^2)\left\{\rho_k(\sigma; f) - \rho_k(\sigma; 3)\right\} \qquad (14)$$

with $\rho_k(\sigma; f) = \Im\bar{\alpha}_s^k(-\sigma, f)$ defines, according to (9) and (11), the smooth global

$$\mathcal{A}_k(Q^2) = \int_0^\infty \frac{d\sigma}{\sigma + x}\, \rho_k(\sigma) \qquad (15)$$

[4]The matching point in the $\overline{\rm MS}$ scheme is just M_f^2 , instead of a "more natural" (mirror reflection of) threshold value $4M_f^2$.

[5]Any massless scheme is an approximation that can be controlled by the related mass–dependent scheme [15]. Using such a scheme, one can devise [16] a smooth transition across the heavy quark threshold. Nethertheless, from the practical point of view, it is sufficient (besides the case of data lying in close vicinity of the threshold) to use the spline–type matching (12) and forget about the smooth threshold crossing.

and spline–continuous

$$\mathfrak{A}_k(s) = \int\limits_s^\infty \frac{d\sigma}{\sigma}\rho_k(\sigma) \tag{16}$$

functions [6].

This means that the role of the input perturbative invariant coupling $\bar{\alpha}_s(Q^2)$ is twofold. It provides us not only with spectral density (14) but with matching conditions relating Λ_f with Λ_{f+1} as well.

Note that the matching condition (12) is tightly related [17, 16] to the renormalization procedure. Just for this profound reason we keep it untouched (compare with Ref. [6]).

Shift constants. As a practical result, we now observe that the "global" s–channel coupling $\tilde{\alpha}(s)$ and other functions $\mathfrak{A}_k(s)$, generally, differs of effective coupling with fixed flavour number f value $\tilde{\alpha}(s;f)$ and $\mathfrak{A}_k(s;f)$ by a constants. For example, at $M_5^2 \le s \le M_6^2$

$$\tilde{\alpha}(s) = \int\limits_s^\infty \frac{d\sigma}{\sigma}\rho(\sigma) = \int\limits_s^{M_6^2} \frac{d\sigma}{\sigma}\rho(\sigma;5) + \int\limits_{M_6^2}^\infty \frac{d\sigma}{\sigma}\rho(\sigma;6) = \tilde{\alpha}(s;5) + c(5)\,.$$

Generally,

$$\tilde{\alpha}(s) = \tilde{\alpha}(s;f) + c(f) \qquad \text{at} \qquad M_f^2 \le s \le M_{f+1}^2 \tag{17}$$

which can be easily calculated in terms of integrals over $\rho(\sigma; f+n)$ $n \ge 1$ with additional reservation related to the asymptotic freedom condition. More specifically,

$$c(6) = 0\ , \quad c(f-1) = \tilde{\alpha}(M_f^2; f) - \tilde{\alpha}(M_f^2; f-1) + c(f)\,.$$

These *shift constants* reflect the $\tilde{\alpha}(s)$ continuity at the matching points M_f^2.

Analogous shift constants

$$\mathfrak{A}_k(s) = \mathfrak{A}_k(s;f) + \mathfrak{c}_k(f) \quad \text{at} \quad M_f^2 \le s \le M_{f+1}^2 \tag{18}$$

are responsible for continuity of higher expansion functions. In particular, $\mathfrak{c}_2(f)$ describes the discontinuities of the "main" spectral function (14).

[6]Here, by eqs.(13),(15) and (16) we introduced new "global" effective couplings and higher functions different from the previous ones with fixed f value.

The one-loop estimate with $\beta_{[f]}\rho(\sigma; f) = \left\{\ln^2(\sigma/\Lambda_f^2) + \pi^2\right\}^{-1}$,

$$c_{f-1} - c_f = \frac{1}{\pi\beta_{[f]}} \arctan \frac{\pi}{\ln \frac{M_f^2}{\Lambda_f^2}} - \frac{1}{\pi\beta_{[f-1]}} \arctan \frac{\pi}{\ln \frac{M_f^2}{\Lambda_{f-1}^2}} \simeq \frac{17-f}{54}\tilde{\alpha}_s^3(M_f^2)$$

and traditional values of the scale parameter $\Lambda_3, \Lambda_4 \sim 300 - 250$ MeV reveals that these constants $c(5) \simeq 3.10^{-4}$; $c(4) \simeq 3.10^{-3}$; $c(3) \simeq 0.01$, $\mathfrak{c}_2(4) \simeq 0.02$ and $\mathfrak{c}_2(3) \simeq -0.02$ are essential at a few per cent level for $\tilde{\alpha}$ and at ca 10% level for the $\mathfrak{A}_2$.

At the same time, if one takes into account some increasing in $\Lambda_{\rm an}$ values due to the nonpower structure of the modified IAA expansion (see, the following section), then the shift constant $c(3)$ could reach the level of 0.02. This means that the quantitative analysis of all s–channel events at moderate energies like, e.g., e^+e^- annihilation [3], τ–lepton decay [4] and charmonium width [11] should be influenced by these constants.

The spacelike region. On the other hand, in the Q^2–channel, instead of the spline-type function $\tilde{\alpha}_s(Q^2)$, eq.(13), we have now continuous, analytic in the whole $Q^2 > 0$ domain, invariant coupling $\alpha_{\rm an}(Q^2)$ defined via the spectral integral

$$\alpha_{\rm an}(Q^2) = \frac{1}{\pi}\int_0^\infty \frac{d\sigma}{\sigma + Q^2}\,\rho(\sigma) \tag{19}$$

with the discontinuous density $\rho(\sigma)$ (14).

Unhappily, here, unlike for the timelike region, there is no possibility of enjoying any more explicit expressions for $\alpha_{\rm an}(Q^2)$ even in the one-loop case. Moreover, the Q^2–channel functions $\alpha_{\rm an}$ and $\mathcal{A}_k$, being considered in the particular region $M_{f-1}^2 \leq Q^2 \leq M_f^2$, do depend on all $\Lambda_3, \ldots, \Lambda_6$ values simultaneously.

Nevertheless, the real difference from the $f = 3$ case, numerially, is not big and, for practical reasons, one can use an approximate formula[7]

$$\alpha_{\rm an}^{\rm appr}(Q^2) = \alpha_{\rm an}(Q^2; 3) + 0.02 \ln \frac{M_5^2 + Q^2}{M_4^2 + Q^2} \tag{20}$$

valid in the whole Q^2 domain.

[7] This and some other numerical estimates are based upon analytic calculation with exact two-loop solution expressed in terms of the Lambert function — see refs.[22, 23, 24]. Details of these calculations will be published elswhere. The assistance of D.S. Kurashev, B. Magradze and A.V. Nesterenko in calculations with the Lambert functions is gratefully acknowledged.

3 Correlation of experiments

Another quantitative effect stems from the nonpower structure of the IAA perturbative expansion. It is more emphasized at the few GeV region.

3.1 The s–channel

To illustrate the qualitative difference between our global scheme and usual practice of data analysis, we first consider the $f = 3$ region.

Inclusive τ decay. The IAA scheme was used in Ref. [4] for analysis of the inclusive τ–decay. Here, the observed quantity, the τ lepton time of half–decay, depends on the integral of the s–channel matrix element over the region $0 < s < M_\tau^2$. As a result of the 2–loop IAA analysis of the experimental input $R_\tau = 3.633$ [18], the value $\tilde{\alpha}^{(2)}(M_\tau^2) = 0.378$ has been obtained that has to be compared with related result of usual analysis $\bar{\alpha}_s^{(3)}(M_\tau^2) = 0.337$. This shift $\Delta\alpha \simeq 0.04$ resulted in a rather big change in the extracted Λ value. Meanwhile, an essential part of this shift can be "absorbed" by the shift constant $c(3) \simeq 0.01 - 0.02$.

The process of Inclusive e^+e^- hadron annihilation provides us with an important piece of information on the QCD parameters. In the usual treatment, (see, e.g., Refs.[18, 19]) the basic relation looks like

$$\frac{R(s)}{R_0} = 1 + r(s)\,; \quad r(s) = \frac{\bar{\alpha}_s(s)}{\pi} + r_2\,\bar{\alpha}_s^2(s) + r_3\,\bar{\alpha}_s^3(s)\,. \tag{21}$$

Here, the numerical coefficients $r_1 = 1/\pi = 0.318\,,\ r_2 = 0.142\,,\ r_3 = -0.413$ (related to the $f = 5$ case) are not diminishing. However, a rather big negative r_3 value comes mainly from the $-r_1\pi^2\beta_{[4]}^2/3$ contribution equal to -0.456. Instead of (21), with due account of (4), we now have

$$r(s) = 1 + \frac{\tilde{\alpha}(s)}{\pi} + d_2\,\mathfrak{A}_2(s) + d_3\,\mathfrak{A}_3(s)\,; \tag{22}$$

with rapidly decreasing coefficients $d_1 = 0.318\,;\ d_2 = 0.142\,;\ d_3 = 0.043\,,$ the mentioned π^2 term of c_3 being "swallowed" by $\tilde{\alpha}(s)$.

Now, the main difference of the last expression from (21) is due to the term $d_2\,\mathfrak{A}_2$ standing in the place of $d_2\,\tilde{\alpha}^2$. The difference can be approximated by adding into (21) the structure $c_4\,\alpha^4$ with $c_4 = d_2\,\beta_{[4]}^2\pi^2 \simeq -0.62$. This effect could be essential in the region of $\tilde{\alpha}(s) \simeq 0.20 - 0.25$.

3.2 The Q^2–channel

The Q^2–channel: Bjorken and GLS sum rules. In the paper [5], the IAA has been applied to the Bjorken sum rules. Here, one has to deal with the Q^2–channel at small transfer momentum squared $Q^2 \lesssim 10\,\mathrm{GeV}^2$. Due to some controversy of experimental data, we give here only a part of the results of [5]. For instance, using data of the SMC Collaboration [20] for $Q_0^2 = 10\,\mathrm{GeV}^2$ the authors obtained $\alpha_{\mathrm{an}}^{(3)}(Q_0^2) = 0.301$ instead of $\alpha_{\mathrm{pt}}^{(3)}(Q_0^2) = 0.275$.

In the Q^2–channel, instead of power expansion like (2), we typically have

$$d(Q^2) = \frac{\alpha_{\mathrm{an}}(Q^2)}{\pi} + d_2\,\mathcal{A}_2(Q^2) + d_3\,\mathcal{A}_3(Q^2)\,. \qquad (23)$$

Here, the modification is related to nonperturbative power structures behaving like Λ^2/Q^2 at $Q^2 \gg \Lambda^2$. These corrections are essential only in a few GeV region. To estimate the effect of transition from $\alpha_{\mathrm{an}}(Q^2; f)$ to our global $\alpha_{\mathrm{an}}(Q^2)$ defined by eq.(19), we use the approximation (20). This gives $\Delta\alpha = \alpha_{\mathrm{an}}(Q_0^2) - \alpha_{\mathrm{an}}(Q_0^2; 3) = 0.02$.

The same comment could be made with respect to analysis of the Gross–Llywellin-Smith sum rules of [7].

Some lessons. Few comments are in order:

- We see that, generally, the extracted values of α_{an} and of $\tilde{\alpha}$ are both slightly greater (by about 10 % in a few GeV region) than the relevant values of $\bar{\alpha}_s$ for the same experimental input. This corresponds to the above-mentioned nonpower character of new asymptotic expansions with a suppressed higher-loop contribution.

- At the same time, for equal values of $\alpha_{\mathrm{an}}(x_*) = \tilde{\alpha}(x_*) = \bar{\alpha}_s(x_*)$, the analytic scale parameter Λ_{an} values extracted from α_{an} and $\tilde{\alpha}$ are a bit greater than that one taken from $\bar{\alpha}_s$. This feature is related to a "smoother" behavior of both the regular functions α_{an} and $\tilde{\alpha}$ as compared to the singular $\bar{\alpha}_s$.

3.3 Conclusion

To summarize, we repeat once more our main points.

- We have discussed the **self-consistent scheme** for analysing data **both in the spacelike and timelike regions.** The fundamental equation connecting these regions is the dipole spectral relation (1) between renormalization–group invariant nonpower expansions $D_{\rm an}(Q^2)$ and $R_\pi(s)$.

  Just this equation (1), equivalent to the Källen–Lehmann representation, is responsible for nonperturbative terms in the Q^2–channel involved into nonpower expansion functions $\{\mathcal{A}_k(s)\}$. These terms, nonanalytic in the coupling constant α, are a counterpart to the perfectly perturbative π^2–terms effectively summed in the s–channel expressions $\{\mathfrak{A}_k(s)\}$.

  An operant algorithm is based upon the Källen–Lehmann spectral density $\rho(\sigma)$ (14) for an invariant coupling and nonpower expansion for an observable.

- The second issue relates to the correlation between regions with different values of the effective flavour number f. Dealing with the massless $\overline{\rm MS}$ renormalization scheme, we argue that the usual perturbative QCD expansion provides our scheme with step–discontinuous spectral density (14) depending *simultaneously* on different scale parameters Λ_f ; $f = 3, 4, 5$ connected by usual matching relations.

  This step–discontinuous spectral density yields, on the one hand, **smooth analytic coupling** $\alpha_{\rm an}(Q^2)$ and **higher functions** $\mathcal{A}_k(Q^2)$ **in the spacelike region** — eq.(15). On the other hand, it produces **the spline–continuous invariant coupling** $\tilde{\alpha}(s)$ **and expansion functions** $\{\mathfrak{A}_k(s)\}$ **in the time-like region.** — eq.(16).

  As a result, our "global" expansion functions $\{\mathcal{A}_k(Q^2)\}$ and $\{\mathfrak{A}_k(s)\}$ differ from the corresponding ones $\{\mathcal{A}_k(Q^2; f)\}$, $\{\mathfrak{A}_k(s; f)\}$ with a fixed value of a flavour number.

- Thus, our global self-consistent scheme uses the popular perturbative invariant coupling $\bar{\alpha}_s(Q^2, f)$, together with the usual matching relations, only as an input. Practical calculation for an observable now involves expansions over the sets $\{\mathcal{A}_k(Q^2)\}$ and $\{\mathfrak{A}_k(s)\}$, that is nonpower series with usual numerical coefficients d_k obtained by calculation of the relevant Feynman diagrams.

- This means that, generally, one should *check the accuracy of the bulk of extractions of the QCD parameters* from diverse experimental data. Our preliminary estimate shows that such a revision could influence the rate of their corellation.

Acknowledgements

The author is indebted to D.Yu. Bardin, N.V. Krasnikov, S.V. Mikhailov, A. V. Radyushkin, I.L. Solovtsov and O.P. Solovtsova for useful discussion and comments. This work was partially supported by grants of the Russian Foundation for Basic Research (RFBR projects Nos 99-01-00091 and 00-15-96691), by INTAS grants No 96-0842 and INTAS-CERN No 2000-377.

References

[1] D.V. Shirkov and I.L. Solovtsov, "Analytic QCD running coupling with finite IR behaviour and universal $\bar{\alpha}_s(0)$ value", *JINR Rapid Comm.* No.2[76]-96, pp 5-10; hep-ph/9604363

[2] D.V. Shirkov and I.L. Solovtsov, *Phys.Rev.Lett.* **79** (1997) 1209-12; hep-ph/9704333;.

[3] I.L. Solovtsov and D.V. Shirkov, *Theor. Math. Phys.* **120** (1999) 1210–1243; hep-ph/9909305.

[4] K.A. Milton, I.L. Solovtsov and O.P. Solovtsova, *Phys. Lett.* **B 415** (1997) 104.

[5] K.A. Milton, I.L. Solovtsov and O.P. Solovtsova, *Phys. Lett. B* **439** (1998) 421–427. hep–ph/9809510.

[6] K.A. Milton and O.P. Solovtsova, Phys. Rev. **D 57**, 5402-5409 (1998).

[7] K.A. Milton, I.L. Solovtsov and O.P. Solovtsova, *Phys. Rev.* **D 60**, 016001 (1999); hep–ph/9809513.

[8] N.G. Stefanis, W. Schroers and H.-Ch. Kim, *Phys. Lett. B* **449** (1999) 299–305. hep–ph/9812280.

[9] D.V. Shirkov, *Lett. Math. Physics* **48** (1999) 135-144; *Theor. Math. Phys.* **119** (1999) 438–447; hep-th/9810246;

[10] A. Radyushkin, Dubna JINR preprint E2-82-159 (1982); see also *JINR Rapid Comm.* No. 4[78]-96 (1996) pp 9-15 and hep–ph/9907228.

[11] N.V. Krasnikov, A.A. Pivovarov, *Phys. Lett.* **116 B** (1982) 168–170.

[12] K.A. Milton and I.L. Solovtsov, Phys. Rev. **D 55**, 5295-5298 (1997).

[13] A.A. Pivovarov, *Z. Phys. C — Particles and Fields* **53** (1992) 461–463.

[14] B.V. Geshkenbein, B.L. Ioffe, *Pis'ma v ZhETP* **70** (1999) 167-170.

[15] D.V. Shirkov, *Theor. Math. Fiz.* **49** (1981) 1039 – 1042; *Nuclear Physics* **B 371** (1992) 467 – 481.

[16] D.V. Shirkov and I.L. Mikhailov, *Z. f. Phys.* **C 63** (1994) 463–469.

[17] W. Bernreuther and W. Wetzel, *Nucl. Phys.* **B 197** (1982) 228-239; W. Marciano, *Phys. Rev.* **D 29** (1984) 580-589.

[18] C. Caso *et al.*, *European Phys.J.* **C 3** (1998) 1.

[19] D.Yu. Bardin and G. Passarino, The Standard model in the making Clarendon Press, 1999.

[20] SMC Collaboration, A.Adams et al., *Phys. Rev.* **D 56** (1997) 5330.

[21] S.A. Larin and J.A.M. Vermaseren, *Phys. Lett.* **259 B** (1991) 345–351.

[22] B.A. Magradze, "The gluon propagator in Analytic Perturbation theory", talk presented at the Intern. Conf. "Quarks-98", Suzdal, May 1998, hep-ph/9808247;

[23] E. Gardi, G. Grunberg and M. Karliner, JHEP 07 (1998) 007; hep-ph/9806462 .

[24] R.M. Corless et al., *Adv. in Comput. Math.* **5**, (1996) 329.

[25] B.A. Magradze, "Analytic Approach to Perturbative QCD", November 1999, hep-ph/9911456;

Contents

Received by Publishing Department
on March 7, 2000.

Ширков Д.В. E2-2000-46
К коррелированному анализу наблюдаемых пертурбативной КХД

Комбинация приема с подсуммированием π^2-членов в инвариантной функции связи и наблюдаемых КХД во времени-подобной области со свежими результатами по «аналитизированным» $\alpha_{\rm an}(Q^2)$ и наблюдаемым в пространственно-подобной области приводит к самосогласованной схеме, свободной от нефизических сингулярностей. Центральным местом этой единой конструкции является «дипольное спектральное представление», вытекающее из аксиом локальной КТП.

Мы рассматриваем вопрос о порогах тяжелых кварков и формулируем модифицированную глобальную схему для анализа опытных данных во всех доступных пространственно- и времени-подобных областях. Такие данные представляются в форме нестепенных разложений с улучшенными свойствами сходимости.

Предварительные численные оценки показывают, что новая глобальная схема приводит к результатам, которые — на уровне нескольких процентов для $\bar{\alpha}_s$ — могут отличаться от обычных, модифицируя таким образом общую картину корреляции параметров КХД.

Работа выполнена в Лаборатории теоретической физики им. Н.Н.Боголюбова ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна, 2000

Shirkov D.V. E2-2000-46
Towards the Correlated Analysis of Perturbative QCD Observables

Combining the trick of resummation of the π^2-terms for the invariant QCD coupling and observables in the timelike region with fresh results on the «analyticized» $\alpha_{\rm an}(Q^2)$ and observables in the spacelike domain yields a self-consistent scheme, free of ghost troubles. The basic point of this joint construction is the «dipole spectral relation» emerging from axioms of local QFT.

We consider the issue of the heavy quark thresholds and devise a modified global IAA scheme for the experimental data analyses in the whole accessible spacelike and timelike domain. Such data in both the regions are presented in a form of nonpower perturbation series with improved convergence properties.

Preliminary numerical estimates indicate that this global IAA scheme produces results a bit different — on a few per cent level for $\bar{\alpha}_s$ — from the usual one, thus influencing the total picture of the QCD parameters correlation.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna, 2000

Макет Т.Е.Попеко

Подписано в печать 14.03.2000
Формат 60 × 90/16. Офсетная печать. Уч.-изд. листов 1,68
Тираж 425. Заказ 51903. Цена 2 р.

Издательский отдел Объединенного института ядерных исследований
Дубна Московской области