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$N = 4$ SUPERCONFORMAL MECHANICS AND THE POTENTIAL STRUCTURE OF AdS SPACES

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1 Introduction

Supersymmetric quantum mechanics which underlies the dynamics of non–relativistic and relativistic spinning particles and superparticles is one of the simplest examples of supersymmetric sigma–models and it has attracted a great deal of attention as a laboratory for studying problems appearing in more complicated supersymmetric field and string theories. For instance, one–dimensional [1] and multidimensional [2, 3] $N = 4$ supersymmetric quantum mechanics (SUSY QM) can be associated with $N = 1$, $D = 4$ supersymmetric field theories (including supergravity) subject to an appropriate dimensional reduction down to $D = 1$.

A recent revival of interest in superconformal mechanics [4, 5] has been caused, in particular, by an observation made in the context of the AdS/CFT correspondence conjecture that the dynamics of a superparticle near the AdS horizon of an extreme Reissner–Nordström black hole of a large mass is described by a superconformal mechanics [6]. Applications of supersymmetric mechanics to the theory of black holes and to other problems have been reviewed in [7, 8]. In [8] conditions on geometry of curved backgrounds, in which $N = 1, 2$ and 4 superconformal invariant models of *non-relativistic* spinning particles can exist, have been studied in the $N = 1$ superfield formalism.

The superconformal group of the particle superworldline is a subgroup of the group of its superdiffeomorphisms. It becomes manifest in the worldline superfield formulations of *relativistic* spinning particles [9] and superparticles [10], which can thus be regarded as examples of quantum mechanics with manifest superconformal symmetry.

The superconformal invariance and, in more general case, worldvolume superdiffeomorphisms impose restrictions on the geometry of the background also in models of relativistic particles and branes. For instance, in the case of superbranes it requires that a target–superspace background obeys superfield supergravity constraints (see [11] for a recent review).

In the case of spinning particles this problem is connected with the problem of selfconsistent field theoretical description of interacting particles with spin higher than 2. It is well known that the theory of interacting higher spin fields should be formulated in an anti–de–Sitter background (see [12] for a review).

In [14] it was shown that difficulties with constructing a model of a spinning particle moving in a gravitational background arise already for spins 3/2 and 2. These difficulties have been overcome in [15], where an action for spinning particles with spin higher than one were constructed in backgrounds of constant curvature (such as the AdS spaces)[1].

The spin 2 particle model of [13, 14, 15] is based on a so called "large" $N = 4$ superconformal algebra containing $SO(4)$ as the subalgebra of local internal symmetries. It is well known that there exists another (so called "small") $N = 4$ superconformal algebra with $SU(2)$ as the subalgebra of local internal symmetries.

[1] We thank Sergey Kuzenko for bringing these papers to our attention.

It is therefore tempting to study whether in a superfield formulation of a spin 2 particle dynamics, which is manifestly invariant under "small" $N = 4$ superconformal symmetry, conditions imposed on a curved background can be less restrictive than in the case of [13, 14, 15].

In this paper we present results of this study. We consider relativistic spinning particle mechanics invariant under local $N = 4$ supersymmetry with $SU(2)_{local} \times SU(2)_{global}$ internal symmetries, which is associated with the "small" $N = 4$ superconformal algebra. In a flat background this model has been constructed and studied in [2]. It was shown that (in four dimensions) its first quantized spectrum consists of three scalar and one spin 2 states corresponding to the linearized limit of a conformal gravity model. The superfield action for this $N = 4$ spinning particle is a localized (or superconformal) version of the action for $N = 4$ supersymmetric quantum mechanics [1, 3] with a quadratic superpotential. This correspondence prompts us how to generalize the free $N = 4$ superconformal spinning particle action to the description of a particle propagating in a gravitational background. For this one should consider supersymmetric quantum mechanics with an arbitrary superpotential [3] and make it invariant under the $N = 4$ superconformal transformations [2].

In [3] it has been shown that the $N = 4$ superfield formulation of multidimensional $N = 4$ SUSY QM leads to a supersymmetric nonlinear sigma–model with a target–space metric being a second derivative of a single real–valued function (superpotential) $A(x)$

$$g_{MN}(x) = \frac{\partial^2 A(x)}{\partial x^M \partial x^N}, \tag{1.1}$$

i.e. for an arbitrary dimension D and signature of the sigma–model manifold, parametrized by *real* scalar fields x^M ($M = 0, 1, ..., D-1$), its metric should have a "Kähler–like" structure. The metric of a similar type appeared also as a metric of black hole moduli spaces considered recently in [16].

As has been announced in [2], the $N = 4$ superconformal generalization of the model of [3] in a manifold of Minkowski signature describes a relativistic spinning particle propagating in a gravitational background with the metric (1.1).

It has been known for a long time that supersymmetry requires sigma–model manifolds of chiral superfields to be Kähler, hyper–Kähler [17], special Kähler [18], [19] or special Lagrangian manifolds [20]. The geometrical structure of these manifolds has been under intensive investigation because of its relation to the compactification of string theory on Calabi–Yau manifolds and to duality symmetries of corresponding supergravity models (see [21] for a review).

The essential difference of the metric (1.1) from a Kähler metric

$$g_{MN}(z, \bar{z}) = \frac{\partial^2 K(z, \bar{z})}{\partial z^M \partial \bar{z}^N} \tag{1.2}$$

is that the latter is a Hermitian metric on a complex manifold, while the former is a *real manifold* metric. The reason why a real sigma–model manifold appears in the

case of $N = 4$ SUSY QM under consideration is that we construct the supersymmetric sigma model with the use of constrained *real* superfields and not with chiral ones as one usually do.

Some Kähler manifolds mentioned above also admit real-valued representation for the metric (1.1). For example, this is so for a metric of the special Kähler manifolds in a flat Darboux coordinate system [19, 20]. However, the class of the manifolds with the metric (1.1) is more general and includes manifolds which do not have complex structure.

In particular, we have found that in a certain coordinate system the metric on an anti–de–Sitter space of an arbitrary dimension D (AdS_D) can be represented in the form (1.1). Other examples are hyperbolic manifolds of negative curvature on which M–theory and string theories can compactify [22, 23]. To the best of our knowledge this observation is a novel one. This result can presumably be useful for better understanding the structure of string and supergravity theories in AdS superbackgrounds and AdS/CFT correspondence.

The paper is organized as follows. In Section 2 we review the $N = 4$ superconformal particle model of ref. [2]. In Section 3 we generalize it to describe a spinning particle in a curved background with the metric (1.1). Properties of the AdS_D space which follow from the potential structure of its metric are considered in Section 4. In Conclusion we discuss open problems and outlook.

2 The free $N = 4$ superconformal particle model

We begin with a brief description of free spinning particle mechanics with $SU(2)_{local} \times SU(2)_{global}$ $N = 4$ superconformal symmetry [2].

To construct the superfield action in the worldline superspace $(\tau, \theta^a, \bar\theta_a)$ (with τ being a time parameter, and θ^a and $\bar\theta_a = (\theta^a)^*$, $(a = 1, 2)$ being two complex (or four real) Grassmann–odd coordinates) one introduces D real "matter" superfields $\Phi^M(\tau, \theta^a, \bar\theta_a)$ $(M = 0, 1, \ldots, D-1)$ and a worldline supereinbein $E(\tau, \theta^a, \bar\theta_a)$ which have the following properties with respect to the $SU(2)$ $N = 4$ superconformal transformations of the worldline superspace [2]

$$
\begin{aligned}
\delta\tau &= \Lambda - \frac{1}{2}\theta^a D_a\Lambda - \frac{1}{2}\bar\theta_a \overline{D}^a\Lambda, \\
\delta\theta^a &= i\overline{D}^a\Lambda, \quad \delta\bar\theta_a = iD_a\Lambda, \qquad (2.1) \\
\delta\Phi^M &= -\Lambda\dot\Phi^M + \dot\Lambda\Phi^M - i(D_a\Lambda)(\overline{D}^a\Phi^M) - i(\overline{D}^a\Lambda)(D_a\Phi^M), \qquad (2.2) \\
\delta E &= -\Lambda\dot E - \dot\Lambda E - i(D_a\Lambda)(\overline{D}^a E) - i(\overline{D}^a\Lambda)(D_a E), \qquad (2.3)
\end{aligned}
$$

where dot denotes the time derivative $\frac{d}{d\tau}$. The transformation law (2.3) for the superfields Φ^M shows that these superfields are vector superfields in the one dimensional

[2] Our conventions for spinors are as follows: $\theta_a = \theta^b\varepsilon_{ba}$, $\theta^a = \varepsilon^{ab}\theta_b$, $\bar\theta_a = \bar\theta^b\varepsilon_{ba}$, $\bar\theta^a = \varepsilon^{ab}\bar\theta_b$, $\bar\theta_a = (\theta^a)^*$, $\bar\theta^a = -(\theta_a)^*$, $(\theta\theta) \equiv \theta^a\theta_a = -2\theta^1\theta^2$, $(\bar\theta\bar\theta) \equiv \bar\theta_a\bar\theta^a = (\theta\theta)^*$, $(\bar\theta\theta) \equiv \bar\theta_a\theta^a$, $\varepsilon^{12} = -\varepsilon^{21} = 1$, $\varepsilon_{12} = 1$.

$N = 4$ superspace, while the superfields $E\Phi^M$ are scalars.

The superfields Φ^M and E obey the quadratic constraints

$$\begin{aligned} [D_a, \overline{D}^a]\Phi^M &= 0, \\ D^a D_a \Phi^M &= 0, \\ \overline{D}_a \overline{D}^a \Phi^M &= 0, \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} [D_a, \overline{D}^a]\frac{1}{E} &= 0, \\ D^a D_a \frac{1}{E} &= 0, \\ \overline{D}_a \overline{D}^a \frac{1}{E} &= 0, \end{aligned} \tag{2.5}$$

where

$$D_a = \frac{\partial}{\partial\theta^a} - \frac{i}{2}\overline{\theta}_a \frac{\partial}{\partial\tau}, \quad \overline{D}^a = \frac{\partial}{\partial\overline{\theta}_a} - \frac{i}{2}\theta^a \frac{\partial}{\partial\tau}, \tag{2.6}$$

are the supercovariant derivatives, and the infinitesimal superfield

$$\begin{aligned} \Lambda(\tau,\theta,\overline{\theta}) &= a(\tau) + \theta^a \overline{\alpha}_a(\tau) - \overline{\theta}_a \alpha^a(\tau) + \theta^a (\sigma_i)_a{}^b \overline{\theta}_b b_i(\tau) \\ &\quad -\frac{i}{2}(\theta^a \dot{\overline{\alpha}}_a(\tau) + \overline{\theta}_a \dot{\alpha}^a(\tau))\overline{\theta}\theta + \frac{1}{8}(\overline{\theta}\theta)^2 \ddot{a}(\tau) \end{aligned} \tag{2.7}$$

contains the parameters of local reparametrizations $a(\tau)$, local supertranslations $\alpha^a(\tau)$, $\overline{\alpha}_a(\tau)$ and local $SU(2)$ rotations $b_i(\tau)$ of the worldline superspace. It is constrained by the same relations (2.4) as $1/E$ and Φ^M ($(\sigma_i)_a{}^b$ are the Pauli matrices, $i = 1, 2, 3$).

The constraints (2.4)–(2.5) can be explicitly solved [2, 24], the solution being described by the superfields

$$\begin{aligned} \Phi^M(\tau,\eta,\overline{\eta}) &\doteq \frac{1}{e(\tau)} x^M(\tau) + \theta^a \overline{\psi}_a'^M(\tau) - \overline{\theta}_a \psi'^{Ma}(\tau) + \theta^a (\sigma_i)_a{}^b \overline{\theta}_b T_i'^M(\tau) \\ &\quad -\frac{i}{2}(\theta^a \dot{\overline{\psi}}_a'^M(\tau) + \overline{\theta}_a \dot{\psi}'^{Ma}(\tau))\overline{\theta}\theta + \frac{1}{8}(\overline{\theta}\theta)^2 \frac{d^2}{d\tau^2}\left(\frac{1}{e(\tau)} x^M\right), \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} \frac{1}{E}(\tau,\theta,\overline{\theta}) &= \frac{1}{e(\tau)} + \theta^a \overline{\lambda}'_a(\tau) - \overline{\theta}_a \lambda'^a(\tau) + \theta^a (\sigma_i)_a{}^b \overline{\theta}_b t_i'(\tau) \\ &\quad -\frac{i}{2}(\theta^a \dot{\overline{\lambda}}'_a(\tau) + \overline{\theta}_a \dot{\lambda}'^a(\tau))\overline{\theta}\theta + \frac{1}{8}(\overline{\theta}\theta)^2 \frac{d^2}{d\tau^2}\frac{1}{e(\tau)}. \end{aligned} \tag{2.9}$$

The leading components $x^M(\tau)$ of the superfields Φ^M are associated with coordinates of the particle trajectory in a D–dimensional flat target space–time, the

Grassmann–odd vectors $\psi'^{Ma}(\tau)$ and $\overline{\psi'}_a^M(\tau)$ correspond to particle spin degrees of freedom and $T_i'^M(\tau)$ are auxiliary fields. The superfield $1/E$ describes an $N = 4$ worldline supergravity multiplet consisting of the einbein ("graviton") $e(\tau)$, two complex "gravitini" $\lambda'^a(\tau)$ and $\overline{\lambda}'_a(\tau)$, and the $SU(2)$ gauge field $t'_i(\tau)$. Upon an appropriate field redefinition (see eqs. (3.2) of the next section) we shall pass from "primed" to "unprimed" component fields.

The $N = 4$ superfield action for a relativistic spinning particle in a flat target space has the following form [2]

$$S = -8 \int d\tau d^2\theta d^2\overline{\theta} E \Phi^M \Phi^N \eta_{MN} \tag{2.10}$$

where $\eta_{MN} = diag\ (-,+,\ldots,+)$ is the Minkowski metric. The components of E play the role of Lagrange multipliers. Their presence implies that the dynamics of the particle is subject to relativistic constraints, in particular, the particle is massless ($p_M p^M = 0$) . The Dirac quantization of the model (2.10) shows that its quantum spectrum consists of one spin 2 and three spin 0 particle states and it can be regarded as a linearized spectrum of a conformal gravity [2].

3 The spinning particle in a curved background

Let us now generalize the model of the previous section to describe a spinning particle propagating in the gravitational background. To this end we replace (2.10) with the most general action functional which respects the $N = 4$ superconformal symmetry

$$S = -8 \int d\tau d^2\theta d^2\overline{\theta} E^{-1} A(E\Phi^M), \tag{3.1}$$

where $A(E\Phi^M)$ is an arbitrary function (called the superpotential) of $E\Phi^M$. Recall that $E\Phi^M$ transform as scalar superfields with respect to (2.3), while Φ^M and $\frac{1}{E}$ are vectors. Note also that $E^{-1}A(E\Phi^M)$ can be regarded as a rank one homogeneous function in a $D+1$ dimensional space with $x^D = E^{-1}$. A consequence of such a structure of the superfield action (3.1) is the fact that only D of the bosonic coordinates in the $D+1$ dimensional space describe dynamical degrees of freedom. The einbein $e(\tau)$ and its superpartners are auxiliary fields as in the free particle case (2.10).

Integrating (3.1) over the Grassmann coordinates θ^a, $\overline{\theta}_a$ and making the following redefinition of the component fields

$$\lambda^a = e^{\frac{3}{2}}\lambda'^a, \quad \overline{\lambda}_a = (\lambda^a)^*, \quad t_i = 2e(t'_i + e\lambda'^b(\sigma_i)_b{}^a\overline{\lambda}'_a), \quad \psi^{Ma} = \sqrt{e}(\psi'^{Ma} - x^M\lambda'^a),$$

$$\overline{\psi}_a^M = (\psi^{Ma})^*, \quad T_i^M = 2\sqrt{e}(T_i'^M - x^M t'_i + \frac{\sqrt{e}}{2}\lambda'^b(\sigma_i)_b{}^a\overline{\psi}_a^M + \frac{\sqrt{e}}{2}\psi^{Mb}(\sigma_i)_b{}^a\overline{\lambda}'_a), \tag{3.2}$$

one obtains the component action

$$S = \int d\tau (K - V), \tag{3.3}$$

where

$$K = \frac{1}{2e} g_{MN}(\dot{x}^M - i\overline{\lambda}_a \psi^{Ma} + i\overline{\psi}_a^M \lambda^a)(\dot{x}^N - i\overline{\lambda}_b \psi^{Nb} + i\overline{\psi}_b^N \lambda^b) + i g_{MN}(\overline{\psi}_a^M \dot{\psi}^{aN} + \psi^{aM} \dot{\overline{\psi}}_a^N) \tag{3.4}$$

is the kinetic term and

$$V = -\frac{1}{2} g_{MN} T_i^M T_i^N + 2\sqrt{e}\Gamma_{LMN}\psi^{Mb}(\sigma_i)_b{}^a \overline{\psi}_a^L T_i^N - t_i g_{MN}\psi^{Nb}(\sigma_i)_b{}^a \overline{\psi}_a^M + 2\Gamma_{LMN}(\lambda^a \overline{\psi}_a^L \overline{\psi}_b^M \psi^{bN} + \overline{\lambda}_b \psi^{Lb}\psi^{Ma}\overline{\psi}_a^N) + e(\partial_L \Gamma_{MNP})(\overline{\psi}_a^L \overline{\psi}^{Ma})(\psi^{Nb}\psi_b^P) \tag{3.5}$$

describes fermionic interactions. In eqs. (3.4) and (3.5)

$$g_{MN}(x) = \frac{\partial^2}{\partial x^M \partial x^N} A(x) \equiv \partial_{MN}^2 A(x), \quad A(x^M) = A(E\Phi^M)|_{\theta,\bar{\theta}=0} \tag{3.6}$$

is the metric of a sigma–model D–dimensional manifold parametrized by the world-line scalar fields $x^M(\tau)$ and

$$\Gamma_{LMN}(x) = \frac{1}{2}\partial_{LMN}^3 A(x) \tag{3.7}$$

is the (totally symmetric) Christoffel connection associated with g_{MN} (i.e. $\mathcal{D}_L g_{MN} = \partial_L g_{MN} - \Gamma_{LM}^P g_{PN} - \Gamma_{LN}^P g_{PM} = 0$).

The Riemann curvature of this manifold has the form

$$R_{LM,NP} = \Gamma_{LP}^Q \Gamma_{QMN} - \Gamma_{LN}^Q \Gamma_{QMP}. \tag{3.8}$$

Upon solving for the equations of motion of the auxiliary fields T_i^M, substituting the solution back into eqs. (3.3)–(3.5) and performing Legendre transformations one arrives at the first order form of the spinning particle action

$$S = \int d\tau \left[p_M \dot{x}^M + i(\psi_M^a \dot{\overline{\psi}}_a^M + \overline{\psi}_{Ma}\dot{\psi}^{Ma}) - H \right], \tag{3.9}$$

where p_M is the momentum canonically conjugate to x^M, and the Hamiltonian H of the system has the following structure

$$H = e(\tau)H_0 + i\lambda^a(\tau)\overline{Q}_a + i\overline{\lambda}_a(\tau)Q^a - t_i(\tau)L_i, \tag{3.10}$$

with

$$H_0 = \frac{1}{2} g^{MN} p_M p_N + R_{LN,PM}(\overline{\psi}_a^L \overline{\psi}^{Ma})(\psi^{Nb}\psi_b^P) + R_{MP,NL}(\overline{\psi}_a^L \psi^{Ma})(\overline{\psi}_b^N \psi^{Pb}) + \mathcal{D}_L G_{MNP}(\overline{\psi}_a^L \overline{\psi}^{Ma})(\psi^{Nb}\psi_b^P), \tag{3.11}$$

$$\overline{Q}_a = \overline{\psi}_a^M p_M + i\Gamma_{LMN}\overline{\psi}_c^L \overline{\psi}^{Mc}\psi_a^N, \tag{3.12}$$

$$Q^b = \psi^{Ib} p_I + i\Gamma_{LMN}\overline{\psi}^{Lb}\psi^{Mc}\psi^N_c \tag{3.13}$$

and

$$L_i = g_{MN}\psi^{Nb}(\sigma_i)_b{}^a\overline{\psi}^M_a \tag{3.14}$$

being associated with constraints on the dynamics of the relativistic particle caused by the worldline superreparametrization invariance of the model. The constraints are of the first class since they form a closed $N = 4$ supersymmetry algebra

$$\{\overline{Q}_a, Q^b\} = -i\delta^b_a H_0, \quad [L_i, L_j] = \epsilon_{ijk}L_k,$$

$$[L_i, \overline{Q}_a] = \frac{i}{2}(\sigma_i)_a{}^c\overline{Q}_c, \quad [L_i, Q^a] = -\frac{i}{2}(\sigma_i)_c{}^a Q^c \tag{3.15}$$

with respect to the following graded Dirac brackets [3] (which are obtained upon solving for the second class constraints on the canonical fermionic momenta $\pi_{Ma} = -i\overline{\psi}_{Ma}$ and $\overline{\pi}^a_M = -i\psi^a_M$ derived from eq. (3.9))

$$[x^M, p_N] = \delta^M_N, \quad \{\psi^{aM}, \overline{\psi}^N_b\} = -\frac{i}{2}\delta^a_b g^{MN}, \quad [p_M, p_N] = 2iR_{MN,PL}\overline{\psi}^P_a\psi^{aL},$$

$$[p_M, \psi^a_N] = \Gamma_{MNP}\psi^{aP}, \quad [p_M, \overline{\psi}^a_N] = \Gamma_{MNP}\overline{\psi}^{aP}, \tag{3.16}$$

We observe that p_M have properties of covariant momenta when acting on fermionic variables ψ^{Ma} and $\overline{\psi}^M_a$.

The superalgebra (3.15) of the constraints (3.11)–(3.14) generates the $SU(2)_{local} \times SU(2)_{global}$ $N = 4$ superconformal transformations (2.3) of the components of the superfields Φ^M.

We have thus shown that the $N = 4$ worldline superfield action (3.1), which reduces to (3.9)–(3.14) upon integrating over Grassmann–odd coordinates and eliminating auxiliary fields, describes the dynamics of an $N = 4$ superconformal spinning particle in a curved background whose geometry is characterized by eqs. (3.7)–(3.8).

We should note that the last terms in (3.11)–(3.13), containing the Christoffel connection, are non-covariant with respect to general coordinate transformations of the background. The reason is that background diffeomorphisms acting on the superfields Φ^M, in general, are incompatible with the constraints (2.4)–(2.5). This, in particular, means that if a background metric (3.6) admits isometries, not all of them will be symmetries of the actions (3.1) and (3.9). It is an interesting open problem to study whether the model under consideration can be modified in such a way that only target–space covariant terms remain in the action.

4 The potential structure of the anti–de–Sitter metric

It is curiously enough that the anti–de–Sitter spaces belong to the class of the manifolds whose metric in a certain coordinate system acquires the form (1.1). To

show this consider first a coordinate system

$$X^M = (X^\mu, \rho), \quad \mu = 0, ..., D-2 \tag{4.1}$$

in which the metric of a D–dimensional AdS space has a conformally flat form (for simplicity we put the AdS radius to one)

$$ds^2 = \frac{1}{\rho^2}\left(\eta_{\mu\nu} dX^\mu dX^\nu + d\rho^2\right), \tag{4.2}$$

where $\eta_{\mu\nu} = (-1, 1, ..., 1)$.

Now perform a coordinate transformation to the new set of variables

$$x^M = (x^\mu, r) \tag{4.3}$$

such that

$$X^\mu = \frac{x^\mu}{r}, \quad \rho = \frac{1}{\sqrt{r}}. \tag{4.4}$$

The passage from ρ to r has proved to be convenient for the analysis of the properties of the potential $A(x)$ considered below.

In the coordinate system (4.3) the AdS metric g_{MN} takes the form

$$g_{\mu\nu} = \frac{\eta_{\mu\nu}}{r}, \quad g_{\mu r} = -\frac{\eta_{\mu\nu} x^\nu}{r^2}, \quad g_{rr} = \frac{x^\mu x^\nu \eta_{\mu\nu}}{r^3} + \frac{1}{4r^2}, \tag{4.5}$$

where the index r of the metric tensor components corresponds to the coordinate r.

One can easily check that the metric (4.5) is a second derivative of the following function

$$A(x) = \frac{x^\mu \dot{x}^\nu \eta_{\mu\nu}}{2r} - \frac{1}{4} \ln r. \tag{4.6}$$

Thus we have shown that AdS_D is one of the manifolds of the type (1.1), where the $N = 4$ superconformal spinning particle can live.

By passing note that if in the action (3.1) we take $A(E\Phi^M)$ in the form (4.6) and put $\Phi^\mu = 0$ and $\Phi^r = 1$ we shall arrive at the action

$$S = 2 \int d\tau d^2\theta d^2\bar{\theta} \, \frac{\ln E}{E}$$

which describes a one–dimensional $N = 4$ superconformal mechanics considered in [5].

The potential (4.6) generating the metric on AdS_D is not unique. Another form of the potential arises when one performs the following change of variables (4.1)

$$X^\mu = \left(\frac{x^\mu}{r}\right)^{m_\mu} \quad \rho = \frac{1}{\sqrt{r}}, \tag{4.7}$$

where $m_\mu \neq 0, \frac{1}{2}$ is a set of real numbers, namely

$$A = -\frac{m_0^2}{2m_0(2m_0-1)}\frac{(x^0)^{2m_0}}{r^{2m_0-1}} + \sum_{i=1}^{D-2}\frac{m_i^2}{2m_i(2m_i-1)}\frac{(x^i)^{2m_i}}{r^{2m_i-1}} - \frac{1}{4}\ln r. \tag{4.8}$$

More generally, we could make, for instance, a "logarithmic transformation"

$$X^\mu = \ln(\frac{x^\mu}{r}) \quad \rho = \frac{1}{\sqrt{r}}, \tag{4.9}$$

for which the corresponding potential has the form

$$A = r\ln\frac{x^0}{r} - \sum_{i=1}^{D-2} r\ln\frac{x^i}{r} - \frac{1}{4}\ln r. \tag{4.10}$$

The coordinate transformation (4.4) is singled out by the requirement that it is a single–valued and that a Lorentz subgroup $SO(1, D-2)$ of the AdS_D isometry group $SO(2, D-1)$ acts linearly on both the "old" coordinates X^μ of (4.1) and the "new" coordinates x^μ of (4.3), (4.4).

So we shall further discuss some amusing properties of AdS_D associated with its potential structure (4.6) in the coordinate system (4.3).

The group $SO(2, D-1)$ of the isometry transformations of AdS coordinates, which leave the form of the AdS metric invariant, is known to act as a conformal group on a $(D-1)$-dimensional boundary of AdS_D. In the coordinate system (4.1) the boundary (which is a $(D-1)$–dimensional Minkowski space) is associated with the coordinates X^μ. Under infinitesimal $SO(2, D-1)$ transformations X^μ and ρ vary as follows

$$\begin{aligned} \delta X^\mu &= a^\mu + a^{\mu\nu}X^\lambda\eta_{\nu\lambda} + a_D X^\mu + a_K^\mu X^\nu X^\lambda\eta_{\nu\lambda} - 2(a_K^\nu X^\lambda\eta_{\nu\lambda})X^\mu + a_K^\mu\rho^2, \\ \delta\rho &= -(2a_K^\mu X^\nu\eta_{\mu\nu} - a_D)\rho, \end{aligned} \tag{4.11}$$

where the $SO(2, D-1)$ parameters a^μ, $a^{\mu\nu}$, a_D, a_K^μ are, respectively, the parameters of $D-1$ translations, $SO(1, D-2)$ rotations, dilatation and conformal boosts, acting as conformal transformations in a $(D-1)$–dimensional slice of AdS_D parametrized by X^μ.

From (4.4) and (4.11) one gets the infinitesimal $SO(2, D-1)$ transformations of x^μ and r (4.3)

$$\begin{aligned} \delta x^\mu &= a^\mu r + a^{\mu\nu}x^\lambda\eta_{\nu\lambda} - a_D x^\mu + a_K^\mu\frac{x^\nu x^\lambda\eta_{\nu\lambda}}{r} + 2(a_K^\nu x^\lambda\eta_{\nu\lambda})\frac{x^\mu}{r} + a_K^\mu, \\ \delta r &= 4a_K^\mu x^\nu\eta_{\mu\nu} - 2a_D r. \end{aligned} \tag{4.12}$$

Under (4.12) the potential (4.6) varies as follows

$$A(x') = A(x) + \delta A(x),$$

$$\delta A(x) = \delta x^M \partial_M A(x) = a^\mu x^\nu \eta_{\mu\nu} + \frac{1}{2} a_D + a_K^\mu x^\nu \eta_{\mu\nu} \frac{x^\lambda x^\sigma \eta_{\lambda\sigma}}{r^2}. \qquad (4.13)$$

One can check that the form of the metric (4.4) remains invariant under the action of the $SO(2, D-1)$ transformations (4.12), so that they are indeed the isometries of this AdS metric. However, the superfield action (3.1) is invariant only under the subgroup of $SO(2, D-1)$ generated by $D-1$ translations a^μ, $SO(1, D-2)$ Lorents rotations $a^{\mu\nu}$ and dilatations a_D which transform the superfields $E\Phi^M$ in the same way as x^M in (4.12). As can be seen from the form of the variation of x^M (and respectively of $E\Phi^M$) with respect to conformal boosts a_K^μ, the corresponding term does not satisfy the superfield constraints (2.4) and (2.5), and, hence, the transformed Φ^M will not do so as well. This is the reason of the appearance of noncovariant terms depending on the Christoffel connection in the component actions (3.3) and (3.9)–(3.14).

An interesting property of the potential (4.6) is that the contraction of its partial derivatives with the coordinates (4.3) are constants starting from the second derivative

$$x^M x^N \partial^2_{MN} A(x) = x^M x^N g_{MN} = \frac{1}{4}$$

$$x^L x^M x^N \partial^3_{LMN} A(x) = 2 x^L x^M x^N \Gamma_{LMN} = -\frac{1}{2}$$

$$x^{M_1} \ldots x^{M_{n+1}} \partial^{n+1}_{M_1 \ldots M_{n+1}} A(x) = (-1)^{n+1} \frac{n+1}{4}, \quad n = 1, \ldots, \infty. \qquad (4.14)$$

To get the relation (4.14) one should note that under the following rescaling of the coordinates (4.3) $x^M \to (1+\epsilon) x^M$ (where ϵ is a numerical parameter)[3] the potential (4.6) takes the form

$$\begin{aligned} A_\epsilon &= (1+\epsilon) \frac{x^\mu x^\nu \eta_{\mu\nu}}{2r} - \frac{1}{4} \ln r - \frac{1}{4} \ln (1+\epsilon) \\ &= (1+\epsilon) \frac{x^\mu x^\nu \eta_{\mu\nu}}{2r} - \frac{1}{4} \ln r - \frac{1}{4} \epsilon + \frac{1}{4} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n+1}{(n+1)!} \epsilon^{n+1}, \end{aligned} \qquad (4.15)$$

where on the right hand side of (4.15) we have expanded $\ln(1+\epsilon)$ in series of ϵ.

On the other hand

$$A_\epsilon = A(x+\epsilon x) = A(x) + \epsilon x^M \partial_M A(x) + \sum_{n=1}^{\infty} \frac{\epsilon^{n+1}}{(n+1)!} x^{M_1} \ldots x^{M_{n+1}} \partial^{n+1}_{M_1 \ldots M_{n+1}} A(x). \qquad (4.16)$$

Comparing (4.15) with (4.16) we get (4.14).

A local basis in a tangent space of the AdS_D manifold can be described by the following vielbein one–form $e^A = dx^M e_M^{\hat{A}}(x)$ $(A = 0, 1 \ldots, D-1)$

$$e^\alpha = dx^\mu \, \delta_\mu^\alpha r^{-\frac{1}{2}}, \quad e^r = -dx^\mu \, x_\mu r^{-\frac{3}{2}} + dr \frac{1}{2r} \qquad (4.17)$$

[3] One should not confuse this rescaling with dilatation isometry (4.12) which acts as follows $x^\mu \to (1+\epsilon) x^\mu$ and $r \to (1+\epsilon)^2 r$.

determined such that $g_{MN} = e_M^A e_N^B \eta_{AB}$ and $\eta_{AB} = (-, + \ldots, +)$.

Using (4.17) it is easy to calculate the determinant of the metric (4.4)

$$\det g_{MN} = -(\det e_M^A)^2 = -\frac{1}{4r^{D+1}}. \tag{4.18}$$

One more observation concerns the form of the covariant derivative of the AdS_D Christoffel connection (3.7) appeared in (3.11). A direct computation results in the following relation

$$\mathcal{D}_L \Gamma_{MNP} = \frac{1}{8}\partial^4_{LMNP} A(x) - g_{LM}g_{NP} - g_{MN}g_{LP} - g_{NL}g_{MP}. \tag{4.19}$$

We see that noncovariance of (4.19) is in a certain sense "concentrated" in the fourth partial derivative of $A(x)$.

Note that the results of this section do not depend on the signature of the metric $\eta_{\mu\nu}$ in (4.2). For instance, we could equally well choose $\eta_{\mu\nu}$ to be Euclidean. Then we would deal with Euclidean AdS, or hyperbolic spaces considered recently in the context of string and M–theory compactifications [22, 23].

5 Discussion

To conclude, in this paper we have considered the classical dynamics of a spinning particle governed by the action invariant under the $SU(2)_{local} \times SU(2)_{global}$ $N = 4$ superconformal transformations of the particle superworldline. We have shown that the $N = 4$ superconformal invariance allows the particle to propagate in a curved background with a "Kähler–like" metric generated by a real superpotential A, and we have found that the anti–de–Sitter and hyperbolic spaces belong to this class of manifolds.

There are several directions of the extension of the results of this paper. One of them is the quantum description of the $N = 4$ superconformal particle model, which can be carried out following either the lines of [2, 3] or using path integral quantization methods. The latter procedure seems to be more attractive, since it may lead to deeper understanding of the model, for instance, in the context of the AdS/CFT correspondence conjecture.

In particular, it is interesting to study both the classical and quantum dynamics of the $N = 4$ superconformal spinning particle moving in backgrounds which are direct products of AdS_D and Kähler manifolds. Particle motion on the Kähler manifolds can be described by making a multidimensional generalization of the $N = 4$ supersymmetric quantum mechanics considered in [25]. For this, in addition to Φ^M, one should introduce a number of chiral superfields $\Psi^n(\tau, \theta, \bar{\theta})$ $(\overline{D}^a \Psi^n = 0)$

$$\begin{aligned} \Psi^n(\tau, \theta, \bar{\theta}) &= z^n(\tau) + \theta^a \chi^n_a(\tau) + \frac{i}{2}\bar{\theta}\theta \dot{z}^n(\tau) + \theta\theta F^n(\tau) \\ &\quad - \frac{i}{4}\theta\theta\bar{\theta}_a \dot{\chi}^{na}(\tau) - \frac{1}{16}\bar{\theta}\bar{\theta}\theta\theta \ddot{z}^n(\tau), \end{aligned} \tag{5.1}$$

and their complex conjugate antichiral superfields $\overline{\Psi}^n(\tau, \theta, \overline{\theta})$. The superfields Ψ and $\bar{\Psi}$ transform as scalars under the $N = 4$ superconformal transformations (2.1).

We can add to the action (3.1) the following $N = 4$ superconformal invariant action constructed from Ψ^n and $\bar{\Psi}^n$

$$S_K = 2 \int d\tau d^2\theta d^2\overline{\theta} \frac{1}{E} K(\Psi, \overline{\Psi}), \tag{5.2}$$

where K is a Kähler superpotential.

When the superpotential $A(E\Phi^M)$ is chosen in the form (4.6), the sum of the actions (3.1) and (5.2) describes a spinning particle propagating in an $AdS_D \times K_{2n}$ background, where K_{2n} is a Kähler manifold with a metric (1.2). For instance, the case $n = 1$ and $K_2 = \ln(1 + \Psi\overline{\Psi})$ corresponds to a two-dimensional sphere S^2, which is known to be a Kähler manifold. A detailed analysis of these models will be given elsewhere.

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$N = 4$ суперконформная механика
и потенциальная структура пространства AdS

Описана динамика спиновой частицы на искривленном фоне с использованием суперполевого формализма. Из требования $SU(2)_{local} \times SU(2)_{global}$ $N = 4$ суперконформной симметрии действия частицы следует, что данный фон должен быть вещественным «кэлеро-подобным» многообразием. Метрика данного многообразия генерируется сигма-модельным суперпотенциалом. Показано, что пространства анти-де-Ситтера принадлежат к данному классу многообразий.

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$N = 4$ Superconformal Mechanics
and the Potential Structure of AdS Spaces

The dynamics of an $N = 4$ spinning particle in a curved background is described using the $N = 4$ superfield formalism. The $SU(2)_{local} \times SU(2)_{global}$ $N = 4$ superconformal symmetry of the particle action requires the background to be a real «Kähler-like» manifold whose metric is generated by a sigma-model superpotential. The anti-de-Sitter spaces are shown to belong to this class of manifolds.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

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