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ANALYTIC PERTURBATION THEORY IN ANALYZING SOME **QCD** OBSERVABLES

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1 Preamble

Usually, the perturbative QCD part of the theoretical contribution to observables in both the space– and time–like channels is presented in the form of two– or three–term power expansion

$$\frac{O(x)}{O_0} = 1 + r(x)\,; \quad r(x) = c_1\,\bar{\alpha}_s(x) + c_2\,\bar{\alpha}_s^2 + c_3\,\bar{\alpha}_s^3 + \ldots\,; \quad x = Q^2 \ \text{or} = s \tag{1}$$

(our coefficients are normalized $c_k = C_k\,\pi^{-k}$ differently from the commonly adopted C_k, like in Refs.[1, 2, 3]) over powers of the effective QCD coupling $\bar{\alpha}_s$ which is supposed *ad hoc* to be of the same form as in both the channels, e.g., in the massless three–loop case

$$\bar{\alpha}_s^{(3)}(x) = \frac{1}{\beta_0 L} - \frac{b_1}{\beta_0^2}\frac{\ln L}{L^2} + \frac{1}{\beta_0^3 L^3}\left[b_1^2(\ln^2 L - \ln L - 1) + b_2\right] +$$

$$+ \frac{1}{\beta_0^4 L^4}\left[b_1^3\left(-\ln^3 L + \frac{5}{2}\ln^2 L + 2\ln L - \frac{1}{2}\right) - 3b_1 b_2 \ln L + \frac{b_3}{2}\right]. \tag{2}$$

Here, $L = \ln(x/\Lambda^2)$, and for the beta-function coefficients we use the normalization

$$\beta(\alpha) = -\beta_0\,\alpha^2 - \beta_1\,\alpha^3 - \beta_2\,\alpha^4 + \ldots = -\beta_0\,\alpha^2\left(1 + b_1\,\alpha + b_2\,\alpha^2 + \ldots\right),$$

that is also free of π powers. Numerically, they all are of an order of unity

$$\beta_0(f) = \frac{33 - 2f}{12\pi}\,; \quad b_1(f) = \frac{153 - 19f}{2\pi(33 - 2f)}$$

$$\beta_0(4 \pm 1) = 0.875 \pm 0.005\,; \ b_1(4 \pm 1) = 0.490^{-0.089}_{+0.076}\,.$$

Meanwhile, the RG notion of invariant coupling was first introduced in QED [4] in the space-like region in terms of a real constant z_3 of the finite Dyson renormalization transformation. Just this QED Euclidean invariant charge $\bar{e}(Q)$ is the Fourier transform of the space distribution $e(r)$ of electric charge (arising due to vacuum fluctuations around a point "bare" electron) discussed by Dirac [5] in 30s — see Appendix IX in the textbook [6].

Generally, in the RG formalism (for details, see, e.g., the chapter "Renormalization group" in the monograph [7] and/or Section 1 in Ref.[8])

the notion of invariant coupling $\bar{g}(Q)$ is *defined only in the space–like domain.*

In particular, this means that if some observable is physically a function of one kinematic Lorentz–invariant space–like argument Q^2, then, due to its renormalization invariance, it should be a function of RG invariants only. E.g., in the one–coupling massless case

$$O(Q^2/\mu^2, g_\mu) = F\left(\bar{g}(Q^2/\mu^2, g_\mu)\right) \quad \text{with} \quad F(g) = O(1, g)\,.$$

Due to this important property, in the weak coupling case we deal with the functional expansion of F in powers of invariant coupling $\bar{g}$. This is a real foundation of QCD power expansion (1) in the Euclidean case with $x = Q^2$. At the same time, inside the RG formalism, there is no natural means for defining invariant coupling $\tilde{g}(s)$ and perturbative expansion for an observable $O(s)$ in the time–like region.

Nevertheless, in modern practice, people commonly use the same singular expression for the QCD effective coupling $\bar{\alpha}_s$, like (2), in both the space– and time–like domains. The only price usually paid for this transferring from the Euclidean to Minkowskian region is the change of numerical expansion coefficients. The time–like ones $c_{k\geq 3} = d_k - \delta_k$ include negative "π^2 terms" proportional to π^2 and lower c_k

$$\delta_3 = \frac{(\pi\beta_0(f))^2}{3}\, c_1\,, \quad \delta_4 = (\pi\beta_0)^2\,(c_2 + \frac{5}{6}\, b_1\, c_1) \;\ldots\,. \tag{3}$$

These (rather essential, as far as $\pi^2\beta_0^2(4\pm 1) = 4.340^{-.666}_{+.723}$) structures δ_k arise [9] — [12] in the course of analytic continuation from the Euclidean to Minkowskian region. The coefficients d_k should be treated as genuine k th–order ones. Just they have to be calculated with the help of the relevant Feynman diagrams.

Table 1

Expansion coefficients in the Euclidean d_i and Minkowskian c_k regions and their differences.

Process	f	c_1/d_1	c_2/d_2	c_3	d_3	δ_3	δ_4
τ decay	3	$1/\pi$	.53	0.852	1.39	0.54	5.01
e^+e^-	4	.318	.16	- 0.35	0.11	0.46	2.45
e^+e^-	5	.318	.14	- 0.41	- 0.02	0.39	1.75
Z_0 decay	5	.318	.09	- 0.48	- 0.09	0.39	1.58

with $\qquad d_3 = c_3 + \delta_3$

To demonstrate the importance of "π^2 terms", we have considered the three–flavor case for τ–decay, the $f = 4\,,5$ cases for $e^+e^- \to$ hadron annihilation and the Z_0 decay (with $f = 5$) — see Table 1 in which we also give values for the π^2–terms. In the normalization (1), all coefficients c_k, d_k and δ_k are of an order of unity. In the $f = 4, 5$ region the contribution δ_3 prevails in c_3 and $|d_3| \ll |c_3|$ (see also Table II in Bjorken's review [11]).

Concerning the fundamental inconsistency of current perturbative QCD practice connected with unphysical singularities, take the well-known relation between the so–called Adler function D and total cross–section ratio R of a related process

$$D(Q^2) = Q^2 \int_0^\infty \frac{R(s)\, ds}{(s+Q^2)^2}\,. \tag{4}$$

In the case of inclusive e^+e^- annihilation into hadrons, $R(s)$ is the ratio of cross–sections presented in the form $R(s) = 1 + r(s)$ with a function r expandable in powers of $\bar{\alpha}_s(s)$ like in Eq.(1). At the same time, the Adler function is also used to be presented in the form $D = 1 + d$ with d expanded in powers of $\bar{\alpha}_s(Q^2)$.

Here, we face two paradoxes. First, $\bar{\alpha}_s(Q^2)$ (and its powers) and, hence, the perturbative $D(Q^2)$ obeys non-physical singularity at $Q^2 = \Lambda^2$ in evident contradiction with representation (4). Second, the integrand $R(s)$, being expressed via powers of $\bar{\alpha}_s(s)$, obeys non-integrable singularities at $s = \Lambda^2$, which makes the r.h.s. of eq.(4) senseless.

This second problem is typical of inclusive cross–sections, e.g., for the τ hadronic decay. Generally, in the current literature it is treated in a very strange way — by shifting the contour of integration from the real axis with strong singularities to a complex plane. However, such a "physical" trick cannot be justified within the theory of complex variable.

Meanwhile, as it is known from the early 80s, the perturbation representation (1) for the Minkowskian observable with the coefficients modified by the π^2–terms is valid only at a small parameter $\pi^2/\ln^2(s/\Lambda^2)$ values, that is in the region of sufficiently high energies $W \equiv \sqrt{s} \gg \Lambda e^{\pi/2} \simeq 2\,\mathrm{GeV}$.

Here, it is appropriate to remind the construction devised by Radyushkin [9] and Krasnikov—Pivovarov [10] (RKP procedure) about 20 years ago. These authors used the integral transformation

$$R(s) = \frac{i}{2\pi} \int_{s-i\varepsilon}^{s+i\varepsilon} \frac{dz}{z}\, D_{\mathrm{pt}}(-z) \equiv \mathbf{R}\left[D(Q^2)\right] \tag{5}$$

reverse to the Adler relation (4) (that is treated now as integral transformation)

$$R(s) \to D(Q^2) = Q^2 \int_0^\infty \frac{R(s)\,ds}{(s+Q^2)^2} \equiv \mathbf{D}\left\{R(s)\right\} \tag{6}$$

for defining modified expansion functions

$$\mathfrak{A}_k(s) = \mathbf{R}[\alpha_s^k(Q^2)] \tag{7}$$

for the perturbative QCD contribution

$$r(s) = d_1\mathfrak{A}_1(s) + d_2\mathfrak{A}_2(s) + d_3\mathfrak{A}_3(s) \tag{8}$$

to an observable in the time–like region.

At the one-loop level, with the effective coupling $\bar{\alpha}_s^{(1)} = \left[\beta_0 \ln(Q^2/\Lambda^2)\right]^{-1}$ one has

$$\mathfrak{A}_1^{(1)}(s) = \mathbf{R}\left[\bar{\alpha}_s^{(1)}\right] = \frac{1}{\pi\beta_0}\arccos\frac{L}{\sqrt{L^2+\pi^2}} = \frac{1}{\beta_0}\left[\frac{1}{2} - \frac{1}{\pi}\arctan\frac{L}{\pi}\right] \tag{9}$$

(with $L = \ln(s/\Lambda^2)$) and for higher functions

$$\mathfrak{A}_2^{(1)}(s) = \frac{1}{\beta_0^2\left[L^2+\pi^2\right]}\,; \quad \mathfrak{A}_3^{(1)}(s) = \frac{L}{\beta_0^3\left[L^2+\pi^2\right]^2}\,, \tag{10}$$

which *are not powers* of $\mathfrak{A}_1^{(1)}(s)$.

The r.h.s of (9) at $L \geq 0$ can also be presented in the form

$$\mathfrak{A}_1^{(1)}(s) = \frac{1}{\pi\beta_0}\arctan\frac{\pi}{L} \tag{9a}$$

convenient for the UV analysis. Just this form was discovered in the early 80s in Refs.[13] and [9], while eqs.(10) in Refs.[9] and [10]. All these papers dealt with HE behavior and did not pay proper attention to the region $L \leq 0$. In particular, expression (9) was first discussed only 15 years later by Milton and Solovtsov [14]. These authors made an important observation that expression (9) represents a continuous monotone function without unphysical singularity at $L = 0$ and proposed to use it as an effective "Minkowskian QCD coupling" $\tilde{\alpha}(s) \equiv \mathfrak{A}_1(s)$ in the time–like region.

For the two–loop case, to the popular approximation

$$\beta_0\bar{\alpha}_{s,pop}^{(2)}(Q^2) = \frac{1}{l} - b_1(f)\frac{\ln l}{l^2}\,; \quad l = \ln\frac{Q^2}{\Lambda^2}$$

there corresponds [9, 15]

$$\tilde{\alpha}^{(2)}_{pop}(s) \equiv \mathfrak{A}_1^{(2,pop)}(s) = \left(1 + \frac{b_1 L}{L^2+\pi^2}\right)\tilde{\alpha}^{(1)}(s) - \frac{b_1}{\beta_0}\frac{\ln\left[\sqrt{L^2+\pi^2}\right]+1}{L^2+\pi^2}\,. \tag{11}$$

At $L \gg \pi$, by expanding this expression and $\mathfrak{A}_2$ from (10) in powers of π^2/L^2 we arrive at the π^2–terms (3).

Both the functions (9) and (11) are monotonically decreasing with a finite IR value $\tilde{\alpha}(0) = 1/\beta_0(f=3) \simeq 1.4$. They have no singularity at $L=0$. Higher functions go to zero $\mathfrak{A}_k(0) = 0$ in the IR limit.

As it has first been noticed in [16, 17], by applying the transformation $\mathbf{D}$ (6) to functions $\mathfrak{A}_k(s)$, instead of $\bar{\alpha}_s(Q^2)$ powers, we obtain expressions $\mathbf{D}[\mathfrak{A}_k(s)] = \mathcal{A}_k(Q^2)$ that are also free of unphysical singularities. These functions have been discussed recently [18] — [23] in the context of the so–called "Analytic approach" to perturbative QCD.

Therefore, this Analytic approach in the Euclidean region and the RKP formulation for Minkowskian observables can be united in the single scheme, the "Analytic Perturbation Theory" — APT, that has been formulated quite recently in our papers [16] and [17]. In the next Section, we give a short resume of this APT construction and then, in Sections 3 and 4, present the results of its practical applications.

2 The APT — a closed theoretical scheme

The APT scheme closely relates two ghost–free formulations of modified perturbation expansion for observables.

2.1 Relation between Euclidean and Minkowskian

The first one, that was initiated in the early eighties [9, 10] and outlined above, changes the standard power expansion (1) in the time-like region into the nonpower one (8). It uses operation Eq.(5), that is reverse $\mathbf{R} = [\mathbf{D}]^{-1}$ to the one defined by the "Adler relation" (6) and transforms a real function $R(s)$ of a positive (time–like) argument into a real function $D(Q^2)$ of a positive (space–like) argument.

By operation $\mathbf{R}$, one can define [14] the RG–invariant Minkowskian coupling $\tilde{\alpha}(s) = \mathbf{R}\left[\bar{\alpha}_s\right]$, and its "effective powers" (7) that are free of ghost singularities. Some examples are given by expressions (9), (10) and

(11). At the one–loop level, they are related by the differential recursion relation $k\beta_0 \mathfrak{A}^{(1)}_{k+1} = -(d/dL)\mathfrak{A}^{(1)}_k$ and *are not powers* of $\mathfrak{A}^{(1)}_1$.

By applying $\mathbf{D}$ to $\mathfrak{A}_k(s)$, one can "try to return" to the Euclidean domain. However, instead of α_s powers, we arrive at some other functions

$$\mathcal{A}_k(Q^2) = \mathbf{D}\left[\mathfrak{A}_k\right] , \tag{12}$$

analytic in the cut Q^2-plane and free of ghost singularities. At the one–loop case

$$\beta_0 \mathcal{A}^{(1)}_1(Q^2) = \frac{1}{\ln(Q^2/\Lambda^2)} - \frac{\Lambda^2}{Q^2-\Lambda^2} ,$$

$$\beta_0^2 \mathcal{A}^{(1)}_2(Q^2) = \frac{1}{\ln^2(Q^2/\Lambda^2)} + \frac{Q^2\Lambda^2}{(Q^2-\Lambda^2)^2} , \quad \ldots . \tag{13}$$

These expressions have been originally obtained by other means [18, 19] in the mid–90s. The first function $\mathcal{A}_1 = \alpha_{\rm an}(Q^2)$, an analytic invariant Euclidean coupling, should now be treated as a *counterpart* of the invariant Minkowskian coupling $\tilde{\alpha}(s) = \mathfrak{A}_1(s)$. Both $\alpha_{\rm an}$ and $\tilde{\alpha}$ are real monotonically decreasing functions with the same maximum value

$$\alpha_{\rm an}(0) = \tilde{\alpha}(0) = 1/\beta_0(f=3) \simeq 1.4$$

in the IR limit[1].

All higher functions vanish $\mathcal{A}_k(0) = \mathfrak{A}_k(0) = 0$ in this limit. For $k \geq 2$, they oscillate in the IR region and form [24, 25] an asymptotic sequence à lá Erdélyi.

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[1]Note that the transition from the usual invariant $\overline{\rm MS}$ coupling α_s to the Minkowskian $\tilde{\alpha}$ and Euclidean $\alpha_{\rm an}$ ones can be understood as a transformation to new renormalization schemes. At the one–loop case

$$\alpha_s \to \tilde{\alpha}^{(1)} = \frac{1}{\pi\beta_0}\arctan(\pi\beta_0\alpha_s) \quad \text{and} \quad \alpha_s \to {\alpha_{\rm an}}^{(1)} = \alpha_s + \frac{1}{\beta_0}\left(1 - e^{1/\beta_0\alpha_s}\right)^{-1} . \tag{14}$$

Here, the first transition looks "quite usual" as $\tilde{\alpha}$ can be expanded in powers of α_s , while the second one in the weak coupling case behaves like the identity transformation as far as the second nonperturbative term $e^{-1/\beta_0\alpha_s}$ leaves no "footsteps" in the power expansion.

For both $\tilde{\alpha}^{(1)}$ and ${\alpha_{\rm an}}^{(1)}$ the corresponding β functions have zero at $\alpha = 1/\beta_0$ and are symmetric under reflection $[\alpha - 1/2\beta_0] \to -[\alpha - 1/2\beta_0]$. Moreover, the β function for $\tilde{\alpha}(s)$ turns out to be equal to the spectral function for $\alpha_{\rm an}(Q^2)$ – see below Eq.(18) at $k=1$.

The same properties remain valid for a higher–loop case. Explicit expressions for $\mathcal{A}_k$ and $\mathfrak{A}_k$ at the two–loop case can be written down (see, Refs. [26] and [27]) in terms of a special Lambert function. They are illustrated below in Figs 1a and 1b. Note here that to relate Euclidean and Minkowskian functions, instead of integral expressions (5) and (6) one can use simpler relations, in terms of spectral functions $\rho(\sigma) = \Im \mathcal{A}(-\sigma)$,

$$\mathcal{A}_k(Q^2; f) = \frac{1}{\pi} \int_0^\infty \frac{d\sigma}{\sigma + Q^2} \, \rho_k(\sigma; f) ; \quad \mathfrak{A}_k(s; f) = \frac{1}{\pi} \int_s^\infty \frac{d\sigma}{\sigma} \rho_k(\sigma; f) , \quad (15)$$

equivalent to expressions $\mathcal{A}_k(Q^2) = \mathbf{D}\,[\mathfrak{A}_k]$, and $\mathfrak{A}_k(s) = \mathbf{R}[\mathcal{A}_k]$.

Remarkably enough, the mechanism of liberation of unphysical singularities is quite different. While in the space-like domain it involves nonperturbative, power in Q^2, structures, in the time-like region it is based only upon resummation of the "π^2 terms". Figuratively, (non-perturbative !) *analyticization* [18, 19, 25] in the Q^2–channel can be treated as a quantitatively distorted reflection (under $Q^2 \to s = -Q^2$) of (perfectly perturbative) π^2–resummation in the s–channel. This effect of "distorting mirror", first discussed in [14] and [28], is clearly seen in the figures 1a,b mentioned above.

This means also that introduction of nonperturbative $1/Q^2$ structures now has got *another motivation, Eq.(12), independent of the analyticization prescription.*

2.2 Global APT

In reality, a physical domain includes regions with various "numbers of active quarks", i.e., with different flavor numbers $f = 3, 4, 5$ and 6. In each of these regions, we deal with a different amount of quark quantum fields, that is with different QFT models and different Lagrangians. To combine them into a joint picture, the procedure of the threshold matching is in use. It establishes relations between renormalization procedures for a model with different f values.

For example, in the $\overline{\text{MS}}$ scheme the matching relation has a simple form

$$\bar{\alpha}_s(Q^2 = M_f^2; f-1) = \bar{\alpha}_s(Q^2 = M_f^2; f) . \quad (16)$$

It defines a "global effective coupling"

$$\bar{\alpha}_s(Q^2) = \bar{\alpha}_s(Q^2; f) \quad \text{at} \quad M_{f-1}^2 \le Q^2 \le M_f^2 ,$$

continuous in the space-like region of positive Q^2 values with discontinuity of derivatives at matching points $Q^2 = M_f^2$. To this global $\bar{\alpha}_s$, there corresponds a discontinuous spectral density

$$\rho_k(\sigma) = \rho_k(\sigma;3) + \sum_{f\geq 4} \theta(\sigma - M_f^2) \{\rho_k(\sigma;f) - \rho_k(\sigma;f-1)\} \tag{17}$$

with $\rho_k(\sigma;f) = \Im\, \bar{\alpha}_s^k(-\sigma, f)$ which yields [16, 17] via relations analogous to (15)

$$\mathcal{A}_k(Q^2) = \frac{1}{\pi}\int_0^\infty \frac{d\sigma}{\sigma + Q^2}\, \rho_k(\sigma)\,; \quad \mathfrak{A}_k(s) = \frac{1}{\pi}\int_s^\infty \frac{d\sigma}{\sigma}\rho_k(\sigma)\,, \tag{18}$$

the smooth global Euclidean and spline–continuous global Minkowskian expansion functions.

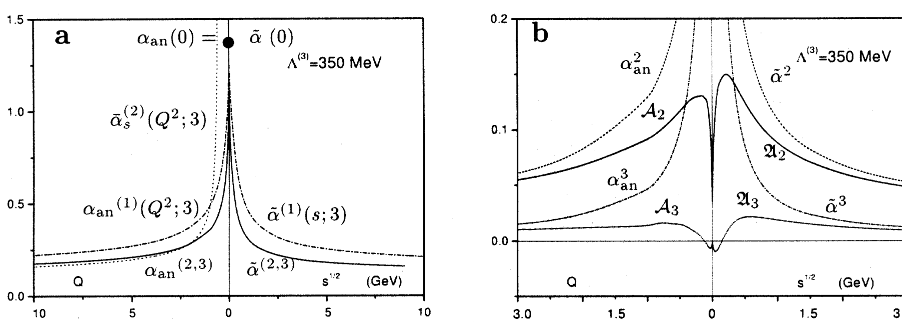

Figure 1: **a** – *Space-like and time-like global analytic couplings in a few GeV domain with $f = 3$ and $\Lambda^{(3)} = 350$MeV;* **b** – *"Distorted mirror symmetry" for global expansion functions. All the curves in* **1b** *correspond to exact two–loop solutions expressed in terms of Lambert function.*

Here, in Fig.1a, by the dotted line we give a usual two-loop effective QCD coupling $\bar{\alpha}_s(Q^2)$ with a singularity at $Q^2 = \Lambda^2$. On the other hand, the dash–dotted curves represent the one-loop APT expressions (9) and (13). The solid APT curves are based on the exact two-loop solutions of RG equations and approximate three–loop solutions in the $\overline{\rm MS}$ scheme. Their remarkable coincidence (within the 2–4 per cent) demonstrates reduced sensitivity of the APT approach (see, also Refs. [19, 20, 21]) with respect to higher–loop effects in the whole Euclidean and Minkowskian regions from IR to UV limits. Fig.1b shows higher two–loop functions in comparison with $\alpha_{\rm an}$ and $\tilde{\alpha}$ powers.

Generally, functions $\mathfrak{A}_k$ and $\mathcal{A}_k$ differ from the local ones with a fixed f value. Minkowskian global functions $\mathfrak{A}_k$ can be presented via $\mathfrak{A}_k(s,f)$ by relations

$$\tilde{\alpha}(s) = \tilde{\alpha}(s;f) + c(f)\,; \quad \mathfrak{A}_2(s) = \mathfrak{A}_2(s;f) + \mathfrak{c}_2(f) \quad \text{at} \qquad M_f^2 \le s \le M_{f+1}^2 \tag{19}$$

with *shift constants* $c(f)$, $\mathfrak{c}_2(f)$ representable via integrals over $\rho_k(\sigma; f+n)$, $n \ge 1$ with additional reservations, like $c(6) = 0$, related to the asymptotic freedom condition.

Numerical estimate performed in Ref.[17] (see also Table 6 in Ref.[26]) for traditional values of the QCD scale parameter $\Lambda_3 \sim 300 - 400$ MeV

$$c(3) \sim 0.02\,, \; c(4) \simeq 3.10^{-3}\,, \; c(5) \simeq 3.10^{-4}\,; \qquad \mathfrak{c}_2(f) \simeq 3\,\alpha(M_f^2)\,c\,(f)$$

reveals that these constants are essential in the $f = 3, 4$ region at a few per cent level for $\tilde{\alpha}$ and at ca 10% level for $\mathfrak{A}_2$.

Meanwhile, global Eucledean functions $\mathcal{A}_k(Q^2)$ cannot be related to the local ones $\mathcal{A}_k(Q^2, f)$ by simple relations. Nevertheless, numerical calculation shows [26, 27] that in the $f = 3$ region one has approximately at $M_3^2 \le s \le M_4^2$

$$\alpha_{\rm an}(Q^2) = \alpha_{\rm an}(Q^2;3) + c(3)\,; \quad \mathcal{A}_2(Q^2) = \mathcal{A}_2(Q^2);3) + \mathfrak{c}_2(3)\,. \tag{20}$$

3 The APT applications

3.1 General comments

To illustrate a quantitative difference between global APT scheme and common practice of data analysis, consider a few examples.

In the usual treatment — see, e.g., Ref. [1] — the (QCD perturbative part of) Minkowskian observable, like e^+e^- annihilation or Z_0 decay cross–section ratio, is presented in the form

$$R(s) = R_0\,(1 + r(s))\,; \quad r_{PT}(s) = c_1\,\bar{\alpha}_s(s) + c_2\,\bar{\alpha}_s^2(s) + c_3\,\bar{\alpha}_s^3(s) + \dots \tag{21}$$

Here, the coefficients c_1, c_2 and c_3 are not diminishing numerically — see Table 1. A rather big negative c_3 value comes mainly from the $-c_1\pi^2\beta_0^2/3$ term. In the APT, we have instead

$$r_{APT}(s) = d_1\tilde{\alpha}(s) + d_2\,\mathfrak{A}_2(s) + d_3\,\mathfrak{A}_3(s)\;+\ldots \qquad (22)$$

with reasonably decreasing coefficients $d_{1,2} = c_{1,2}$ and $d_3 = c_3 + c_1\pi^2\beta_0^2/3$, the mentioned π^2 term of c_3 being "swallowed" by $\tilde{\alpha}(s)$.

In the Euclidean channel, instead of power expansion similar to (21), we typically have

$$d_{APT}(Q^2) = d_1\alpha_{\rm an}(Q^2) + d_2\,\mathcal{A}_2(Q^2) + d_3\,\mathcal{A}_3(Q^2)\;+\ldots \qquad (23)$$

with the same coefficients d_k. Here, the modification is related to nonperturbative, power in Q^2, structures like in (13).

Table 2

Relative contributions (in %) of 1– , 2– and 3–loop terms to observables

Process	Q or $\sqrt{s}$	f	*PT*			*APT*		
GLS sum rule	1.73 GeV	4	65	24	11	75	21	**4**
Bjorken. s.r.	1.73 GeV	3	55	26	19	80	19	**1**
Incl. τ-decay	0 - 2 GeV	3	55	29	16	88	11	**1**
$e^+e^- \to$ hadr.	10 GeV	4	96	8	-4	92	7	**.5**
$Z_o \to$ hadr.	89 GeV	5	98.6	3.7	-2.3	96.9	3.5	**-.4**

In Table 2, we give values of the relative contribution of the first, second, and third terms of the r.h.s. in (21),(22) and (23) for the Gross–Llywellin-Smith [29] and Bjorken [30] sum rules, τ – decay in the vector channel [31], as well as for e^+e^- and Z_0 inclusive cross-sections. As it follows from this Table, in the APT case, the three–loop (last) term is very small, and being compared with data errors, numerically inessential. This means that, in practice,

one can use the APT expansions (22) and (23) without the last term.

3.2 Semi–quantitative estimate

This conclusion can be valuable for the case when the three–loop contribution, i.e., d_3 is unknown. Here, people use the so-called NLLA approximation, that is common practice in the $f = 5$ region. For the Minkowskian observable, e.g., in the event–shape (see, e.g., Ref. [33]) analysis there corresponds the two-term expression

$$r(s) = c_1 \alpha_s(s) + c_2 \alpha_s^2(s)\,. \tag{24}$$

On the basis of the numerical estimates of Table 1, in such a case, we recommend instead *to use the two-term APT representation*

$$r^{(2)}_{APT}(s) = d_1 \tilde{\alpha}(s) + d_2\, \mathfrak{A}_2(s) \tag{25}$$

which, at $L^2 \gg \pi^2$, is equivalent to the three-term expression

$$r_3^{\Delta}(s) = d_1 \left\{ \bar{\alpha}_s - \frac{\pi^2 \beta_0^2}{3} \bar{\alpha}_s^3 \right\} + d_2 \bar{\alpha}_s^2 = c_1\, \bar{\alpha}_s + c_2 \bar{\alpha}_s^2 - \underline{\boldsymbol{\delta_3\, \bar{\alpha}_s^3}}\,, \tag{26}$$

i.e., to take into account the known predominant π^2 part of the next coefficient c_3. As it follows from the comparison of the last expression with the previous, two–term one (24), the $\bar{\alpha}_s$ numerical value extracted from eq.(26), for the same measured value r_{obs}, will differ mainly by a positive quantity (e.g., in the $f = 5$ region with $\bar{\alpha}_s \simeq 0.12 \div 0.15$)

$$(\Delta \bar{\alpha}_s)_3 = \left. \frac{\pi \delta_3\, \bar{\alpha}_s^3}{1 + 2\pi d_2 \bar{\alpha}_s} \right|^{f=5}_{20\div 100\mathrm{GeV}} = \frac{1.225\, \bar{\alpha}_s^3}{1 + 0.90\, \bar{\alpha}_s} \simeq 0.002 \div 0.003 \tag{27}$$

that turns out to be numerically important.

Moreover, in the $f = 4$ region, where the three-loop (NNLLA) approximation is commonly used in the data analysis, the π^2 term δ_4 of the next order turns out also to be essential. Hence, we propose there, instead of (21), *to use the APT three–term expression*

$$r^{(3)}_{APT}(s) = d_1 \tilde{\alpha}(s) + d_2\, \mathfrak{A}_2(s) + d_3\, \mathfrak{A}_3(s) \tag{28}$$

approximately equivalent to the four-term one

$$r_4^{\Delta}(s) = d_1\, \bar{\alpha}_s + d_2\, \bar{\alpha}_s^2 + c_3\, \bar{\alpha}_s^3 - \underline{\boldsymbol{\delta_4\, \bar{\alpha}_s^4}}\,; \quad c_3 = d_3 - \delta_3 \tag{29}$$

or to

$$r_4^{\Delta}(s) = d_1 \left\{ \bar{\alpha}_s - \frac{\pi^2 \beta_0^2}{3} \bar{\alpha}_s^3 - b_1 \frac{5}{6}\, \pi^2 \beta_0^2\, \bar{\alpha}_s^4 \right\} + d_2 \left\{ \bar{\alpha}_s^2 - \pi^2 \beta_0^2\, \bar{\alpha}_s^4 \right\} + d_3\, \bar{\alpha}_s^3$$

with δ_3 and δ_4 defined[9, 12] in eq.(3).

The three– and two–term structures in braces are related to specific expansion functions $\tilde{\alpha} = \mathfrak{A}_1$ and $\mathfrak{A}_2$ defined above (18) and entering into the non-power expansion (28).

To estimate roughly the numerical effect of using this last modified expression (29), we take the e^+e^- inclusive annihilation. For $\sqrt{s} \simeq 3 \div 5\,\mathrm{GeV}$ with $\bar{\alpha}_s \simeq 0.28 \div 0.22$ one has

$$(\Delta\bar{\alpha}_s)_4 = \frac{\pi\delta_4\,\bar{\alpha}_s^4}{1+2\pi d_2\bar{\alpha}_s}\Bigg|_{3\div 5\mathrm{GeV}}^{f=4} = \frac{1.07\,\bar{\alpha}_s^4}{1+0.974\,\bar{\alpha}_s} \simeq 0.005 \div 0.002$$

— an important effect on the level of ca $1 \div 2\%$.

Moreover, the $(\Delta\bar{\alpha}_s)_4$ correction turns out to be noticeable even in the lower part of the $f=5$ region! Indeed, to $\sqrt{s} \simeq 10 \div 40\,\mathrm{GeV}$ with $\bar{\alpha}_s \simeq 0.20 \div 0.15$ there corresponds

$$(\Delta\bar{\alpha}_s)_4\Big|_{10\div 40\ \mathrm{GeV}}^{f=5} \simeq 0.71\,\bar{\alpha}_s^4 \simeq (1.1 \div 0.3)\cdot 10^{-3} \quad (\lesssim 0.5\%)\,.$$

3.3 Important warning

It is essential to note that approximate expressions eqs.(26) and (29) are equivalent to the exact ones (25) and (28) only in the region $L = \ln(s/\Lambda^2) \gg \pi$ as it is shown on Fig. 2.

One can see that the curve for approximate Minkowskian coupling

$$\tilde{\alpha}_{appr}(s) = \bar{\alpha}_s(s) - (\pi^2\beta_0^2/3)\bar{\alpha}_s^3\,, \tag{30}$$

that precisely corresponds to the popular approximation (21) (and gives rise to the π^2 term in the α_s^3 coefficient) has a rather peculiar behavior. In the region $L > \pi$ it goes rather close to the curve for $\tilde{\alpha}$. For instance, at $L \simeq \pi$ the relative error of approximation is about 5 per cent. On the other hand, below $L \simeq 0.8\,\pi$ (i.e., $W \simeq 1.0 - 1.4\,\mathrm{GeV}$) the distance $\tilde{\alpha} - \tilde{\alpha}_{\mathrm{appr}}$ between curves (error of approximation) increases and at $L \simeq 0.7\,\pi$ it blows up (better to say "comes down").

In particular, at $s \leq 2\,\mathrm{GeV}^2$ it is rather desorienting to refer to $\bar{\alpha}_s(s)$ and it is erroneous to use $\tilde{\alpha}_{appr}(s)$ and common expansion (21).

This means that below $s = 2\,\mathrm{GeV}^2$ it is nonadequate to use common $\bar{\alpha}_s(s)$ and power expansion eq.(21).

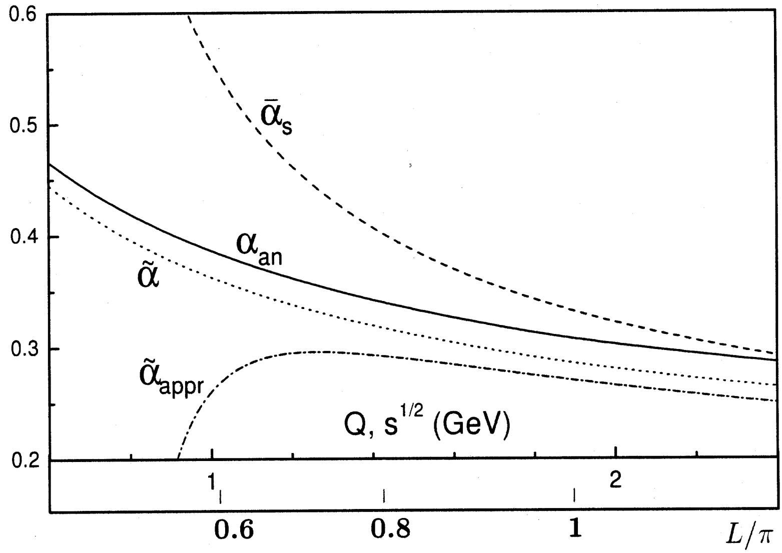

Figure 2: Comparison of common QCD coupling $\bar{\alpha}_s$ with the APT global ones ($\tilde{\alpha}$, $\alpha_{\rm an}$) in the $Q, \sqrt{s} < 3\,\mathrm{GeV}$ region at $\Lambda_3 = 400$ MeV . By dash-dotted line we give an approximate Minkowskian coupling (30). All the curves are taken (see Tables 1,5 and 6 in Ref.[27]) for the 3-loop global case.

In other words, we claim that below $s = 2\,\mathrm{GeV}^2$ it is an intricate business to analyze data in terms of the "old good" (but singular) α_s [2]. Here, approximate relation (30) does not work as it is illustrated in Fig.2.

In this, low–energy Minkowskian/Euclidean region data have to be analyzed in terms of nonpower expansion (22)/(23) and extracted parameter should be $\alpha_{\rm an}(s)\,/\tilde{\alpha}(Q^2)$ or $\Lambda^{(3)}$. In Table 3 we give few numerical examples for the chain

$$\alpha_{\rm an}(M_\tau) \leftrightarrow \tilde{\alpha}(M_\tau) \leftrightarrow \Lambda^{(3)} \rightarrow \Lambda^{(5)} \leftrightarrow \bar{\alpha}_s(M_Z)$$

[2] In particular, this relates to analysis of τ decay. In this connection we would like to attract attention to the important paper [31] that treats the τ decay within the APT approach with results $\Lambda^{(3)} = 420$ MeV that corresponds to $\alpha_{\rm an}(M_\tau^2) = 0.32$ or $\tilde{\alpha}(M_\tau^2) = 0.30$. At the same time, attempts to interpret results of APT for τ decay in terms of α_s, like, e.g., in Ref.[32], needs some special precaution — see next footnote. A more detailed comment on the τ decay theoretical analysis will be published elsewhere.

that allows to study the QCD theoretical compatibility of LE data with the HE ones in the APT analysis.

Table 3 : Numerical chain related LE with HE regions

$\tilde{\alpha}(M_\tau)$	$\alpha_{\rm an}(M_\tau)$	$\Lambda^{(3)}$	$\Lambda^{(5)}$	$\bar{\alpha}_s(M_Z)$
0.309	0.332	450 MeV	303 MeV	0.125
0.292	0.314	400 MeV	260 MeV	0.121
0.278	0.299	350 MeV	218 MeV	0.119
0.266	0.286	300 MeV	180 MeV	0.116

Here, the main element of correlation is the chain $\Lambda^{(3)} \leftrightarrow \Lambda^{(3)} \leftrightarrow \Lambda^{(5)}$ that follows from the matching condition (16) [3].

4 Quantitative illustration

Consider now a few cases in the $f = 5$ region.

Υ decay. According to the Particle Data Group (PDG) overview (see their Fig.9.1 on page 88 of Ref.[1]), this is (with $\alpha_s(M_\Upsilon^2) \simeq 0.170$ and $\bar{\alpha}_s(M_Z^2) = 0.114$) one of the most "annoying" points of their summary of $\bar{\alpha}_s(M_Z^2)$ values. It is also singled out theoretically. The expression for the ratio of decay widths starts with the cubic term

$$R(\Upsilon) = R_0\, \alpha_s^3(M_\Upsilon^2)(1 + e_1\, \alpha_s) \quad \text{with} \quad e_1 \simeq 1 \,.$$

Due to this, the π^2 correction[4] is rather big here

$$\mathfrak{A}_3 \simeq \alpha_s^3 \left(1 - 2(\pi\beta_0)^2\alpha_s^2\right) \,.$$

Accordingly,

$$\Delta\alpha_s(M_\Upsilon^2) = \frac{2}{3}\,(\pi\beta_0)^2\, \alpha_s^3(M_\Upsilon^2) \simeq 0.0123 \,,$$

[3] Generally, it is possible to use correspondence between $\alpha_{\rm an}$, $\tilde{\alpha}$ and α_s as expressed by relations (14). However, the use of $\alpha_s^{\overline{\rm MS}}(\mu^2)$ at $\mu \lesssim 1$ *GeV* as a QCD parameter could be misleading due to vicinity to singularity. For example, at $\Lambda^{(3)} = 400$ MeV one has $\alpha_s(M_\tau^2) \simeq 0.35$ and $\alpha_s(1\ \text{GeV}^2) \simeq 0.55$ to be compared with $\alpha_{\rm an}(M_\tau^2) \simeq 0.31$ and $\alpha_{\rm an}(1\ \text{GeV}^2) \simeq 0.40$.

[4] First proposal of taking into account this effect in the Υ decay was discussed[10] more than a quarter of century ago. Nevertheless, in current practice it is completely forgotten.

which corresponds to

$$\Delta\bar{\alpha}_s(M_Z^2) = 0.006 \quad \text{with resulting} \quad \bar{\alpha}_s(M_Z^2) = 0.120\,. \tag{31}$$

The NNLO case. Now, let us turn to a few cases analyzed by the three-term expansion formula (1). For the first example, take e^+e^- hadron annihilation at $\sqrt{s} = 42\,\mathrm{GeV}$ and $11\,\mathrm{GeV}$.

A common form (see, e.g., Eq.(15) in Ref.[2]) of theoretical presentation of the QCD correction in our normalization looks like

$$r_{e^+e^-}(\sqrt{s}) = 0.318\bar{\alpha}_s(s) + 0.143\,\bar{\alpha}_s^2 - 0.413\,\bar{\alpha}_s^3\,.$$

In the standard PT analysis, one has (see, e.g., Table 3) $\bar{\alpha}_s(42^2) = 0.144$ that corresponds to $r_{e^+e^-}(42) \simeq 0.0476$. Along with the APT prescription, one should use

$$r_{e^+e^-}(\sqrt{s}) = 0.318\,\tilde{\alpha}(s) + 0.143\,\mathfrak{A}_2(s) - 0.023\,\mathfrak{A}_3(s)\,, \tag{32}$$

which yields $\tilde{\alpha}(42^2) = 0.142 \;\rightarrow\; \alpha_s(42^2) = 0.145$ and $\bar{\alpha}_s(M_Z^2) = 0.127$ to be compared with $\bar{\alpha}_s(M_Z^2) = 0.126$ under a usual analysis.

Quite analogously, with $\bar{\alpha}_s(11^2) = 0.200$ and $r_{e^+e^-}(11) \simeq 0.0661$ we obtain via (32) $\tilde{\alpha}(11^2) = 0.190$ that corresponds to $\bar{\alpha}_s(M_Z^2) = 0.129$ instead of 0.130.

For the next example, take the Z_0 inclusive decay. The observed ratio

$$R_Z = \Gamma(Z_0 \rightarrow \mathit{hadrons})/\Gamma(Z_0 \rightarrow \mathit{leptons}) = 20.783 \pm .029$$

can be written down as follows: $R_Z = R_0\,(1 + r_Z(M_Z^2))$ with $R_0 = 19.93$. A common form (see, e.g., Eq.(15) in Ref.[2]) of presenting the QCD correction r_Z looks like

$$r_Z(M_Z) = 0.3326\bar{\alpha}_s + 0.0952\,\bar{\alpha}_s^2 - 0.483\,\bar{\alpha}_s^3\,.$$

To $[r_Z]_{obs} = 0.04184$ there corresponds $\bar{\alpha}_s(M_Z^2) = 0.124$ with $\Lambda^{(5)}_{\overline{\mathrm{MS}}} = 292\,\mathrm{MeV}$. In the APT case, from

$$r_Z(M_Z) = 0.3326\,\tilde{\alpha}(M_Z^2) + 0.0952\,\mathfrak{A}_2(M_Z^2) - 0.094\,\mathfrak{A}_3(M_Z^2) \tag{33}$$

we obtain $\tilde{\alpha}(M_Z^2) = 0.122$ and $\bar{\alpha}_s(M_Z^2) = 0.124$. Note that here the three-term approximation (8) gives the same relation between the $\bar{\alpha}_s(M_Z^2)$ and $\tilde{\alpha}(M_Z^2)$ values.

Nevertheless, in accordance with our preliminary estimate for the $(\Delta\bar{\alpha}_s)_4$ role, even the so-called NNLO theory needs some π^2 correction in the $W = \sqrt{s} \lesssim 50$ GeV region.

The NLO case. Now, turn to those experiments in the HE ($f = 5$) Minkowskian region (mainly with a shape analysis) that usually are confronted with the two-term expression (24). As it has been shown above (27), the main theoretical error here can be expressed in the form

$$(\Delta\bar{\alpha}_s(s)|^{f=5}_{20\div 100\mathrm{GeV}} \simeq 1.225\,\bar{\alpha}^3_s(s) \simeq 0.002 \div 0.003\,. \tag{34}$$

An adequate expression for the equivalent shift of the $\bar{\alpha}_s(M_Z^2)$ value is

$$[\Delta\bar{\alpha}_s(M_Z^2)]_3 = 1.225\bar{\alpha}_s(s)\bar{\alpha}_s(M_Z^2)^2\,. \tag{35}$$

We give the results of our approximate APT calculations, mainly by Eqs.(34) and (35), in the form of Table 4 and Figure 3. In the last column of Table 3 in brackets, we indicate the difference between the APT and usual analysis. The results of the three–loop analysis are marked by bold figures. Dots in the lower part of the Table correspond to shape–events data for energies $W = 133, 161, 172$ and 183 GeV with the same positive shift 0.002 for the the extracted $\bar{\alpha}_s$ values.

Table 4 : The APT revised[a] part ($f = 5$) of Bethke's[2] Table 6

Process	$\sqrt{s}$ GeV	loops No	$\bar{\alpha}_s$ (s) ref.[2]	$\bar{\alpha}_s(m_z^2)$ ref.[2]	$\bar{\alpha}_s$ (s) APT	$\bar{\alpha}_s(m_z^2)$ APT
Υ-decay [b]	9.5	2	.170	.114	.182	.120 (+6)
$e^+e^-[\sigma_{had}]$	10.5	**3**	.200	.130	.198	.129(**-1**)
$e^+e^-[j\,\&\,sh]$	22.0	2	.161	.124	.166	.127(+3)
$e^+e^-[j\,\&\,sh]$	35.0	2	.145	.123	.149	.126(+3)
$e^+e^-[\sigma_{had}]$	42.4	**3**	.144	.126	.145	.127(+**1**)
$e^+e^-[j\,\&sh]$	44.0	2	.139	.123	.142	.126(+3)
$e^+e^-[j\,\&sh]$	58	2	.132	.123	.135	.125(+2)
$Z_0 \to$ **had.**	91.2	**3**	.124	.124	.124	.124(**0**)
$e^+e^-[j\,\&\,sh]$	91.2	2	.121	.121	.123	.123(+2)
-"-	....	2	...	...	...	...(+2)
$e^+e^-[j\,\&\,sh]$	189	2	.110	.123	.112	.125(+2)

Averaged $< \bar{\alpha}_s(M_z^2) >_{f=5}$ values 0.121; 0.124

[a] "j & sh" = jets and shapes; Figures in brackets in the last column give the difference $\Delta\bar{\alpha}_s(M_Z^2)$ between common and APT values.

[b] Taken from Ref.[1].

In Fig.3, by open and hatched circles we give two– and three–loops data from Fig.10 of paper [2]. The only exclusion is the Υ decay taken from Table X of the same paper. By crosses, we marked the "APT values" calculated approximately by Eq.(34).

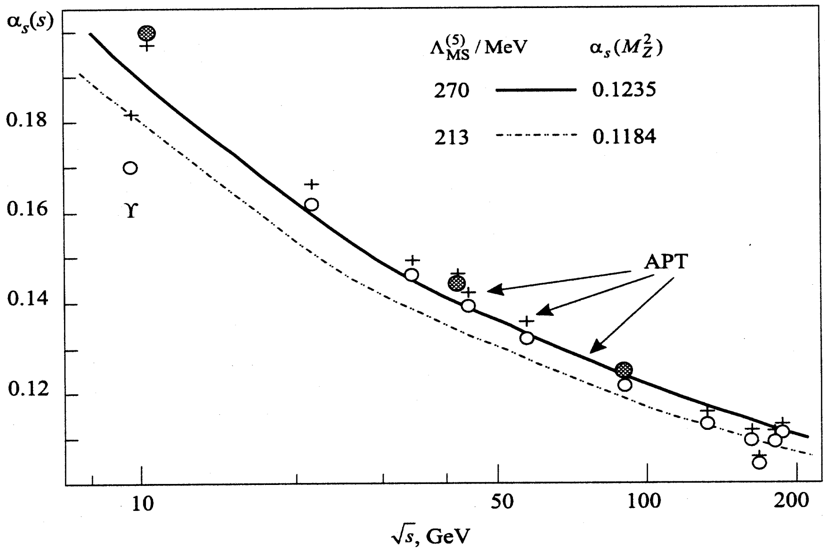

Figure 3: The new APT analysis for $\bar{\alpha}_s$ in the five-flavor time–like region. Crosses (+) differ from circles by the π^2 correction (34). Solid APT curve relates to $\Lambda^{(5)}_{\overline{MS}} = 270\,\mathrm{MeV}$ and $\bar{\alpha}_s(M_Z^2) = 0.124$. To compare, we give also the standard (dot-and-dash curve) $\bar{\alpha}_s$ (at $\Lambda^{(5)} = 213\,\mathrm{MeV}$ and $\bar{\alpha}_s(M_Z^2) = 0.118$) taken from Fig.10 of paper [2].

For clearness of the π^2 effect, we skipped the error bars. They are the same as in the mentioned Bethke's figure and we used them for calculating χ^2.

Let us note that our average 0.121 over events from Table 6 of Bethke's review [2] nicely correlates with recent data of the same author (see Summary of Ref.[34]). The best χ^2 fit yields $\bar{\alpha}_s(M_Z^2)_{[2]} = 0.1214$ and[5]

$$\bar{\alpha}_s(M_Z^2)_{APT} = 0.1235\,.$$

This new $\chi^2_{\rm APT}$ is smaller $\chi^2_{APT}/\chi^2_{PT} \simeq 0.73$ than the usual one. This

[5]This value, corresponding to $\Lambda^{(5)} = 290\,\mathrm{MeV}$, is supported by recent analysis [31] of τ decay that gives $\Lambda^{(3)} = 420\,\mathrm{MeV}$; compare with Table 3.

illustrates the effectiveness of the APT procedure in the region far enough from the ghost singularity.

5 Conclusion

It is a common standpoint that in QCD it is legitimate to use the power in α_s expansion for observables in the low energy (low momentum transfer) region. At the same time, there exist rather general (and old [35]) arguments in favor of nonanalyticity of the S matrix elements at the origin of the complex plane of the α variable, with α being an expansion parameter[36]. This, in turn, implies that common perturbation expansion has no domain of convergence. Technically, this corresponds to the factorial growth ($\sim n!$) of expansion coefficients (like d_n or r_n) at large n [37, 38]. In QCD, with its "not small enough" α_s values in the region below 10 GeV it is a popular belief that one does face an asymptotic nature of perturbation expansion by observing approximate equality of relative contributions of the second (α_s^2) and the third (α_s^3) terms into observable, like in all PT columns of Table 2.

Our first qualitative result consists in observation that convergence properties of the APT expansions drastically differ from the usual PT ones.

The evidently better practical convergence of the APT series for the Euclidean observable, as it has been demonstrated in the right part of Table 2, probably means that essential singularity at $\alpha_s = 0$ is adequately taken into account by new expansion functions $\mathcal{A}_k(Q^2)$. On the other hand, in the time–like region the improved approximation property of the APT expansion over $\mathfrak{A}_k(s)$ has a bit different nature, being related, in our opinion, to the nonuniform convergence of the standard PT expansion for Minkowskian observables. In any case, from a practical point of view:

1. In the APT approach one can use the nonpower expansions (22) and (23) without the last term.

The next point, discussed in Section 3.3, refers to a more specific issue connected with current practice of the Minkowskian observable analysis in the low–energy ($s \lesssim 3\,\mathrm{GeV}^2$) region (like, e.g., inclusive τ decay). As it has been shown – see Fig. 2 —

2. Below 2 GeV^2 it is impossible to use the common power expansion (1) for a time–like observable.

Second group of our results is of a quantitative nature:

3. Effective positive shift $\Delta\bar{\alpha}_s = +0.002$ in the upper half ($\geq$ 50 GeV) of the $f = 5$ region for all time-like events that have been analyzed up to now in the NLO mode.

4. Effective shift $\Delta\bar{\alpha}_s \simeq +0.003$ in the lower half ($10 \div 50$ GeV) of the $f = 5$ region for all time-like events that have been analyzed in the NLO mode.

5. The new value

$$\bar{\alpha}_s(M_Z^2) = 0.124\,, \tag{36}$$

obtained by averaging new APT results over the $f = 5$ region.

The quantitative results are based on the new APT nonpower expansion (8) and plausible hypothesis on the π^2–term prevalence in common expansion coefficients for observables in the Minkowskian domain. The hypothesis has some preliminary support — see Table 1 — but needs to be checked in more detail.

Nevertheless, our result (36) being taken as granted raises two physical questions:

– The issue of self-consistency of QCD invariant coupling behavior between the "medium ($f = 3, 4$)" and "high ($f = 5, 6$)" regions.

– The new "enlarged value" (36) can influence various physical speculations in the several hundred GeV region.

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Аналитическая теория возмущений для анализа наблюдаемых КХД

Работа посвящена применению недавно развитой, свободной от нефизических особенностей, аналитической теории возмущений (АТВ) к анализу некоторых наблюдаемых КХД. Она начинается с обсуждения главной проблемы пертурбативной КХД — наличию призрачных сингулярностей и с краткого обзора решения этой проблемы в рамках АТВ.

На нескольких примерах в различных областях (с числом ароматов $f = 3$, 4 и 5) энергии и переданного импульса мы демонстрируем эффект улучшенной сходимости для АТВ модифицированной теории возмущений в КХД. Наше первое наблюдение состоит в том, что трехпетлевой вклад (порядка α_s^3) в АТВ-разложении, как правило, оказывается численно несущественным. Это порождает надежду на практическое решение известной проблемы асимптотического характера (отсутствия убывания коэффициентов) разложений теории возмущений в КТП.

Второй результат заключается в том, что обычный пертурбативный анализ наблюдаемых во времениподобной области с большими π^2-составляющими в коэффициенте при α_s^3-члене оказывается неадекватным при $s \leq 2$ ГэВ^2. В частности, это относится к τ-распаду.

Установлено, что в области «высоких» энергий (10 ГэВ $\leq \sqrt{s} \leq$ 170 ГэВ) общепринятое при анализе «профилей» двухпетлевое (NLO, NLLA) приближение приводит к систематической теоретической погрешности в 1–2 процента для извлекаемых значений $\bar{\alpha}_s^{(2)}$.

Наш физический результат состоит в том, что усредненное по АТВ-данным в области $f = 5$ значение $\langle \bar{\alpha}_s(M_Z^2) \rangle_{\text{АТВ};\, f=5} \cong 0{,}124$ значительно отличается от общепринятого «мирового среднего» ($= 0{,}118$).

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Analytic Perturbation Theory in Analyzing Some QCD Observables

The paper is devoted to application of recently devised ghost-free Analytic Perturbation Theory (APT) for analysis of some QCD observables. We start with the discussion of the main problem of the perturbative QCD — ghost singularities and with the resume of this trouble solution within the APT.

By a few examples in the various energy and momentum transfer regions (with the flavor number $f = 3$, 4 and 5) we demonstrate the effect of improved convergence of the APT modified perturbative QCD expansion. Our first observation is that in the APT analysis the three-loop contribution (of an order of α_s^3) is as a rule numerically inessential. This raises hope for practical solving the well-known problem of asymptotic nature of common QFT perturbation series.

The second conclusion is that a common perturbative analysis of time-like events with the big π^2 term in the π^2 coefficient is not adequate at $s \leq 2$ GeV^2. In particular, this relates to τ decay.

Then, for the «high» ($f = 5$) region it is shown that the common two-loop (NLO, NLLA) perturbation approximation widely used there (at 10 GeV $\leq \sqrt{s} \leq$ 170 GeV) for analysis of shape/events data contains a systematic negative error of a 1–2 per cent level for the extracted $\bar{\alpha}_s^{(2)}$ values.

Our physical conclusion is that the $\bar{\alpha}_s(M_Z^2)$ value averaged over the $f = 5$ data $\langle \bar{\alpha}_s(M_Z^2) \rangle_{\text{APT};\, f=5} \cong 0.124$ appreciably differs from the currently accepted «world average» ($= 0.118$).

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

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