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AN ANALOG OF THE FOURIER TRANSFORM ASSOCIATED WITH A NONLINEAR ONE-DIMENSIONAL SCHRÖDINGER EQUATION

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1 Introduction. Notation. Formulation of results

In the present paper, we continue investigations begun in [1]. We consider the equation

$$-u'' + f(u^2)u = \lambda u, \quad u = u(x), \quad x \in (0, +\infty), \tag{1}$$

supplied with the following boundary conditions:

$$u(0) = p, \; u'(0) = 0, \; \sup_{x>0} |u(x)| < \infty. \tag{2}$$

Hereafter all quantities are real, $\lambda \in \mathbb{R}$ is a spectral parameter, f is a given function such that $f(u^2)u$ is continuously differentiable with respect to $u \in \mathbb{R}$, and p is an arbitrary positive parameter fixed throughout the paper. In this paper, it is proved that an arbitrary infinitely differentiable function defined on the half-line $x \geq 0$, satisfying some conditions at $x = 0$ and rapidly decaying as $x \to +\infty$, can be uniquely expanded into an integral over eigenfunctions of this problem similar to the representation of this function by the Fourier transform.

In view of our assumption on the function f, for the Cauchy problem for equation (1) with arbitrary initial data standard local existence, uniqueness and continuous dependence theorems take place. The case $p < 0$ can obviously be reduced to the considered one by the change of variables $u(x) \to -u(x)$. In addition, since the boundary conditions (2) contain Cauchy data as a part, for each value of the parameter λ at most one function $u(x)$ satisfying the problem (1),(2) can exist. If a pair (λ, u) consisting of a real number λ and a twice continuously differentiable function $u = u(x)$ of the argument $x \geq 0$ satisfies the problem (1),(2), then we call λ an *eigenvalue* and $u(x)$ the corresponding *eigenfunction* of this problem. We call the set Λ of all its eigenvalues the *spectrum* of this problem. We shall also denote by $u(\lambda, x)$ eigenfunctions indicating explicitly their dependence on $\lambda \in \Lambda$.

Throughout the paper, the assumptions about the function f are the following.

(f) *Let $f(s)$ be a real-valued continuous nondecreasing function of the argument $s \geq 0$, and let $f(u^2)u$ be a ten-time continuously differentiable function of the argument $u \in \mathbb{R}$.*

We introduce the following notation. By $L_2(a,b)$, where $-\infty \leq$ $\leq a < b \leq +\infty$, we denote the usual Lebesgue space, consisting of real-valued functions of the argument $x \in (a,b)$, square integrable between a and b, with the scalar product $(u,v)_{L_2(a,b)} = \int\limits_a^b u(x)v(x)dx$ and the norm $||u||_{L_2(a,b)} = (u,u)^{\frac{1}{2}}_{L_2(a,b)}$. We set $e(k,x) = p\cos\frac{\pi x}{2k}$ so that $-e''_{xx}(k,x) =$ $= z(k)e(k,x)$, $e(k,0) = p$, $e'_x(k,0) = 0$ where $z(k) = \left(\frac{\pi}{2k}\right)^2$. Also, let $l(x) = 2x+1$. We denote by S the Schwartz space of functions g infinitely differentiable in $\mathbb{R}$ and such that $\sup\limits_{x\in\mathbb{R}} \left|x^m \frac{d^n g}{dx^n}\right| < \infty$ for all $m,n = 0,1,2,...$, and by S_e, the space of restrictions onto $[0,+\infty)$ of even functions from S. For any $g(\cdot) \in S_e$, let $\hat{g}(\cdot) \in S_e$ be its renormalized Fourier transform so that

$$g(x) = \int\limits_0^\infty \hat{g}(r)e(r^{-1},x)dr, \quad x \in [0,+\infty).$$

Finally, by $C, C_1, C_2, C', C'', ...$ we shall denote positive constants.

Questions of expansions on a segment of functions from spaces, containing as a part the set of all continuous functions, over eigenfunctions of nonlinear boundary-value problems with denumerable spectra are considered in a number of papers (see, for example, [1-8]). In monograph [2], some interesting results in this direction are established. The author's papers [3-8] are devoted to proving the property of being a basis, in L_2 and other spaces, of systems of eigenfunctions of nonlinear Schrödinger-type eigenvalue problems on a segment; below we exploit some of these results. In [1], it is shown that the set of functions g representable by the integral (3) (see below) with finite functions $\overline{g}$ is dense in $L_2(0,\infty)$. In addition, in this paper, some corrections to the paper [3] are presented.

In the present paper, our main result is the following.

Theorem *Under the hypothesis* (f)
(a) $\Lambda = [f(p^2),+\infty)$ (*we denote* $\overline{\lambda} = f(p^2)$);
(b) *for each* $\lambda \in (\overline{\lambda},+\infty)$ *there exists a minimal positive zero* $k = k(\lambda)$ *of* $u(\lambda,x)$ *as a function of* $x \in (0,+\infty)$; *the function* $k : (\overline{\lambda},+\infty) \to (0,+\infty)$ *is continuously differentiable,* $k'(\lambda) < 0$ *for any* $\lambda \in (\overline{\lambda},+\infty)$, *and* $\lim\limits_{\lambda\to\overline{\lambda}+0} k(\lambda) = +\infty$, $\lim\limits_{\lambda\to+\infty} k(\lambda) = 0$. *By* $\lambda = \lambda(k)$ *we denote the*

function with the domain $(0,+\infty)$ *inverse to* $k(\lambda)$;
(c) *the family of eigenfunctions* $\{u(\lambda,x)\}_{\lambda\in\Lambda}$ *is uniformly bounded with respect to* $x\geq 0$;
(d) *for any function* $g(\cdot)\in S_e$ *there exists a unique continuous function* $\overline{g}(\cdot)$ *such that for any* $x\geq 0$ *the following equality takes place:*

$$g(x)=\int_0^\infty \overline{g}(k)u(\lambda(k),x)dk \tag{3}$$

and that there exist $\gamma\in(0,1)$ *and* $C>0$ *for which*

$$|\overline{g}(k)|\leq Ck^{\gamma-1}(1+k)^{-1-\gamma},\quad k>0. \tag{4}$$

Remark 1. The expansion (3) is obviously an analog in our nonlinear case of the representation of functions by the Fourier transform; as it is well known, the latter is associated with a linear self-adjoint eigenvalue problem.

Remark 2. The author wants to note that, with the above theorem, he did not strive to obtain a maximally strong result, but he aimed only to demonstrate a possibility to expand an "arbitrary function" as in (3). In this connection, it should be said that probably the assumption that $f(u^2)u$ is a ten-time continuously differentiable function of $u\in\mathbb{R}$ is technical and a sufficient assumption is the single continuous differentiability of this function.

Remark 3. As it is proved in earlier papers, for any $k>0$ the functions $\left\{u\left(\lambda\left(\frac{k}{l(n)}\right),\cdot\right)\right\}_{n=0,1,2,\ldots}$ form a basis in $L_2(0,k)$. In view of this fact, a proof of (3) may be reduced to the justification of formal relations (19) (see Section 3). One of the main difficulties the author met in this way was the problem how to estimate coefficients in the expansions of $e(k,\cdot)$ over the above basis: as it is shown in what follows (see Lemma 1), the functions $e(k,x)$ and $u(\lambda(k),x)$ are arbitrary close to each other for sufficiently small $k>0$ and in this case the problem is not difficult so far. At the same time, the behavior of these functions is different as $k\to+\infty$

and for large $k > 0$ the estimating of the indicated coefficients becomes sufficiently difficult.

In Section 2, we present preliminary results being used in proving the above Theorem. Section 3 contains the last part of the proof of Theorem. In a short Section 4, we consider a nonlinear eigenvalue problem similar to (1),(2) and establish a modification of the Theorem for this new problem.

Everywhere below we accept that hypothesis (f) is valid.

2 Auxiliary results

Consider the following Cauchy problem:

$$-y'' + f(y^2)y = \lambda y, \quad y = y(x), \quad x > 0, \tag{5}$$

$$y(0) = p, \ y'(0) = 0. \tag{6}$$

Proposition 1.

(a) *For any* $\lambda > \overline{\lambda}$ *the corresponding solution* $y(x)$ *of the problem* (5),(6) *can be continued onto the entire real line and there exists a continuously differentiable function* $x_0(\lambda) > 0$ *of the argument* $\lambda > \overline{\lambda}$ *such that* $x_0'(\lambda) < 0$ *for all* $\lambda > \overline{\lambda}$, $y(x) > 0$ *in* $(0, x_0(\lambda))$ *and* $y(x_0(\lambda)) = 0$;
(b) *if* $\lambda = \overline{\lambda}$, *then* $y(x) \equiv p$ *and for* $\lambda < \overline{\lambda}$ *the solution* $y(x)$ *of the problem* (5),(6) *is unbounded in the maximal interval of its existence from* $(0, +\infty)$;
(c) *for any* $a, b : \ \overline{\lambda} < a < b$ *for the corresponding solutions* $y_a(x)$ *and* $y_b(x)$ *of the problem* (5),(6) *taken with* $\lambda = a$ *and* $\lambda = b$, *respectively, for all* $x \in [0, k(b)]$ *the following inequality takes place:* $y_a(x) \geq y_b(x)$. *In addition,* $|y(x)| \leq p$ *for all* $\lambda \in \Lambda$ *and all* $x \in \mathbb{R}$;
(d) *any solution* $y(x)$ *of equation* (5) *is odd with respect to an arbitrary its zero* $\overline{x}$ *and is even with respect to any point* $\tilde{x}$ *such that* $y'(\tilde{x}) = 0$, *i. e.* $y(\tilde{x} - x) = y(\tilde{x} + x)$ *and* $y(\overline{x} - x) = -y(\overline{x} + x)$ *for all* x;
(e) $\lim\limits_{\lambda \to +\infty} x_0(\lambda) = 0$ *and* $\lim\limits_{\lambda \to \overline{\lambda}+0} x_0(\lambda) = +\infty$;
(g) *for any* $\lambda > \overline{\lambda}$ *zeros of the corresponding solution* $y(x)$ *of the problem* (5),(6) *are precisely the points* $l(m)x_0(\lambda)$ *where* $m = 0, \pm 1, \pm 2, \ldots$.

For the Proof, we refer readers to papers [5,7].

Remark 4. Proposition 1 immediately implies statements (a) and (b) of our Theorem with $k(\lambda) = x_0(\lambda)$.

In what follows, for simplicity of the notation we rename by $u(k,x)$ the eigenfunction $u(\lambda(k),x)$. Further, in view of Proposition 1, for any $k>0$

$$\lambda\left(\frac{k}{l(0)}\right) < ... < \lambda\left(\frac{k}{l(n)}\right) < ...$$

are all values of the parameter λ that are larger than $\overline{\lambda}$ and for which the corresponding solutions $y(x) = y_n(k,x)$ of the problem (5),(6) become zero at $x = k$. In addition, for any $n = 0,1,2,...$ and a fixed $k > 0$ the function $y_n(k,x)$ has precisely n zeros in the interval $(0,k)$.

Lemma 1. *For any $\overline{k} > 0$ there exists $C > 0$ such that for all $k \in$ $\in (0,\overline{k}]$ one has*

$$|e(k,x) - u(k,x)| \le Ck^2 \quad \text{and} \quad \left|\frac{d^m}{dx^m}[u(k,kx) - e(k,kx)]\right| \le Ck^2,$$

$$x \in \mathbb{R}, \quad m = 1,2,...,10.$$

Proof of Lemma 1 in fact repeats the proof of a similar statement for a linear problem in [9] (see Lemma 1.7 in [9]). We sketch this proof for the convenience of readers.

The uniform boundedness of the family of functions $\{u(k,x)\}_{k>0}$ with respect to $x \in \mathbb{R}$, stated by Proposition 1, and the standard comparison theorem imply the existence of $D > 0$ such that

$$|\lambda(k) - z(k)| \le D \tag{7}$$

for all $k > 0$. Take an arbitrary $\overline{k} > 0$. One can easily verify that for any sufficiently small $k > 0$ the corresponding eigenfunction $u(k,x)$ of the problem (1),(2) for any $x > 0$ satisfies the equation

$$u(k,x) = p\cos(\lambda^{\frac{1}{2}}(k)x) + \lambda^{-\frac{1}{2}}(k)\int_0^x \sin\{\lambda^{\frac{1}{2}}(k)(x-t)\}f(u^2(k,t))u(k,t)dt \tag{8}$$

(since due to Proposition 1 $\lim_{k\to+0} \lambda(k) = +\infty$, the right-hand side in (8) is well-defined for all sufficiently small $k > 0$). Making in equation (8) the changes of variables $x = ky$ and $t = ks$, due to the uniform boundedness of the family of functions $\{u(k,x)\}_{k>0}$, we get for all sufficiently small $k > 0$ and all $y \in [0,1]$:

$$|u(k,ky) - p\cos(\lambda^{\frac{1}{2}}(k)ky)| \leq C_1 k \lambda^{-\frac{1}{2}}(k)$$

with a constant $C_1 > 0$ independent of the above y and k. Hence, since due to (7) $\lambda(k) \geq C_2 k^{-2}$ and $|\lambda^{\frac{1}{2}}(k) - z^{\frac{1}{2}}(k)| \leq C_2 k$ for all sufficiently small $k > 0$, we get

$$|e(k,ky) - u(k,ky)| \leq C_3 k^2$$

with a constant $C_3 > 0$ independent of sufficiently small $k > 0$ and $y \in [0,1]$. Due to the uniform boundedness of the family $\{u(k,x)\}_{k>0}$, this estimate is also valid for any $k \in (0, \overline{k}]$. Finally, in view of Proposition 1 the latter estimate holds for all $y \in \mathbb{R}$. The other inequalities in Lemma 1 follow from (8) by similar arguments. Thus, Lemma 1 is proved.□

One can easily see that for an arbitrary $k > 0$ and integer $n \geq 0$ the functions $\left\{e(\frac{k}{l(n)l(m)}, x\right\}_{m=0,1,2,...}$ form an orthogonal basis in the space $L_2\left(0, \frac{k}{l(n)}\right)$. Hence, we have

$$u\left(\frac{k}{l(n)}, \cdot\right) = \sum_{m=0}^{\infty} b_{n,m}(k) e\left(\frac{k}{l(m)}, \cdot\right), \quad n = 0, 1, 2, ..., \tag{9}$$

in $L_2\left(0, \frac{k}{l(n)}\right)$ where $b_{n,m}(k) = 0$ if $l(m) \neq l(n)l(r)$ for all $r = 0, 1, 2, ...$ Since due to Proposition 1 the function $u\left(\frac{k}{l(n)}, x\right)$ is even and since $e\left(\frac{k}{l(n)l(m)}, x\right)$ are even, too, these expansions also hold in $L_2\left(-\frac{k}{l(n)}, \frac{k}{l(n)}\right)$. Since according to Proposition 1 the functions $u\left(\frac{k}{l(n)}, x\right)$ are odd with respect to their zeros and since the functions $e\left(\frac{k}{l(n)l(m)}, x\right)$ are odd with respect to these zeros, too, the expansions (9) hold in $L_2(a,b)$ with arbitrary finite $a < b$.

So, for any $k > 0$ we have the sequence of expansions (9) held in $L_2(0,k)$ and in $L_2(a,b)$ with arbitrary $a, b: -\infty < a < b < \infty$, where $b_{n,m}(k) = 0$ if $l(m) \neq l(n)l(r)$ for all $r = 0, 1, 2, ...$ Note also that

$B(k) = (b_{n,m}(k))_{n,m=0,1,2,...}$ is an upper triangular matrix and, for any n,

$$b_{n,n}(k) = \left\| e\left(\frac{k}{l(n)}, \cdot\right) \right\|^{-2}_{L_2(0,k)} \left(u\left(\frac{k}{l(n)}, \cdot\right), e\left(\frac{k}{l(n)}, \cdot\right) \right)_{L_2(0,k)} > 0$$

because the functions $u\left(\frac{k}{l(n)}, x\right)$ and $e\left(\frac{k}{l(n)}, x\right)$ are of the same sign. So, in (9)

$$b_{n,m}(k) = 0 \text{ if } l(m) \neq l(n)l(r) \text{ for all } r = 0, 1, 2, ... \text{ and } b_{n,n}(k) > 0$$
$$\text{for all } n = 0, 1, 2, ... \qquad (10)$$

We also remark that generally speaking the properties (10) do not yield the completeness of the system of functions $\left\{ u\left(\frac{k}{l(n)}, \cdot\right) \right\}_{n=0,1,2,...}$ in the space $L_2(0,k)$ (see a counterexample in [8]).

The following statement is proved in [5,7].

Proposition 2. *For any fixed $k > 0$ the system of functions $\{y_n(k,x)\}_{n=0,1,2,...}$ is a basis in the space $L_2(0,k)$, i. e. for any $g \in L_2(0,k)$ there exists a unique sequence $\{a_n\}_{n=0,1,2,...}$ of real numbers a_n such that $g(\cdot) = \sum_{n=0}^{\infty} a_n y_n(k,\cdot)$ in $L_2(0,k)$. In addition, clearly $y_n(k,x) = u\left(\frac{k}{l(n)}, x\right)$.*

According to Proposition 2, for any $k > 0$ there exists a unique sequence of real numbers $\{d_n(k)\}_{n=0,1,2,...}$ such that

$$e(k,\cdot) = \sum_{n=0}^{\infty} d_n(k) u\left(\frac{k}{l(n)}, \cdot\right) \qquad (11)$$

in $L_2(0,k)$. As in the case of the expansions (9), the equality (11) holds in $L_2(a,b)$ with arbitrary $a, b: \ -\infty < a < b < +\infty$.

Lemma 2. *For any $k > 0$ $u(k,x)$ regarded as a function of $x \in (0,k)$ is concave and there exist $0 < c < C$ such that*

$$ck^{\frac{1}{2}} \leq ||u(k,\cdot)||_{L_2(0,k)} \leq Ck^{\frac{1}{2}}$$

for all $k > 0$.

Proof follows from Proposition 1 which in view of equation (1) implies in particular the concavity of $u(k,x)$ for $x \in (0,k)$ that yields the estimates in Lemma 2.□

Corollary 1. *There exist* $0 < c < C$ *such that* $c \le b_{n,n}(k) \le C$ *for all* $k > 0$ *and* n.

Proof is obvious.□

In the next part of this section, we shall derive estimates for the coefficients $b_{n,m}(k)$ and $d_m(k)$ of two types. According to Lemma 1, $u(k,x)$ and $e(k,x)$ regarded as functions of the argument $x \in (0,k)$ become arbitrary close to each other as $k \to 0$. Using this fact, we shall obtain estimates that show, in particular, a sufficiently rapid decay, as $m \to \infty$, of the coefficients $d_m(k)$ uniform with respect to k from an arbitrary interval $(0,\overline{k}]$. At the same time, the behavior of these functions for large $k > 0$ is different which does not allow us to extend this approach to get smallness of these coefficients uniform with respect to all $k > 0$. Below, applying another method, we shall only obtain certain upper bounds, uniform with respect to $k > 0$, of the absolute values of the indicated coefficients. The estimates we shall get are rough enough, but they suffice our goals.

Lemma 3.
1) *For any* $\overline{k} > 0$ *there exists* $C = C(\overline{k}) > 0$ *such that* $|b_{n,m}(k)| \le$ $\le Ck^2 l^{10}(n) l^{-12}(m)$ *for all numbers* $n < m$ *and all* $k \in (0, l(n)\overline{k}]$.
2) *For any* $\overline{k} > 0$ *there exists* $C_1 = C_1(\overline{k}) > 0$ *such that* $|b_{n,m}(k)| \le$ $\le C_1 k^{12} l^{-12}(m)$ *for all* $k \ge \overline{k} l(n)$ *and all numbers* $n < m$.

Proof. Let us prove the first statement. In view of (10), it suffices to consider coefficients $b_{n,m}(k)$ where $l(m) = l(n)l(r)$ for some nonnegative integer n, m and $r \ge 1$. By Lemma 1, for any $\overline{k} > 0$ we have $|\frac{d^m}{dx^m}[u(k,kx) - e(1,x)]| \le C_2(\overline{k})k^2$ for all $k \in (0,\overline{k}]$, all $m = 1,2,...,10$ and all $x \in \mathbb{R}$. Hence, it easily follows from equation (1) that

$$\left| \frac{d^{12}}{dx^{12}} u(k,kx) - d(k) \cos \frac{\pi x}{2} \right| \le C_3(\overline{k})k^2, \quad x \in [0,1],$$

for a function $d(k)$ of $k > 0$, any $\overline{k} > 0$ and $k \in (0,\overline{k}]$. Therefore, using also Proposition 1, we deduce:

$$|b_{n,m}(k)| = p\left|\left|e\left(\frac{k}{l(m)},\cdot\right)\right|\right|^{-2}_{L_2(0,k)}\left|\int\limits_0^k u\left(\frac{k}{l(n)},x\right)\cos\frac{\pi l(m)x}{2k}dx\right| =$$

$$= \frac{2}{p}\left|\int\limits_0^1 u\left(\frac{k}{l(n)},\frac{k}{l(n)}x\right)\cos\frac{\pi l(m)x}{2l(n)}dx\right| \leq$$

$$\leq \frac{2^{13}l^{12}(n)}{\pi^{12}pl^{12}(m)}\left|\int\limits_0^1 \frac{d^{12}}{dx^{12}}\left[u\left(\frac{k}{l(n)},\frac{k}{l(n)}x\right)\right]\cos\frac{\pi l(m)x}{2l(n)}dx\right| \leq C(\overline{k})\frac{k^2 l^{10}(n)}{l^{12}(m)} \tag{12}$$

where $k \in (0, l(n)\overline{k}]$.

To prove statement (b) of our lemma, let us multiply equation (1) by $2u'_x(k,x)$ and integrate the result from 0 to x. We get the identity

$$-[u'_x(k,x)]^2 + F(u^2(k,x)) - F(p^2) = \lambda(k)[u^2(k,x) - p^2], \quad k > 0, \ x \in \mathbb{R},$$

where $F(s) = \frac{1}{2}\int\limits_0^s f(t)dt$. This identity yields the existence of $C_4 > 0$ such that $|u'_x(k,x)| \leq C_4$ for all $k \geq \overline{k}$ and $x \in \mathbb{R}$. Therefore, it easily follows step by step from equation (1) that

$$\left|\frac{d^m}{dx^m}u(k,x)\right| \leq C_5, \quad m = 2,3,...,12,$$

and hence

$$\left|\frac{d^{12}}{dx^{12}}u(k,kx)\right| \leq C_6 k^{12}$$

for all $k \geq \overline{k}$ and $x \in \mathbb{R}$. Proceeding now as in (12), we get the second statement of Lemma.□

Lemma 4. *$b^{-1}_{n,n}(k)|b_{n,m}(k)| \leq \frac{l(n)}{l(m)}$ for all $k > 0$ and all numbers n and m.*

Proof. In view of (10), we have only to prove the statement of Lemma 4 when $l(m) = l(n)l(r)$ for some nonnegative integer r, n and m. Let $\overline{b}_n =$
$= ||e(l^{-1}(n),\cdot)||^{-2}_{L_2(0,1)}\left|\int\limits_0^1 e(l^{-1}(n),x)dx\right| = \frac{4}{p\pi}l^{-1}(n)$ where $n = 0,1,2,....$

If we prove that $b_{0,0}^{-1}(k)|b_{0,n}(k)| \le (\bar{b}_0)^{-1}|\bar{b}_n|$, where $(\bar{b}_0)^{-1}|\bar{b}_n| = l^{-1}(n)$, for all $k > 0$ and all numbers n, then, since $b_{n,m}(k) = b_{0,r}\left(\frac{k}{l(n)}\right)$ if $l(m) = l(n)l(r)$ for some nonnegative integer r and $b_{n,m}(k) = 0$ otherwise, we get the statement of Lemma 4.

So, let us denote here by b_n the coefficients $b_{0,n}(k)$ and take an arbitrary $k > 0$ and a number $n > 0$. Let $x_m = \frac{2m-1}{2n+1}$ and $r_m = \frac{2(m-1)}{2n+1}$ where in both cases $m = 1, 2, ..., n+1$. Then, x_m are zeros and r_m are the points of extremum of $e(l^{-1}(n), x) = p\cos\frac{\pi l(n)x}{2}$ regarded as a function of $x \in [0,1]$ and $0 = r_1 < x_1 < r_2 < x_2 < ... < r_{n+1} < x_{n+1} = 1$.

(a) Let $\psi(x) = u(k, kx)$ and $I(\varphi, z_1, z_2) = p\int\limits_{z_1}^{z_2} \varphi(x)\cos\frac{\pi l(n)x}{2}dx$. Then, we state that, for any $m = 2, 3, ..., n$, the absolute value of $I(\psi, r_{m-1}, r_m)$ is not larger than the absolute value of $I(\psi, r_m, r_{m+1})$ and that the signs of these two quantities are opposite; in addition, $I(\psi, r_1, r_2) > 0$.

Let us prove this statement. Here, we exploit geometric arguments that are more clear visually on a picture. First of all, for any constant $c \in \mathbb{R}$ we have $I(\psi, r_m, r_{m+1}) = -I(\psi(\cdot + r_2) + c, r_{m-1}, r_m)$. Take an arbitrary $m = 2, 3, ..., n$ and set $c = -\psi(x_{m+1}) + \psi(x_m)$. Let also for the definiteness $I(\psi, r_{m-1}, r_m) \ge 0$. Then,

$$I(\psi, r_m, r_{m+1}) = I(-[\psi(\cdot + r_2)) + c], r_{m-1}, r_m) \text{ and } \psi(x_m + r_2) + c = \psi(x_m).$$

Clearly, since $\psi(x)$ and $\psi'(x)$ are respectively positive and negative decreasing functions in $(0,1)$, in (r_{m-1}, r_m) $\psi(x + r_2) + c$ is larger than $\psi(x)$ from the left of x_{m-1} and is smaller than $\psi(x)$ from the right of x_{m-1}, where x_{m-1} is the middle of the interval (r_{m-1}, r_m). Furthermore, in view of the hypothesis (f) and equation (1), $\psi''(x)$ is a nonincreasing negative function of $x \in (0,1)$, therefore $\psi_1(x) = \psi(x + r_2) + c - \psi(x)$ is a concave function in (r_{m-1}, r_m), hence $I(\psi_1, r_{m-1}, r_m) \le 0$. By analogy $I(\psi, r_1, r_2) > 0$. So, the statement (a) is proved.

(b) A simple corollary of the statement (a) and the arguments in (a) is that $I(\psi, 0, 1) = I(\psi, \bar{x}, 1)$ for some $\bar{x} \in [r_{n+1}, 1)$ so that in particular sign $I(\psi, 0, 1) =$ sign $I(\psi, r_{n+1}, 1)$.

(c) Let

$$\int_{\overline{x}}^{1} \psi(x) \left|\cos\frac{\pi l(n)x}{2}\right| dx = \psi(x_1) \int_{\overline{x}}^{1} \left|\cos\frac{\pi l(n)x}{2}\right| dx$$

and

$$\int_{\overline{x}}^{1} \psi(x) \cos\frac{\pi x}{2} dx = \psi(x_2) \int_{\overline{x}}^{1} \cos\frac{\pi x}{2} dx$$

where $x_1, x_2 \in (\overline{x}, 1)$. We state that $x_1 > x_2$.

Let us prove this statement. For this aim, let us show that

$$\frac{\int_{\overline{x}}^{1} \psi(x) \left|\cos\frac{\pi l(n)x}{2}\right| dx}{\int_{\overline{x}}^{1} \psi(x) \cos\frac{\pi x}{2} dx} < \frac{\int_{\overline{x}}^{1} \left|\cos\frac{\pi l(n)x}{2}\right| dx}{\int_{\overline{x}}^{1} \cos\frac{\pi x}{2} dx}. \tag{13}$$

Let $J(q,n,r_1,r_2) = \int_{r_1}^{r_2} [\psi(x)]^q \left|\cos\frac{\pi l(n)x}{2}\right| dx$, where $q = 0, 1$. Simple calculations with the use of Taylor expansions and the fact that due to (1) and (f) $\psi_x^{(3)}(1) > 0$ show that

$$\frac{J(1,n,x_1,1)}{J(1,0,x_1,1)} < \frac{J(0,n,x_1,1)}{J(0,0,x_1,1)}$$

for all x_1 from a sufficiently small left half-neighborhood of 1. To prove (13), it suffices to prove the latter estimate for all $x_1 \in [\overline{x}, 1)$. In contradiction, suppose the existence of $\hat{x} \in [\overline{x}, 1)$ such that

$$\frac{J(1,n,x_1,1)}{J(1,0,x_1,1)} < \frac{J(0,n,x_1,1)}{J(0,0,x_1,1)} \tag{14}$$

for all $x_1 \in (\hat{x}, 1)$ and

$$\frac{J(1,n,\hat{x},1)}{J(0,0,\hat{x},1)} = \frac{J(0,n,\hat{x},1)}{J(0,0,\hat{x},1)}. \tag{15}$$

Observe that the values of the left-hand and right-hand sides in (14) and (15) do not vary if we change $\psi(x)$ by $\psi_1(x) = r\psi(x)$ where r is an arbitrary positive constant. We take $r > 0$ so that, if we change $\psi(x)$ by $\psi_1(x)$, then $J(1,0,\hat{x},1) = J(0,0,\hat{x},1)$. Consequently, $J(1,n,\hat{x},1) = J(0,n,\hat{x},1)$ after this change. Everywhere in the proof of (13) we accept that $\psi(x)$ is changed by $\psi_1(x)$. Note also that obviously $\psi_1(\hat{x}) > 1$.

It can be easily verified that $\left|\cos\frac{\pi l(n)x}{2}\right| : \cos\frac{\pi x}{2}$ is a strongly increasing positive continuous function in $[\overline{x},1)$. Hence, there exists a unique $\tilde{x} \in (\hat{x},1)$ such that the left-hand side in (15) (with ψ_1 in place of ψ) is equal to $\left|\cos\frac{\pi l(n)\tilde{x}}{2}\right| : \cos\frac{\pi\tilde{x}}{2}$. Let $a_q(x_1) = J(q,n,x_1,\tilde{x})$, $c_q = J(q,n,\tilde{x},1)$, $b_q(x_1) = J(q,0,x_1,\tilde{x})$ and $d_q = J(q,0,\tilde{x},1)$. For any sufficiently small $\epsilon > 0$ we have the following Taylor expansions:

$$\frac{a_1(\hat{x}+\epsilon)+c_1}{b_1(\hat{x}+\epsilon)+d_1} =$$

$$= \frac{a_1(\hat{x})+c_1}{b_1(\hat{x})+d_1}\left(1+\frac{\epsilon\psi_1(\hat{x})\cos\frac{\pi\hat{x}}{2}}{b_1(\hat{x})+d_1} - \frac{\epsilon\psi_1(\hat{x})\left|\cos\frac{\pi l(n)\hat{x}}{2}\right|}{a_1(\hat{x})+c_1}\right) + O(\epsilon^2)$$

and

$$\frac{a_0(\hat{x}+\epsilon)+c_0}{b_0(\hat{x}+\epsilon)+d_0} = \frac{a_0(\hat{x})+c_0}{b_0(\hat{x})+d_0}\left(1+\frac{\epsilon\cos\frac{\pi\hat{x}}{2}}{b_0(\hat{x})+d_0} - \frac{\epsilon\left|\cos\frac{\pi l(n)\hat{x}}{2}\right|}{a_0(\hat{x})+c_0}\right) + O(\epsilon^2).$$

Therefore, due to our supposition and the above arguments, we get for sufficiently small $\epsilon > 0$:

$$\frac{\epsilon\psi_1(\hat{x})\cos\frac{\pi\hat{x}}{2}}{b_1(\hat{x})+d_1} - \frac{\epsilon\psi_1(\hat{x})\left|\cos\frac{\pi l(n)\hat{x}}{2}\right|}{a_1(\hat{x})+c_1} < \frac{\epsilon\cos\frac{\pi\hat{x}}{2}}{b_0(\hat{x})+d_0} - \frac{\epsilon\left|\cos\frac{\pi l(n)\hat{x}}{2}\right|}{a_0(\hat{x})+c_0} + O(\epsilon^2)$$

that easily implies

$$\frac{a_1(\hat{x})+c_1}{b_1(\hat{x})+d_1} \le \frac{\left|\cos\frac{\pi l(n)\hat{x}}{2}\right|}{\cos\frac{\pi\hat{x}}{2}} \ne \frac{\left|\cos\frac{\pi l(n)\tilde{x}}{2}\right|}{\cos\frac{\pi\tilde{x}}{2}},$$

which is a contradiction. Thus, the statement (c) is proved.

Now, in view of (a)-(c) we have for some $x' \in (0,\overline{x})$:

$$\frac{\overline{b}_0}{b_0} = \frac{\left(\int_0^{\overline{x}} + \int_{\overline{x}}^1\right)\cos\frac{\pi x}{2}dx}{\psi(x')\int_0^{\overline{x}} cos\frac{\pi x}{2}dx + \psi(x_2)\int_{\overline{x}}^1 \cos\frac{\pi x}{2}dx} < \frac{1}{\psi(x_2)} < \frac{1}{\psi(x_1)} =$$

$$= \frac{\int\limits_{\overline{x}}^{1} \left|\cos \frac{\pi l(n)x}{2}\right| dx}{\int\limits_{\overline{x}}^{1} \psi(x) \left|\cos \frac{\pi l(n)x}{2}\right| dx} \leq \left|\frac{\overline{b}_n}{b_n}\right|,$$

and the proof of Lemma 4 is complete.□

For an arbitrary positive integer n the number $l(n)$ can be uniquely represented as $p_{i_1}^{r_1}...p_{i_m}^{r_m}$ where m, r_j and p_{i_j} are positive integers and $3 = p_1 < p_2 < ... < p_i < ...$ is the sequence of all prime numbers except 2. For a given positive integer s, denote by $\theta(s,n)$ the number of all possible positive integer $m_1 < m_2 < ... < m_s$, where $m_1 \geq 1$ and $m_s < n$, such that $\frac{l(m_i)}{l(m_{i-1})}$ are positive integers for each i and, also, $\frac{l(n)}{l(m_s)}$ is a positive integer. Let $\theta(n) = \sum\limits_{s=1}^{n} \theta(s,n)$. Then, for a given n, $\theta(n)$ is the number of all possible sets $\{s, m_1, m_2, ..., m_s\}$ satisfying the above conditions.

Lemma 5. *$\theta(n) \leq [l(n)]^{\sigma} \log_3 l(n)$, where $\sigma = \log_3 e \in (0,1)$, for all $n = 1,2,3,...$ so that, in particular, for any $\epsilon > 0$ there exists $C = C(\epsilon) >$ > 0 such that $\theta(n) \leq C[l(n)]^{\sigma+\epsilon}$ for all n.*

Proof. Note first that $l(1) = 3$. Therefore, admissible values of s, for each of which numbers $m_1, ..., m_s$ can exist, can be estimated from above as follows:

$$s \leq \log_3 l(n). \tag{16}$$

Further, for a given s, the number m_s can be chosen as $p_{i_1}^{r_1^s}...p_{i_m}^{r_m^s}$ where $0 \leq r_j^s \leq r_j$ are arbitrary positive integers such that there exist j' and j'' for which $0 < r_{j'}^s$ and $r_{j''}^s < r_{j''}$. By analogy, m_{s-1} can be chosen as $p_{i_1}^{r_1^{s-1}}...p_{i_m}^{r_m^{s-1}}$ where $0 \leq r_j^{s-1} \leq r_j^s$ and again there exist j' and j'' such that $0 < r_{j'}^{s-1}$ and $r_{j''}^{s-1} < r_{j''}^s$. Continuing these arguments, we come to the estimate

$$\theta(s,n) \leq \sum_{r_1^s=0}^{r_1} \sum_{r_2^s=0}^{r_2} \cdots \sum_{r_m^s=0}^{r_m} \times \sum_{r_1^{s-1}=0}^{r_1^s} \sum_{r_2^{s-1}=0}^{r_2^s} \cdots \sum_{r_m^{s-1}=0}^{r_m^s} \cdots \sum_{r_1^2=0}^{r_1^3} \sum_{r_2^2=0}^{r_2^3} \cdots \sum_{r_m^2=0}^{r_m^3} \times$$

$$\times \sum_{r_1^1=0}^{r_1^2} \sum_{r_2^1=0}^{r_2^2} \cdots \sum_{r_m^1=0}^{r_m^2} 1 \leq \int\limits_0^{r_1+1} dx_1^s \int\limits_0^{x_1^s+1} dx_1^{s-1} \cdots \int\limits_0^{x_1^2+1} dx_1^1 \times$$

$$\times \int_0^{r_2+1} dx_2^s \int_0^{x_2^s+1} dx_2^{s-1} \dots \int_0^{x_2^2+1} dx_2^1 \times \int_0^{r_m+1} dx_m^s \int_0^{x_m^s+1} dx_m^{s-1} \dots \int_0^{x_m^2+1} dx_m^1 =$$

$$= \left(1 + \frac{r_1}{1!} + \frac{(r_1)^2}{2!} + \dots + \frac{(r_1)^s}{s!}\right)\left(1 + \frac{r_2}{1!} + \frac{(r_2)^2}{2!} + \dots + \frac{(r_2)^s}{s!}\right) \times$$

$$\times \left(1 + \frac{r_m}{1!} + \frac{(r_m)^2}{2!} + \dots + \frac{(r_m)^s}{s!}\right) \le e^{r_1 + r_2 + \dots + r_m}.$$

Hence, since obviously $r_1 + r_2 + \dots + r_m \le \log_3 l(n)$, we obtain:

$$\theta(s,n) \le [l(n)]^\sigma.$$

Finally, due to (16) $\theta(n) \le [l(n)]^\sigma \log_3 l(n)$.□

Remark 5. One can easily verify that if $l(n) = 3^d$ with a positive integer d, then precisely $\theta(n) = \sum_{s=1}^{d-1} \binom{n}{s} = 2^d - 2 = [l(n)]^{\log_3 2} - 2$ so that in any estimate $\theta(n) \le h(l(n))$ it should be $h(l(n)) \ge C'[l(n)]^b$ for some constants $C' > 0$ and $b > 0$ and some values of n.

Proposition 3. *Let again* $\sigma = \log_3 e \in (0,1)$.
(a) *For any* $\epsilon > 0$ *there exists* $C = C(\epsilon) > 0$ *such that for any nonnegative integer* n *and any* $k > 0$ *the following estimate takes place:* $|d_n(k)| \le$ $\le C(1+k)^{12} l^{\sigma+\epsilon-12}(n)$.
(b) *For any* $\epsilon > 0$ *there exists* $C = C(\epsilon) > 0$ *such that* $|d_n(k)| \le$ $\le C l^{\sigma+\epsilon-1}(n)$ *for all* $k > 0$ *and all* $n = 0, 1, 2, \dots$.

Proof. One can easily see that, since the matrix $B(k)$ is upper triangular and the elements of its principal diagonal are nonzero, each coefficient $d_n(k)$ is equal to the element $t_{0,n}(k)$ of the matrix $T^n(k) = [B^n(k)]^{-1}$ where $B^n(k) = (b_{r,m}(k))_{r,m=0,1,\dots,n}$. Let $B_{m,r}(k)$ be the matrix obtained from $B^n(k)$ by putting away the mth row and the rth column of the latter. Then,

$$d_n(k) = (-1)^n \det[B_{n,0}(k)] \times \det[T^n(k)] \tag{17}$$

where

$$B_{n,0}(k) = \begin{pmatrix} b_{0,1} & b_{0,2} & \dots\dots & b_{0,n-1} & b_{0,n} \\ b_{1,1} & b_{1,2} & \dots\dots & b_{1,n-1} & b_{1,n} \\ 0 & b_{2,2} & \dots\dots & b_{2,n-1} & b_{2,n} \\ \dots & \dots & \dots\dots & \dots & \dots \\ 0 & 0 & \dots\dots & b_{n-1,n-1} & b_{n-1,n} \end{pmatrix}.$$

In this matrix, let us, for each $m = 0, 1, ..., n-2$, subtract the last row, multiplied by $b^{-1}_{n-1,n-1} b_{m,n-1}$, from the mth row and then, subtract the next to the last column, multiplied by $b^{-1}_{n-1,n-1} b_{n-1,n}$, from the last column. Further, let us, for each $m = 0, 1, ..., n-3$, subtract the next to the last row, multiplied by $b^{-1}_{n-2,n-2} b_{m,n-2}$, from the mth row and then, subtract the column with the first index $(n-2)$, multiplied by $b^{-1}_{n-2,n-2}(b_{n-2,n} - - b^{-1}_{n-1,n-1} b_{n-1,n} b_{n-2,n-1})$, from the last column, and so on. As a result of this procedure, we obtain a matrix, all elements in the last column of which, except the element in the upper row, are equal to zero, and the element $b'_{0,n}$ in the upper row is equal to $b_{0,n} - S$ where S is a sum of terms of the kind

$$b^{-1}_{m_1,m_1} b^{-1}_{m_2,m_2} ... b^{-1}_{m_s,m_s} b_{0,m_1} b_{m_1,m_2} ... b_{m_{s-1},m_s} b_{m_s,n}$$

with coefficients $+1$ or -1, where $s, m_1, ..., m_s$ are arbitrary positive integers satisfying $s \leq n$ and $0 < m_1 < m_2 < ... < m_s < n$.

Let us prove the second statement of Proposition 3. We obviously have:

$$|\det\ (B_{n,0}(k))| = |b'_{0,n}| \prod_{r=1}^{n-1} b_{r,r},$$

therefore, in view of Corollary 1 and (17), $|d_n(k)| \leq C_1 |b'_{0,n}|$ where the constant $C_1 > 0$ does not depend on $k > 0$ and n. Now, in view of (10) and Lemma 5, for any $\epsilon > 0$ there exists $C_2 > 0$ such that for any $n = 1, 2, 3, ...$ the number of nonzero terms in the sum S is not larger than $C_2 l^{\sigma+\epsilon}(n)$. Also, due to Lemma 4 and Corollary 1, the absolute value of each term in this sum is not larger than $C_3 l^{-1}(n)$ with a constant $C_3 > 0$ independent of $k > 0$ and nonnegative integer n, and thus, we get the second statement of Proposition 3.

Let us prove the first claim. Fix an arbitrary $\overline{k} \in (0,1)$ such that $C\overline{k} < 1$ where $C = C(\overline{k}) > 0$ is the constant from the first statement of Lemma 3. Take arbitrary $k > 0$ and a number n and let b be the absolute value of the term

$$\pm b_{0,m_1}(k) b^{-1}_{m_1,m_1}(k) b_{m_1,m_2}(k) b^{-1}_{m_2,m_2}(k) b_{m_2,m_3}(k) \ldots b^{-1}_{m_s,m_s}(k) b_{m_s,n}(k)$$

contained in the sum S. First, we study the case when there exists $s_0 \in \{0, 1, \ldots, s\}$ such that $k < \overline{k} l(m_{s_0+1})$ and $k \geq \overline{k} l(m_{s_0})$ where $m_0 = 0$ and $m_{s+1} = n$. In this case, in accordance with Lemma 3, we estimate the cofactors $b_{m_r,m_r}^{-1}|b_{m_r,m_{r+1}}|$ for $r = s_0+1,\ldots,s$ by $C\overline{k}^2\left(\frac{l(m_r)}{l(m_{r+1})}\right)^{12}$, the cofactor $b_{m_{s_0},m_{s_0}}^{-1}|b_{m_{s_0},m_{s_0+1}}|$ by $C_4k^{12}l^{-12}(m_{s_0+1})$, and, for $r = 0,1,\ldots,s_0-1$, we estimate the cofactors $b_{m_r,m_r}^{-1}|b_{m_r,m_{r+1}}|$ by 1. We get: $b \le C_4k^{12}l^{-12}(n)$.

Now, suppose that the number s_0 defined above does not exist. If $k < \overline{k}$, then we estimate all cofactors $b_{m_r,m_r}^{-1}|b_{m_r,m_{r+1}}|$ from above by $C\overline{k}^2\left(\frac{l(m_r)}{l(m_{r+1})}\right)^{12}$ and, consequently, arrive at the estimate $b \le C_5l^{-12}(n)$. If $k > \overline{k}l(m_s)$, then, in accordance with Lemma 3, we apply the estimates $b_{m_s,m_s}^{-1}|b_{m_s,n}| \le C_6k^{12}l^{-12}(n)$ and $b_{m_r,m_r}^{-1}|b_{m_r,m_{r+1}}| \le 1$ for $r = 0,1,\ldots,s-1$; we get $b \le C_6k^{12}l^{-12}(n)$.

In view of the above estimates of the nonzero terms contained in the sum S, applying Lemma 5, we obtain the first claim of Proposition 3.□

Corollary 2. *Coefficients $d_n(k)$ are continuous in k.*

Proof. On the contrary, let for some $k_0 > 0$ and n $d_n(k) \not\to d_n(k_0)$ as $k \to k_0$. Let $k_r \to k_0$ $(r = 1,2,3,\ldots)$ be such a sequence that $d_n(k_r) \to \overline{d}_n \ne d_n(k_0)$ as $r \to \infty$. Then, due to Proposition 3(a)

$$\sum_{m=0}^{\infty}\left[d_m(k_r)u\left(\frac{k_r}{l(m)},\cdot\right) - d_m(k_0)u\left(\frac{k_0}{l(m)},\cdot\right)\right] \to 0$$

in $L_2(0,k_0)$ as $r \to \infty$, and we get

$$\sum_{m=0}^{\infty}[\overline{d}_m - d_m(k_0)]u\left(\frac{k_0}{l(m)},\cdot\right) = 0 \quad \text{in} \quad L_2(0,k_0),$$

where $\overline{d}_n \ne d_n(k_0)$, which contradicts Proposition 2.□

3 Proof of Theorem

Let $g \in S_e$ and let $\hat{g}(\cdot) \in S_e$ be its renormalized Fourier transform so that

$$g(x) = \int_0^\infty \hat{g}(r)e(r^{-1},x)dr, \quad x \ge 0, \tag{18}$$

where the improper integral in the right-hand side converges absolutely uniformly with respect to $x \ge 0$. We shall make a procedure which is formally the following. First, we make the change of variables $k = r^{-1}$ obtaining from (18):

$$g(x) = \int_0^\infty \frac{\hat{g}(k^{-1})}{k^2} e(k,x)dk,$$

where again the improper integral in the right-hand side converges absolutely uniformly with respect to $x \geq 0$. Representing the function $e(k,x)$ as in (11), we get formally

$$g(x) = \int_0^\infty \frac{\hat{g}(k^{-1})}{k^2} \sum_{n=0}^\infty d_n(k) u\left(\frac{k}{l(n)}, x\right) dk =$$

$$= \sum_{n=0}^\infty \int_0^\infty \frac{\hat{g}(k^{-1})}{k^2} d_n(k) u\left(\frac{k}{l(n)}, x\right) dk =$$

$$= \sum_{n=0}^\infty \int_0^\infty \frac{\hat{g}(l^{-1}(n)k^{-1})}{l(n)k^2} d_n(l(n)k) u(k,x)dk =$$

$$= \int_0^\infty \sum_{n=0}^\infty \frac{\hat{g}(l^{-1}(n)k^{-1})}{l(n)k^2} d_n(l(n)k) u(k,x)dk =$$

$$= \int_0^\infty \overline{g}(k) u(k,x)dk. \tag{19}$$

In this section, to prove that (3) and (4) take place, we shall justify these formal relations. This justification easily follows from the following two lemmas.

<u>Lemma 6.</u> *For any $R > 0$ the series $\sum_{n=0}^{\infty} \frac{\hat{g}(k^{-1})}{k^2} d_n(k)$ converges absolutely uniformly with respect to $k \in (0, R]$.*

<u>Proof</u> in view of the estimate $|\hat{g}(k^{-1})| \leq C(1 + k^{-1})^{-2}$ follows from Corollary 1 and Proposition 3(a).□

Lemma 7. *The series* $\sum_{n=0}^{\infty} \frac{\hat{g}(k^{-1}l^{-1}(n))}{k^2 l(n)} d_n(kl(n))$ *converges absolutely uniformly with respect to* $k \in (a,b)$ *with arbitrary* $a, b: \ 0 < a < b < \infty$ *and is continuous in* $k > 0$; *the sum* $\sum_{n=0}^{\infty} \frac{|\hat{g}(k^{-1}l^{-1}(n))|}{k^2 l(n)} |d_n(kl(n))|$ *satisfies estimate* (4) *with some* $C > 0$ *and* $\gamma \in (0,1)$.

Proof. The first two claims follow from Corollary 2 and Proposition 3(b). Let us prove the last claim.

First, in view of Proposition 3(b), for $k \geq 1$ this estimate holds. Consider the case $k \in (0,1)$. Due to Corollary 1 and Proposition 3 and the estimate $|\hat{g}(s)| \leq \frac{C}{(1+s)^{12}}$, we get for an arbitrary $\delta > 0$ and a sufficiently small $\epsilon > 0$:

$$S(k) = \left(\sum_{n:\ 0<l(n)\leq k^{-1-\delta}} + \sum_{n:\ l(n)>k^{-1-\delta}} \right) \frac{|\hat{g}(k^{-1}l^{-1}(n))|}{k^2 l(n)} |d_n(kl(n))| \leq$$

$$\leq C_1 \sum_{n:\ 0<l(n)\leq k^{-1-\delta}} \frac{(1+kl(n))^{12}}{k^2 l^{13-\sigma-\epsilon}(n)(1+k^{-1}l^{-1}(n))^{12}} +$$

$$+C_2 \sum_{n:\ l(n)>k^{-1-\delta}} \frac{1}{k^2 l^{2-\sigma-\epsilon}(n)} \leq C_3[1 + k^{10-(1+\delta)(\sigma+\epsilon)} + k^{-2+(1+\delta)(1-\sigma-\epsilon)}].$$

Now, we observe that, since $\log_3 e < \frac{11}{12}$, there exist $\delta > 0$ and a sufficiently small $\epsilon > 0$ satisfying the conditions $10 - (1+\delta)(\sigma+\epsilon) > -1$ and $-2 + (1+\delta)(1-\sigma-\epsilon) > -1$. Taking these values of δ and ϵ, we get from the above estimate:

$$S(k) \leq C_4 k^{-\gamma}, \quad k \in (0,1),$$

with some $C_4 > 0$ and $\gamma \in (0,1)$. So, the proof of Lemma 7 is complete.□

So, for any $g(\cdot) \in S_e$, the existence of a continuous function $\overline{g}(\cdot)$ satisfying (3) and (4) is proved.

Now, we turn to proving the uniqueness of the function $\overline{g}(k)$ satisfying (3) and (4). Suppose the opposite. Then, clearly there exists a continuous function $\overline{g}(\cdot) \not\equiv 0$ satisfying (4) with some $\gamma \in (0,1)$ and $C > 0$ and such that

$$0 \equiv \int_0^{\infty} \overline{g}(k) u(k,x) dx, \quad x \geq 0. \tag{20}$$

In what follows, we rename the coefficients $b_{0,n}(k)$ by $b_n(k)$. Substituting the expansion of $u(k,x)$, regarded as a function of x, over the functions $e\left(\frac{k}{l(n)},\cdot\right)$, where $n=0,1,2,...$, into the right-hand side of (20) and taking into account (4) and Lemma 3, we come to the following:

$$0 \equiv \lim_{R\to+\infty} \int_0^R \overline{g}(k) \sum_{n=0}^{\infty} b_n(k) e\left(\frac{k}{l(n)}, x\right) dk \equiv$$

$$\equiv \lim_{R\to+\infty} \sum_{n=0}^{\infty} \int_0^R \overline{g}(k) b_n(k) e\left(\frac{k}{l(n)}, x\right) dk \equiv$$

$$\equiv \lim_{R\to+\infty} \sum_{n=0}^{\infty} \int_0^{\infty} \chi_{[0,R]}(kl(n)) l(n) \overline{g}(kl(n)) b_n(kl(n)) e(k,x) dk \tag{21}$$

where $\chi_{[0,R]}(r)=1$ if $r\in[0,R]$ and 0 otherwise.

Lemma 8. *The right-hand side in* (21) *is identically equal to* $\int_0^{\infty} g_1(k)e(k,x)dk$, *where* $g_1(k) = \sum_{n=0}^{\infty} l(n)b_n(kl(n))\overline{g}(kl(n))$; *the function* $g_1(\cdot)$ *is continuous and it satisfies the estimate*

$$|g_1(k)| \le Ck^{\gamma-1}(1+k)^{-1-\gamma} \tag{22}$$

with some constants $C>0$ *and* $\gamma\in(0,1)$ *independent of* $k>0$.

Proof. Consider the function $S_1(k) = \sum_{n=0}^{\infty} l(n)|b_n(kl(n))\overline{g}(kl(n))|$. To prove Lemma, it suffices to show the existence of $C>0$ and $\gamma\in(0,1)$ such that

$$S_1(k) \le Ck^{\gamma-1}(1+k)^{-1-\gamma}, \quad k>0.$$

Let first $k\ge 1$. Then, using the fact that $\overline{g}$ satisfies (4) and applying Corollary 1 and Lemma 4, we come to the estimate:

$$S_1(k) \le C\sum_{n=0}^{\infty} k^{-2}l^{-2}(n) \le C_1 k^{-2}.$$

Second, let $k\in(0,1)$. Then, due to Lemmas 3 and 4 and Corollary 1, we have

$$S_1(k) \le \left(\sum_{n:\ 0<l(n)\le k^{-1}} + \sum_{n:\ k^{-1}<l(n)\le k^{-\frac{4}{3}}} + \sum_{n:\ l(n)>k^{-\frac{4}{3}}} \right) \times$$

$$\times l(n)|b_n(kl(n))|\ |\overline{g}(kl(n))| \le$$

$$\le C\Big(k^{\gamma+1} \sum_{n:\ 0<l(n)\le k^{-1}} l^{\gamma-10}(n) + \sum_{n:\ k^{-1}<l(n)\le k^{-\frac{4}{3}}} k^{10} l^{-1}(n) +$$

$$+ \sum_{n:\ l(n)>k^{-\frac{4}{3}}} \frac{1}{k^2 l^2(n)}\Big) \le C' + C'' k^{-\frac{2}{3}}.$$

Thus, Lemma 8 is proved.□

In view of Lemma 8, making in the integral in the right-hand side of the identity

$$0 \equiv \int_0^\infty g_1(k) e(k,x) dk, \quad x \ge 0,$$

the change of variables $r = k^{-1}$, we get

$$0 \equiv \int_0^\infty \hat{g}(r) e(r^{-1}, x) dr, \quad x \ge 0,$$

where $\hat{g}(k) = \frac{g_1(k^{-1})}{k^2}$ so that $\hat{g}(\cdot) \in L_2(0,\infty)$ and $\hat{g}$ is continuous. By the well-known property of the Fourier transform we have $\hat{g}(r) \equiv 0$.

Now, to prove our Theorem, it suffices to show the existence of $k > 0$ such that $g_1(k) \ne 0$. Let us do this. In view of Lemma 8 and Corollaries 1 and 2, one can easily see that for any $\delta \in (1,2)$ there exists a $\overline{k} > 0$ such that

$$\overline{k}^\delta |\overline{g}(\overline{k})|\ b_0(\overline{k}) \ge \sup_{k \ge \overline{k}} [k^\delta |\overline{g}(k)|\ b_0(k)] > 0.$$

Using this fact and the expression for $g_1(k)$, we deduce:

$$|\overline{g}(\overline{k})| b_0(\overline{k}) \le |\overline{g}(\overline{k})| b_0(\overline{k}) \sum_{n=1}^\infty l^{-\delta}(n) <$$

$$< |\overline{g}(\overline{k})| b_0(\overline{k}) \int_{\frac{1}{2}}^\infty \frac{dz}{(2z+1)^\delta} < |\overline{g}(\overline{k})| b_0(\overline{k}).$$

So, we get a contradiction and, thus, our Theorem is completely proved.

4 A variant of Theorem

A result similar to our Theorem takes place for the following problem:

$$-u'' + f(u^2)u = \lambda u, \quad u = u(x), \quad x \in (0, \infty), \tag{23}$$

$$u(0) = 0, \; u'(0) = p, \; \sup_{x>0} |u(x)| < \infty \tag{24}$$

where again all quantities are real, $p > 0$ is an arbitrary fixed parameter, and λ is a spectral parameter. We also denote eigenfunctions by $u(\lambda, x)$. The result for this problem we establish here is the following.

<u>Theorem′</u> *Under the hypothesis* (f)
(a) *there exists* $\overline{\lambda} \geq f(0)$ *such that the spectrum of the problem* (23),(24) *is either* $(\overline{\lambda}, +\infty)$ *or* $[\overline{\lambda}, +\infty)$;
(b) *for each* $\lambda \in (\overline{\lambda}, +\infty)$ *there exists a minimal positive zero* $k = k(\lambda)$ *of* $u(\lambda, x)$ *as a function of* $x \in (0, +\infty)$; *the function* $k : \; (\overline{\lambda}, +\infty) \to (0, +\infty)$ *is continuously differentiable,* $k'(\lambda) < 0$ *for any* $\lambda \in (\overline{\lambda}, +\infty)$, *and* $\lim_{\lambda \to \overline{\lambda}+0} k(\lambda) = +\infty$, $\lim_{\lambda \to +\infty} k(\lambda) = 0$. *By* $\lambda = \lambda(k)$ *we denote the function with the domain* $(0, +\infty)$ *inverse to* $k(\lambda)$;
(c) *let* $u_1(\lambda, x) = [\max_{x \in [0, k(\lambda)]} u(\lambda, x)]^{-1} u(\lambda, x)$. *Then, for any odd function* $g(\cdot) \in S$ *there exists a unique continuous function* $\overline{g}(\cdot)$, *defined in* $(0, \infty)$, *such that for any* $x \geq 0$ *the following equality takes place:*

$$g(x) = \int_0^\infty \overline{g}(k) u_1(\lambda(k), x) dk$$

and that there exist $\gamma \in (0, 1)$ *and* $C > 0$ *for which*

$$|\overline{g}(k)| \leq C k^{\gamma - 1} (1 + k)^{-1-\gamma}, \quad k > 0.$$

<u>Proof.</u> Statements similar to Propositions 1 and 2 for the problem (23),(24) are proved in [7]. They imply the statements (a) and (b) of Theorem′. The further proof of Theorem′ in fact repeats the above proof of Theorem.□

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Жидков П.Е. E5-2001-63

Аналог преобразования Фурье,
связанный с нелинейным одномерным уравнением Шредингера

Рассматривается задача на собственные значения, включающая нелинейное уравнение Шредингера на полупрямой $[0, \infty)$ и некоторые граничные условия. Показано, что спектр этой задачи заполняет некоторую полупрямую и что каждой точке спектра соответствует единственная собственная функция. Главный результат работы состоит в доказательстве того, что произвольная бесконечно дифференцируемая функция, которая быстро убывает при $x \to \infty$ и удовлетворяет подходящим граничным условиям в точке $x=0$, может быть единственным образом разложена в интеграл по собственным функциям подобно представлению функций при помощи преобразования Фурье (последнее, очевидно, ассоциировано с линейной задачей на собственные значения).

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Zhidkov P.E. E5-2001-63

An Analog of the Fourier Transform Associated
with a Nonlinear One-Dimensional Schrödinger Equation

We consider an eigenvalue problem which includes a nonlinear Schrödinger equation on the half-line $[0, \infty)$ and certain boundary conditions. It is shown that the spectrum of this problem fills a half-line and that to each point of the spectrum there corresponds a unique eigenfunction. The main result of the paper is that an arbitrary infinitely differentiable function $g(x)$ rapidly decaying as $x \to \infty$ and satisfying suitable boundary conditions at the point $x=0$ can be uniquely expanded into an integral over eigenfunctions similar to the representation of functions by the Fourier transform (the latter is obviously associated with a linear self-adjoint eigenvalue problem).

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

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