E2-2003-95

D. I. Kazakov[1,2], V. S. Popov[1]

ON THE ASYMPTOTICS OF THE GELL-MANN–LOW FUNCTION IN QUANTUM FIELD THEORY

Submitted to «Письма в ЖЭТФ»

[1]Institute of Theoretical and Experimental Physics, 117218 Moscow, Russia

[2]Joint Institute for Nuclear Research, 141980 Dubna, Russia;
E-mail: kazakovd@thsun1.jinr.ru

1. Asymptotics of the Gell-Mann-Low (GML) function $\beta(g)$ as $g \to \infty$, which defines the behaviour of the effective coupling at small distances is of considerable interest in quantum field theory (QFT) [1, 2]. The available information is based on perturbation theory (PT) in the coupling constant g and is given by the asymptotic series

$$\beta(g) \sim \sum_{n=2}^{\infty} \beta_n (-g)^n, \tag{1}$$

where we know the first few terms calculated from Feynman diagrams and also the higher order asymptotics $\bar{\beta}_n$ as $n \to \infty$ which can be chosen as

$$\bar{\beta}_n = \Gamma(n + 1/2) a^n n^b c, \tag{2}$$

where the constants a, b and c can be found with the help of the steepest descent method [3].

Though the available information is strictly speaking insufficient for reconstruction of the function $\beta(g)$, originally there were hopes that matching the asymptotical coefficients $\bar{\beta}_n$ with the exact first ones β_n will allow one to reconstruct the function with reasonable accuracy in a wide range of the coupling g and even in the strong coupling regime, i.e. $g \to \infty$.

Such attempts have been undertaken by several authors with the help of various methods of summation of divergent series, however all of them have shown that the realization of this programme is only possible in some limited interval of g and not for $g \to \infty$ (see, e.g. Refs. [4]-[8] and references therein.). As far as we know, this conclusion is commonly accepted in the literature.

In a series of recent papers by Suslov [9]-[13] there is an attempt to revise these results. These papers are based on the interpolation of PT coefficients β_n for the intermediate values of n: "A reasonable formulation of the problem corresponds to approximately setting all β_n, after which $\beta(g)$ can be reconstructed with certain precision. Thus, a necessary stage in solving the problem consists in interpolating the coefficient function, which naturally implies that the function is analytical" (see [9], p.11). Then the interpolation procedure is optimized with respect to parameters thus, according to [9], defining the asymptotics of the desired function as $g \to \infty$. As a result of the application of this procedure in Refs. [9]-[12]

the asymptotics of the GML function in several models of QFT, including the scalar model $\phi^4_{(4)}$, QED and QCD was found. This allowed the author to declare the absence of a "zero charge" in the $\phi^4_{(4)}$ theory and in QED that contradicts the conclusions based on summation of the asymptotic series made by the other authors [6]-[8].

As it has already been mentioned in [14], the method used in Refs. [9]-[12] is not well-justified to make any definite conclusions. Moreover, we think that it is reasonable to talk about the reconstruction of the function $\beta(g)$ starting with its asymptotic series (assuming its analytical properties) only in some extended, compared to the usual PT, range of the coupling g but not as $g \to \infty$. The corresponding arguments were given in our paper [14]. The new paper by Suslov [12] dedicated to the asymptotics of the GML function in QCD induces us to turn back to this question[1].

2. The characteristic feature of QCD is that PT series (1) is sign non-alternating and hence is not Borel summable, the latter being the basis of practically all the approaches to the summation of asymptotic series. To avoid this difficulty, the author of [12] makes the substitution $g \to -g$ in (1) and uses the assumption that the asymptotics of $\beta(g)$ for $g \to \pm\infty$ coincide. There is no need to tell that this assumption in not justified. Moreover, physically it is clear that upon changing the sign of the coupling the system undergoes a complete change: it becomes unstable and simply decays. This circumstance is responsible for the fact that the point $g = 0$ is essentially singular in the complex g-plane and, therefore, the PT series is asymptotic [16]. As is well known, the sign non-alternating series in QCD reflects the degeneracy of the vacuum and hence the contributions proportional to the exponent of the inverse coupling which are not reproducible by PT. At the same time, they are not necessarily suppressed for the strong coupling. That is why, the approach based on the change of the sign of the coupling, consideration of the sign alternating series and subsequent transition to the sign non-alternating series, seems absolutely unjustified and in general incorrect.

To illustrate this point we consider the following example. Let the two functions $f_1(g)$ and $f_2(g)$ be given by

$$f_1(g) \sim \frac{e^g \sum a_n g^n + e^{-g} \sum b_n g^n}{e^g + e^{-g}}, \tag{3}$$

[1]The main results of this paper are presented in Ref.[15]

$$f_2(g) \sim \frac{e^{-g}\sum a_n(-g)^n + e^g\sum b_n(-g)^n}{e^{-g}+e^g}, \tag{4}$$

where the series are asymptotic: $a_n \sim \Gamma(n+a)$, $b_n \sim \Gamma(n+b)$. Then the functions $f_1(g)$ and $f_2(g)$ at $g \to 0$ have the same asymptotic series but with the change of the sign of the coupling

$$f_1(g) \sim \sum f_n g^n, \quad f_2(g) \sim \sum f_n(-g)^n,$$

where $f_n \simeq a_n + b_n$ for $n \gg 1$. Nevertheless, as $g \to \infty$ they may have completely independent behaviour

$$f_1(g) \sim \sum a_n g^n \sim g^\alpha, \quad f_2(g) \sim \sum b_n(-g)^n \sim g^\beta. \tag{5}$$

Remind that $f_2(g) \neq f_1(-g)$, as it may seem, since the function $f_1(-g)$ may not even exist and the analytical continuation from the positive semi-axis usually leads to an imaginary part [16] which is absent in $f_2(g)$.

The number of such examples can be easily increased. The asymptotics of an analytical function at infinity is not as a rule an analytical function and possess discontinuities (Stocks phenomenon), which is well known in the theory of special functions. One can mention also the quasiclassical wave functions changing their form when going from the classically allowed into the under-barrier region, and in the complex plane while crossing the Stocks line [17].

Thus, the assumption that the two functions have the same behaviour at infinity even if they have the same asymptotic series up to the sign of the coupling is not obvious and needs additional arguments, which are absent in Ref. [12].

However, even after such an assumption, the author's approach to the sign alternating series gives, from our point of view, rather ambiguous results (see Fig.1 in [12]): the presence of a large number of minima of χ^2 looks like an artefact of the given procedure with a small number of the original terms of PT. Note that the working interval $20 \leq n \leq 40$ for the interpolation of the coefficient function lies in the region where the exact coefficients are unknown and the asymptotics may not be established. The estimate of the coefficient of the leading asymptotics $\beta_\infty \sim 10^5$ with the accuracy of several orders (practically it changes within the interval from 1 to 10^{10}, see Fig.2b in [12]) also indicates that the asymptotics is unreliable.

3. One can give other arguments against the asymptotics advocated in [12]. Indeed, let us consider the function given by the asymptotic series

$$f(g) \sim \sum_{n=0}^{\infty} f_n(-g)^n, \quad f_n \sim \Gamma(n+b) \quad \text{as} \quad n \to \infty. \tag{6}$$

Let us assume[2] that it is Borel summable and apply the Borel transformation

$$f(g) = \int_0^{\infty} dx\; e^{-x} \sum_{n=0}^{\infty} \frac{f_n}{n!}(-gx)^n = \int_0^{\infty} dx\; e^{-x} B(gx), \tag{7}$$

where the function $B(x)$, given by the convergent series, is called the Borel transform of $f(g)$.

Without loss of generality let us assume that the Borel transform $B(x)$ has a power law behaviour $B(x) \sim x^{\alpha}$ as $x \to \infty$. Then, depending on the value of the exponent α, we get the following behaviour of the function $f(g)$ as $g \to \infty$:

$$f(g) \sim \begin{cases} \Gamma(\alpha+1) g^{\alpha}, & \alpha > -1, \\ \log g/g, & \alpha = -1, \\ c_1\, g^{-1}, & \alpha < -1, \end{cases} \tag{8}$$

where $c_1 = \int_0^{\infty} dx B(x) < \infty$. The last equality is easy to get by making the substitution $x = t/g$ in the integral (7) and tending g to infinity.

At first sight, it seems that the function $f(g)$ cannot decrease faster than $1/g$. However, this is not so. In the case when the first N moments of the Borel transform are equal to zero[3]:

$$c_i = \int_0^{\infty} dx\; x^{i-1} B(x) = 0, \quad i = 1, ..., N, \tag{9}$$

one has

$$f(g) \sim c_{N+1}\, g^{-(N+1)}, \quad g \to \infty \tag{10}$$

if $0 < |c_{N+1}| < \infty$. Otherwise (i.e. when $c_{N+1} = \infty$) the exponent of the asymptotics of $f(g)$ is within $-N$ и $-(N+1)$ or $f(g) \sim \ln g/g^{N+1}$.

Hence, to get a decrease of the order of g^{-13} (see Ref. [12]), one needs 12 first moments of $B(x)$ to vanish. Bearing in mind that today we know only four terms of the PT series (1), this statement seems unjustified.

[2] This is quite natural since without it the PT series in quantum mechanics and field theory are not well defined. This assumption is also used in Refs. [9]-[12].

[3] This also means that the Borel transform must oscillate and have N zeros in the interval $0 < x < \infty$.

The same property of the function can be seen in the modified Borel transformation used, in particular, in [12]:

$$f(g) = \int_0^\infty dx\ e^{-x} x^{\beta-1} \sum_{n=0}^{\infty} \frac{f_n}{\Gamma(n+\beta)}(-gx)^n = \int_0^\infty dx\ e^{-x} x^{\beta-1} B_\beta(gx), \tag{11}$$

for $\beta > 1$. The asymptotics of the Borel transform in this case depends on the value of β. Thus, if one chooses $\beta = b$ from (6), one gets $B_b(x) \sim c_1/x$ and

$$f(g) \sim c_1 \Gamma(b-1) g^{-1}, \quad \text{if } \ c_1 \neq 0.$$

Otherwise, performing the substitution $x \to x/g$ in the integral (11) we get[4]

$$f(g) \ = \ \frac{1}{g^\beta} \int_0^\infty dx\ e^{-x/g} x^{\beta-1} B_\beta(x) \ \underset{g\to\infty}{\sim} \ \frac{1}{g^\beta} \int_0^\infty dx\ x^{\beta-1} B_\beta(x). \tag{12}$$

Allowing here the parameter β to take the values $\beta = 1, 2, ..., N$, one can see that to get the asymptotics $f(g) \sim 1/g^N$ it is necessary that the lower moments vanish:

$$\int_0^\infty dx\ x^{j-1} B_k(x) = 0, \quad k \le j \le N;\ k = 1, 2, ..., N. \tag{13}$$

Altogether, this gives $N(N+1)/2$ conditions (for more details see Appendix). Note that these conditions are imposed not on a single function, but on N different (but dependent) functions $B_1(x), ..., B_N(x)$.

Thus, a fast decrease of the GML function $\beta(g) \sim g^{-N}$ is in principle possible; however, it requires a large number of conditions (N or $N^2/2$ for (9) and (13), respectively) to be imposed on the Borel transform. Clearly, the only known 4-5 coefficients β_n cannot guarantee the fulfillment of these conditions. On the other hand, as it follows from the above analysis, the asymptotics increasing or decreasing slower than $1/g$ happens to be much less limiting.

4. Let us make a few remarks concerning the comment [13] to our paper [14]. They are related to the zero-dimensional model $\phi^4_{(0)}$

$$J(g) = \frac{1}{\sqrt{2\pi}} \int_\infty^\infty d\phi\ exp\{-\frac{1}{2}\phi^2 - \frac{g}{4!}\phi^4\} \ \sim \sum_{k=0}^{\infty} (-g)^k \frac{\Gamma(2k+1/2)}{\Gamma(k+1)\Gamma(1/2)6^k}. \tag{14}$$

[4]We assume here that the integral converges, that imposes an upper bound on β.

It is stated [13] that taking into account just one (!) coefficient of PT series, one can get, with the help of the method advocated in [9], the exponent of $J(g) \propto g^{\alpha}$ with 10% accuracy: $-0.271 < \alpha < -0.218$ (the exact value is $\alpha = -1/4$). Further, it is said that "this result disproves the main statement of Ref. [14] of the necessity of large number of PT coefficients". However, this example is rather specific for the following reasons:

a) already the first PT coefficients quickly tend to their asymptotic values (see the column $D = 0$ in Table 1) that is true for neither quantum mechanics (the column $D = 1$), nor field theory (see, e.g. Refs. [7],[14] and $D = 3, 4$ in the Table 1);

b) if one takes into account 50 PT coefficients instead of one, the result is almost the same [9]: $\alpha = -0.235 \pm 0.025$, Therefore, the allowance made for a large number of coefficients - the only new information about the function - does not improve the accuracy of evaluation of the exponent α, which indicates weak convergence of the method proposed in [9];

c) the model (14) does not have the property that is characteristic of the field theory, namely, the dependence of the ratio $\bar{\beta}_n/\beta_n$ on the renormalization scheme (MOM or $\overline{MS}$, see Table 1), which also suggests that the calculated coefficients β_n are far from their asymptotics as $n \to \infty$.

Thus, the zero-dimensional model (14) is too simplified to make any definite conclusions about the number of coefficients needed to reproduce the GML function beyond the boundaries of PT.

5. It should be noted that the situation in the Yang-Mills theory is even more ambiguous than in the scalar field theory. Here there are 4 known coefficients of the GML function [18] which grow very fast in absolute value:

$$\beta_2 = -11, \ \beta_3 = -102, \ \beta_4 = -\frac{2857}{2} = 1428.5,$$
$$\beta_5 = -\left[\frac{149753}{6} + 3564\zeta(3)\right] \approx -29243$$

n	D=0	1	3	4(MOM)	4($\overline{MS}$)
2	1.0317	2.005	0.019	0.0978	0.0075
3	1.0210	1.897	0.085	0.659	0.0505
4	1.0157	1.718	0.166	1.072	0.097
5	1.0126	1.562	0.252	1.554	0.128
6	1.0104	1.443	0.322	–	0.139
7	1.0090	1.354	0.379	–	–
10	1.0063	1.203	–	–	–
20	1.0031	1.078			
30	1.0021	1.049			
50	1.0013	1.028			
75	1.0008	1.018			
a	2/3	3	0.1477	1	1
b	-1/2	0	4	4	4

Table 1: The ratios $\rho_n = \bar{\beta}_n/\beta_n$ for the model $\phi^4_{(D)}$. The case $D = 0$ corresponds to the integral (14), $D = 1$ - to the ground state energy of the anharmonic oscillator [26], $D = 3$ and 4 - to the GML function in a scalar field theory. In the last case, the values of ρ_n are given in two different renormalization schemes (MOM and $\overline{MS}$). The last two lines of the table contain the values of the parameters of the asymptotic formula (2).

and the asymptotics as $n \to \infty$ [19, 13] is[5]

$$\beta_n \sim c\,\Gamma(n + 35/2). \tag{15}$$

Since in this case the coefficient c is unknown, to illustrate the convergence we show in Table 2 the ratios $\sigma_n = \rho_{n+1}/\rho_n$, where $\rho_n = \bar{\beta}_n/\beta_n$, for the model (14), the anharmonic oscillator, $\phi^4_{(4)}$ theory and the Yang-Mills theory. As follows from the asymptotic expression

$$\beta_n = \bar{\beta}_n(1 + \frac{\bar{c}_1}{n} + \frac{\bar{c}_2}{n^2} + ...) \quad \text{as } n \to \infty, \tag{16}$$

the ratio σ_n (which does not depend on c) behaves as

$$\sigma_n = 1 + \frac{\bar{c}_1}{n^2} - \frac{\bar{c}_1^2 + \bar{c}_1 - 2\bar{c}_2}{n^3} + O(1/n^4). \tag{17}$$

[5]As in [12], we consider the case $N_c = 3$, $N_f = 0$, or pure gluodynamics without quarks.

n	D=0	$D=1$	$D=4$ MOM	$D=4$ ($\overline{MS}$)	YM ($\overline{MS}$)
2	0.9896	0.9459	6.738	6.733	177.089
3	0.9948	0.9063	1.627	1.921	24.935
4	0.9969	0.9090	1.450	1.320	7.810
5	0.9978	0.9235	–	1.086	–
10	0.99943	0.9769	–	–	–
50	0.99998	0.9994	–	–	–
75	0.99999	0.9998	–	–	–

Table 2: The ratios $\sigma_n = \rho_{n+1}/\rho_n$ for the model $\phi^4_{(D)}$ and for the Yang-Mills theory with $N_c = 3$ и $N_f = 0$.

This means that if the asymptotics of β_n is established, all σ_n as $n \to \infty$ have to approach unity and faster than $\rho_n = 1 - \bar{c}_1/n + \ldots$. Indeed, it takes place for the zero-dimensional model (14) for which

$$\begin{aligned} \rho_n &= \frac{\Gamma(n+1/2)\Gamma(n+1)}{\Gamma(n+1/4)\Gamma(n+3/4)\sqrt{n}} = 1 + \frac{1}{16n} + \frac{1}{512n^2} - \frac{5}{8192n^3} + \ldots, \\ \sigma_n &= \frac{(n+1/2)\sqrt{n(n+1)}}{(n+1/4)(n+3/4)} = 1 - \frac{1}{16n^2} + \frac{1}{16n^3} + \ldots, \end{aligned}$$

as well as for the anharmonic oscillator (see Table 2). However, in scalar field theory the approach of σ_n to unity just begins, and in the Yang-Mills theory the known values of σ_n are still far from unity.

In this case, the parameters of asymptotics (2) are $a = 1$, $b = 17$. For $n \ll b$ there is an essential dependence on the form of presentation of the asymptotic coefficients. For instance,

$$\delta_n = \frac{\Gamma(n+b+1/2)}{\Gamma(n+1/2)n^b} = 1 + \frac{b^2}{2n} + \frac{3b^4 - 4b^3 + b}{24n^2} + \ldots . \tag{18}$$

In particular, for $b = 17$ one gets $\delta_2 \sim 2 \cdot 10^{11}, \delta_3 \sim 10^9$ и $\delta_5 \sim 6 \cdot 10^6$. It is obvious that the asymptotics of β_n is not yet established and the $1/n$-corrections, which are the basis of the algorithm proposed in [9]-[12], strongly depend on the form of this asymptotics.

6. For comparison we show in Fig.1 the ratios $\rho_n = \tilde{\beta}_n/\beta_n$ and $\tilde{\tilde{\beta}}_n/\beta_n$, where

$$\tilde{\beta}_n = n!\, n^{7/2} c, \quad \tilde{\tilde{\beta}}_n = \Gamma(n+9/2)c, \tag{19}$$

in the case of the scalar $\phi^4_{(4)}$ theory in MOM scheme. The curves for $n > 5$ correspond to the first power correction in eq.(16) taken in the following form:

$$(a) \quad \rho_n = (1 + c_1/n)^{-1}, \quad (b) \quad \rho_n = 1 - c_1/n. \tag{20}$$

One can see that the asymptotic coefficients $\tilde{\beta}_n$ and $\tilde{\tilde{\beta}}_n$ for $n < 10$ are far from each other and do not approach the $1/n$-corrections, though the first parametrization $\tilde{\beta}_n$ looks more preferable than $\tilde{\tilde{\beta}}_n$. For parametrization

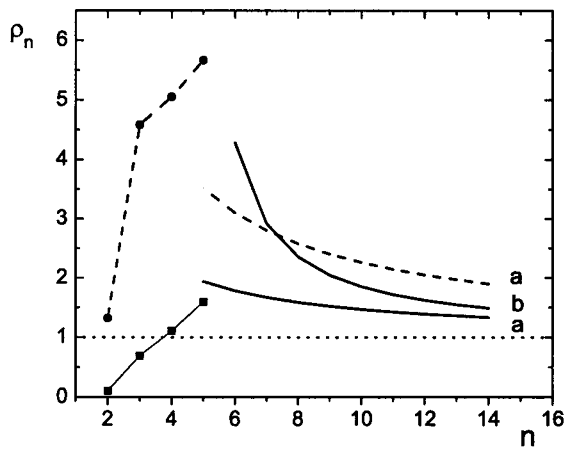

Figure 1: The ratios $\rho_n = \tilde{\beta}_n/\beta_n$ (solid lines) and $\tilde{\tilde{\beta}}_n/\beta_n$ (dashed lines) for the scalar $\phi^4_{(4)}$ theory. The curves a and b correspond to the first power corrections, see eq.(20).

of a general type

$$\beta_n \approx \Gamma(n + d + 1/2) n^{b-d} (1 + \frac{c_1}{n} + \frac{c_2}{n^2} + ...), \tag{21}$$

and the power corrections are related by

$$c_1(d) = c_1(0) - \frac{1}{2}d^2, \; c_2(d) = c_2(0) - \frac{1}{2}d^2 c_1(0) + \frac{1}{24}d(d+1)(3d^2+d-1), \; ... \tag{22}$$

Note that $c_1(d) \leq c_1(0)$ for any d. For $d = 0, 1/2$ and b, respectively, one gets the parametrizations $\bar{\beta}_n, \tilde{\beta}_n$ and $\tilde{\tilde{\beta}}_n$ widely used in a summation procedures. They are equivalent at $n \to \infty$, but may considerably differ for small n. For example, in the case of the $\phi^4_{(4)}$ theory, according to [8], $\tilde{c}_1 = -4.7$ that gives: $\bar{c}_1 = -4.6$ and $\tilde{\bar{c}}_1 = -12.6$.

It is instructive to compare the results of reconstruction of the GML function by different methods. In Refs.[4, 5], the so-called "improved PT" was used. In this case the reconstruction is made by the formula

$$\beta(g) = \tilde{\beta}(g) + \sum_{n=2}^{N} (\beta_n - \tilde{\beta}_n)(-g)^n, \tag{23}$$

where the sum $\tilde{\beta}(g) = \Sigma_{n=2}^{\infty} \tilde{\beta}_n(-g)^n$ with the asymptotic coefficients $\tilde{\beta}_n$ is calculated analytically. With allowance made for three coefficients β_2, β_3 and β_4 known at that time this permitted one to find the GML function $\beta(g)$ for $0 < g \precsim 1$.

In paper [6], the additional coefficient β_5 calculated in [20], was taken into account and summation of PT series was performed with the help of the modified Borel transformation

$$f(g) \;=\; \int_0^\infty \frac{dx}{g} e^{-x/g} (x\frac{d}{dx})^5 B(x), \tag{24}$$

$$B(x) \;\sim\; \sum_{k=2} \frac{\beta_k}{k!\; k^{7/2}} \,(-x)^k, \tag{25}$$

which corresponds to the parametrization $\tilde{\beta}_n$. To the convergent series (25) the conformal mapping was then applied which performed the analytical continuation beyond the radius of convergence. This allowed one to reconstruct the GML function with 10% accuracy for $g \precsim 40$ (see Fig.2).

At last, in [8] besides the four known coefficients the first $1/n$-correction was included and the Sommerfeld-Watson transformation was used. As a result, the GML function was reconstructed in extended region up to $g \approx 50$. What is essential, in the overlapping region all these methods give compatible results and being extrapolated lead to the asymptotic $\beta(g) \propto g^\alpha$ with the exponent $\alpha = 1.9 \pm 0.1$ which is close to the one-loop behaviour (in contradiction with [9]).

7. That is why we keep our opinion [14] as to unreliability of the statements about the asymptotics of the GML function as $g \to \infty$ in the field theory made in [9]-[12]. As it was shown in [14], to get reliable reconstruction of the GML function in the strong coupling regime ($g \gg 1$, but not as $g \to \infty$), it is necessary to get a large number of PT terms

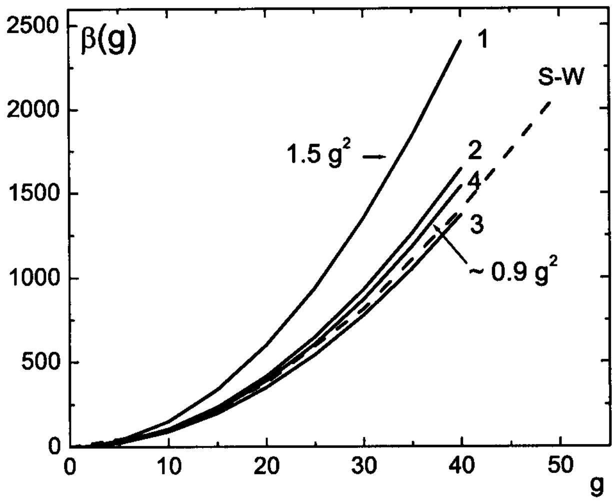

Figure 2: The β function for the scalar field theory with $L_{int} = -16\pi^2/4!\ g\phi^4$ reconstructed with the help of the Borel transformation with conformal mapping. The solid curves correspond to 1,2,3 and 4 loop contributions, respectively [6]. The dashed line is the result of the Sommerfeld-Watson summation procedure in 4 loops [8].

which have already approached their asymptotic values. The situation is more complicated if the function has intermediate asymptotics [14] or if the asymptotics contains log g. In these cases, the asymptotic regime may be essentially delayed. A good example is given by the problem of a hydrogen atom in a strong electric [14, 21] or magnetic field [22].

In conclusion it should be noted, that while for the sign alternating series the Borel summation and similar methods give quite reliable results (see, e.g. the calculation of the critical exponents of the second order phase transitions [23, 24, 8]), for the sign non-alternating series such a method is absent at the moment. The reason is that the sign non-alternating series indicates the degeneracy of the ground state and the presence of the contributions not reproducible in perturbation theory. To reconstruct the function, here one needs additional information which is absent in perturbative field theory. An instructive example is the degenerate anharmonic oscillator considered in [25].

We would not like that the readers of the papers [9]-[12] got an impression that the problem of asymptotics of the GML function in quantum field theory as $g \to \infty$ can find its solution with the help of any algorithm based on the first few terms of PT without additional information beyond PT which is absent at the present time.

Appendix A

We give here the derivation of eqs.(9) and (13). Assume that the function $f(g)$ defined by divergent PT series decreases faster than $1/g^N$ at infinity. Substituting $\beta = 1$ in (12) and expanding the exponent $\exp(-x/g)$ up to $g^{-(N+1)}$ one gets:

$$\int_0^\infty dx \{\frac{1}{g} - \frac{x}{g^2} + \frac{x^2}{g^3} - ... + \frac{(-1)^{N-1}}{(N-1)!}\frac{x^{N-1}}{g^N} + ...\} B(x) = o(g^{-N}). \quad (A.1)$$

This equality can be valid as $g \to \infty$ only if the first N moments of the Borel transform $B(x)$ vanish

$$\int_0^\infty dx B(x) = \int_0^\infty dx\ x B(x) = ... = \int_0^\infty dx\ x^{N-1} B(x) = 0, \qquad (A.2)$$

which coincides with eq.(9). This assumes that $B(x)$ decreases at infinity faster than x^{-N} and oscillates in the interval $0 < x < \infty$.

For the modified Borel transform (11) proceeding in the same way for $\beta = 2, 3, ..., N$ we get for any $\beta = k \leq N$

$$\int_0^\infty dx \{\frac{1}{g^k} - \frac{x}{g^{k+1}} + ... + \frac{(-1)^{N-k}}{(N-k)!}\frac{x^{N-k}}{g^N} + ...\} x^{k-1} B_k(x) = o(g^{-N}). \quad (A.3)$$

This means that

$$\int_0^\infty dx\ x^{k-1} B_k(x) = \int_0^\infty dx\ x^k B_k(x) = ... = \int_0^\infty dx\ x^{N-1} B_k(x) = 0, \quad (A.4)$$

and the last equation takes the form

$$\int_0^\infty dx\ x^{N-1} B_N(x) = 0. \qquad (A.5)$$

Altogether this gives $N(N+1)/2$ conditions (13) needed to get the behaviour of $f(g) = o(g^{-N})$ as $g \to \infty$. This conditions are imposed at N functions $B_k(x)$, $1 \leq k \leq N$. Notice that the higher moments may not exist since the integral $\int_0^\infty dx\ x^{\beta-1} B_\beta(x)$ for $\beta > N$ is usually divergent.

Let us illustrate the above discussion by a concrete example. Consider the function

$$f_N(g) = \frac{d^N}{dg^N}[g^N f(g,\nu)], \quad N < \nu < N+1, \tag{A.6}$$

where

$$f(g,\nu) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\nu)}{\Gamma(\nu)}(-g)^n = g^{-\nu} e^{1/g}\, \Gamma(1-\nu, 1/g), \tag{A.7}$$

and $\Gamma(\alpha, x)$ is the incomplete Γ-function. For $g \to \infty$ it has an expansion

$$g^{-\nu}\Gamma(1-\nu,1/g) = \Gamma(1-\nu)g^{-\nu} + \sum_{k=1}^{\infty} \frac{(-1)^k}{(k-1)!(k-\nu)} g^{-k}. \tag{A.8}$$

Hence, in eq.(A.6) one gets for non-integer ν

$$\frac{d^N}{dg^N}(g^{N-\nu}) = \frac{\Gamma(N+1-\nu)}{\Gamma(1-\nu)}\ g^{-\nu}, \quad N = 1,2,3,...,$$

while for integer ν

$$\frac{d^N}{dg^N}(g^{N-\nu}) = \begin{cases} 0, & 1 \le \nu \le N, \\ (-1)^N \dfrac{(\nu-1)!}{(\nu-N)!}\ g^{-\nu}, & \nu > N. \end{cases}$$

This means that $f_N(g) \sim g^{-\nu}$ as $g \to \infty$, i.e. decreases faster than g^{-N}. Thus, this simple trick allows one to construct a function that is described by an asymptotic series and has arbitrary power low decrease at infinity.

In conclusion we check that the conditions (9) are really satisfied in this case. The Borel transform corresponding to (A.6) is

$$B(x) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\nu)(n+N)!}{\Gamma(\nu)(n!)^2}\ (-x)^n = N!\ {}_2F_1(\nu, N+1; 1; -x). \tag{A.9}$$

It can be written also as

$$B(x) = N!\ (1+x)^{-\nu} {}_2F_1(\nu, -N; 1; \frac{x}{1+x}) \ \sim\ x^{-\nu}, \quad x \to \infty. \tag{A.10}$$

Hence the integral (9) is convergent if $i \le N$ and divergent otherwise. The conditions (9) mean that

$$\begin{aligned} J_{iN} &= \int_0^{\infty} dx\ x^{i-1}\ {}_2F_1(\nu, N+1; 1; -x) \\ &= \int_0^1 dt\ t^{i-1}(1-t)^{\nu-i-1}\ {}_2F_1(\nu, -N; 1; t) = 0, \quad t = x/(1+x) \end{aligned} \tag{A.11}$$

for $i = 1, 2, ..., N$ and $\nu > N$. One can check the validity of (A.11) by explicit substitution for small N. Thus, for $N = 1$ one has: $i = 1$, ${}_2F_1(\nu, -1; 1; t) = 1 - \nu t$,

$$J_{11} = \int_0^1 dt\ (1-t)^{\nu-2}(1-\nu t) = 0, \quad \text{for } \nu > 1,$$

while

$$J_{21} = -\frac{1}{(\nu-1)(\nu-2)} \neq 0.$$

Here $i = 2 > N = 1$. The same is true for the higher values of N.

One can also make sure that the Borel transform (A.10) is an oscillating function having N zeros on the positive real axis. For illustration in Fig.3 we show its behaviour in the cases $N = 4, \nu = 5$ and $N = 6, \nu = 7$.

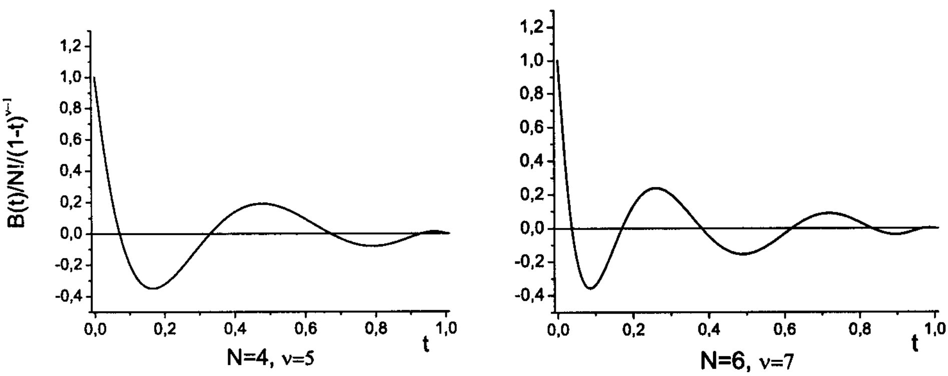

Figure 3: The normalized Borel transform $B(t)/N!/(1-t)^{\nu-1}$, $t = x/(1+x)$ defined by eq.(A.10) for the cases $N = 4, \nu = 5$ and $N = 6, \nu = 7$.

Acknowledgments

The authors are grateful to A.V.Bakulev, S.V.Mikhailov, V.D.Mur, V.A.Novikov and D.V.Shirkov for useful discussions. The paper has been done with the partial support from the RFBR (grants # 01-02-16850, 02-02-16889 и 00-15-96691).

References

[1] M.Gell-Mann and F.E.Low, Phys.Rev. **95**, 1300 (1954).

[2] N.N.Bogoliubov and D.V.Shirkov, *Introdution to the theory of the quantized fileds*, Ch. IX, Moscow, Nauka, 1984.

[3] L.N.Lipatov, Pis'ma v ZhETF **25**, 116 (1977); ZhETF **72**, 411 (1977).

[4] V.S.Popov, V.L.Eletsky, and A.V.Turbiner, Phys. Lett. **72B**, 99 (1977); ZhETF **74**, 445 (1978).

[5] V.L.Eletsky and V.S.Popov, Phys.Lett. **77B**, 411 (1978).

[6] D.I.Kazakov, O.V.Tarasov, and D.V.Shirkov, Theor.Math.Phys. **38**, 15 (1979).

[7] D.I.Kazakov and D.V.Shirkov, Fortsch. der Phys. **28**, 465 (1980).

[8] Yu.A.Kubyshin, Theor.Math.Phys. **58**, 137 (1984).

[9] I.M.Suslov, Pis'ma v ZhETF **71**, 315 (2000); ZhETF **120**, 5 (2001).

[10] I.M.Suslov, Pis'ma v ZhETF **74**, 211 (2001).

[11] I.M.Suslov, ZhETF **117**, 659 (2000).

[12] I.M.Suslov, Pis'ma v ZhETF **76**, 387 (2002).

[13] I.M.Suslov, ZhETF **122**, 696 (2002).

[14] D.I.Kazakov and V.S.Popov, ZhETF **122**, 675 (2002).

[15] D.I.Kazakov and V.S.Popov, Pis'ma v ZhETF **77**, 347 (2003).

[16] F.J.Dyson, Phys.Rev. **85**, 631 (1952).

[17] J.Heading, *An Introduction to Phase Integral Methods*, Methuen, London, 1962;

[18] T. van Ritbergen, J.A.M. Vermaseren, and S.A.Larin, Phys.Lett. **B400**, 379 (1997).

[19] E.B.Bogomolny and V.A.Fateev, Phys.Lett. **B71**, 93 (1977).

[20] F.M.Dittes, Yu.A.Kubyshin and O.V.Tarasov, Theor.Math.Phys. **37**, 66 (1978).

[21] L.Benassi, V.Grecchi, E.Harrell, and B.Simon, Phys.Rev.Lett. **42**, 704, 1430 (1979).

[22] H. Hasegawa and R.E.Howard, J.Phys.Chem.Solids **21**, 179 (1961).

[23] J.C..Le Guillou and J.Zinn-Justin, Phys.Rev.Lett. **39**, 95 (1977); Phys.Rev. **B21**, 3976 (1980); Ref.[25], Ch.25.

[24] A.A.Vladimirov, D.I.Kazakov and O.V.Tarasov, ZhETF **77**, 1035 (1979).

[25] J.Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, Ch.40, Clarendon Press, Oxford 1989, 1993.

[26] C.M.Bender and T.T.Wu, Phys.Rev. **184**, 1231 (1969).

Received on May 21, 2003.

Казаков Д. И., Попов В. С. E2-2003-95
Об асимптотике функции Гелл-Манна–Лоу
в квантовой теории поля

Обсуждается задача о восстановлении функции Гелл-Манна–Лоу в квантовой теории поля по ее асимптотическому ряду, первые члены которого вычислены по теории возмущений. И хотя математически однозначно это не осуществимо, при разумных предположениях об искомой функции оказывается возможным восстановить ее в некотором конечном интервале значений константы связи g, однако попытки определить поведение функции при $g \to \infty$ являются, на наш взгляд, необоснованными. Получены условия, при выполнении которых сумма расходящегося ряда ТВ может быстро убывать на бесконечности.

Работа выполнена в Лаборатории теоретической физики им. Н. Н. Боголюбова ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна, 2003

Kazakov D. I., Popov V. S. E2-2003-95
On the Asymptotics of the Gell-Mann–Low Function
in Quantum Field Theory

The problem of reconstructing the Gell-Mann–Low function in quantum field theory starting with its asymptotic series with the first terms calculated by perturbation theory is discussed. And though in a strict mathematical sense this is not unambiguously realizable, under reasonable assumptions about the function it appears to be possible to reconstruct it in some finite interval of g. However, any attempts to find its asymptotics as $g \to \infty$ from our point of view are not justified. We also present the conditions under which the sum of the asymptotic series may decrease at infinity.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna, 2003

Макет *Т. Е. Попеко*

Подписано в печать 02.06.2003.
Формат 60 × 90/16. Бумага офсетная. Печать офсетная.
Усл. печ. л. 1,18. Уч.-изд. л. 1,28. Тираж 415 экз. Заказ № 53930.

Издательский отдел Объединенного института ядерных исследований
141980, г. Дубна, Московская обл., ул. Жолио-Кюри, 6.
E-mail: publish@pds.jinr.ru
www.jinr.ru/publish/