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GENERALIZATION OF **BLM** PROCEDURE AND ITS SCALES IN ANY ORDER OF **pQCD**: A PRACTICAL APPROACH

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Обобщение процедуры БЛМ для любого порядка теории возмущений КХД и ее масштабы: практический подход

Известная процедура Бродского–Лепажа–Макензи (БЛМ) последовательно обобщена для любого фиксированного порядка теории возмущений (ТВ) КХД. Преобразованный ряд ТВ может быть представлен в форме цепной дроби. Обобщенная процедура интерпретируется с помощью интегрального представления для ТВ-рядов и скелетными диаграммами. Эта процедура обобщается и далее включением механизма «наибыстрейшей подходящей сходимости» (G. Grunberg). Последняя обобщенная процедура БЛМ применяется к разложению функции Адлера D в приближении БЛМ для функции D и наблюдаемой $R_{e^+e^-}$.

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Mikhailov S. V. E2-2005-25
Generalization of BLM Procedure and Its Scales in Any Order of pQCD: a Practical Approach

Brodsky–Lepage–Mackenzie procedure is sequentially extended for any fixed order of the perturbation QCD. The reformed perturbation series looks like a continued fraction. A generalization of this procedure which provides one with a certain mechanism of the Fastest Apparent Convergence (FAC) prescription is developed. This generalized BLM procedure is applied to Adler function D in N^{3}LO and N^{4}LO. The final effect of this generalized BLM improvement for D and $R_{e^+e^-}$ functions is discussed.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

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INTRODUCTION

The first goal of this work is to extend the well-known Brodsky, Lepage and Mackenzie (BLM) procedure [1] of scale setting for any order of pQCD. Let me start with an appropriate citation: «...One, therefore, has to address the question of what is the «best» choice for μ^2 within a given scheme, usually $\overline{\mathrm{MS}}$. There is no definite answer to this question – higher-order corrections do not «fix» the scale, rather they render the theoretical predictions less sensitive to its variation» (I. Hinchliffe, Particle Data Group booklet [2]). It will show that higher orders of pQCD in the $\overline{\mathrm{MS}}$ scheme unambiguously determine the new scales in the BLM prescription sense in contrast to the premonitory citation. Namely, the effects of the coupling renormalization encoded in β-function coefficients are absorbed into a set of proper scales μ_i^2 of the couplings $a_i = a_s(\mu_i^2)$ at any fixed order of pQCD. To simplify the analysis of the structure of radiation corrections, the renormalization-group invariant quantities like Adler function (D) are considered below. The procedure of this absorbtion is constructed in Secs. 3, 4 and includes as a partial case the «bubble approximation» elaborated in [3, 4, 5]. The corresponding new perturbation series is obtained in Sec. 4, so that the initial BLM suggestion [1] appears to be completed by the sequential BLM (sBLM) procedure. Quantities like heavy quark potential V_Q [6]; Bjorken sum rules, Gross-Levellin Smith sum rule and so on (see [8, 7] for review), can be considered in the same manner. The name «practical» means that the *empirical relation* between the QCD β–function coefficients b_i, $b_i \sim b_0^{i+1}$ has been used. The hierarchy of the contributions of coupling renormalization to the perturbation coefficients at every order of a_s is based on this power law for b_i. This detailed hierarchy requires a matrix representation for the perturbation expansion (PE) rather than the standard series. The mentioned power relation works at least up to the last known coefficient b_3 (at $N_f = 0 \div 5$), instead of the usually discussed proposition of the so-called «large–b_0» limit (at $-N_f \gg 1$, e. g., $b_0 \sim b_1$) (see for review [9]). This power law, $b_i \sim b_0^{i+1}$, should of course fail somewhere in the higher orders of the PE when its expected factorial explosion starts.

Different kinds of extensions of BLM approach has been discussed in a number of interesting articles [3–5, 7, 10–13] appeared in last decade. Our approach differs from them mainly in two items: i) the generalized scheme is formulated explicitly for any fixed order of PE; ii) all the sources of the coupling renormalization are taken into account and absorbed into the coupling scales. The important issues about the scheme ambiguity of BLM procedure [14] as well as the role of anomalous dimensions (for the corresponding quantities) in the optimization procedures [15] are not discussed here, all the calculations are performed at the fixed

$\overline{\rm MS}$ scheme for the massless QCD. Our procedure can be formulated in terms of the dynamic characteristics only, the β-function coefficients, rather than in terms of certain $SU(3)_c$ Casimir operators that may appear at an intermediate stage. This β-function expansion is performed for the 4-loop D function (Appendix C), then the sBLM procedure is applied to this D function in Sec. 5 to highlight the advantages and disadvantages of the procedure in the case of this physical quantity.

The sBLM procedure cannot be related to the improvement of the perturbation series. The next goal here is to supply this procedure with the mechanism *a'la* Fast Apparent Convergence (FAC) [16] to improve the convergence of the series. This machinery, using the proper scales of the sBLM, is considered in Secs. 5 and 6. This generalization of the sBLM procedure that has been named xBLM is formulated in Sec. 6. In Sec. 1, a convolution representation is proposed which can be useful for an interpretation of the BLM procedure.

1. CONVOLUTION REPRESENTATION FOR THE AMPLITUDES

Here we rewrite the standard perturbation power expansion for an amplitude in the form of a formal integral representation. This representation in contrast to the similar ones in [3,4] and in [13] does not involve integration over an intrinsic momentum k. It is not related to Feynman integral in momenta. The properties of this representation are discussed. We shall use this form in the next section as a convenient «perturbative tool» to interpret the BLM task from the point of view of average virtuality flows.

Let us consider the formal perturbation series $s(a)$ for the two-point amplitudes at the external momentum Q^2. The coupling $a \equiv a_s(\mu^2) = \alpha_s(\mu^2)/(4\pi)$ is normalized at the same external scale $\mu^2 = Q^2$. In this case the coefficients of the expansion d_n are numbers in the MS-like schemes, due to cancellation of the logarithms $\ln(Q^2/\mu^2)$ there. However, the constant parts ($\ln(C)$) of these logarithms accompanied by the β-function coefficients are left in d_n and we shall be interested in these traces. For further convenience we introduce a new scaled expansion parameter $A = |b_0|a$,

$$s(a) = d_0 + \sum_{n=1} a^n d_n \equiv S(A) = d_0 + \frac{d_1}{b_0} \cdot \sum_{n=1} A^n D_n \;\; \text{with} \quad D_1 = 1, \qquad (1)$$

and new coefficients $D_i = \dfrac{d_i}{d_1 b_0^{i-1}}$ that simplify intermediate calculations and will help us to maintain contact with the «large b_0» limit, $b_0 \gg 1, \quad A \lesssim 1$. Note that in the real world, below the c-quark threshold (at $N_f = 3$), we have $b_0 = 9 \gg 1$ and $A(\mu^2) \equiv \alpha(\mu^2)\frac{b_0}{4\pi} \approx 0.32 < 1$ at the NLO level at μ^2 = 1 GeV^2.

The generation function $\mathcal{P}(\alpha)$ can be introduced for these expansion coefficients,

$$D_n = \int_0^\infty \mathcal{P}(\alpha)\alpha^{n-1}d\alpha \equiv \langle\alpha^{n-1}\rangle, \quad (2)$$

with the normalization condition

$$\int_0^\infty \mathcal{P}(\alpha)d\alpha = 1.$$

At large α the behaviour $\mathcal{P}(\alpha) \sim \alpha^{\gamma+1}\mathrm{e}^{-\alpha/c}$ corresponds to the expected asymptotic behaviour for the expansion coefficients D_n [17],

$$\mathcal{P}(\alpha) \sim \alpha^{\gamma+1}\mathrm{e}^{-\alpha/c} \to D_n \sim c^n\Gamma(n+1+\gamma), \quad (3)$$

the familiar behaviour exhibits purely renormalon divergence that is proportional to $n!$.

The running of the coupling $A \to \bar{A}(t)$ (or $a \to \bar{a}(t)$) follows the renormalization group (RG) equation

$$\frac{d}{dt}\bar{A} \equiv B(\bar{A}) = -\left(\bar{A}^2 + c_1\bar{A}^3 + c_2\bar{A}^4 + \ldots\right), \quad (4)$$

where

$$c_i = \frac{b_i}{b_0^{i+1}}, \quad (5)$$

$B(A)$ is the modified β function and $t = \ln\left(\frac{Q^2}{\Lambda^2}\right)$ is a natural variable for MS-like schemes. The $\bar{A}(t)$ has the following behaviour at large t, $\bar{A}(t) \simeq \frac{1}{t}$. Below we shall construct the representation for (1) based on Eq. (4).

1.1. 1-Loop Integral Representation. For the 1-loop case the evolution (4) leads to

$$\frac{1}{n!}\left(-\frac{d}{dt}\right)^n \bar{A}_{(1)} = \left(\bar{A}_{(1)}\right)^{n+1}. \quad (6)$$

Substituting (2) and (6) into the definition (1) and *changing* the order of the sum and the integration one obtains the formal integral representation

$$S(\bar{A}_1) = d_0 + \frac{d_1}{b_0}\cdot\int_0^\infty \mathcal{P}(\alpha)\left[\exp\left(-\alpha\frac{d}{dt}\right)\bar{A}_{(1)}(t)\right]d\alpha = \\ = d_0 + \frac{d_1}{b_0}\cdot\int_0^\infty \mathcal{P}(\alpha)\bar{A}_{(1)}(t-\alpha)d\alpha \quad (7)$$

that is *linearized* in the coupling $\bar{A}_1$ as a convolution

$$\int_0^\infty \mathcal{P}(\alpha)\bar{A}_{(1)}(t-\alpha)d\alpha \equiv \langle\bar{A}_{(1)}(t-\alpha)\rangle. \quad (8)$$

This representation seems to be close to that invented by Neubert [3]. V. Braun, M. Beneke, and collaborators [4, 5] also widely used a similar kind of the convolution basing on the Borel representation. An essential difference between these representations and the one like Eq. (7) will be demonstrated in the next subsection. The representation in Eq. (7) is «formal» because the change of the order in the derivation has not been proved. Nevertheless, the integration of the Taylor expansion of factor $\bar{A}_{(1)}(t-\alpha)$ in the integrand in Eq. (7) leads to the initial series, Eq. (1), up to any finite order of the expansion. We shall *call* just this Eq. (7) as the «sum» of the perturbative series (1) in the one-loop running approximation. The integrand in (7) has pole-singularity at $t=\alpha$ which reflects the fact that the representation is ill-defined. One can discuss how the contour should be deformed to give a sense to this integral [18, 19]. The residue of the pole can be taken as a measure of the uncertainty of the asymptotic series and it reflects the factorial grows of the perturbation coefficients D_n (see [3, 9, 18]).

We would not discuss here the non-perturbative interpretations of Eq. (7). Let me only mention that quantum field models with the integrable running coupling $\bar{A}_{(1)}(t)$ lead to a finite expression for convolution in Eq. (8). Therefore, the finiteness of the coupling of the analytic perturbation theory (APT) [20, 21], $A^{\rm APT}_{(1)}(t) \leqslant 1$ provides an example of the «convergent» perturbation series in the sense of the convolution representation (in weak sense). The brief discussion of this observation is presented in Appendix A.

1.2. *N*-Loop Generalization of the Integral Representation. Let $\bar{A}_{(N)}$ be the solution of the RG equation at the N-loop approximation in the r.h.s. of Eq. (4). Then the initial series, Eq. (4), with the coupling $\bar{A}_{(N)}$ can be represented as

$$S(\bar{A}_{(N)}) = d_0 + \frac{d_1}{b_0}\cdot\int_0^\infty \mathcal{P}(\alpha)\,\bar{A}_{(N)}\left(t+\alpha\cdot\frac{\bar{A}^2_{(N)}}{B(\bar{A}_{(N)})}\right)d\alpha \equiv d_0 + \frac{d_1}{b_0}\cdot\mathcal{P}\ast\bar{A}_{(N)}. \tag{9}$$

This representation can be proved in the same way as Eq. (7); this is outlined in Appendix A. Consider the factor $\bar{A}_{(N)}\,(t+\ldots)$ in the integrand of Eq. (9). Its Taylor expansion in the second term of the argument generates PE in Eq. (1) for the case $A=\bar{A}_{(N)}$. Further we shall consider $S(\bar{A}_{(N)})$ in Eq. (9) as a «sum» of the perturbative series in the N-loop running approximation.

2. AN ILLUSTRATION: 1-LOOP BLM PROCEDURE

Following the Neubert's proposal [3] it is convenient to consider the integration in Eqs. (7) and (9) as an average of the corresponding coupling A over $\mathcal{P}$ and then use its linear property,

$$S(\bar{A}_1) = d_0 + \frac{d_1}{b_0}\cdot\langle\bar{A}_{(1)}(t-\alpha)\rangle. \tag{10}$$

Let (t_0, A_0) belong to the RG trajectory. Based on the RG law $\bar{A}(t) = A(t - t_0, A_0)$ that leads to $\bar{A}_{(1)}(t) = A_0/\left(1 + A_0(t-t_0)\right)$ in the one-loop case, one can expand the average $\langle \bar{A}_{(1)}(t-\alpha)\rangle$

$$\langle \frac{A_0}{1 + A_0(t - t_0 - \alpha)} \rangle = A_0 - A_0^2(t - t_0 - \langle\alpha\rangle) + \\ + A_0^3 \left((t-t_0)^2 - 2(t-t_0)\langle\alpha\rangle + \langle\alpha^2\rangle\right) + \dots \quad (11)$$

Recall that $\langle\alpha^n\rangle \equiv D_{n+1}$ in accordance with the definition (2); D_2 is proportional to the coefficient d_2 of the initial series (1) which contains the term proportional to b_0 appearing due to the coupling renormalization. Introducing the representation

$$d_2 = d_1 \cdot (b_0 d_2[1] + d_2[0]) \quad (12)$$

we get

$$\langle\alpha\rangle \equiv D_2 = \frac{d_2}{d_1 b_0} = d_2[1] + \frac{d_2[0]}{b_0}. \quad (13)$$

Suppose the NLO order coefficient (at A_0^2) in the expansion (11) satisfies the conditions:

• The coefficient is nullified in Eq. (11), $t_0 = t - D_2 \Rightarrow \mu^2 = Q^2 \exp(-D_2)$. This term is accumulated as a whole by the new expansion parameter A_0 – this is the Fastest Apparent Convergence (FAC) procedure [16],

$$\langle \bar{A}_{(1)}(t-\alpha)\rangle = A_0 + A_0^2 \cdot 0 + O(A_0^3). \quad (14)$$

Here the structure of the $O(A_0^3)$ tail looks like $A_0^3\left(D_3 - D_2^2\right) + O(A_0^4)$ [3] and the distribution $\mathcal{P}$ can be written as $\mathcal{P}(\alpha) \to P_1^{\text{FAC}} = \delta(\alpha - D_2)$ in this approximation.

• The contribution $d_2[1]$ to the coefficient is nullified, so that $t_0 = t - d_2[1] \Rightarrow \mu^2 = Q^2 \exp(-d_2[1])$. Only the «$a$-renormalization» responsible term, $d_2[1]$, is accumulated by the new expansion parameter A_0 – this is the BLM procedure [1, 7],

$$\langle \bar{A}_{(1)}(t-\alpha)\rangle = A_0 + A_0^2 \cdot \frac{d_2[0]}{b_0} + O(A_0^3). \quad (15)$$

The remainder term of this procedure at A_0^2 is suppressed by the inverse power of the «large b_0». The N^{2}LO term in Eq. (15) looks like $A_0^3\left(D_3 - 2D_2 d_2[1] + (d_2[1])^2\right)$.

Following the BLM procedure one deals with those parts of the coefficients D_i (d_i) that are responsible for the «a-renormalization». Therefore, it is convenient to introduce the partial distribution P_1 just for these parts, $\langle\alpha^{n-1}\rangle_1$,

$$\langle\alpha^{n-1}\rangle_1 \equiv \int_0^\infty P_1(\alpha)\alpha^{n-1} d\alpha \text{ and } \langle\alpha\rangle_1 = d_2[1]. \quad (16)$$

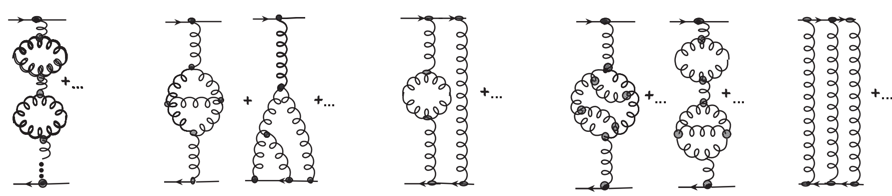
At the considered LO approximation for the BLM scale fixing one can put $P_1(\alpha) = \delta(\alpha - d_2[1])$ that determines an intrinsic «scale», $d_2[1]$, and shifts the normalization scale from t to t_0. Let us compare the distributions $\mathcal{P}$ and P_1 from the standard PE point of view. In this case $\mathcal{P}$ can be represented as a formal series

$$\mathcal{P}(\alpha) = \sum_{n=0} \frac{(-1)^n}{n!} \delta^{\{n\}}(\alpha) \cdot D_{n+1}. \tag{17}$$

Suppose $\mathcal{P}$ is a smooth function, then the series, Eq. (17), provides an insufficient approximation to $\mathcal{P}$ except the case when $\mathcal{P}$ is strongly concentrated near the origin $\alpha = 0$. Indeed, for any other cases one should take into account a lot more terms of the expansion in Eq. (17) or even its infinite subseries to approximate the real behaviour anywhere not close to the origin, say, near the next extremum in α. The distribution $\delta(\alpha - d_2[1])$ looks preferable from this point of view, the approximation takes into account at once (admittedly in a rather crude manner) the behaviour of $\mathcal{P}(\alpha)$. What parts of PE in next orders should one involve in the procedure to improve P_1? Which are diagram classes that generate these parts? To clarify this (and to define next approximations for $\mathcal{P}$) one should analyze the structure of d_n coefficients at N^{n}LO of the PE.

3. BLM TASK, THE FIRST STAGE OF GENERALIZATION

3.1. The β-Function Structure of the Perturbation Coefficients. In N^{2}LO the a-renormalization coming from one-gluon line and vertices generates contributions proportional to $a^3 b_0^2,\ a^3 b_1$, see Fig. 1, $a), b)$.



a *b* *c* *d* *e*

Fig. 1. The diagram insertions for α_s-radiative corrections to, e. g., photon polarization operator Π. *a)* The chain of gluon bubbles, two-bubble chain contributes to the b_0^2-term in d_3. *b)* The diagrams with the renormalization of gluon fields and vertices contribute to the b_1-term in d_3. *c)* The diagrams that contribute to the b_0-term in d_3. *d)* The diagrams generating by b_2- and $b_0 b_1$-terms in d_4. *e)* The diagrams without intrinsic renormalization contribute to the «genuine» term, $d_3[0]$

A contribution like $a^3 b_0$ is generated by two-gluon exchange with the renormalization of one of these gluon lines/vertices, see Fig. 1, c. The final representation for d_3 seems similar to that in Eq. (12) and looks like an expansion in power series in $b_0, b_1, \ldots$

$$d_3 = d_1 \cdot \left(b_0^2 d_3[2,0] + b_1 d_3[0,1] + b_0 d_3[1,0] + d_3[0]\right), \tag{18}$$

where the first argument n_0 of the expansion coefficients $d_n[n_0, n_1, \ldots]$ corresponds to the power of b_0; and the second one n_1 — to the power of b_1, etc. The coefficient $d_n[0]$ corresponds to the so-called [14] «genuine» corrections with $n_i = 0$ for all possible b_i powers. One of the diagram sources of these contributions is presented in Fig. 1, e. If all the arguments of the coefficient $d_n[\ldots, m, 0, \ldots, 0]$ to the right of m are equal to zero, then we shall omit these arguments for simplicity and write $d_n[\ldots, m]$ hereinafter. In N^{3}LO the a-renormalization generates contributions proportional to $a^4 b_0^3,\ a^4 b_0 b_1,\ a^4 b_2$ originating from one-gluon lines/vertices; contributions proportional to $a^4 b_0^2,\ a^4 b_1,\ a^4 b_0$ originate from the mixing of the a-renormalization from different gluon lines/vertices; and contribution like a^4 appears from the «genuine» corrections. The d_4 coefficient looks in this notation like

$$\begin{aligned} d_4 = d_1 \cdot (b_0^3 d_4[3] + b_1 b_0 d_4[1,1] + b_2 d_4[0,0,1] + \\ + b_0^2 d_4[2] + b_1 d_4[0,1] + b_0 d_4[1] + d_4[0]). \end{aligned} \tag{19}$$

The same ordering of the β-function elements holds for all the next d_n. It is convenient for our purposes to present this «β-structure» for the «normalized» variables $\bar{A}$ and D_i. The D_i coefficients have an evident form

$$\begin{aligned}
&\bar{A}^1(t) \quad D_1 = \mathbf{1}, \\
&\bar{A}^2(t) \quad D_2 = \boldsymbol{d_2}[\mathbf{1}] + \tfrac{1}{b_0} \cdot d_2[0], \\
&\bar{A}^3(t) \quad D_3 = \boldsymbol{d_3}[\mathbf{2}] + \boldsymbol{c_1 d_3}[\mathbf{0},\mathbf{1}] + \frac{1}{b_0} \cdot \left(d_3[1] + \frac{1}{b_0} d_3[0]\right), \\
&\bar{A}^4(t) \quad D_4 = \boldsymbol{d_4}[\mathbf{3}] + \boldsymbol{c_1 d_4}[\mathbf{1},\mathbf{1}] + \boldsymbol{c_2 d_4}[\mathbf{0},\mathbf{0},\mathbf{1}] + \\
&\qquad \frac{1}{b_0} \cdot \left(d_4[2] + c_1 d_4[0,1] + \frac{1}{b_0} \cdot \left(d_4[1] + \frac{1}{b_0} d_4[0]\right)\right), \\
&\bar{A}^5(t) \quad D_5 = \boldsymbol{d_5}[4] + \boldsymbol{c_1 d_5}[\mathbf{2},\mathbf{1}] + \boldsymbol{c_1^2 d_5}[\mathbf{0},\mathbf{2}] + \boldsymbol{c_2 d_5}[\mathbf{1},\mathbf{0},\mathbf{1}] + \\
&\qquad \boldsymbol{c_3 d_5}[\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{1}] + \frac{1}{b_0} \cdot \Bigg(d_5[3] + c_1 d_5[1,1] + c_2 d_5[0,0,1] + \\
&\qquad \frac{1}{b_0} \cdot \left(d_5[2] + c_1 d_5[0,1] + \frac{1}{b_0} \cdot \left(d_5[1] + \frac{1}{b_0} d_5[0]\right)\right)\Bigg), \\
&\ldots \ \ldots
\end{aligned} \tag{20}$$

where c_i are defined in Eq. (5). Here we do not discuss how to derive this representation for the known multi-loop results. We suppose that the elements of the structure in Eq. (20) have already been obtained. The first column of the coefficients $\boldsymbol{d_n}[\boldsymbol{n-1}]$ in Eq. (20) corresponds to the «bubble approximation» that includes the contributions from the diagrams with the maximum numbers of the bubbles, see Fig. 1, a. These «bubble» contributions involved into the extended BLM procedure were considered in [3, 4, 5, 7]. However, there are other unsuppressed contributions emphasized in the bold type in the «table» of Eq. (20). Really, the known c_i are of the order of 1 for the QCD $\overline{\text{MS}}$ scheme (below their certain values are obtained at $N_f = 3$),

$$c_1 \approx 0.79; \quad c_2 \approx 0.88; \quad c_3 \approx 1.9; \quad c_4 = ? \tag{21}$$

Therefore, one has no reasons to neglect the other terms emphasized in Eq. (20) in the bold type. These terms, see the c_1-term in D_3 in Eq. (20), originate in part from the diagrams in Fig. 1, b; the c_2- and c_1-terms in D_4 there – from Fig. 1, d, and so on. To extend our final results far more broadly, we suggest the same estimates, $c_i = O(1)$, for the unknown required coefficients as well (see the discussion in Introduction).

We face two different expansion parameters in the representation (20), coupling $\bar{A}$ for the lines and b_0^{-1} for the horizontal direction. To simplify the handling of these parameters, it is convenient to introduce the notation $\bar{A}^i \cdot y_{ij} \cdot b_0^{-j+1}$ for the contributions and $D_i = y_{ij} \cdot b_0^{-j+1}$ for their coefficients. The $Y = ||y_{ij}||$ is a triangular matrix with «genuine» terms $y_{nn} \equiv d_n[0]$ in the diagonal. These diagonal terms are maximally suppressed by the b_0^{-1} powers in D_n, while the unsuppressed terms are contained in the first column of the matrix Y

$$y_{11} \equiv \mathbf{1}; \tag{22}$$
$$y_{21} = \boldsymbol{d_2}[\mathbf{1}]; \tag{23}$$
$$y_{31} = \boldsymbol{d_3}[\mathbf{2}] + \boldsymbol{c_1 d_3}[\mathbf{0}, \mathbf{1}]; \tag{24}$$
$$y_{41} = \boldsymbol{d_4}[\mathbf{3}] + \boldsymbol{c_1}\ \boldsymbol{d_4}[\mathbf{1}, \mathbf{1}] + \boldsymbol{c_2}\ \boldsymbol{d_4}[\mathbf{0}, \mathbf{0}, \mathbf{1}]; \tag{25}$$
$$\ldots$$

All these terms originate from the renormalization of a single coupling/gluon line. For details see the skeleton diagram in Fig. 2, a which accumulates all the diagrams like in Fig. 1, a, b, d.

3.2. BLM Scheme for n Loop. Let us find a new pair $(t_1, A(t_1))$ to nullify all the «bold type» contributions y_{i1} and to accumulate them into the new expansion parameter $A(t_1)$

$$\begin{aligned} &\bar{A}_(t) \to \bar{A}(t_1) \equiv A_1, \\ &\qquad \mathrm{t} - \mathrm{t}_1 \equiv \Delta_1 = \Delta_{1,0} + A_1 \cdot \Delta_{1,1} + A_1^2 \cdot \Delta_{1,2} + \ldots \end{aligned} \tag{26}$$

Here the shift Δ_1 of the t to the first intrinsic scale t_1 is found in the form of a perturbation series in A_1 (that has first been suggested in [11]). The corresponding procedure consists in the re-expansion of Exp. (20) in the new coupling A_1 and rearrangement of the power series. Following the RG law for the coupling $\bar{A}(t) = A(\Delta_1, A_1)$ and expanding it in Δ_1 one obtains

$$\bar{A}(t) = A(t - t_1, A_1) = A_1 - B(A_1)\Delta_1 + B'(A_1)B(A_1)\frac{\Delta_1^2}{2} + \dots$$

Substituting this expansion together with the the expansion for Δ_1 in Eq. (26) into Exp. (20) one arrives at the rearranged series

$$\begin{aligned}
\bar{A}^1 D_1 \to& \bar{A}_1^1 \cdot \quad 1; \\
\bar{A}^2 D_2 \to& \bar{A}_1^2 \cdot \quad D_2 - 1\Delta_{1,0}; && (27)\\
\bar{A}^3 D_3 \to& \bar{A}_1^3 \cdot \quad D_3 - 2\Delta_{1,0} \cdot D_2 - \Delta_{1,0}c_1 + \Delta_{1,0}^2 - \Delta_{1,1}; && (28)\\
\bar{A}^4 D_4 \to& \bar{A}_1^4 \cdot \quad D_4 - 3\Delta_{1,0} \cdot D_3 + \left(3\Delta_{1,0}^2 - 2c_1\Delta_{1,0}\right) D_2 - \\
& - c_2\Delta_{1,0} + \frac{5}{2}c_1\Delta_{1,0}^2 - \Delta_{1,0}^3 + \\
& + \left(2\Delta_{1,0} - 2D_2 - c_1\right)\Delta_{1,1} - \Delta_{1,2}; && (29)\\
A^{n+1} D_{n+1} \to& A_1^{n+1} \cdot D_{n+1} - n\Delta_{1,0} \cdot D_n + \dots
\end{aligned}$$

The generalized BLM requires that the y_{i1} *contributions should cancel* at each order $\bar{A}_1^i$ in the set of Eqs. (27), (28), (29), ... This requirement completely determines the partial «scales» $\Delta_{1,i}$ from the set of algebraic equations. The explicit expressions for a few $\Delta_{1,k}$ are presented in Appendix B; below we write a few coefficients important for further discussion

$$\Delta_{1,0} = y_{21} = d_2[1], \tag{30}$$

$$\Delta_{1,1} = y_{31} - (y_{21})^2 - c_1 y_{21} = d_3[2] - d_2^2[1] + c_1\left(d_3[0,1] - d_2[1]\right), \tag{31}$$

$$\begin{aligned}\Delta_{1,2} = y_{41} - 3y_{31}y_{21} - 2(y_{21})^3 - c_1 \cdot \ldots = \\ d_4[3] - 3d_2[1]d_3[2] + 2(d_2[1])^3 + c_1 \cdot \ldots\end{aligned} \tag{32}$$

Note that the NLO BLM correction in Eq. (31) cancels in the particular case $d_3[2] = (d_2[1])^2$, $d_3[0,1] = d_2[1]$. First of them corresponds to the geometric progression for the leading logarithms of the RG law, while the second one corresponds to sub-leading logarithm cancellation. If one applies these conditions to $\Delta_{1,\,2}$ in Eq. (32) (see also Eq. (B.3)), one obtains again the evident «geometric» condition, $d_4[3] = (d_2[1])^3$, for cancellation of the leading logarithm part at the next step and so on. This test demonstrates the self-consistence of the calculations. If the coefficients $d_{n+1}[n]$ in y_{n1} follow the RG law for the leading logarithms, then they are taken into account even by the first partial BLM «scale» $\Delta_{1,0}$. For the sub-leading contributions one needs more subtle conditions. The absorbtion of such contributions into the proper scales, see, e. g., c_1-terms in Eq. (31), distinguishes our procedure from the other BLM extensions [7].

In this way one can rearrange the first column y_{i1} into Δ_1 step by step for any fixed order of the PE. As a result of the procedure the initial series, Eq. (20), can be reduced to the new one that contains only one unsuppressed term $\bar{A}_1 \cdot 1$, the first diagonal term in Eq. (33). All the other terms are suppressed by the powers of b_0^{-1},

$$\begin{aligned}
&\bar{A}^1 D_1 \to \bar{A}_1^1 \cdot 1, \\
&\bar{A}^2 D_2 \to \bar{A}_1^2 \cdot 0 + \frac{y_{22}}{b_0}, \\
&\bar{A}^3 D_3 \to \bar{A}_1^3 \cdot 0 + \frac{1}{b_0}\left(y_{32} - 2y_{21}\ y_{22}\right) + \frac{y_{33}}{b_0^2}, \\
&\bar{A}^4 D_4 \to \bar{A}_1^4 \cdot 0 + \frac{1}{b_0}(y_{42} - 3y_{21}\ y_{32} + y_{22}[5y_{21}^2 - 2y_{31}]) + \\
&\qquad\qquad\qquad \frac{1}{b_0^2}(y_{43} - 3y_{21}y_{33}) + \frac{y_{44}}{b_0^3}, \\
&A^n D_n \to A_1^n \cdot 0 + \frac{1}{b_0}(y_{n2} - \ldots)\ldots
\end{aligned} \tag{33}$$

At this stage the matrix Y transforms into the new matrix $Y^{(1)}$, the first column of which is now $y_{1i}^{(1)} = \delta_{1i}$, and the other few elements are presented in Eq. (33). The first BLM stage result can be rewritten in the form of the matrix representation, $\sum_{i>j} \bar{A}^i \cdot y_{ij} \cdot b_0^{-j+1} \equiv \bar{A}\mathcal{A}^+ Y \mathcal{B}$, where $\mathcal{A} = \left(1, \bar{A}, \bar{A}^2, \ldots\right), \mathcal{A}_i = \left(1, \bar{A}_i, \bar{A}_i^2, \ldots\right)$, $\mathcal{B} = \left(1, b_0^{-1}, b_0^{-2}, \ldots\right)$,

$$\bar{A}\left(\mathcal{A}^+ Y \mathcal{B}\right)_n \stackrel{\text{1 stage}}{\longrightarrow} \bar{A}_1 \left(\mathcal{A}_1^+ Y^{(1)} \mathcal{B}\right)_n = \bar{A}_1 \left(\mathbf{1} + \bar{A}_1 (\mathcal{A}_1^+ Y^{(1)} \mathcal{B})_{n-1}\right). \tag{34}$$

The single unsuppressed (diagonal) term, $\mathbf{1}$, is picked out in the r.h.s. in the parentheses while the second term there is formed by the power b_0^{-1}-suppressed minor of the matrix $Y^{(1)}$.

4. SEQUENTIAL BLM PROCEDURE

4.1. Next Stages of the BLM Generalization. Let us continue to put the matrix $Y^{(1)}$ into the diagonal form by reforming its *second column*. Now we single out its $(Y^{(1)})_{(n-1)}$-minor part in Eq. (34) which generates the b_0^{-1}-suppressed terms

$$A_1 (\mathcal{A}_1^+ Y^{(1)} \mathcal{B})_n \to \bar{A}_1 \left(\bar{A}_1 (\mathcal{A}_1^+ Y^{(1)} \mathcal{B})_{n-1}\right).$$

Table 1. The structure of $\bar{A}\ \ \cdot\bar{A}^{i-1}y_{ij}^{(1)}b_0^{1-j}$ contributions

$A^1D_1\to$	$\bar{A}_1\cdot$	1	1
$A^2D_2\to$	$\bar{A}_1\cdot$	$\bar{A}^1(t_1)$	$0+\dfrac{d_2[0]}{b_0}\cdot\mathbf{1}$
$A^3D_3\to$	$\bar{A}_1\cdot$	$\bar{A}^2(t_1)$	$0+\dfrac{d_2[0]}{b_0}\tilde{\mathbf{y}}_{\mathbf{32}}^{(\mathbf{1})}+\dfrac{d_3[0]}{b_0^2}$
$A^4D_4\to$	$\bar{A}_1\cdot$	$\bar{A}^3(t_1)$	$0+\dfrac{d_2[0]}{b_0}\tilde{\mathbf{y}}_{\mathbf{42}}^{(\mathbf{1})}+\dfrac{1}{b_0^2}y_{43}^{(1)}+\dfrac{d_4[0]}{b_0^3}$
$A^5D_5\to$	$\bar{A}_1\cdot$	$\bar{A}^4(t_1)$	$0+\dfrac{d_2[0]}{b_0}\tilde{\mathbf{y}}_{\mathbf{52}}^{(\mathbf{1})}+\dfrac{1}{b_0^2}y_{53}^{(1)}+\dfrac{1}{b_0^3}y_{54}^{(1)}+\dfrac{d_5[0]}{b_0^4}$
	$\ldots$	$\ldots$	$\ldots$

The elements of this minor are represented in the right column of Table 1. The first power b_0^{-1}-suppressed terms of this minor are emphasized in the bold type there. Repeating the same procedure as at the first BLM stage (see the previous section) with the column $y_{i2}^{(1)}$, we rearrange again these terms into new expansion parameter $A(t_2)$ at the new «scale» t_2

$$\bar{A}(t_1)\to\bar{A}(t_2)\equiv A_2,$$
$$t_1-t_2\equiv\Delta_2=\Delta_{2,0}+A_2\cdot\Delta_{2,1}+A_2^2\cdot\Delta_{2,2}+\ldots \tag{35}$$

The first equalities in Eqs. (30)–(32) remain valid also for the partial scales $\Delta_{2,m}$ appearing from the $y_{i2}^{(1)}$ with the evident shifting of all the indices by 1.

Therefore, using Eqs. (33) to determine $y_{i2}^{(1)}$ and taking the common factor $\dfrac{d_2[0]}{b_0}$ to normalize the elements ($\tilde{\mathbf{y}}_{\mathbf{n2}}^{(\mathbf{1})}=y_{n2}^{(1)}/y_{22}$) one arrives at

$$\Delta_{2,0}=\tilde{\mathbf{y}}_{\mathbf{32}}^{(\mathbf{1})}=\frac{y_{32}}{y_{22}}-2y_{21}=\frac{d_3[1]}{d_2[0]}-2d_2[1], \tag{36}$$

$$\Delta_{2,1}=\tilde{\mathbf{y}}_{\mathbf{42}}^{(\mathbf{1})}-\left(\tilde{\mathbf{y}}_{\mathbf{32}}^{(\mathbf{1})}\right)^{\mathbf{2}}-c_1\tilde{\mathbf{y}}_{\mathbf{32}}^{(\mathbf{1})}, \tag{37}$$

$$\ldots$$

So column by column the chain of transformations $Y\to Y^{(1)}\to Y^{(2)}\to\ldots\to Y^{(n-1)}$ leads to the diagonalization of Table 1. At each stage one will obtain a new coupling $A(t_i)$. The final result of this successive BLM procedure reduces the initial perturbation series, Eq. (1), to the special form*

$$S(A) = d_0 + \frac{\bar{A}(t_1)}{b_0} \cdot d_1 \left\{ 1 + \frac{\bar{A}(t_2)}{b_0} d_2[0] \left\{ 1 + \frac{\bar{A}(t_3)}{b_0} d_3[0] \{1 + \ldots\} \right\} \right\} \quad (38)$$

containing only the «genuine» coefficients $d_i[0]$ accompanied by the corresponding coupling at its proper scale t_i, $t_i = t_1 - \Delta_1 - \ldots - \Delta_i$. Moreover, the series in powers $\bar{a}^n(t)$ transforms to the series in products $\prod_{i=1}^{n} \bar{a}(t_i)$. Note that Eq. (38) can be easily presented in the form *a'la* the continued fraction

$$S(A) = d_0 + \cfrac{\bar{a}_1 d_1[0]}{1 - \cfrac{\bar{a}_2 d_2[0]}{1 + \bar{a}_2 d_2[0] - \cfrac{\bar{a}_3 d_3[0]}{1 + \bar{a}_3 d_3[0] - \cfrac{\bar{a}_4 d_4[0]}{\ldots,}}}} \quad (39)$$

where $\bar{a}_i = \bar{a}(t_i)$. The latter may be the source of a new approximation for the $S(A)$: one can construct an associated continued fraction to Eq. (39) that possesses better convergent properties or apply Pade approximation to this representation. Of course, the final results, Eq. (38) (or Eq. (39)), look rather formal because the sBLM procedure constructed above disregards the perturbative applicability constraints for both the pairs $(A(t_i),\ t_i)$ and the new expansion coefficients $d_i[0]$. We apply this result to known N^{3}LO calculation of the D function in Section 5.

4.2. Distribution Sense of sBLM. It is instructive to illustrate the discussed procedure from the «distribution» point of view based on the convolution representation, Eq. (9), as it has been demonstrated in Eq. (11) for the one-loop running. One can rederive the results of the sBLM first stage using these terms. Really, expanding the factor $\bar{A}_{(N)}$ in the integrand of the convolution $\mathcal{P} * \bar{A}$ in Eq. (9) in the variable α, and then following the line in Sec. 3 one can arrive at the same Eqs. (30)–(32) for the partial scales. For that, one should specify the partial distribution P_1 in Eq. (16) as the distribution for the first column of Y, $\int_0^\infty P_1(\alpha)\alpha^{n-1} d\alpha \equiv \langle \alpha^{n-1} \rangle_1 = y_{n1}$, and substitute P_1 instead of $\mathcal{P}$ at the final stage of the procedure.

As the result of the first sBLM stage, Eq. (34), the $P_1 * \bar{A}$ contribution reduces to

$$P_1 * \bar{A} \to \bar{A}_1 \equiv \bar{A}(t - \Delta_1) = \int_0^\infty \delta(\alpha - \Delta_1)\bar{A}(t - \alpha) d\alpha \ .$$

*Let me stress here that the form of this series differs from those suggested in [7, 10].

Therefore, the distribution P_1 *reduces to* $\delta(\alpha - \Delta_1)$ that is similar to the standard BLM result, $\delta(\alpha - d_2[1])$, in Sec. 2. This Δ_1 accumulates all the «α_s–renormalizations» associated with the single dressed gluon propagator with two dressed vertices and the connected quark fields in the skeleton diagrams in Fig. 2, a. The common normalizing factor d_1 originates from the first tree diagrams of these skeleton diagrams.

At the second stage of sBLM we deal with the contribution from the second column of $Y^{(1)}$; a normalized distribution P_2 can be introduced for $y^{(1)}_{n2}$, $\int_0^\infty P_2(\alpha)\alpha^{n-1}d\alpha \equiv \langle\alpha^{n-1}\rangle_2 = \tilde{y}^{(1)}_{n2}$. These contributions correspond to «two-gluon» skeleton diagrams in Fig. 2, b (see also Fig. 1, c). By now the effective

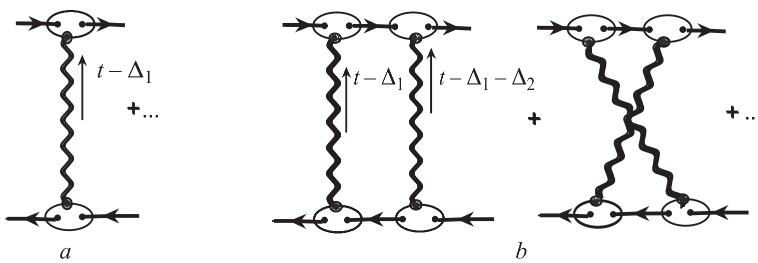


Fig. 2. The elements of skeleton diagrams: oval is dressed vertex; thick wavy strip is dressed gluon propagator, and thick fermion line is dressed quark propagators. The arrows with caption denote the value of the sBLM «scale» for the corresponding effective charge. *a)* Skeleton diagrams for the first stage of sBLM. *b)* Skeleton diagrams with two dressed gluon lines for the second stage of sBLM

scale of the coupling, $\bar{A}(t - \Delta_1)$, is already fixed at the previous stage. As a result of this stage, $P_2 \to \delta(\alpha - \Delta_2)$ and the normalization scale one of the couplings appears to be shifted, $\bar{A}(t - \Delta_1) \to \bar{A}(t - \Delta_1 - \Delta_2)$. The normalizing factor for the contribution, $d_2[0]/b_0$, originates from the undressed diagrams, which corresponds to the skeleton diagrams in Fig. 2, b.

Executing a number of sBLM stages for $\mathcal{P} * \bar{A}$ one arrives at the representation,

$$\mathcal{P} * \bar{A} \to \int_0^\infty d\alpha\biggl(\delta(\alpha - \Delta_1) + \frac{d_2[0]}{b_0}\bar{A}(t - \Delta_1)\times \\ \times\delta(\alpha - \Delta_1 - \Delta_2) + \ldots\biggr)\bar{A}(t - \alpha). \quad (40)$$

The kernel of this convolution can be compared with the PE representation in Eq. (17),

$$\mathcal{P}(\alpha) = \sum_n \frac{(-1)^n}{n!}\delta^{\{n\}}(\alpha)\cdot D_n \to \delta(\alpha-\Delta_1)+$$
$$+\, d_2[0]/b_0\bar{A}(t-\Delta_1)\cdot\delta(\alpha-\Delta_1-\Delta_2)+\ldots$$

The final sBLM series, Eq. (38), corresponds to the expansion in the skeleton diagrams. The coefficients of this expansion, $d_n[0]$ (y_{nn}), originate from the first undressed diagrams of the corresponding skeleton diagrams while the scales Δ_n appear due to the renormalizations in these skeleton diagrams.

5. sBLM PROCEDURE AND ITS IMPROVEMENT FOR THE D FUNCTION

5.1. sBLM Procedure for the *D* Function. The initial well-known series for D [22] can be rewritten by means of the β-function coefficients

$$D = 3\sum_f Q_f^2\left\{d_0 + d_1\left[a + a^2 d_2 + a^3 d_3 + a^4 d_4 + \ldots\right]\right\},\ d_0 = 1;\ d_1 = 3C_F,$$

$$\begin{aligned} d_2 &= b_0\cdot d_2[1] + d_2[0],\\ d_3 &= b_0^2\cdot d_3[2] + b_1\cdot d_3[0,1] + b_0\cdot d_3[1] + d_3[0];\\ d_4 &= b_0^3\cdot d_4[3] + b_0 b_1\cdot d_4[1,1] + b_2\cdot d_4[0,0,1] + b_0^2\cdot d_4[2] + \ldots \end{aligned} \qquad (41)$$

A separate problem is to recast D into this form; it is solved in Appendix C on the basis of the results obtained in [23] and on the partial results for d_4 in [24]. Note that the expressions for the expansion elements in (41) remain valid for the inclusion of light gluinos that contribute to the β function (Appendix A). The explicit expressions for the $d_3[m,\, n]$ are presented in Appendix C while below they are written numerically

$$d_2 = b_0\cdot 0.69\ +\ \frac{1}{3}, \qquad (42)$$

$$d_3 = b_0^2{\cdot}3.104\ -\ b_1{\cdot}1.2\ \ +b_0{\cdot}55.70 + \left(-573.96 - 19.83\frac{(\sum_f Q_f)^2}{3(\sum_f Q_f^2)}\right). \qquad (43)$$

We substitute the value $b_0(N_f=3)=9$, $b_1(N_f=3)=64$ in (43) for an illustration

$$d_3 = 251.1 - 76.8 +\ 501.3 + (-573.96 - 0) \approx \mathbf{101.9} \qquad (44)$$

to compare the contributions from the different sources. Further we shall apply the sBLM procedure to D step by step to remove, respectively, the b_0-contribution in N^{2}LO, b_0^2 and b_1-contributions in N^{3}LO, and so on. The results of sBLM will be analyzed at every step.

At the first standard step the BLM scale setting transforms the coefficients $d_2,\ d_3$ (compare with expressions (42)–(43)) and the coupling as follows:

$$d_2 \to \tilde{d}_2 = b_0 \cdot 0 + \frac{1}{3}, \tag{45}$$

$$d_3 \to \tilde{d}_3 = b_0^2 \left(d_3[2] + d_3[0,1]c_1 - d_2^2[1] - d_2[1]c_1\right) + b_0(d_3[1] - 2d_2[0]d_2[1]) + \\ + d_3[0] \tag{46}$$

$$= b_0^2\ (2.1555 - 1.0251) + b_0(55.70 - 0.46) + \ldots \approx \mathbf{14.7}, \tag{47}$$

$$A(t) \to A(t_1);\ t - t_1 = \Delta_{1,0} = d_2[1] \approx 0.69. \tag{48}$$

We can see that the value of $b_0^2\ y_{31}$ reduced approximately by a factor of 2 at the first step (at the same condition as for Eq. (44)); the value of $b_0\ y_{32}$ practically does not change; the value of the total coefficient reduced to 14.7 in comparison with the initial value $d_3 \approx 101.9$ in Eq. (44). This strong cancellation as well as the other features of the BLM steps appear due to the large and negative value of the genuine term $d_3[0]$.

At the next step of the stage the modified $\tilde{y}_{31}$ term in Eq. (46) is transferred into Δ_1 by following Eqs. (30)–(31):

$$\tilde{d}_3 \to \tilde{\tilde{d}}_3 = b_0^2 \cdot 0 + b_1 \cdot 0 + b_0 \cdot (d_3[1] - 2d_2[0]d_2[1]) + d_3[0] \approx \mathbf{-77}, \tag{49}$$

$$A(t) \to A(t_1);\ t - t_1 = \Delta_1 = d_2[1] + A(t_1) \cdot \left(d_3[2] + d_3[0,1]c_1 - d_2^2[1] - d_2[1]c_1\right), \tag{50}$$

$$\Delta_1 \approx 0.69 + A(t_1) \cdot 1.13. \tag{51}$$

Here in Eq. (51) one can put $t_1 \approx t - d_2[1]$ for the A argument rather than solve Eq. (50) with respect to t_1. The new value of d_3, $d_3 \to \tilde{\tilde{d}}_3 \approx -77$, is noticeably larger in absolute value than the value of this coefficient at the first step $\tilde{d}_3 \approx 14.7$. At the same time, the first perturbation correction to Δ_1 in Eq. (51) looks rather moderate and admissible. The contents of d_4 in Eq. (19) also transforms by following Eq. (32). Based on the results in [24], which lead to $d_4[3] \approx 2.18$, one can predict the modification of the «bubble part» $d_4[3]$ of the d_4,

$$d_4[3] \approx 2.18 \to d_4[3] - d_3[2]d_2[1] - 2d_2[1](d_3[2] - d_2[1]^2) \approx -3.3$$

that is not also improved in itself. One can conclude that though the next step of sBLM is admissible due to the moderate size of correction to Δ_1, it does not improve the convergence of the perturbation series.

Let us consider the second stage of the sBLM procedure

1) t_2 goes out beoynd the pQCD domain because $t_1 - t_2 = \Delta_{2,0} = d_3[1]/d_2[0]$ $-2d_2[1] \approx 166(!)$;

2) it does not lead to the decrease in the $\tilde{\tilde{d}}_3$ term $\tilde{\tilde{d}}_3 \to \tilde{\tilde{\tilde{d}}}_3 = d_3[0] \approx -574$ due to the large value of the genuine term (compare the contributions of the different terms in Eq. (44)).

Here we meet the case when the sBLM procedure does not improve the convergence of the perturbation series or even cannot be applied (at the second stage) because this procedure ignores peculiar properties of certain series. This possibility, typical of this formal procedure, has been mentioned in Sec. 4.

5.2. How to Improve the sBLM Procedure. It would be better not to perform the second stage at all and try another way to optimize the value of $\tilde{d}_3$ after the first step. It is tempting not to remove the contribution $y_{31} = d_3[2] + c_1 d_3[0,1]$ completely, as we did in Eq. (49) at the second step above, but rearrange a part of it into the coupling renormalization and keep the other positive part in the rest to compensate the large and negative $d_3[0]$ contribution. It is convenient to introduce an $\boldsymbol{x}$-part of y_{31}, $\boldsymbol{x} y_{31}$, to absorb this into the partial scale, see Eq. (53), while its $(\mathbf{1}-\boldsymbol{x})$-part, $(\mathbf{1}-\boldsymbol{x})y_{31}$, is kept to cancel $d_3[0]$ in (52). This trick leads to the $\boldsymbol{x}$-dependent BLM, (xBLM),

$$\tilde{d}_3 \to \tilde{\tilde{d}}_3 = b_0^2 \cdot (\mathbf{1}-\boldsymbol{x})(d_3[2] + c_1 d_3[0,1]) + b_0 \cdot (d_3[1] - 2d_2[0]d_2[1]) + d_3[0], \quad (52)$$

$$t - t_1 = \Delta_1 = d_2[1] + A(t_1) \cdot \left(\boldsymbol{x}(d_3[2] + d_3[0,1]c_1) - d_2^2[1] - d_2[1c_1]\right). \quad (53)$$

Let us set an «optimization» condition, say FAC, $\tilde{\tilde{d}}_3 = 0$, to fix a certain value of $\boldsymbol{x}$. One has make sure that the perturbation corrections are improved for both $\tilde{\tilde{d}}_3$ and $\tilde{\Delta}_1$ (see the 5–6th columns in Table 2) in comparison with that in Eqs. (49), (51). The final result for D is reduced to

$$D = 3 \sum_f Q_f^2 \left\{ 1 + 3C_F \left[a(\tilde{t}_1) + \frac{1}{3} \cdot a^2(\tilde{t}_1) + 0 \cdot a^3(\tilde{t}_1) \right] \right\}, \quad (54)$$

where $t - \tilde{t}_1 = \tilde{\Delta}_1$ are presented in Table 2.

It is instructive to apply a similar procedure also to the observable quantity $R(s) = \sigma(e^+e^- \to h)/\sigma(e^+e^- \to \mu^+\mu^-)$ associated with D

$$R(s) = D(s) - d_1 \frac{\pi^2}{3} \cdot b_0^2 \bar{a}^3 = 3 \sum_f Q_f^2 \left\{ 1 + 3C_F \left[\bar{a} + r_2 \bar{a}^2 + r_3 \bar{a}^3 \right] \right\},$$

Table 2. The results of xBLM at the $\tilde{\tilde{d}}_3 = 0$ and $\tilde{\tilde{r}}_3 = 0$ conditions

N_f	$b_0(N_f)$		x	$t - \tilde{t}_1 = \tilde{\Delta}_1$	$\tilde{\Delta}_1(Q^2)$		x	$s - \tilde{s}_1 = \tilde{\Delta}_1$
3	9		0.56	$d_2[1] + a(\tilde{t}_1) b_0 \cdot 0.18$			1.84	$d_2[1] - a(\tilde{s}_1) b_0 \cdot 3.1$
4	$\frac{25}{3}$	$\tilde{\tilde{d}}_3 = 0$	0.24	$d_2[1] - a(\tilde{t}_1) b_0 \cdot 0.45$	$Q^2 = 3$ [GeV^2] 0.58	$\tilde{\tilde{r}}_3 = 0$	2.56	$d_2[1] - a(\tilde{s}_1) b_0 \cdot 3.7$
5	$\frac{23}{3}$		-0.11	$d_2[1] - a(\tilde{t}_1) b_0 \cdot 1.19$	$Q^2 = 26$ [GeV^2] 0.52		3.63	$d_2[1] - a(\tilde{s}_1) b_0 \cdot 4.48$

where $r_1 = d_1,\ r_2 = d_2,\ r_3 = d_3 - \dfrac{\pi^2}{3} \cdot b_0^2$ (see, e. g., [22]). The large and negative π^2 term arising due to an analytic continuation makes r_3 also negative (compare with Eq. (43)). As a result of the xBLM procedure $\tilde{d}_2 \to r_2$, $\tilde{r}_3 \to r_3$, and x-dependent term in $\tilde{r}_3$ transforms to $b_0^2 \cdot (1-x)(d_3[2] + c_1 d_3[0,1] - \pi^2/3)$. To obtain a positive compensative term to cancel $d_3[0]$ at the next step of the transformation (that leads to $\tilde{\tilde{r}}_3 = 0$), one should take $x > 1$, see two right columns in Table 2.

6. GENERALIZED BLM PROCEDURE TO IMPROVE THE PERTURBATION SERIES

To generalize the sBLM procedure in the way mentioned in Sec. 5 as the xBLM procedure, let us introduce a lower triangular matrix $X = ||x_{ij}||,\ x_{ii} \equiv 0$ associated with the matrix Y. The element x_{ij} is the part of the contribution $y_{ij},\ y_{ij}\ x_{ij}$, that should be absorbed into the coupling renormalization, while the remainder of the contribution $y_{ij}\ \bar{x}_{ij}$, where $\bar{x}_{ij} \equiv 1 - x_{ij}$, is kept in PE coefficient D_i. The choice $X = 0$ returns us to the initial series before transformations, while $X = \{x_{21} = 1,\ \text{others}\ x_{i>j} = 0\}$ corresponds to the standard BLM. The matrix $X = \{x_{i>j} = 1\}$ corresponds to the sBLM; the first column $x_{i1} = 1$ leads to the first stage of sBLM, compare second column in Table 3 with Eq. (33). For the discussed xBLM procedure the first column transforms into $\{1,\ A_1 y_{21}\bar{x}_{i1},\ A_1^2 y_{31}\bar{x}_{31},\ \ldots\}$ instead of $\{1,\ 0,\ 0,\ \ldots\}$ in sBLM. The formulae for the Δ_1 proper scale, (30), (31), ... remain valid in this case with the obvious changes $y_{i1} \to y_{i1}x_{i1}$, see Eqs. (56), (57). These additional free parameters x_{ij}, altogether $n(n-1)/2$ parameters in N^{n}LO of PE, allow one to perform a «fine tuning» of the coefficients of the series. A more complicated structure of the final PE series is the price one should pay for such an improvement of the convergence of the series. The schematic sketch of the first stage of xBLM is demonstrated in Table 3 and Eqs. (55)–(57).

$$\bar{A}(t) \to \bar{A}(t_1) \equiv A_1, t - t_1 \equiv \Delta_1 = \Delta_{1,0}(X) + A_1 \cdot \Delta_{1,1}(X) + \ldots \tag{55}$$
$$\Delta_{1,0}(X) = y_{21}x_{21}, \tag{56}$$
$$\Delta_{1,1}(X) = y_{31}x_{31} - 2(y_{21}x_{21})y_{21} + (y_{21}x_{21})^2 - (y_{21}x_{21})c_1. \tag{57}$$

Following the xBLM we keep $y_{i2}\bar{x}_{i2}$ parts in the second column at the second stage. All other parts of this column should be absorbed into the new coupling A_2. The result of this stage is illustrated in Table 4 and in Eq. (58) for the proper scale $\Delta_2(X)$. Further stages of xBLM are similar to the first ones and to that in sBLM, but the final result differs from the later case. We lose the clear diagonal

Table 3. The first stage of the xBLM procedure

$\bar{A}^1 D_1 \to \bar{A}_1 \cdot$	1	$+\,0$	
$\bar{A}^2 D_2 \to \bar{A}_1 \cdot$	$\bar{A}_1^1 y_{21} \bar{x}_{21}$	$+\bar{A}_1^1 \dfrac{y_{22}}{b_0}$	
$\bar{A}^3 D_3 \to \bar{A}_1 \cdot$	$\bar{A}_1^2 y_{31} \bar{x}_{31}$	$+\,\bar{A}_1^2 \dfrac{y_{22}}{b_0} \left(\dfrac{y_{32}}{y_{22}} - 2 y_{21}\ x_{21} \right)$	$+\bar{A}_1^2 \dfrac{y_{33}}{b_0^2}$
$\ldots$	$\ldots$	$+\ldots$	
$\bar{A}^n D_n \to \bar{A}_1 \cdot$	$\bar{A}_1^{n-1} y_{n1} \bar{x}_{n1}$	$+\ldots$	

form like Eq. (38), but obtain more flexibility to fix the values of the $A^i D_i$ terms in expansion.

Table 4. The second stage of the xBLM procedure

$\bar{A}^1 D_1 \to \bar{A}_1 \cdot$	1	$+\,0$	
$\bar{A}^2 D_2 \to \bar{A}_1 \cdot$	$\bar{A}_1^1 y_{21} \bar{x}_{21}$	$+\bar{A}_2^1 \dfrac{y_{22}}{b_0}$	
$\bar{A}^3 D_3 \to \bar{A}_1 \cdot$	$\bar{A}_1^2 y_{31} \bar{x}_{31}$	$+\bar{A}_2^2 \dfrac{y_{22}}{b_0} \left(\dfrac{y_{32}\ \bar{x}_{32}}{y_{22}} \right)$	$+\bar{A}_2^2 \dfrac{y_{33}}{b_0^2}$
$\ldots$	$\ldots$	$+\ldots$	$+\ldots$
$\bar{A}^n D_n \to \bar{A}_1 \cdot$	$\bar{A}_1^{n-1} y_{n1} \bar{x}_{n1}$	$+\bar{A}_2^{n-1} \dfrac{y_{22}}{b_0} \left(\dfrac{y_{n2} \bar{x}_{n2}}{y_{22}} \right)$	$+\ldots$

$$\bar{A}(t_1) \to \bar{A}(t_2) \equiv A_2;\ t_1 - t_2 \equiv \Delta_2 = \Delta_{2,0}(X) + A_1 \cdot \Delta_{2,1}(X) + \ldots$$
$$\Delta_{2,0}(X) = \frac{y_{32}}{y_{22}} x_{32} - 2 y_{21}\ x_{21}. \quad (58)$$

It is instructive to consider a few partial cases: i) The FAC setting in N^{2}LO corresponds to the condition $(\bar{A}_1)^2 (y_{21} \bar{x}_{21} + \dfrac{y_{22}}{b_0}) = 0$ which has been considered in Eq. (14); ii) If we restrict ourselves, say, to the N^{3}LO (see Table 4) then we have 3 parameters, x_{21}, x_{31}, x_{32} to optimize the contributions of $A^2 D_2$ and $A^3 D_3$, respectively,

$$A^2 D_2 \to C_2 = \bar{A}_1 \left[\bar{A}_1 y_{21}\bar{x}_{21} + \bar{A}_2 \frac{y_{22}}{b_0}\right], \tag{59}$$

$$A^3 D_3 \to C_3 = \bar{A}_1 \left[\bar{A}_1^2 y_{31}\bar{x}_{31} + \bar{A}_2^2 \frac{y_{22}}{b_0}\left(\frac{y_{32}\bar{x}_{32}}{y_{22}}\right) + \bar{A}_2^2 \frac{y_{33}}{b_0^2}\right], \tag{60}$$

where

$$\bar{A}_1 = \bar{A}(t - \Delta_1), \;\; \bar{A}_2 = \bar{A}(t - \Delta_1 - \Delta_2). \tag{61}$$

The case discussed in Sec. 5 in Table 2 corresponds to the partial solution of the above equations at $C_3 = 0$, $A_1 = A_2$ with $x_{21} = 1$, $x_{31} = \boldsymbol{x}$, $x_{32} = 0$. A complete set of solutions to Eqs. (59)–(61) at $C_2 = C_3 = 0$ with respect to x_{ij} can be obtained and analyzed numerically.

CONCLUSION

The hierarchy of the contributions to every order of the perturbation expansion is obtained using peculiarities of the β-function coefficients in QCD. Following this hierarchy we construct a sequential generalization of the well-known BLM procedure [1] to any fixed order of the perturbation expansion for the case of two-point functions. This sBLM procedure leads to new expansion series in the new couplings, Eq. (38), that can also be rewritten in the form *a'la* continued fraction, see Eq. (39).

The advantages and disadvantages of this sBLM are exemplified in four-loop Adler D-function. One has to be sure that the sBLM procedure fails to improve perturbation expansion for the D-function at its second stage. To improve the convergence of this series the sBLM is generalized in the spirit of the Fast Apparent Convergence procedure [16] by the introducing of the number of additional free parameters, see Eq. (54) and Table 2 in the second part of Sec. 5. This next generalization of the BLM, xBLM procedure parameterized by a matrix X, is briefly discussed in Sec. 6. The xBLM procedure looks like the most complete generalization of the initial BLM one.

APPENDIX A
PROOF OF THE CONVOLUTION REPRESENTATION

Let $\bar{A} \equiv \bar{A}_{(N)}$ be a solution to the RG equation with the N-loop B-function. Let us introduce the operator $\hat{D}_t = \dfrac{\bar{A}^2}{B(\bar{A})}\dfrac{d}{dt}$, then $\hat{D}_t \bar{A} = \bar{A}^2$ and

$$\frac{1}{n!}\left(\hat{D}_t\right)^n \;\bar{A} = \left(\bar{A}\right)^{n+1}. \tag{A.1}$$

Substituting this Eq. (A.1) and Eq. (2) into Eq. (1) and collecting corresponding terms in the exponent,

$$\sum_{n=1} D_n \bar{A}^n = \int_0^\infty \mathcal{P}(\alpha) \left(\sum_{m=0} \frac{\left(\alpha \hat{D}_t\right)^m}{m!} \right) \; \bar{A}(t) d\alpha = \tag{A.2}$$

$$\int_0^\infty \mathcal{P}(\alpha) \left[\exp\left(\alpha \hat{D}_t\right) \bar{A}(t) \right] d\alpha = \int_0^\infty \mathcal{P}(\alpha) \; \bar{A} \left(t + \alpha \cdot \frac{\bar{A}^2}{B(\bar{A})} \right) d\alpha, \tag{A.3}$$

one arrives at the representation, Eq. (9). In the same way one can obtain this representation for the nonpower expansion series that appears in the analytic perturbation theory (APT) [20, 21]. The one-loop APT coupling constant is a bounded function of t, $A_1^{\mathrm{APT}}(t) = \dfrac{1}{t} - \dfrac{1}{\mathrm{e}^t - 1} \leqslant 1$. The next order one-loop constants A_n^{APT} are not the powers of A_1^{APT}, they can be obtained from A_1^{APT} by means of the same relation Eq. (6), see [21],

$$A_{n+1}^{\mathrm{APT}}(t) = \frac{1}{n!} \left(-\frac{d}{dt} \right)^n A_1^{\mathrm{APT}}(t) \neq (A_1^{\mathrm{APT}}(t))^{n+1}. \tag{A.4}$$

The APT expansion for $S(A)$ in (1) can be written as $S(A) \to d_0 + \dfrac{d_1}{b_0} \cdot \sum\limits_{n=1} D_n A_n^{\mathrm{APT}}$. This leads to the same convolution Eq. (7) with $A_1^{\mathrm{APT}}(t)$ in the integrand. The finiteness of $A_1^{\mathrm{APT}}(t)$ guarantees the convergence of this integral representation.

The required β-function coefficients with the MSSM light gluinos [25] are the following:

$$b_0\left(N_f, N_g\right) = \frac{11}{3} C_A - \frac{4}{3} \left(T_R N_f + \frac{N_g C_A}{2} \right), \tag{A.5}$$

$$b_1\left(N_f, N_g\right) = \frac{34}{3} C_A^2 - \frac{20}{3} C_A \left(T_R N_f + \frac{N_g C_A}{2} \right) - 4 \left(T_R N_f C_{\mathrm{F}} + \frac{N_g C_A}{2} C_A \right), \tag{A.6}$$

$$\begin{aligned} b_2\left(N_f, N_g\right) = {} & \frac{2857}{54} C_A^3 - N_f T_R \left(\frac{1415}{27} C_A^2 + \frac{205}{9} C_A C_F - 2 C_F^2 \right) + \\ & + (N_f T_R)^2 \left(\frac{44}{9} C_F + \frac{158}{27} C_A \right) - \frac{988}{27} N_g C_A (C_A^2) + \\ & + N_g C_A N_f T_R \left(\frac{22}{9} C_A C_F + \frac{224}{27} C_A^2 \right) + (N_g C_A)^2 \frac{145}{54} C_A. \end{aligned} \tag{A.7}$$

APPENDIX B
EXPLICIT FORMULAE FOR THE COEFFICIENTS $\Delta_{i,k}$

$$\Delta_{1,0} = y_{21} = d_2[1], \tag{B.1}$$

$$\Delta_{1,1} = y_{31} - (y_{21})^2 - c_1 y_{21} = d_3[2] - d_2^2[1] + c_1\left(d_3[0,1] - d_2[1]\right), \tag{B.2}$$

$$\begin{aligned}\Delta_{1,2} &= y_{41} - 3y_{31}y_{21} - 2(y_{21})^3 - c_1 y_{31} + \frac{3}{2}c_1(y_{21})^2 + (c_1^2 - c_2)y_{21} = \\ &= d_4[3] - 3d_2[1]d_3[2] + 2(d_2[1])^3 + \\ &\quad c_1\left(d_4[1,1] - 3d_3[0,1]d_2[1] + \frac{3}{2}(d_2[1])^2 - d_3[2]\right) + \\ &\quad c_1^2\left(d_2[1] - d_3[0,1]\right) + c_2(d_4[0,0,1] - d_2[1]),\end{aligned} \tag{B.3}$$

$$\begin{aligned}\Delta_{2,0} &= \frac{y_{32}}{y_{22}} - 2y_{21} = \frac{d_3[1]}{d_2[0]} - 2d_2[1], \\ \Delta_{2,1} &= \frac{y_{42}}{y_{22}} - 3y_{21}\frac{y_{32}}{y_{22}} + 5y_{21}^2 - 2y_{31} - \Delta_{2,0}^2 - c_1\Delta_{2,0} \\ &\dots\end{aligned} \tag{B.4}$$

APPENDIX C
REPRESENTATION FOR THE D FUNCTION

The Adler D function is known [23] for the MSSM with N_g light gluinos, $D(a, N_f, N_g)$. On the other hand, one can obtain the explicit functions $N_f = N_f(b_0, b_1)$ and $N_g = N_g(b_0, b_1)$ solving the set of equations (A.5), (A.6) with respect to $N_f,\ N_g$. Substituting this solution into $D(a, N_f, N_g)$, one arrives at the expansions Eqs. (C.1), (C.2) and Eqs. (C.4)–(C.8)

$$D(a) = 3\sum_f Q_f^2\left\{d_0 + d_1\left(a + d_2a^2 + d_3a^3 + \ldots\right)\right\},\ d_0 = 1;\ d_1 = 3C_F,$$

$$d_2 = b_0 d_2[1] + d_2[0], \tag{C.1}$$

$$d_3 = b_0^2 d_3[2] + b_1 \cdot d_3[0,1] + b_0 d_3[1] + d_3[0], \tag{C.2}$$

$$d_4 = b_0^3 d_4[3] + b_2 \cdot d_4[0,0,1] + b_0 b_1 d_4[1,1] + b_0^2 d_4[2] + b_1 d_4[1] + b_0 d_4[1] + d_4[0]. \tag{C.3}$$

The N_f^2–terms of d_4 have recently been calculated in [24], but this cannot be used in our approach. It is impossible to separate the terms $b_2 d_4[0,0,1]$ and

$b_0b_1d_4[1,1]$ that are of an order of $O(b_0^3)$ from the b_0^2-term, $b_0^2d_4[2]$ that also contributes to the «N_f^2 projection».

$$d_2[1] = \frac{11}{2} - 4\zeta_3 \approx 0.691772, \qquad d_2[0] \;=\; \frac{C_A}{3} - \frac{C_F}{2} = \frac{1}{3}, \tag{C.4}$$

$$d_3[2] = \frac{302}{9} - \frac{76}{3}\zeta_3 \approx 3.10345, \qquad d_3[0,1] = \frac{101}{12} - 8\zeta_3 \approx -1.19979, \tag{C.5}$$

$$d_3[1] = C_A\left(\frac{3}{4} + \frac{80}{3}\zeta_3 - \frac{40}{3}\zeta_5\right) - C_F\left(18 + 52\zeta_3 - 80\zeta_5\right) \approx 55.7005, \tag{C.6}$$

$$d_3[0] = \frac{1}{36}(523C_A^2 + 852C_AC_F - 414C_F^2) - 72C_A^2\zeta_3 +$$

$$+\frac{5}{24}(\frac{176}{3} - 128\zeta_3)\frac{(\sum_f Q_f)^2}{3(\sum_f Q_f^2)} \tag{C.7}$$

$$\approx -573.9607 - 19.8326\frac{(\sum_f Q_f)^2}{3(\sum_f Q_f^2)}. \tag{C.8}$$

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