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CONVERGENCE TOWARDS EQUILIBRIUM IN URN MODELS

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Сходимость к равновесию в урновых моделях

Выведены явные распределения для двух простых урновых моделей. Показано, что дисперсия неравновесных распределений практически достигает своего равновесного значения уже на кинетической стадии эволюции. Диффузионная стадия эволюции содержит только медленный дрейф средних значений к равновесию.

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Patrick A. E., Priezzhev V. B. E17-2006-13
Convergence towards Equilibrium in Urn Models

We derive explicitly nonstationary distributions for two simple urn models. We show that the variance of nonequilibrium distributions approaches its equilibrium value during the kinetic stage of evolution. The diffusion stage of evolution contains only a «slow» drift of expected values towards equilibrium.

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1. INTRODUCTION

Over the last century various urn models were used to gain understanding of convergence towards equilibrium in many-particle systems. One of the most popular classes of urn models comprises various modifications of the Ehrenfest model [1]. For recent activity in the area see [2, 3, 6, 7]. The less popular Bernoulli–Laplace model (see Ch. XV of the book [11]) is closely related to the Ehrenfest model.

Within the Ehrenfest model we are looking at balls numbered $1, 2, \ldots, N$ that can occupy urn A and urn B. In the initial state all balls occupy urn A. At regular time intervals, $t_1, t_2, t_3, \ldots$, a number from 1 to N is chosen at random (from the uniform distribution), and the corresponding ball changes its urn. The «macroscopic» state of the system of N balls at time t is identified with the number of balls, $n(t)$, in urn A. The evolution of the distribution of the state variable, $P_k(t_l) \equiv \Pr[n(t_l) = k]$, is described by the master equation

$$P_k(t_{l+1}) = \frac{k+1}{N} P_{k+1}(t_l) + \frac{N-k+1}{N} P_{k-1}(t_l). \tag{1}$$

Within the Bernoulli–Laplace model we have $2N$ balls of two colours (say, N black balls numbered $1, 2, \ldots, N$ and N red balls numbered $N+1, N+2, \ldots, 2N$). Urns A and B always contain exactly N balls of either color. In the initial state, all black balls occupy urn A and all red balls — urn B. At regular time intervals, $t_1, t_2, t_3, \ldots$, two balls (one ball from urn A, another from urn B) are chosen at random and permuted. The «macroscopic» state of the system of $2N$ balls at time t is identified with the number of black balls, $n(t)$, in urn A. The evolution of the distribution of this variable, $P_k(t_l) \equiv \Pr[n(t_l) = k]$, is described by the master equation

$$P_k(t_{l+1}) = \frac{2k}{N}\left(1 - \frac{k}{N}\right) P_k(t_l) + \frac{(k+1)^2}{N^2} P_{k+1}(t_l) + \left(1 - \frac{k-1}{N}\right)^2 P_{k-1}(t_l).$$

Generating functions are frequently used to find the equilibrium distribution of the state variable in the Ehrenfest model, and the evolution of the expected values $\boldsymbol{E}(n(t))$, $\boldsymbol{E}(n^2(t))$, etc. [5]. In particular, for the initial condition $n(t_0) = N$ the formulae derived by Kac [5] yield

$$\boldsymbol{E}(n(t_m)) = \frac{N}{2}\left[1 + \left(1 - \frac{2}{N}\right)^m\right] \tag{2}$$

and

$$\boldsymbol{E}(n^2(t_m)) = \frac{N(N+1)}{4} + \frac{N^2}{2}\left(1-\frac{2}{N}\right)^m + \frac{N(N-1)}{4}\left(1-\frac{4}{N}\right)^m .$$

Despite the relative simplicity of the method, it is difficult to find a manageable expression for nonequilibrium distribution of the state variable $n(t)$ using generating functions.

The analytical structure of the Ehrenfest model is closely related to that of a chain of noninteracting quantum spins [4]. It turns out that the analytical structure of the Bernoulli–Laplace model is related to that of the mean-field hard-core lattice gas [8, 9] and of the Lieb–Mattis (anti)ferromagnet [10]. Therefore, technically, the Bernoulli–Laplace models is significantly more complicated than the Ehrenfest model, although, as we will see, the main features of the two models are very similar.

Indeed, Figs. 1 and 2 show evolutions of the Ehrenfest and Bernoulli–Laplace models generated by a computer using pseudo-random numbers, together with the evolution of $\boldsymbol{E}(n(t))$ given by Eq. (2). From the pictures one can guess

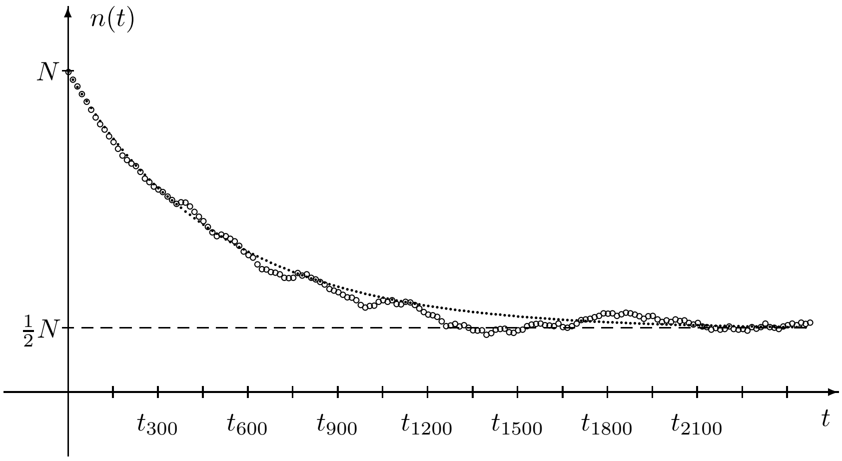

Fig. 1. Evolution of the state variable $n(t)$ (circles) in the discrete-time Ehrenfest model for $N = 1000$. The dots show the evolution of the expected value $\boldsymbol{E}n(t)$, see Eq. (2)

that expected values of $n(t)$ within the Ehrenfest and Bernoulli–Laplace models coincide. In Sec. 3 we will confirm the coincidence of the expected values by a direct calculation. The only difference between the paths in Figs. 1 and 2 is noticeably smaller fluctuations around the expected value in the Bernoulli–Laplace model.

In this paper, we actually look at a continuous-time version of the Ehrenfest model, where the balls are no longer restricted to jump at regular time intervals. Note that setting $\triangle t \equiv t_{l+1} - t_l = (\lambda N)^{-1}$ and dividing master equation (1) by $(\lambda N)^{-1}$ we obtain

$$\frac{P_k(t_{l+1}) - P_k(t_l)}{\triangle t} = \lambda(k+1)P_{k+1}(t_l) + \lambda(N-k+1)P_{k-1}(t_l) - \lambda N P_k(t_l).$$

Therefore, it is reasonable to define the master equation for the continuous-time Ehrenfest model as follows:

$$\frac{1}{\lambda}\frac{dP_k(t)}{dt} = (k+1)P_{k+1}(t) + (N-k+1)P_{k-1}(t) - NP_k(t). \tag{3}$$

The analytic solution of the last equation is well known in the theory of continuous-time Markov processes, see, e.g., Eq. (7.26) in Ch. XVII of the book [11]. Nevertheless, in Sec. 2 we rederive the solution of Eq. (3) by a probabilistic argument.

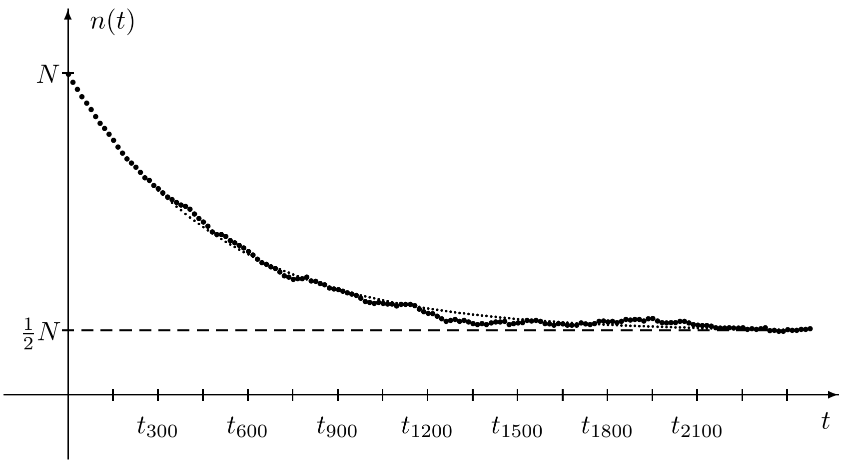

Fig. 2. Evolution of the state variable $n(t)$ (discs) in the Bernoulli–Laplace model for $N = 1000$. To isolate structural differences between the two models, the same sequence of «underlying» pseudo-random numbers as in Fig. 1 was used in the simulations. The dots show the evolution of the expected value $\boldsymbol{E}n(t)$, see Eq. (2)

To find a microscopic model which evolution is described by master Eq. (3), we consider a system of N independent balls which jump from urn A to urn B and back driven by a Poisson process with parameter λ. We stress that each of the N balls is driven by an independent Poisson process. In the initial state all the balls occupy urn A. The «macroscopic» state of the system of N balls at time t is still the number of balls, $n(t)$, in urn A.

A short calculation shows that the master equation for the distribution of the state variable $n(t)$ for the system of independent balls coincides with Eq. (3). On the other hand, this system is simple enough to allow a direct calculation of the distribution $P_k(t) \equiv P[n(t) = k]$ of the state variable.

The rest of the paper is organized as follows. In Sec. 2 we derive and analyze the nonequilibrium distribution for the state variable $n(t)$ within the continuous-time version of the Ehrenfest model. In Sec. 3 we derive and analyze the distributions for the state variable $n(t)$ within the Bernoulli–Laplace model. In Sec. 4 we present numerical results and discuss the overall picture.

2. THE EHRENFEST MODEL

To find the distribution $P_k(t)$ of the state variable $n(t)$ within the continuous-time Ehrenfest model, let us call «Success» («Success(t)» would be more precise at the expense of being more cumbersome) the event «A given ball is in urn A at time t». Note that we can treat the N independent balls as a sequence of N Bernoulli trials with success probability $p = \Pr[\text{«Success»}]$. Then the probability of having exactly k balls at time t in urn A is given by the binomial distribution (the probability of exactly k successes in N trials)

$$P_k(t) = \left(\begin{array}{c} N \\ k \end{array} \right) p^k (1-p)^{N-k}.$$

To find the success probability p, recall that the number of jumps, J, made by a single ball over the time interval $[t_0, t]$ is given by the Poisson distribution

$$\Pr[J = m] = \frac{[\lambda(t-t_0)]^m}{m!} \exp[-\lambda(t-t_0)].$$

Since all balls are in urn A at time t_0, a given ball is in urn A at time t if it made an even number of jumps over the interval $[t_0, t]$. Therefore the success probability is given by

$$p = \sum_{j=0}^{\infty} \frac{[\lambda(t-t_0)]^{2j}}{(2j)!} \exp[-\lambda(t-t_0)] = \frac{1+\exp[-2\lambda(t-t_0)]}{2}.$$

On substitution of the success probability in the binomial distribution, we obtain

$$P_k(t) = \left(\begin{array}{c} N \\ k \end{array} \right) \left(\frac{1+\exp[-2\lambda(t-t_0)]}{2} \right)^k \left(\frac{1-\exp[-2\lambda(t-t_0)]}{2} \right)^{N-k}. \quad (4)$$

Substitution in Eq. (3) shows that $P_k(t)$ is a solution of the master equation for the continuous-time Ehrenfest model.

The reason for simplicity of the distribution of the state variable $n(t)$ is not just the continuity of time, but rather the independence of the balls $1, 2, \ldots, N$. Indeed, it is possible to obtain a similar result for a system of independent balls restricted to jump at regular time intervals only (like in the original Ehrenfest model).

Now we pass to our main task of finding the asymptotic behaviour of the distribution $P_k(t) \equiv \Pr[n(t) = k]$. First of all, note that

$$\lim_{t\to\infty} P_k(t) = \frac{1}{2^N} \begin{pmatrix} N \\ k \end{pmatrix},$$

which is the stationary (binomial) distribution found by Kac [5].

If N is large (as it is usually assumed to be) a binomial distribution with parameters N and p is virtually identical to a normal distribution. Therefore, we can replace all our binomial random variables by, technically much more convenient, normal random variables as follows:

$$n(t) \approx \boldsymbol{E}[n(t)] + Z(t)\sqrt{\mathrm{Var}[n(t)]},$$

where $Z(t)$ is a standard normal random variable. Note that the above formula is asymptotically exact in the limit of large N. For the state variable in the continuous-time Ehrenfest model we obtain

$$n(t) \approx \frac{1 + \exp[-2\lambda(t - t_0)]}{2} N + Z(t)\sqrt{\frac{1 - \exp[-4\lambda(t - t_0)]}{4} N}.$$

The asymptotically normal distribution of the state variable $n(t)$ was obtained as a solution of the Fokker–Planck equation in the recent paper [3].

From the above formula we see that the evolution of $n(t)$ contains two components: a deterministic evolution of the expected value and a stochastic «jitter» of order $\sqrt{N}$. As usual, it is convenient to separate the *kinetic* (initial) and *diffusion* (quasi-equilibrium) stages of evolution. In the kinetic stage, changes in expected values dominate over stochastic «jitter». In the diffusion stage, the magnitude of change in expected values, over a unit time-interval, and the magnitude of stochastic «jitter» are of the same order, $\sqrt{N}$ in our case.

In the kinetic stage of the Ehrenfest model both the mean value of the state variable and the variance of stochastic «jitter» race toward their equilibrium values, $\frac{1}{2}N$ and $\frac{1}{4}N$, respectively. To get to the diffusion stage of evolution, we have to shift time from t_0 (the beginning of evolution) by an amount $T(N)$ such that

$$\mathrm{e}^{-2\lambda T(N)} N = c\sqrt{N}.$$

The last equation makes sure that the rate of change in expected value is comparable with the magnitude of stochastic «jitter». For instance, we can choose $T(N) = \dfrac{1}{4\lambda}\log N$.

Thus, to pass to the diffusion stage we have to introduce a new (diffusion) time-zone τ, via $t = t_0 + \dfrac{1}{4\lambda}\log N + \tau$. The evolution of the state variable in the diffusion time-zone is given by

$$n(t_0 + \frac{1}{4\lambda}\log N + \tau) \approx \frac{1}{2}N + \mathrm{e}^{-2\lambda\tau}\frac{1}{2}\sqrt{N} + Z(t)\frac{1}{2}\sqrt{N}.$$

Note that, the mean value of the state variable still drifts towards the equilibrium value in the diffusion time-zone, while the variance of stochastic «jitter» has already reached its equilibrium value. The (nonequilibrium) distribution of the deviation of state variable from its equilibrium value (rescaled, to obtain a proper distribution in the large-N limit),

$$\nu(\tau) \equiv \frac{n(t_0 + \frac{1}{4\lambda}\ \log\ N + \tau) - \frac{1}{2}N}{\sqrt{N}},$$

is given by

$$f_\tau(x) = \sqrt{\frac{2}{\pi}}\exp\left[-2\left(x - \frac{1}{2}\,\mathrm{e}^{-2\lambda\tau}\right)^2\right]. \tag{5}$$

The last formula shows explicitly how the distribution of the (rescaled) state variable $\nu(\tau)$ converges to equilibrium at the diffusion stage of evolution.

3. THE BERNOULLI–LAPLACE MODEL

3.1. The Expected Value and the Variance of $n(t)$. The simplest way to find the moments of $n(t)$ within the Bernoulli–Laplace model is to use the tower property of conditional expected values. Of course, the same approach can be used for calculation of moments within the Ehrenfest model. This way it is possible to obtain exact expressions for $f(t) \equiv \boldsymbol{E}n(t)$ and $s(t) \equiv \boldsymbol{E}n^2(t)$ (within the Ehrenfest model) with less effort spent than even if we use the approach proposed by Hess [4].

Within the Bernoulli–Laplace model with N black and N red balls we have the following relationship for the state variable $n(t)$ at two consecutive time instants:

$$n(t_{l+1}) = n(t_l) + X(t_{l+1}),$$

where the conditional distribution of $X(t_{l+1})$ given $n(t_l)$ is given by

$$\Pr[X(t_{l+1}) = 1|n(t_l)] = \left(1 - \frac{n(t_l)}{N}\right)^2,$$
$$\Pr[X(t_{l+1}) = 0|n(t_l)] = 2\frac{n(t_l)}{N}\left(1 - \frac{n(t_l)}{N}\right),$$

and

$$\Pr[X(t_{l+1}) = -1|n(t_l)] = \frac{n^2(t_l)}{N^2}.$$

The tower property yields

$$f(t_{l+1}) = \boldsymbol{E}[n(t_l)] + \boldsymbol{E}\boldsymbol{E}[X(t_{l+1})|n(t_l)] = f(t_l) + \boldsymbol{E}\left(1 - \frac{2n(t_l)}{N}\right).$$

Hence, for the expected value of $n(t_l)$, we have the recurrent relationship

$$f(t_{l+1}) = 1 + \left(1 - \frac{2}{N}\right) f(t_l)$$

with the initial condition $f(t_0) = N$. The solution of the recurrent relationship is given by

$$f(t_k) = \frac{N}{2}\left[1 + \left(1 - \frac{2}{N}\right)^k\right]. \tag{6}$$

Therefore, the first moments of the state variables within the Bernoulli–Laplace model and the Ehrenfest model (see Eq. (2)) coincide.

Calculation of the second moment is almost as simple as the calculation of the first moment. We have

$$n^2(t_{l+1}) = n^2(t_l) + 2n(t_l)X(t_{l+1}) + X^2(t_{l+1}).$$

Application of the tower property yields the recurrent relationship

$$s(t_{l+1}) = \left(1 - \frac{4}{N} + \frac{2}{N^2}\right) s(t_l) + 1 + 2f(t_l)\left(1 - \frac{1}{N}\right)$$

with the initial condition $s(t_0) = N^2$. The solution of the recurrent relationship is given by

$$s(t_k) = \frac{N^2(N-1)}{2(2N-1)}\left(1 - \frac{4}{N} + \frac{2}{N^2}\right)^k + \frac{N^3}{2(2N-1)} + \frac{N^2}{2}\left(1 - \frac{2}{N}\right)^k.$$

Therefore, the variance of the state variable $n(t)$ is given by

$$\mathrm{Var}[n(t_k)] = \\ = \frac{N^2}{4(2N-1)} - \frac{N^2}{4}\left(1-\frac{2}{N}\right)^{2k} + \frac{N^2(N-1)}{2(2N-1)}\left(1-\frac{4}{N}+\frac{2}{N^2}\right)^{k}. \quad (7)$$

The calculation of a few higher moments can be performed using the same method, but the amount of labour it requires increases rapidly. We are not going to do that, instead our aim now is to find a formula for the distribution of the state variable $n(t)$ similar to that found in the papers [4, 5] within the Ehrenfest model.

3.2. The Spectral Properties of the Transition Operator. First, we reformulate the Bernoulli–Laplace model as follows. We do not switch the balls between boxes A and B anymore, instead we place labels on the balls. If a label covers a particular ball (red or black), then we assume that the ball occupies urn A. If a ball is not covered by a label, then it occupies urn B. In the initial state (at time t_0) all labels cover black balls. At regular time-instants we pick a label at random (each of the N labels can be picked with probability N^{-1}), then we pick an uncovered ball at random (again, each of the N uncovered balls can be picked with probability N^{-1}), and shift the label on the picked ball.

It is clear now that the Bernoulli–Laplace model is equivalent to a system of N hard-core labels jumping randomly (one label at a time) on a *complete* graph of $2N$ cites. The hard-core interaction prevents more than one label occupying any particular cite of the graph. Since the graph is complete, any label can jump to any unoccupied cite.

A deterministic state of the system of N labels is specified by a vector $|\varphi\rangle$, where $\varphi \subset \Lambda \equiv \{1, 2, \ldots, 2N\}$. Recall that the black and red balls are numbered $1, 2, \ldots, 2N$. Hence, the set φ contains the numbers of all balls covered by the labels. The initial state of the system (all black balls covered by the labels) is $|\psi_0\rangle \equiv |\{1, 2, \ldots, N\}\rangle$. Linear combinations like $|a\rangle = \alpha|\varphi\rangle + \beta|\psi\rangle$, where $\varphi \subset \Lambda$ and $\psi \subset \Lambda$, are also «possible» states of our labels. If $\alpha \geqslant 0$, $\beta \geqslant 0$, and $\alpha + \beta = 1$, then the state $|a\rangle$ has the obvious probabilistic interpretation. Otherwise, the mixture $|a\rangle$ should appear only in intermediate calculations. We will make a reasonable effort to use Latin (as opposed to Greek) letters to denote linear combinations of deterministic states.

Let us define the scalar product for any two deterministic states $|\varphi\rangle$ and $|\psi\rangle$, where $\varphi \subset \Lambda$ and $\psi \subset \Lambda$, as follows:

$$\langle\psi|\varphi\rangle = \begin{cases} 1, & \text{if } \psi = \varphi, \\ 0, & \text{if } \psi \neq \varphi. \end{cases}$$

Then, the $\binom{2N}{N}$ vectors $|\varphi\rangle$ containing exactly N labels, $|\varphi| = N$, make up an orthonormal basis in the $\begin{pmatrix} 2N \\ N \end{pmatrix}$-dimensional space of their linear combinations.

To find the probability of n-step transition from the initial state to an arbitrary state $|\psi_1\rangle$, we adapt the method used by Penrose [9] for investigation of Bose–Einstein condensation in a system of hard-core particles on a complete graph.

Consider a linear operator $\widehat{T}$ (the transition operator) defined by the relationship

$$\widehat{T}|\varphi\rangle = \sum_{x\in\Lambda\setminus\varphi} \sum_{y\in\varphi} |(\varphi \setminus y) \cup x\rangle.$$

In terms of the operator $\widehat{T}$ the probability of n-step transition from a state $|\varphi\rangle$ to a state $|\psi\rangle$ is given by

$$\langle\psi|(N^{-2}\widehat{T})^n|\varphi\rangle.$$

Hence, we can, in principle, find the transition probabilities once we know the eigenstates and eigenvalues of the operator $\widehat{T}$.

In order to find the eigenstates and eigenvalues, let us define a linear operator $\hat{A}$ as follows:

$$\hat{A}|\varphi\rangle = \sum_{y\in\varphi} |\varphi \setminus y\rangle.$$

The linear operator $\hat{A}^\dagger$ defined by

$$\hat{A}^\dagger|\varphi\rangle = \sum_{x\in\Lambda\setminus\varphi} |\varphi \cup x\rangle,$$

is adjoint to $\hat{A}$, that is, $\langle\psi|\hat{A}\varphi\rangle = \langle\hat{A}^\dagger\psi|\varphi\rangle$, for any two vectors $|\psi\rangle$ and $|\varphi\rangle$.

From the definitions of the operators $\hat{A}$ and $\hat{A}^\dagger$ we obtain

$$\hat{A}^\dagger\hat{A}|\varphi\rangle = \hat{A}^\dagger \sum_{y\in\varphi} |\varphi \setminus y\rangle = \sum_{y\in\varphi} \sum_{x\in\Lambda\setminus\varphi} |(\varphi \setminus y) \cup x\rangle + \sum_{y\in\varphi} |\varphi\rangle.$$

Therefore $\hat{A}^\dagger\hat{A} = \widehat{T} + \widehat{V}$, where $\widehat{V}$ is the linear operator of the number of labels, $\widehat{V}|\varphi\rangle = |\varphi||\varphi\rangle$. Analogously $\hat{A}\hat{A}^\dagger = \widehat{T} + 2N\widehat{E} - \widehat{V}$, where $\widehat{E}$ is the identity operator. Hence, the commutator of $\hat{A}$ and $\hat{A}^\dagger$ is given by $\hat{A}\hat{A}^\dagger - \hat{A}^\dagger\hat{A} \equiv [\hat{A}, \hat{A}^\dagger] = 2(N\widehat{E} - \widehat{V})$. The commutators of $\hat{A}^\dagger$ with $\widehat{V}$ and $\widehat{T}$ are also not difficult to find, they are given by $[\widehat{V}, \hat{A}^\dagger] = \hat{A}^\dagger$ and $[\widehat{T}, \hat{A}^\dagger] = (2N-1)\hat{A}^\dagger - 2\hat{A}^\dagger\widehat{V}$.

The key to the «diagonalization» of the operator $\widehat{T}$ is the existence of an operator $\hat{L} = \widehat{T} + a\widehat{V} + b\widehat{V}^2$ commuting with $\hat{A}^\dagger$, see [9]. We have

$$\hat{L}\hat{A}^\dagger - \hat{A}^\dagger\hat{L} = (2N-1)\hat{A}^\dagger - 2\hat{A}^\dagger\widehat{V} + a\hat{A}^\dagger + b\left(2\hat{A}^\dagger\widehat{V} + \hat{A}^\dagger\right).$$

Therefore, if we set $a = -2N$ and $b = 1$ we obtain $\hat{L}\hat{A}^\dagger - \hat{A}^\dagger\hat{L} = 0$.

Let us now consider n-label subspaces

$$S_n \equiv \{|a\rangle : \widehat{V}|a\rangle = n|a\rangle, \hat{A}|a\rangle = 0\}.$$

For instance, the subspace S_0 contains only one normalized vector: $|\emptyset\rangle$. This vector describes $2N$ balls not covered by labels. The subspace S_1 contains linear combinations of one-label deterministic states

$$|a\rangle = \sum_{x\in\Lambda} \alpha_x |\{x\}\rangle, \quad \text{such that} \quad \hat{A}\sum_{x\in\Lambda} \alpha_x |\{x\}\rangle = \sum_{x\in\Lambda} \alpha_x |\emptyset\rangle = 0. \tag{8}$$

Solutions of the equation $\sum_{x\in\Lambda} \alpha_x = 0$ are all vectors $\{\alpha_x\}_{x\in\Lambda}$ orthogonal to the $2N$-component vector $(1, 1, \ldots, 1)$. Hence, there are exactly $2N-1$ linearly independent solutions of that equation. Thus, S_1 is a $(2N-1)$-dimensional space. In general, see [9], the dimension of the subspace S_n is

$$\begin{pmatrix} 2N \\ n \end{pmatrix} - \begin{pmatrix} 2N \\ n-1 \end{pmatrix}, \quad n = 1, 2, \ldots, N.$$

The importance of the subspaces S_n is due to the following lemma, which can be proven by almost verbatim repetition of the corresponding proof from [9].

Lemma. Let $|a\rangle \in S_l$, where $l = 0, 1, 2, \ldots, N$. Then $(\hat{A}^\dagger)^{N-l}|a\rangle$ is an eigenstate of the operator $\widehat{T}$ with the eigenvalue $\lambda_l = N^2 - (2N+1)l + l^2$.

■

It is not difficult to count the total number of linearly independent eigenstates corresponding to the eigenvalues $\lambda_0, \lambda_1, \ldots, \lambda_N$ described in the Lemma. It is given by

$$\dim S_0 + \dim S_1 + \ldots + \dim S_N = 1 + \sum_{n=1}^{N} \left[\begin{pmatrix} 2N \\ n \end{pmatrix} - \begin{pmatrix} 2N \\ n-1 \end{pmatrix} \right] = \begin{pmatrix} 2N \\ N \end{pmatrix}.$$

Therefore, the eigenstates described in the Lemma make up a complete set of N-label eigenstates of the transition operator $\widehat{T}$.

It is always easier to deal with normalized eigenstates, such that $\langle b|b\rangle = 1$. It is shown in Appendix A that the states

$$|a_l\rangle \equiv \frac{1}{\sqrt{[2(N-l)]!}} (\hat{A}^\dagger)^{N-l}|a\rangle, \tag{9}$$

are normalized, if $|a\rangle$ is a normalized vector from S_l.

3.3. The n-Step Transition Probabilities. The core of the previous paragraph is a mere adaptation to the problem under consideration of the technique used by Penrose in [9]. However, to find manageable expressions for the transition probabilities, we have to go beyond the scope of that paper and investigate the structure of the eigenstates of the transition operator.

The distribution of the state variable $n(t_m)$ is given by

$$\Pr[n(t_m) = k] = \sum_{\varphi:|\varphi\cap\{1,2,\ldots,N\}|=k} \langle\varphi|(N^{-2}\widehat{T})^m|\psi_0\rangle. \tag{10}$$

In terms of the normalized eigenstates, $|a_l^{(j)}\rangle$, and the eigenvalues of the operator $\widehat{T}$ the m-step transition probabilities are given by

$$\langle\psi_1|(N^{-2}\widehat{T})^m|\psi_0\rangle =$$
$$= \sum_{l=0}^{N} \left(1 - \frac{(2N+1)l}{N^2} + \frac{l^2}{N^2}\right)^m \sum_{j=1}^{D(l,N)} \langle\psi_1|a_l^{(j)}\rangle\langle a_l^{(j)}|\psi_0\rangle, \tag{11}$$

where $D(l,N)$ is the degeneracy of the lth eigenvalue. The difficulty of calculating the scalar products $\langle\psi_1|a_l^{(j)}\rangle$ and $\langle a_l^{(j)}|\psi_0\rangle$ increases rapidly with l.

The eigenstate $|a_0\rangle$ corresponding to the maximal (nondegenerate) eigenvalue $\lambda_0 = N^2$ has a simple structure. We have

$$(\hat{A}^\dagger)^N|\emptyset\rangle = (\hat{A}^\dagger)^{N-1}\sum_{x\in\Lambda}|\{x\}\rangle = (\hat{A}^\dagger)^{N-2}\sum_{x\in\Lambda}\sum_{y\in(\Lambda\backslash x)}|\{x,y\}\rangle =$$
$$= 2(\hat{A}^\dagger)^{N-2}\sum_{\{x,y\}\subset\Lambda}|\{x,y\}\rangle = \ldots = N!\sum_{\varphi\subset\Lambda:|\varphi|=N}|\varphi\rangle.$$

Using Eq. (9) we obtain the normalized eigenstate corresponding to the maximal eigenvalue

$$|a_0\rangle = \frac{N!}{\sqrt{(2N)!}}\sum_{\varphi\subset\Lambda:|\varphi|=N}|\varphi\rangle.$$

Therefore, if ψ_1 and ψ_0 are arbitrary deterministic N-label states, then

$$\langle\psi_1|a_0\rangle = \langle a_0|\psi_0\rangle = \frac{N!}{\sqrt{(2N)!}}.$$

Taking the limit $m \to \infty$, we obtain the equilibrium distribution

$$\lim_{m\to\infty}\langle\psi_1|(N^{-2}\widehat{T})^m|\psi_0\rangle = \langle\psi_1|a_0\rangle\langle a_0|\psi_1\rangle = \frac{N!N!}{(2N)!}.$$

That is, in equilibrium all (deterministic) N-label states are equally likely.

There are $\binom{N}{k}$ ways of placing k labels over N black balls, and $\binom{N}{N-k}$ ways of placing the remaining $N-k$ labels over N red balls. Hence, there are

$\binom{N}{k}\binom{N}{N-k}$ subsets φ such that $|\varphi \cap \{1, 2, \ldots, N\}| = k$. Therefore, cf. Eq. (10), the (equilibrium) probability of having exactly k black balls in urn A is given by the hypergeometric distribution (this well-known result can be found in the book by Feller [11])

$$\Pr[n(t_\infty) = k] = \Pr[\text{«Exactly } k \text{ black balls in urn A»}] = \frac{\binom{N}{k}^2}{\binom{2N}{N}}.$$

The structure of the eigenstates $|a_1^{(l)}\rangle$, $l = 1, 2, \ldots, 2N-1$, corresponding to the $(2N-1)$-times degenerate eigenvalue $\lambda_1 = N^2 - 2N$ is slightly more complicated.

$$\begin{aligned}(A^\dagger)^{N-1}|a\rangle \equiv (A^\dagger)^{N-1}\sum_{x\in\Lambda}\alpha_x|\{x\}\rangle = (A^\dagger)^{N-2}\sum_{x\in\Lambda}\alpha_x \sum_{y\in(\Lambda\backslash x)}|\{x,y\}\rangle =\\ = (A^\dagger)^{N-3}\sum_{x\in\Lambda}\alpha_x \sum_{y\in(\Lambda\backslash x)}\sum_{z\in(\Lambda\backslash\{x,y\})}|\{x,y,z\}\rangle =\\ = 2(A^\dagger)^{N-3}\sum_{x\in\Lambda}\alpha_x \sum_{\{y,z\}\subset(\Lambda\backslash x)}|\{x,y,z\}\rangle = \ldots =\\ = (N-1)!\sum_{x\in\Lambda}\alpha_x \sum_{\phi\subset\Lambda: x\in\phi, |\phi|=N}|\phi\rangle.\end{aligned} \quad (12)$$

Therefore, the orthonormal eigenstates $|a_1^{(l)}\rangle$ are given by

$$|a_1^{(l)}\rangle = \frac{(N-1)!}{\sqrt{(2N-2)!}}\sum_{x\in\Lambda}\alpha_x^{(l)} \sum_{\phi\subset\Lambda: x\in\phi, |\phi|=N}|\phi\rangle,$$

where $\{\alpha_x^{(l)}\}_{x\in\Lambda}$, $l = 1, 2, \ldots, 2N-1$ are orthonormal vectors orthogonal to $(1, 1, \ldots, 1)$.

If $|\psi_0\rangle$ is an arbitrary N-label state, then

$$\langle a_1^{(l)}|\psi_0\rangle = \frac{(N-1)!}{\sqrt{(2N-2)!}}\sum_{x\in\psi_0}\alpha_x^{(l)}.$$

To calculate the remaining sum over x, let us choose the vectors $\{\alpha_x^{(l)}\}_{x\in\Lambda}$ as follows. There are $N-1$ orthonormal vectors $\{\alpha_x^{(l)}\}_{x\in\Lambda}$ orthogonal to $(1, 1, \ldots, 1)$ such that $\alpha_x^{(l)} = 0$, if $x \in \psi_0$. There are $N-1$ orthonormal vectors $\{\alpha_x^{(l)}\}_{x\in\Lambda}$ orthogonal to $(1, 1, \ldots, 1)$ such that $\alpha_x^{(l)} = 0$, if $x \notin \psi_0$. The last required orthonormal vector $\{\alpha_x^{(1)}\}_{x\in\Lambda}$ is such that $\alpha_x^{(1)} = (2N)^{-1/2}$ if $x \in \psi_0$, and $\alpha_x^{(1)} = -(2N)^{-1/2}$ if $x \notin \psi_0$. For this choice of the orthonormal vectors $\{\alpha_x^{(l)}\}_{x\in\Lambda}$ we obtain

$$\sum_{x\in\psi} \alpha_x^{(l)} = \begin{cases} \sqrt{\dfrac{N}{2}}, & \text{if } l = 1; \\ 0, & \text{if } l = 2, 3, \ldots, 2N-1. \end{cases}$$

The total contribution to the m-step transition probability (11) from the $2N-1$ eigenstates corresponding to the eigenvalue $\lambda_1 = N^2 - 2N$ is given by

$$\Lambda_1 \equiv \left(\frac{\lambda_1}{N^2}\right)^m \sum_{j=1}^{2N-1} \langle\psi_1|a_1^{(j)}\rangle\langle a_1^{(j)}|\psi_0\rangle =$$

$$= \left(1 - \frac{2}{N}\right)^m \frac{(N-1)!}{\sqrt{(2N-2)!}} \sqrt{\frac{N}{2}} \langle\psi_1|a_1^{(1)}\rangle.$$

If $|\psi_1 \cap \psi_0| = k$, then

$$\langle\psi_1|a_1^{(1)}\rangle = \frac{(N-1)!}{\sqrt{(2N-2)!}} \sum_{x\in\psi_1} \alpha_x^{(1)} =$$

$$= \frac{(N-1)!}{\sqrt{(2N-2)!}} \left(\sum_{x\in\psi_1\cap\psi_0} \alpha_x^{(1)} + \sum_{x\in\psi_1\setminus\psi_0} \alpha_x^{(1)} \right)$$

$$= \frac{(N-1)!}{\sqrt{(2N-2)!}} \left(\frac{k}{\sqrt{2N}} - \frac{N-k}{\sqrt{2N}} \right).$$

Hence

$$\Lambda_1 = \left(1 - \frac{2}{N}\right)^m \frac{(N-1)!(N-1)!}{(2N-2)!} \left(k - \frac{N}{2}\right),$$

where k is the number of labels covering black balls in the state $\langle\psi_1|$.

Let us now find the contribution to the m-step transition probability (11) from the eigenstates corresponding to the eigenvalue $\lambda_2 = N^2 - 4N + 2$. The eigenstates are given by

$$(\hat{A}^\dagger)^{N-2} \sum_{\{x,y\}\subset\Lambda} \alpha_{\{x,y\}} |\{x,y\}\rangle,$$

where the vector $\boldsymbol{\alpha} \equiv \{\alpha_{\{x,y\}}\}$ is a solution of the equation

$$\hat{A} \sum_{\{x,y\}\subset\Lambda} \alpha_{\{x,y\}} |\{x,y\}\rangle = 0.$$

The last equation is equivalent to the following system of $2N$ equations:

$$\sum_{y\in(\Lambda\setminus x)} \alpha_{\{x,y\}} = 0, \quad \text{for all } x \in \Lambda. \tag{13}$$

The dimension (the number of components) of the vector $\boldsymbol{\alpha}$ is $\begin{pmatrix} 2N \\ 2 \end{pmatrix}$. In Eq. (13) we are looking for vectors $\boldsymbol{\alpha}$ orthogonal to $2N$ vectors

$$\boldsymbol{v}^{(z)} \equiv \{v^{(z)}_{\{x,y\}}\},\ z \in \Lambda,\ \text{such that } v^{(z)}_{\{x,y\}} = \begin{cases} 1, & \text{if } z \in \{x,y\}, \\ 0, & \text{if } z \notin \{x,y\}. \end{cases}$$

The $2N$ vectors $\boldsymbol{v}^{(z)}$, $z \in \Lambda$, are linearly independent, hence there are exactly $\begin{pmatrix} 2N \\ 2 \end{pmatrix} - 2N$ linearly independent solutions of the system (13).

Analogously to Eq. (12) we obtain

$$(\hat{A}^\dagger)^{N-2} \sum_{\{x,y\}\subset\Lambda} \alpha_{\{x,y\}} |\{x,y\}\rangle = (N-2)! \sum_{\{x,y\}\subset\Lambda} \alpha_{\{x,y\}} \sum_{\phi\subset\Lambda:\{x,y\}\subset\phi,|\phi|=N} |\phi\rangle.$$

Hence, the orthonormal eigenstates $|a_2^{(l)}\rangle$ are given by

$$|a_2^{(l)}\rangle = \frac{(N-2)!}{\sqrt{(2N-4)!}} \sum_{\{x,y\}\subset\Lambda} \alpha^{(l)}_{\{x,y\}} \sum_{\phi\subset\Lambda:\{x,y\}\subset\phi,|\phi|=N} |\phi\rangle,$$

where $\boldsymbol{\alpha}^{(l)} \equiv \{\alpha^{(l)}_{\{x,y\}}\}$, $l = 1, 2, \ldots, \begin{pmatrix} 2N \\ 2 \end{pmatrix} - 2N$ are orthonormal solutions of the system (13).

The scalar products of an N-label state $|\psi\rangle$ with the eigenstates $\langle a_2^{(l)}|$ are given by

$$\langle a_2^{(l)}|\psi\rangle = \frac{(N-2)!}{\sqrt{(2N-4)!}} \sum_{\{x,y\}\subset\psi} \alpha^{(l)}_{\{x,y\}}. \tag{14}$$

To calculate the remaining sum, note that it is the scalar product of the vector $\boldsymbol{\alpha}^{(l)}$ with the vector

$$\boldsymbol{\tau} \equiv \{\tau_{\{x,y\}}\} \text{ such that } \tau_{\{x,y\}} = \begin{cases} 1, & \text{if } \{x,y\} \subset \psi, \\ 0, & \text{otherwise.} \end{cases}$$

Note also, that the system (13) has a solution $\boldsymbol{\alpha}^{(1)}$ such that

$$\alpha^{(1)}_{\{x,y\}} = \begin{cases} \dfrac{1}{\sqrt{(2N-1)(N-1)}}, & \text{if } \{x,y\} \subset \psi, \text{ or if } \{x,y\} \subset \Lambda \setminus \psi, \\ -\dfrac{1}{N}\sqrt{\dfrac{N-1}{2N-1}}, & \text{otherwise.} \end{cases}$$

The solutions $\boldsymbol{\alpha}^{(k)}$, with $k > 1$ are orthogonal to $\boldsymbol{\alpha}^{(1)}$. The normalized «constant» vector

$$\boldsymbol{u} \equiv \{u_{\{x,y\}}\}, \quad \text{such that} \quad u_{\{x,y\}} = \frac{1}{\sqrt{N(2N-1)}} \quad \text{for all} \quad \{x,y\} \subset \Lambda,$$

can be written as follows:

$$\boldsymbol{u} = \frac{1}{2\sqrt{N(2N-1)}} \sum_{z\in\Lambda} \boldsymbol{v}^{(z)}.$$

Hence, the «constant» vector $\boldsymbol{u}$ is orthogonal to all vectors $\boldsymbol{\alpha}^{(k)}$.

Finally, let us consider the vector $\boldsymbol{w} \equiv \{w_{\{x,y\}}\}$ such that

$$w_{\{x,y\}} = \frac{1}{\sqrt{N(N-1)}} \times \begin{cases} 1, & \text{if } \{x,y\} \subset \psi, \\ -1, & \text{if } \{x,y\} \subset \Lambda \setminus \psi, \\ 0, & \text{otherwise.} \end{cases}$$

The vector $\boldsymbol{w}$ can also be written as a linear combination of the vectors $\boldsymbol{v}^{(z)}$, $z \in \Lambda$. Namely

$$\boldsymbol{w} = \frac{1}{2\sqrt{N(N-1)}} \left(\sum_{z\in\psi} \boldsymbol{v}^{(z)} - \sum_{z\in(\Lambda\setminus\psi)} \boldsymbol{v}^{(z)} \right).$$

Hence, the vector $\boldsymbol{w}$ is also orthogonal to all vectors $\boldsymbol{\alpha}^{(k)}$.

We can write down the vector $\boldsymbol{\tau}$ as a linear combination of $\boldsymbol{\alpha}^{(1)}$, $\boldsymbol{u}$, and $\boldsymbol{w}$. Namely

$$\boldsymbol{\tau} = \frac{N}{2}\sqrt{\frac{N-1}{2N-1}}\,\boldsymbol{\alpha}^{(1)} + \frac{N-1}{2}\sqrt{\frac{N}{2N-1}}\,\boldsymbol{u} + \frac{1}{2}\sqrt{N(N-1)}\,\boldsymbol{w}.$$

Therefore, for our choice of the orthonormal vectors $\boldsymbol{\alpha}^{(k)}$ we obtain

$$\sum_{\{x,y\}\subset\psi} \alpha^{(k)}_{\{x,y\}} = \boldsymbol{\tau}\cdot\boldsymbol{\alpha}^{(k)} = \begin{cases} \dfrac{N}{2}\sqrt{\dfrac{N-1}{2N-1}}, & \text{if } k = 1; \\ 0, & \text{otherwise.} \end{cases}$$

The total contribution to the m-step transition probability (11) from the eigenstates $|a_2^{(j)}\rangle$ corresponding to the eigenvalue $\lambda_2 = N^2 - 4N + 2$ is given by

$$\Lambda_2 \equiv \left(\frac{\lambda_2}{N^2}\right)^m \sum_{j=1}^{N(2N-3)} \langle\psi_1|a_2^{(j)}\rangle\langle a_2^{(j)}|\psi_0\rangle =$$
$$= \left(\frac{\lambda_2}{N^2}\right)^m \frac{(N-2)!}{\sqrt{(2N-4)!}}\,\frac{N}{2}\sqrt{\frac{N-1}{2N-1}}\langle\psi_1|a_2^{(1)}\rangle.$$

If exactly k black balls are covered by the labels in the state $|\psi_1\rangle$, then there are $k(k-1)/2$ pairs $\{x,y\}$ in the set $\psi_1 \cap \psi_0$. For any pair $\{x,y\}$ like that we have $\alpha^{(1)}_{\{x,y\}} = [(2N-1)(N-1)]^{-1/2}$. There are $(N-k)(N-k-1)/2$ pairs $\{x,y\}$ in the set $\psi_1 \setminus \psi_0$. For any pair $\{x,y\}$ from $\psi_1 \setminus \psi_0$ we have the same value $\alpha^{(1)}_{\{x,y\}} = [(2N-1)(N-1)]^{-1/2}$. Finally, there are $k(N-k)$ pairs of integers $\{x,y\}$ such that one of the integers belongs to $\psi_1 \cap \psi_0$, and the other belongs to $\psi_1 \setminus \psi_0$. For any pair $\{x,y\}$ like that we have

$$\alpha^{(1)}_{\{x,y\}} = -\frac{1}{N}\sqrt{\frac{N-1}{2N-1}}.$$

Hence Eq. (14) yields

$$\langle \psi_1 | a_2^{(1)} \rangle = \frac{(N-2)!}{\sqrt{(2N-4)!}} \sum_{\{x,y\} \subset \psi_1} \alpha^{(1)}_{\{x,y\}} = \frac{k(k-1)}{2\sqrt{(2N-1)(N-1)}} + \\ + \frac{(N-k)(N-k-1)}{2\sqrt{(2N-1)(N-1)}} - \frac{k(N-k)}{N}\sqrt{\frac{N-1}{2N-1}},$$

and after some algebra we obtain

$$\Lambda_2 = \left(1 - \frac{4}{N} + \frac{2}{N^2}\right)^n \frac{(N-2)!(N-2)!}{(2N-4)!} \frac{1}{2} \left[\left(k - \frac{N}{2}\right)^2 - \frac{N^2}{4}\frac{1}{2N-1}\right].$$

It is possible to find the contributions to the n-step transition probability from the eigenstates corresponding to the eigenvalues $\lambda_3, \lambda_4, \ldots$ in a similar way. For instance, the total contribution from all eigenstates corresponding to the eigenvalue λ_3 is given by

$$\Lambda_3 \equiv \left(\frac{\lambda_3}{N^2}\right)^n \sum_{j=1}^{D(3,N)} \langle \psi_1 | a_3^{(j)} \rangle \langle a_3^{(j)} | \psi_0 \rangle = \left(1 - \frac{6}{N} + \frac{6}{N^2}\right)^n \times \\ \times \frac{(N-3)!(N-3)!}{(2N-6)!} \frac{1}{6} \left(k - \frac{N}{2}\right) \left[\left(k - \frac{N}{2}\right)^2 - \frac{N}{4}\frac{3N-4}{2N-3}\right].$$

The total contribution from all eigenstates corresponding to the eigenvalue λ_4 is given by

$$\Lambda_4 \equiv \left(\frac{\lambda_4}{N^2}\right)^n \sum_{j=1}^{D(4,N)} \langle\psi_1|a_4^{(j)}\rangle\langle a_4^{(j)}|\psi_0\rangle =$$

$$= \left(1-\frac{8}{N}+\frac{12}{N^2}\right)^n \frac{(N-4)!(N-4)!}{(2N-8)!}\times$$

$$\times \frac{1}{24}\left[\left(k-\frac{N}{2}\right)^4 - \frac{3N^2-8N+2}{2(2N-5)}\left(k-\frac{N}{2}\right)^2 + \frac{3}{16}\frac{N^2(N-2)^2}{(2N-3)(2N-5)}\right].$$

The probability of having exactly k black balls in urn A at time t_n is given by

$$\Pr[\text{«Exactly } k \text{ black balls in urn A»}] = \left(\begin{array}{c} N \\ k \end{array}\right)^2 \sum_{j=0}^{N} \Lambda_j,$$

where Λ_j is the total contribution from all eigenstates corresponding to the eigenvalue λ_j.

In the limit $N \to \infty$, setting $m = \dfrac{1}{4}N\ln N + \tau N$ (the diffusion time-zone) and $k = \dfrac{1}{2}N + x\sqrt{N}$, we obtain using the formulae for $\Lambda_1, \ldots, \Lambda_4$

$$\Pr[\text{«Exactly } k \text{ black balls in urn A at time } t_m\text{»}] \sim \frac{2}{\sqrt{\pi N}}\,\mathrm{e}^{-4x^2}\times$$
$$\times \left(1 + 4x\,\mathrm{e}^{-2\tau} + (8x^2-1)\,\mathrm{e}^{-4\tau} + \frac{4x}{3}(8x^2-3)\,\mathrm{e}^{-6\tau} + \right.$$
$$\left. + \frac{1}{6}(64x^4-48x^2+3)\,\mathrm{e}^{-8\tau} + \ldots\right).$$

The last formula corresponds to the first five terms of the Taylor expansion of the normal distribution density with the expected value $\dfrac{1}{2}\,\mathrm{e}^{-2\tau}$ and the variance $\dfrac{1}{8}$,

$$f_\tau(x) = \frac{2}{\sqrt{\pi}}\exp\left[-4\left(x-\frac{1}{2}\,\mathrm{e}^{-2\tau}\right)^2\right]. \tag{15}$$

Therefore, it seems reasonable to conclude that at the diffusion stage of evolution the distribution of the rescaled state variable,

$$\nu(\tau) \equiv \frac{n(t_m)-\frac{1}{2}N}{\sqrt{N}},$$

within the Bernoulli–Laplace model is asymptotically normal (as $N \to \infty$) with the expected value $\dfrac{1}{2}\,\mathrm{e}^{-2\tau}$ and the variance $\dfrac{1}{8}$.

4. DISCUSSION AND NUMERICAL RESULTS

The main results of this paper are explicit formulae for the distribution of the state variable at the diffusion stage of evolution, see Eqs. (5) and (15). Both, within the Ehrenfest model and within the Bernoulli–Laplace model, the distributions are asymptotically normal. To double check this conclusion for the case of the Bernoulli–Laplace model, we ran Monte-Carlo simulations for a model containing 1000 black and 1000 red balls. To get to the diffusion time-zone we made $1700 \approx \frac{1}{4}1000 \ln 1000$ dynamic time-steps. That yields one possible value of the state variable $n(t_{1700})$. To obtain enough data for a histogram we repeated that procedure $100,000$ times. The histogram is shown in Fig. 3, together with theoretical occupation numbers corresponding to the normal distribution with mean and variance given by Eqs. (6) and (7).

The histogram is very close to the theoretical occupation numbers — the size of the sample times the probability of a sample point falling into a particular bin. Of course, there is a discrepancy between the actual and theoretical occupation numbers. To check if the discrepancy statistically significant or not, we compute the χ^2 statistics

$$\chi^2 = \sum_{j=1}^{31} \frac{(A_j - E_j)^2}{E_j},$$

where A_j and E_j are the actual and theoretical occupation numbers, respectively, for the bin number j. The value of the χ^2 statistics for the histogram in Fig. 3 is 29.6 which is comfortably below 43.77 — the 95% critical level for the χ^2 distribution with 30 degrees of freedom. Hence, the χ^2 test does not give us any ground for suspecting a deviation from the normal distribution (15).

Of course, using Monte-Carlo simulations it is possible to investigate the distribution of the state variable $n(t)$ at the kinetic stage of the evolution. For $m = 346$ we have $\boldsymbol{E}n(t_{346}) \approx 750$, which is the half-way between the initial and equilibrium values of the state variable. We repeated the above Monte-Carlo simulations for 346 dynamic time-steps (instead of 1700). The χ^2 statistics in this case is slightly (but systematically) above the 95% critical level. Hence, in this case the χ^2 test detects a deviation from the normal distribution. The detected deviation, however, is only due to a finite-size correction. When we increased the number of balls to 3000 black and 3000 red balls (and the number of time-steps to 1039), we obtained an agreement between the actual and theoretical occupation numbers similar to that in Fig. 3. More precisely, we obtained the value 26.4 for the χ^2 statistics, which is again comfortably below the 95% critical level. Therefore (apparently), the distribution of the state variable $n(t)$ is asymptotically normal already during the kinetic stage of the evolution.

An interesting feature of the Ehrenfest and Bernoulli–Laplace models is the constant variance of the rescaled state variable $\nu(\tau)$ at the diffusion stage of

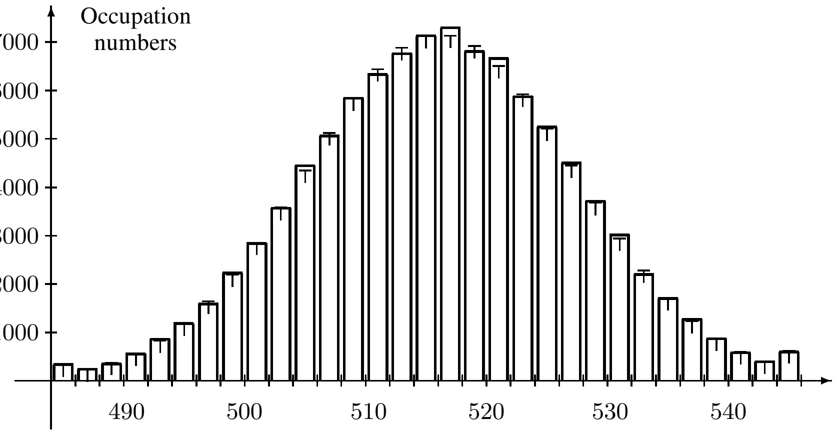

Fig. 3. A histogram for the state variable $n(t_{1700})$ within the Bernoulli–Laplace model with 1000 black and 1000 red balls. A sample of 100,000 values was used to build the histogram. The horizontal lines of T-shape symbols indicate the theoretical occupation numbers for the normal distribution with mean 516.63 and standard deviation 11.135, cf. Eqs. (6) and (7)

evolution. It is possible to interpret this feature as the quasi-stationarity of the corresponding random processes at the diffusion stage. More precisely, if we take the system of N Ehrenfest balls in equilibrium, and remove $O(\sqrt{N})$ balls from, say, urn A, we would push the system out of equilibrium. As a result, the mean value of the rescaled state variable $\nu(\tau)$ would begin changing with time. However, the type of distribution and the value of the second parameter (the variance) would stay the same as they were in equilibrium. Therefore, apparently, one can use the Ehrenfest and Bernoulli–Laplace models not only for investigation of convergence towards equilibrium, but also as microscopic models of quasi-stationary thermodynamic processes.

Finally, we would like to discuss the relationship between the continuous-time (easy to solve) and the discrete-time (not-so-easy to solve) Ehrenfest models. This relationship is similar to the relationship between the canonical and microcanonical ensembles. Indeed, typically, in a microcanonical ensemble we have a macroscopic observable, E, which value is fixed, $E = \varepsilon$. In the corresponding canonical ensemble the observable E is a random variable with only the expected value equal to ε, which often leads to a simplification of the model. If the two ensembles are equivalent, then the (harder to calculate) expected values within the microcanonical ensemble coincide with the corresponding (easier to calculate) expected values in the canonical ensemble. Likewise, in the discrete-time Ehrenfest model exactly one ball changes its urn at any of the time instances $t_1, t_2, t_3, \ldots$ In the corresponding continuous-time Ehrenfest model only the expected number of balls which change their urns over a time interval $(t_l, t_l + 1]$ is equal to 1, and the N-particle process «factorizes» into N independent one-particle processes.

The expected values of the state variable in the continuous-time and the discrete time versions of the Ehrenfest model, see Eqs. (2) and (4), coincide in the limit $N \to \infty$. Undoubtedly, the asymptotic distribution of the rescaled state variable $\nu(\tau)$ in the discrete-time Ehrenfest model is also normal. However, if we set $m = N(t - t_0)$, then the variance in the discrete-time model,

$$\mathrm{Var_d}[n(t_m)] = \frac{N}{4}\left[1 - \left(1 - \frac{4}{N}\right)^m\right] - \frac{N^2}{4}\left[\left(1 - \frac{2}{N}\right)^{2m} - \left(1 - \frac{4}{N}\right)^m\right],$$

has the following large-N asymptotics

$$\mathrm{Var_d}[n(t_m)] \sim \frac{N}{4}\left[1 - (1 + 4(t - t_0))\exp[-4(t - t_0)]\right].$$

The last asymptotics is smaller than the variance of the state variable in the continuous-time model

$$\mathrm{Var_c}[n(t)] = N\frac{1 - \exp[-4(t - t_0)]}{4}.$$

The reason for the greater variance in the continuous-time model is, of course, the extra disorder due to uncertainty in the number of jumps over a time-interval $(t_l, t_{l+1}]$.

APPENDIX
NORMALIZED EIGENSTATES OF THE OPERATOR $\widehat{T}$

The eigenstates of the operator $\widehat{T}$ are given by

$$|\Phi_k\rangle = \left(\hat{A}^\dagger\right)^{N-k} \sum_{\{x,y,\ldots,w\}\subset\Lambda} \alpha_{\{x,y,\ldots,w\}} |\{x, y, \ldots, w\}\rangle \equiv \left(\hat{A}^\dagger\right)^{N-k} |\chi_k\rangle,$$

where the summation runs over all subsets of Λ containing exactly k elements, the (real) coefficients $\alpha_{\{x,y,\ldots,w\}}$ satisfy the system of equations

$$\hat{A} \sum_{\{x,y,\ldots,w\}\subset\Lambda} \alpha_{\{x,y,\ldots,w\}} |\{x, y, \ldots, w\}\rangle = 0,$$

and the normalization condition

$$\sum_{\{x,y,\ldots,w\}\subset\Lambda} \alpha^2_{\{x,y,\ldots,w\}} = 1.$$

The square of the norm of the states $|\Phi_k\rangle$ is given by

$$\langle\Phi_k|\Phi_k\rangle = \langle\chi_k|\hat{A}^{N-k}\left(\hat{A}^\dagger\right)^{N-k}|\chi_k\rangle.$$

Our strategy is to find a relationship between $Q_l \equiv \langle\chi_k|\hat{A}^l\left(\hat{A}^\dagger\right)^l|\chi_k\rangle$ and Q_{l-1}. Then $\langle\Phi_k|\Phi_k\rangle$ can be found using the obtained relationship and the initial condition $Q_0 = 1$. We have

$$Q_l = \langle\chi_k|\hat{A}^{l-1}\hat{A}\hat{A}^\dagger\left(\hat{A}^\dagger\right)^{l-1}|\chi_k\rangle =$$
$$= \langle\chi_k|\hat{A}^{l-1}\left(\hat{L} + (2N-1)\widehat{V} - \widehat{V}^2 + 2N\widehat{E}\right)\left(\hat{A}^\dagger\right)^{l-1}|\chi_k\rangle.$$

Using the commutation relationships we obtain

$$\left(\hat{L} + (2N-1)\widehat{V} - \widehat{V}^2 + 2N\widehat{E}\right)\left(\hat{A}^\dagger\right)^{l-1} =$$
$$= \left(\hat{A}^\dagger\right)^{l-1}\left(\hat{L} + (2N-1)\widehat{V} + (2N-1)(l-1)\widehat{E} - \right.$$
$$\left. -\widehat{V}^2 - 2(l-1)\widehat{V} - (l-1)^2\widehat{E} + 2N\widehat{E}\right).$$

The identities $\widehat{L} = \hat{A}^\dagger\hat{A} - (2N+1)\widehat{V} + \widehat{V}^2$, $\hat{A}|\chi_k\rangle = 0$, and $\widehat{V}|\chi_k\rangle = k|\chi_k\rangle$, yield

$$Q_l = Q_{l-1}(2(N-k) + 1 - l)l.$$

Hence

$$\langle\Phi_k|\Phi_k\rangle = Q_{N-k} = Q_0\prod_{l=1}^{N-k}(2(N-k) + 1 - l)l = [2(N-k)]!.$$

Thus, the normalized eigenvectors of the operator $\widehat{T}$ are given by

$$\frac{1}{\sqrt{[2(N-k)]!}}\left(\hat{A}^\dagger\right)^{N-k}\sum_{\{x,y,\ldots,w\}\subset\Lambda}\alpha_{\{x,y,\ldots,w\}}|\{x,y,\ldots,w\}\rangle.$$

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