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RECURSIVE OPERATIONS FOR GENERATING VERTICES OF INTEGER PARTITION POLYTOPES

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Шлык В. А. E5-2008-18
Рекурсивные операции порождения вершин политопов
разбиений чисел

Следуя полиэдральному подходу к задаче о разбиениях чисел, мы рассматриваем множество разбиений каждого целого числа как политоп (выпуклый ограниченный многогранник). Тогда все разбиения каждого политопа полностью определяются набором его вершин как их выпуклые комбинации. В работе введены две комбинаторные операции на разбиениях и показано, что с их помощью все вершины каждого политопа разбиений рекурсивно порождаются из подмножества его опорных вершин, имеющего существенно меньшую мощность.

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Recursive Operations for Generating Vertices of Integer
Partition Polytopes

Following the polyhedral approach to integer partitions, we look at the set of partitions of any integer as a polytope (convex bounded polyhedron). So, all partitions of a given number are completely defined by the vertices of this polytope as their convex combinations. In this work, we introduce two combinatorial operations on partitions and show that with their help all vertices of any integer partition polytope can be recursively generated from a considerably smaller subset of its support vertices.

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INTRODUCTION

Integer partitions were a subject of many prominent mathematicians' constant interest over the centuries. The first results on the topic date back to the Middle Ages, though the first significant theorems on the topic belong to Euler [1]. The blow-up of research was caused by the works by Hardy, Littlewood, and Ramanujan at the beginning of the XX century [1, 3]. Partitions turned out to be the source of many new problems and techniques. In the last decades we have witnessed an outburst of new results in the field that is related to the Young diagram technique [2, 7]. Today integer partitions are of great importance in number theory, combinatorics, representation theory, mathematical physics, and statistical mechanics [7].

This work develops the polyhedral approach to integer partitions originated in [5] and significantly enhanced in [6].

A partition of a positive integer n is any representation of n as a sum of positive integers:

$$n = n_1 + n_2 + \ldots + n_k, \quad n_i \in \mathbb{Z}, \quad n_i > 0, \quad i = 1, \ldots, k. \tag{1}$$

The main idea of the new approach is to study the set of partitions of any n as a polytope (i.e., a bounded polyhedral sets). This shift from the set to the polytope brings geometry into arithmetic and gives hope of clarifying a resulting geometrical structure related to convex combinations.

In general, there are two ways of describing any polytope: 1) by means of its facets (faces of the maximal dimension) and 2) via its vertices. In [6], with the use of a representation of P_n as a polytope on a partial algebra, all facets of the partition polytope were described. Along with this, some sufficient and, separately, necessary conditions for a partition to be a vertex of P_n were proved there. In [4], the vertices were studied in more detail. A lifting type method for constructing all vertices of P_n was proposed. The main result is a criterion of whether a given partition is a convex combination of two others. With its use, a great amount of partitions that are not vertices (actually, all for up to $n = 20$) can be recognized. The criterion generalizes all previously obtained necessary conditions for a partition to be a vertex and yields new ones, in particular, an exact upper bound = $\log(n+1)$ on the number of distinct parts in any vertex following from it as a consequence.

Vertices of the partition polytope form a subset of the set of all partitions that generates the whole set, since each partition can be expressed as a convex combination of finite tuples of vertices. Hence, the polyhedral approach allows avoiding enumeration of all partitions of a given n.

In this work, we show that it is sufficient to know only a small subset of vertices because all others can be constructed from these ones with the use of two simple combinatorial operations. The essence of both operations is that some parts of a partition are combined into greater parts. When one operation is applied, the new part is constructed from two different parts, while in the case of the second operation, the new part is built of all entries of one part.

In Sec. 1, we define the polytope of partitions, introduce notation, and formulate the previous results to be used further. In Sec. 2, we define the combinatorial operations of combining the parts of a partition and prove that their application to vertices leads to the vertices. Further on, we introduce the notion of a support vertex and present some experimental data on the number of support vertices for small n. The results of the work as well as some problems it evokes are discussed in Conclusion.

1. BASIC NOTIONS, NOTATION, AND SOME PREVIOUS RESULTS

When the polyhedral approach to partitions is applied, each partition (1) of the integer n is associated with a point $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, whose ith component x_i is a nonnegative integer equal to the number of entries of the part i in the given partition, $1 \leqslant i \leqslant n$. For example, the partition $8 = 1 + 1 + 2 + 4$ corresponds to the point $x = (2, 1, 0, 1, 0, 0, 0, 0) \in \mathbb{R}^8$. We identify partition (1) with the corresponding point x throughout the text. The summands n_i participating in (1) are called the parts of the partition.

The following notation is used: $\mathbb{Z}_+$ denotes the set of positive integers; $[1,\, m]$ denotes the segment $\{1, 2, \ldots, m\}$ of integers, $m \in \mathbb{Z}_+$. Let $x \vdash n$ denote that $x \in \mathbb{Z}_+^n$ is a partition of n, and let $S(x)$ denote the set $\{i \in [1,\, n]\,|x_i > 0\}$; so $S(x)$ is the set of distinct parts of $x \vdash n$. For a given polytope P, let $\mathrm{vert}\ P$ denote the set of its vertices. To this end, we write 0^k for the sequence of k zeroes.

The polytope $P_n \subset \mathbb{R}^n$ of partitions of n is defined as the convex hull of the set

$$T_n = \{x \in \mathbb{Z}^n | x_1 + 2x_2 + \ldots + nx_n = n, \quad x_i \in \mathbb{Z}, \quad x_i \geqslant 0, \quad i = 1, \ldots, n\}$$

of incidence vectors of all partitions of n:

$$P_n = \mathrm{conv}\ T_n.$$

The polytope P_n belongs to the hyperplane $x_1 + 2x_2 + \ldots + nx_n = n$ and is $(n-1)$-dimensional; in fact, it is a pyramid with the point $(0^{n-1}, 1)$ as the apex and the base lying in the hyperplane $x_n = 0$ [6]. Nevertheless, we study it as lying in $\mathbb{R}^n$.

For any polyhedron P, its point x is a vertex if it cannot be represented as a convex combination $x = \sum_{j=1}^{k} \lambda_j y^j$ of some other points $y^j \in P$, $1 \leqslant j \leqslant k$, $k > 0$; here $\lambda_1, \lambda_2, \ldots, \lambda_k$ are real numbers satisfying $\sum_{j=1}^{k} \lambda_j = 1$, $\lambda_j > 0$. The same refers to the polytope P_n. Since every partition of n, which is not a vertex, is a convex combination of some vertices, the vertices of P_n form a kind of basis, regarding the operation of taking a convex combination, for the whole set of partitions of n. This means that the set of all partitions can be reduced to the set of vertices, which is smaller in size. A direct calculation shows that the gap in their sizes is noticeable, though we cannot estimate how big the difference is. Anyway, this reduction could be used, for instance, for solving linear optimization problems on partitions, because these are vertices that provide their optimal solutions.

Let us cite some results from [4]. The next theorem gives a criterion of representability of a given partition as a convex combination of two others.

Theorem A [4]. A partition $x \vdash n$ is a convex combination of two partitions (and hence $x \notin \text{vert } P_n$) if and only if there exist two disjoint subsets of parts of x, $S_1, S_2 \in S(x)$, and two tuples of integers $u = \langle u_j \in \mathbb{Z}_+; j \in S_1 \rangle$ and $v = \langle v_k \in \mathbb{Z}_+; k \in S_2 \rangle$ satisfying the following relation:

$$\sum_{j \in S_1} u_j j = \sum_{k \in S_2} v_k k, \quad 0 < u_j < x_j, \quad 0 < v_k < x_k. \tag{2}$$

The Theorem easily implies the following corollary.

Corollary. For a given $x \in \text{vert } P_n$ none integer $k \in [1, n]$ that can be represented as a nonnegative integer combination

$$k = \sum_{i \in S(x)} \alpha_i i, \quad \alpha_i \in \mathbb{Z}_+, \quad \alpha_i \leqslant x_i, \tag{3}$$

of some distinct parts of x, is a part of x. In other words, any k of form (3) satisfies $k \notin S(x)$.

2. GENERATING VERTICES OF THE PARTITIONS POLYTOPE

Now we introduce two combinatorial operations of combining some parts of a partition and show that if applied to a vertex of the polytope P_n each of them produces a new vertex.

Operation 1. Let $x \vdash n$ and let $u, v \in S(x)$, $u \neq v$, be two distinct parts of x. To be more specific, we consider that $x_u \leqslant x_v$. Then build the point $y = C_{u,v}(x) \in \mathbb{Z}_+^n$ with the components $y_u = 0$, $y_v = x_v - x_u$, $y_{u+v} = x_{u+v} + x_u$, and $y_j = x_j$, for $j \in [1, n]$, $j \neq u, v, u+v$.

Operation 2. Let x be a partition of n such that some part $u \in S(x)$ enters x more than once, i.e., $x_u > 1$. Then build the point $y = C_u(x) \in \mathbb{Z}_+^n$ with the components $y_u = 0$, $y_{au} = x_{au} + 1$, and $y_j = x_j$, for $j \in [1, n]$, $j \neq u, au$.

For brevity, denote $x_u = a$, $x_v = b$.

Theorem 1. Let a vertex x of the polytope P_n contain two distinct parts $u, v \in S(x)$, $u \neq v$. Then, $y = C_{u,v}(x)$ is also a vertex of P_n.

Proof. At first, let us show that $y \vdash n$. Indeed,

$$\sum_{i=1}^{n} y_i i = \sum_{\substack{j=1,\\ j\neq u,v,u+v}}^{n} y_j j + (b-a)v + (x_{u+v} + a)(u+v) =$$
$$= \sum_{\substack{j=1,\\ j\neq u,v,u+v}}^{n} x_j j + bv - av + x_{u+v}(u+v) + a(u+v) = \sum_{i=1}^{n} x_i i = n.$$

Now prove that $y \in \operatorname{vert} P_n$. It follows from the corollary that $x_{u+v} = 0$. Suppose, on the contrary, that $y \notin \operatorname{vert} P_n$. Then, y is a convex combination $y = \sum\limits_{t=1}^{k} \lambda_t y^t$, $\sum\limits_{t=1}^{k} \lambda_t = 1$, $\lambda_t > 0$, of some partitions $y^t \vdash n$, $1 \leqslant t \leqslant k$. It follows from $y_u = 0$ that $y_u^t = 0$ for all t. Define integer points $x^t \in \mathbb{R}^n$, $1 \leqslant t \leqslant k$, with the components

$$\begin{aligned}
&x_u^t = y_{u+v}^t,\\
&x_v^t = y_{u+v}^t + y_v^t,\\
&x_{u+v}^t = 0,\\
&x_j^t = y_j^t, \quad j \neq u, v, u+v.
\end{aligned}$$

All x^t are partitions of n, since

$$\sum_{i=1}^{n} x_i^t i = \sum_{\substack{j=1,\\ j\neq u,v,u+v}}^{n} x_j^t j + x_u^t u + x_v^t v =$$
$$= \sum_{\substack{j=1,\\ j\neq u,v,u+v}}^{n} y_j^t j + y_{u+v}^t u + (y_{u+v}^t + y_v^t)v = \sum_{i=1}^{n} y_i^t i = n,$$

where the last equality follows from $y^t \vdash n$. Now we obtain a representation of x as a convex combination $x = \sum\limits_{t=1}^{k} \lambda_t x^t$, since

$$\begin{aligned}
&\sum_t \lambda_t x_u^t = \sum_t \lambda_t y_{u+v}^t = y_{u+v} = x_u,\\
&\sum_t \lambda_t x_v^t = \sum_t \lambda_t (y_{u+v}^t + y_v^t) = \sum_t \lambda_t y_{u+v}^t + \sum_t \lambda_t y_v^t = \\
&\qquad\qquad\qquad\qquad\qquad\qquad = y_{u+v} + y_v = a + (b-a) = b = x_v,\\
&\sum_t \lambda_t x_{u+v}^t = \sum_t \lambda_t 0 = 0 = x_{u+v},\\
&\sum_t \lambda_t x_j^t = \sum_t \lambda_t y_j^t = y_j = x_j, \text{for } j \neq u, v, u+v.
\end{aligned}$$

However, this contradicts x being a vertex of P_n. Therefore, y is a vertex of P_n and the Theorem is proved.

Theorem 2. Let a vertex x of the polytope P_n contain a part $u \in S(x)$ more than once, i.e., $x_u = a > 1$. Then, $y = C_u(x)$ is also a vertex of P_n.

Proof. It follows from the corollary that $x_{au} = 0$, hence $y_{au} = 1$. As in the previous theorem, one can check that $y \vdash n$

$$\sum_{i=1}^{n} y_i i = \sum_{\substack{j=1,\\ j\neq u,au}}^{n} y_j j + au = \sum_{\substack{j=1,\\ j\neq u,au}}^{n} x_j j + x_u u = \sum_{i=1}^{n} x_i i = n.$$

Now prove that $y \in \text{vert } P_n$. Suppose, on the contrary, that $y \notin \text{vert } P_n$. Then, y is a convex combination $y = \sum\limits_{t=1}^{k} \lambda_t y^t$, $\sum\limits_{t=1}^{k} \lambda_t = 1$, $\lambda_t > 0$, of some partitions y^t, $1 \leqslant t \leqslant k$, of n. Define integer points $x^t \in \mathbb{Z}_+^n$, $1 \leqslant t \leqslant k$, with the components

$$\begin{aligned}
&x_u^t = a y_{au}^t,\\
&x_{au}^t = 0,\\
&x_j^t = y_j^t, \quad j \neq u, au,
\end{aligned}$$

for all t. All x^t are partitions of n, since

$$\sum_{i=1}^{n} x_i^t i = \sum_{\substack{j=1,\\ j\neq u,au}}^{n} x_j^t j + x_u^t u = \sum_{\substack{j=1,\\ j\neq u,au}}^{n} y_j^t j + a y_{au}^t u = \sum_{i=1}^{n} y_i^t i = n,$$

where the last equality follows from $y \vdash n$. Now, for x we have a representation $x = \sum\limits_{t=1}^{k} \lambda_t x^t$, since

$$\sum_t \lambda_t x_u^t = \sum_t \lambda_t a y_{au}^t = a \sum_t \lambda_t y_{au}^t = a y_{au} = a = x_u,$$
$$\sum_t \lambda_t x_{au}^t = \sum_t \lambda_t \cdot 0 = 0 = x_{au},$$
$$\sum_t \lambda_t x_j^t = \sum_t \lambda_t y_j^t = y_j = x_j, \text{ for } j \neq u, au.$$

So, we obtained that the partition x is a convex combination of partitions x^t, $1 \leqslant t \leqslant k$, which contradicts x being a vertex of P_n and ends the proof.

Let us illustrate the application of the operations of combining parts of partitions using the polytope P_6 as an example. Following [6], 7 partitions listed below are all vertices of P_6, while the total number of partitions of 6 is 11:

$$x^1 = (6,0,0,0,0,0), \quad x^2 = (2,0,0,1,0,0), \quad x^3 = (1,0,0,0,1,0),$$
$$x^4 = (0,3,0,0,0,0), \quad x^5 = (0,1,0,1,0,0), \quad x^6 = (0,0,2,0,0,0),$$
$$x^7 = (0,0,0,0,0,1).$$

Applying Operation 1 to x^2 leads to x^3, while applying it to x^3 and x^5 leads to x^7. With the help of Operation 2, we obtain x^5 from the vertex x^2, and x^7 again from the vertices x^4 and x^6. On the other hand, it is not hard to verify that none of the vertices x^1, x^2, x^3, x^6 can be obtained from any other vertex with the use of these two operations. Therefore, all vertices of the polytope P_6 can be obtained from 4 vertices x^1, x^2, x^3, x^6 and this is a minimal set of this kind, relative to inclusion.

The next definition is natural.

Definition. A vertex of a partition polytope is called the *support vertex* if it cannot be obtained as a result of application of any operation of combining parts from any other vertex of the same polytope.

It follows from above that x^1, x^2, x^3, x^6 are support vertices of the polytope P_6.

Numbers of partitions, vertices and support vertices for $n = 6, 10, 20$

n	6	10	20
# partitions	11	42	627
# vertices	7	19	99
# support vertices	4	9	29

Numerical data of the number of partitions, vertices, and support vertices of the partition polytopes for $n = 6, 10$ and 20 is presented in the Table.

We can see from the Table that if part of the support vertices for $n = 6$ constitutes 36% of the whole set of partitions, then in the case of $n = 10$ it decreases to 21%, while in the last case of $n = 20$ it falls down to less than 5%. It is not hard to notice that the ratio of the number of support vertices to the number of vertices in total also definitely decreases, while n grows up.

CONCLUSION

For the most known polytopes, their vertex description demands the knowledge of the full list of their vertices. The results of this work show that in the case of the partition polytope a peculiar situation takes place. It is possible to avoid calculation of all vertices of P_n if one would have constructed its support vertices. All others can be built from these ones with the use of consequent application of the operations of combining parts introduced in the work. The full list of partitions of an integer n can be obtained afterwards from the vertices of P_n by calculating all integer points of $\mathbb{R}^n$ that are convex combinations of the vertices. The questions of how to construct the set of support vertices for a given n and what is its cardinality remain the tasks for the future research.

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