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SOLUTIONS OF RICCATI–ABEL EQUATION IN TERMS OF CHARACTERISTICS OF GENERAL COMPLEX ALGEBRA

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Представление решений уравнения Риккати–Абеля
через характеристические функции обобщенных комплексных чисел

Исследуется уравнение Риккати–Абеля, которое определяется как уравнение между производной первого порядка и кубическим полиномом. Предложен метод решения этого уравнения путем сведения к решению соответствующего алгебраического уравнения. Найдено правило сумм для решений нелинейных уравнений типа Риккати. Установлена связь решений уравнения Риккати–Абеля с характеристическими функциями обобщенной комплексной алгебры третьего порядка.

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Solutions of Riccati–Abel Equation in Terms
of Characteristics of General Complex Algebra

The Riccati–Abel differential equation defined as an equation between the first order derivative and the cubic polynomial is explored. In the case of constant coefficients this equation is reduced into an algebraic equation. Method of derivation of a summation formula for solutions of Riccati–Abel equation is elaborated. The solutions of the Riccati–Abel equation are expressed in terms of the characteristic functions of general complex algebra of the third order.

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1. INTRODUCTION

Consider the first order differential equation

$$f(u,x) = \frac{du}{dx}. \tag{1.1}$$

If we approximate $f(u,x)$, while x is kept constant, we will get

$$Q_0(x) + Q_1(x)u + Q_2(x)u^2 + Q_3(x)u^3 + \ldots = \frac{du}{dx}. \tag{1.2}$$

When the series in the left-hand side is restricted with the second order polynomial, the equation is the Riccati equation [1].

The Riccati equation is one of the widely used equations of mathematical physics. The ordinary Riccati equations are closely related to the second order linear differential equations. For the solutions of the ordinary Riccati equations with constant coefficients, a summation formula can be derived. These solutions are presented by trigonometric functions induced by general complex algebra.

In particular, if $f(u,x)$ is a cubic polynomial, then the equation is called Riccati–Abel equation. Abel's original equation was written in the form

$$(y+s)\frac{dy}{dx} + p + qy + ry^2 = 0. \tag{1.3}$$

This equation is converted into Riccati–Abel equation by transformation $y + s = 1/z$, which yields

$$\frac{dz}{dx} = rz + (q - s' - 2rs)z^2 + (p - qs + rs^2)z^3. \tag{1.4}$$

It is seen that the case $Q_0(u,\phi) = 0$ was actually considered by Abel [2].

When the series in the left-hand side of equation (1.2) is given by the n-order polynomial, we deal with the generalized Riccati equations. The solution of the generalized Riccati equation with constant coefficients can be denominated as a *generalized tangent function*. The generalized Riccati equations are used, for example, in various problems of renorm-group theory [3]. The mean field free energy concept and the perturbation renormalization group theory deal with the first order differential equations with polynomial nonlinearity.

The aim of this paper is to explore solutions of the Riccati–Abel equation with constant coefficients and to derive some kind of summation formula for them. Summation (addition) formulae for solutions of linear differential equations are considered as important features of these functions. Let us mention, for example, a summation formulae for the trigonometric sine–cosine functions, the Bessel functions, the hypergeometric functions and their various generalizations. Whereas solutions of the linear differential equations with constant coefficients admit universal methods of obtaining summation formulas (see, for instance, [4,5]), the solutions of nonlinear equations require special investigations. In this context, let us mention the addition formulae for Jacobi and Weierstrass elliptic functions [6].

In general, the solutions of the generalized Riccati equations with cubic and higher polynomials do not admit any summation formula. Nevertheless, by careful analysis we found a new summation law according to which in order to obtain a summation formula for the solutions of the third order Riccati equation two independent variables should be used. In this way we will establish an interconnection between solutions of Riccati–Abel equation and the characteristic functions of generalized complex algebra of the third order.

The paper is presented by the following sections. Section 2 deals with solution of ordinary Riccati equation with constant coefficients. Summation formula for the solutions is derived and interrelation with solutions of the linear differential equations is underlined. In Sec. 3, the Riccati–Abel equation is integrated, a corresponding algebraic equation for solutions is derived, a summation formula for solutions is established. In Sec. 4, the solutions of Riccati–Abel equation are constructed within generalized complex algebra of the third order. In Sec. 5, it is shown that the Riccati–Abel equation is an evolution equation of the generalized classical dynamics.

2. ORDINARY RICCATI EQUATION, SUMMATION FORMULA AND GENERAL COMPLEX ALGEBRA

2.1. The Ordinary Riccati Equation. Consider the Riccati equation with constant coefficients

$$u^2 - a_1 u + a_0 = \frac{du}{d\phi}. \tag{2.1}$$

If coefficients a_0, a_1 are constants then a great simplification results because it is possible to obtain the complete solution by means of quadratures. Thus, equation (2.1) admits direct integration

$$\int \frac{dx}{x^2 - a_1 x + a_0} = \int d\phi. \tag{2.2}$$

Let $x_1, x_2 \in C$ be roots of the polynomial equation

$$x^2 - a_1 x + a_0 = 0. \tag{2.3}$$

In order to calculate the integral (2.2), the following formula expansion is used:

$$\frac{1}{x^2 - a_1 x + a_0} = \frac{1}{2x_1 - a_1}\frac{1}{x - x_1} + \frac{1}{2x_2 - a_1}\frac{1}{x - x_2}, \tag{2.4}$$

where,

$$2x_1 - a_1 = (x_1 - x_2), \quad 2x_2 - a_1 = (x_2 - x_1).$$

Then the integral (2.2) is easily calculated and the result is given by the logarithmic functions

$$\int_w^u \frac{dx}{x^2 - a_1 x + a_0} = \frac{1}{m_{12}} \left(\log \frac{u - x_1}{u - x_2} - \log \frac{w - x_1}{w - x_2} \right) = \phi(u) - \phi(w), \quad (2.5)$$

where $m_{12} = x_1 - x_2$. Now, let us keep the first logarithm of (2.5) depending on the initial limit of the integral, that is

$$\frac{1}{m_{12}} \log \left[\frac{u - x_1}{u - x_2} \right] = \phi(u).$$

By inverting the logarithm function, we come to the algebraic equation for solution of (2.1),

$$\exp(m_{12}\phi) = \frac{u - x_1}{u - x_2}. \quad (2.6)$$

Let $u(\phi_0) = 0$, then

$$\exp(m_{12}\phi_0) = \frac{x_1}{x_2}. \quad (2.7)$$

As soon as the point $\phi = \phi_0$ is determined, one may calculate the function $u(\phi)$ by making use of algebraic equation (2.6). Since $a_1 = x_1 + x_2$, from (2.7) it follows that

$$a_1 = m_{12} \coth(m_{12}\phi_0/2).$$

Consequently, from (2.6) we obtain

$$u(\phi, \phi_0) = \frac{1}{2} m_{12} \coth(m_{12}\phi_0/2) - \frac{1}{2} m_{12} \coth(m_{12}\phi/2).$$

2.2. Summation (Addition) Formula for Function $u = u(\phi, \phi_0)$. Consider the following integral equation:

$$\int^u \frac{dx}{x^2 - a_1 x + a_0} + \int^v \frac{dx}{x^2 - a_1 x + a_0} = \int^w \frac{dx}{x^2 - a_1 x + a_0}. \quad (2.8)$$

The quantity w is a function of u and v. If the function $w = f(u, v)$ is an algebraic function then this function can be considered as a summation formula. Write (2.8) in the following notations $\phi_u + \phi_v = \phi_w$. Then, $w(\phi_w) = w(\phi_u + \phi_v) = f(u(\phi_u), v(\phi_v))$.

Calculating the integrals in (2.8), we come to the following algebraic equation:

$$\frac{1}{2m}\log\frac{u-x_1}{u-x_2}\frac{v-x_1}{v-x_2}=\frac{1}{2m}\log\frac{w-x_1}{w-x_2}. \tag{2.9}$$

Thus, the function $w(u,v)$ has to satisfy the equation

$$\frac{u-x_1}{u-x_2}\frac{v-x_1}{v-x_2}=\frac{w-x_1}{w-x_2}. \tag{2.10}$$

Multiplying fractions and taking into account the fact that x_1, x_2 obey (2.3), we get

$$\begin{gathered}\frac{uv-x_1(u+v)+a_1x_1-a_0}{uv-x_2(u+v)+a_1x_2-a_0}=\frac{\dfrac{uv-a_0}{u+v-a_1}-x_1}{\dfrac{uv-a_0}{u+v-a_1}-x_2}=\frac{w-x_1}{w-x_2},\\ w=\frac{uv-a_0}{u+v-a_1}.\end{gathered} \tag{2.11}$$

This is a summation formula for solutions of the Riccati equation (2.1).

2.3. Relationship with General Complex Algebra. Like the cotangent function can be defined as a ratio of cosine and sine functions, the solution of the Riccati equation $u(\phi,\phi_0)$ can also be represented as a ratio of modified cosine and sine functions. Firstly, let us construct these functions.

Consider general complex algebra generated by the (2×2) matrix [7]

$$E=\begin{pmatrix}0 & -a_0\\ 1 & a_1\end{pmatrix} \tag{2.12}$$

obeying the quadratic equation (2.3):

$$E^2-a_1E+a_0I=0, \tag{2.13}$$

with I-unit matrix. Expansion with respect to E of the exponential function $\exp(E\phi)$ leads to the Euler formula [8]

$$\exp(E\phi)=g_1(\phi;a_0,a_1)\,E+g_0(\phi;a_0,a_1). \tag{2.14}$$

In terms of the roots x_1, x_2 this matrix equation is separated into two equations

$$\begin{aligned}\exp(x_2\phi)&=x_2\,g_1(\phi;a_0,a_1)+g_0(\phi;a_0,a_1),\\ \exp(x_1\phi)&=x_1\,g_1(\phi;a_0,a_1)+g_0(\phi;a_0,a_1),\end{aligned} \tag{2.15}$$

from which an explicit form of g-functions can be obtained. Apparently, g_0 and g_1 are modified (generalized) cosine–sine functions with the following formulas of differentiation:

$$\begin{aligned} \frac{d}{d\phi} g_1(\phi; a_0, a_1) &= g_0(\phi; a_0, a_1) + a_1\, g_1(\phi; a_0, a_1), \\ \frac{d}{d\phi} g_0(\phi; a_0, a_1) &= -a_0\, g_1(\phi; a_0, a_1). \end{aligned} \tag{2.16}$$

Form a ratio of two equations of (2.15) as follows:

$$\exp\left(m_{21}\phi\right) = \frac{x_2\, g_1(\phi; a_0, a_1) + g_0(\phi; a_0, a_1)}{x_1\; g_1(\phi; a_0, a_1) + g_0(\phi; a_0, a_1)}. \tag{2.17}$$

Let $g_1(\phi; a_0, a_1) \neq 0$. Then,

$$\exp\left(m_{21}\phi\right) = \frac{x_2 + D}{x_1 + D}, \tag{2.18}$$

where

$$D = \frac{g_0(\phi; a_0, a_1)}{g_1(\phi; a_0, a_1)}.$$

Differential equation for function $D(\phi)$ is obtained by using (2.16):

$$D^2 + a_1 D + a_0 = -\frac{dD}{d\phi}. \tag{2.19}$$

Thus, we have proved that the function

$$u(\phi; a_0, a_1) = -D = -\frac{g_0(\phi; a_0, a_1)}{g_1(\phi; a_0, a_1)} \tag{2.20}$$

obeys the Riccati equation.

Summation formulae for g-functions are well-defined (see, for example, [7]). They are

$$\begin{aligned} g_0(a+b) &= g_0(a)g_0(b) - a_0 g_1(a)g_1(b), \\ g_1(a+b) &= g_1(a)g_0(b) + g_0(a)g_1(b) + a_1 g_1(a)g_1(b), \\ \frac{g_0(a+b)}{g_1(a+b)} &= \frac{g_0(a)g_0(b) - a_0 g_1(a)g_1(b)}{g_1(a)g_0(b) + g_0(a)g_1(b) + a_1 g_1(a)g_1(b)}. \end{aligned} \tag{2.21}$$

By taking into account (2.20), we get

$$u(a+b) = -\frac{g_0(a+b)}{g_1(a+b)} = \frac{u(a)u(b) - a_0}{u(a) + u(b) - a_1}, \tag{2.22}$$

which coincides with (2.11).

3. GENERALIZED RICCATI EQUATION WITH CUBIC ORDER POLYNOMIAL

3.1. The Riccati–Abel Equation. Consider the following nonlinear differential equation with constant coefficients:

$$u^3 - a_2 u^2 + a_1 u - a_0 = \frac{du}{d\phi}, \tag{3.1}$$

which admits direct integration by

$$\int\limits_w^u \frac{dx}{x^3 - a_2 x^2 + a_1 x - a_0} = \phi(w) - \phi(u). \tag{3.2}$$

This integral is calculated by making use of the well-known method of the partial fractional decomposition [9]

$$\frac{1}{x^3 - a_2 x^2 + a_1 x - a_0} = \frac{1}{(x-x_3)(x-x_2)(x-x_1)} =$$
$$= \frac{(x_3 - x_2)}{V}\frac{1}{x-x_1} + \frac{(x_1 - x_3)}{V}\frac{1}{x-x_2} + \frac{(x_2 - x_1)}{V}\frac{1}{x-x_3}, \tag{3.3}$$

where V is the Vandermonde determinant [10]

$$V = (x_1 - x_2)(x_2 - x_3)(x_3 - x_1), \tag{3.4}$$

and the distinct constants $x_1, x_2, x_3 \in C$ are roots of the cubic polynomial

$$f(x) = x^3 - a_2 x^2 + a_1 x - a_0 = 0. \tag{3.5}$$

By using expansion (3.3) the integral (3.2) is easily calculated:

$$\int\limits_w^u \frac{dx}{x^3 - a_2 x^2 + a_1 x - a_0} = \frac{(x_3 - x_2)}{V}\log\frac{u - x_1}{w - x_1} +$$
$$+ \frac{(x_1 - x_3)}{V}\log\frac{u - x_2}{w - x_2} + \frac{(x_2 - x_1)}{V}\log\frac{u - x_3}{w - x_3} = \phi(u) - \phi(w). \tag{3.6}$$

Let us introduce the following notations:

$$m_{ij} = (x_i - x_j), \quad i, j = 1, 2, 3, \quad \text{with} \quad m_{21} + m_{32} + m_{13} = 0, \tag{3.7}$$

and write equation (3.6) as follows:

$$\int^u \frac{dx}{x^3 - a_2 x^2 + a_1 x - a_0} = \log\,(u - x_1)^{m_{32}}(u - x_2)^{m_{13}}(u - x_3)^{m_{21}} = V\phi(u), \tag{3.8}$$

and invert the logarithm. This leads to the following algebraic equation:

$$[u - x_1]^{m_{32}}[u - x_2]^{m_{13}}[u - x_3]^{m_{21}} = \exp(V\phi). \tag{3.9}$$

This equation can also be written in the fractional form

$$\left[\frac{u - x_1}{u - x_3}\right]^{m_{32}} \left[\frac{u - x_2}{u - x_3}\right]^{m_{13}} = \exp(V\phi). \tag{3.10}$$

Thus, the problem of solution of differential equation (3.1) is reduced to the problem of solution of the algebraic equation (3.10). Notice, if the roots of cubic equation and function u are defined in the field of real numbers, then this equation is meaningful only for a certain domain of definition of $u(\phi)$.

3.2. Semigroup Property of Fractions of n-Order Monic Polynomials on the Set of Roots of $n+1$-Order Polynomial. In this section, let us recall a semigroup property of the fractions of n-order polynomials defined on the set of roots of $n+1$-order polynomial. Let $F(x, n+1)$ be $(n+1)$-order polynomial with $(n+1)$ distinct roots $x_i, i = 1, \ldots, n+1$. Denote this set of roots by $FX(n+1)$.

Lemma 3.1. *Let $P_a(x_i, n)$ be the n-order polynomial on $x_i \in FX(n+1)$. The product of two n-order polynomials*

$$P_a(x_i, n)P_b(x_i, n)$$

is also an n-order polynomial $P_c(x_i, n)$.

Proof. The product $P_{ab}(x_i, 2n) := P_a(x_i, n)P_b(x_i, n)$ is a polynomial of $2n$-degree with respect to variable x_i. Since x_i obeys $n+1$-order polynomial equation, all monomials with degrees higher than n can be expressed via polynomials of n-degree. Consequently, the polynomial $P_{ab}(x_i, 2n)$ with $x_i \in FX(n+1)$ is reduced into n-degree polynomial.

End of proof.

Consider two monic polynomials of n-degree $P_a(x_i, n)$, $P_b(x_k, n)$ with $x_i \neq x_k \in FX(n+1)$. Form a rational algebraic fraction

$$\frac{P_a(x_i, n)}{P_a(x_k, n)}.$$

The following **Corollary 3.2** holds true:

The product of two fractions formed by two n-order monic polynomials on the roots of $(n+1)$-order polynomial is a fraction of the same order monic polynomials on the variables,

$$\frac{P_a(x_i, n)}{P_a(x_k, n)} \frac{P_b(x_i, n)}{P_b(x_k, n)} = \frac{P_c(x_i, n)}{P_c(x_k, n)}.$$

3.3. Addition Formula for $u(\phi)$. Let $\phi = \phi_0$ be a point where $u(\phi_0) = 0$. Then, (3.10) is reduced to

$$\left[\frac{x_1}{x_3}\right]^{m_{32}} \left[\frac{x_2}{x_3}\right]^{m_{13}} = \exp\left(V\phi_0\right). \tag{3.11}$$

Now, let us make simultaneous translations of the roots $x_k, k = 1, 2, 3$ by some value u. Since the Vandermonde determinant remains invariant under these translations, the parameter ϕ_0 will undergo some translation by $\phi = \phi_0 + \delta$. In this way one may construct the solution of Riccati–Abel equation (3.1) with initial condition $u(\phi_0) = 0$.

Let the triple u, v, w form a set of solutions of equation (3.1) calculated for three variables $\phi_u, \phi_v, \phi_w = \phi_u + \phi_v$, correspondilngly. Then, in accordance with (3.10) we write:

$$\exp\left(V\phi_u\right)\exp\left(V\phi_v\right) = \left\{\left[\frac{u-x_1}{u-x_3}\frac{v-x_1}{v-x_3}\right]^{m_{32}} \left[\frac{u-x_2}{u-x_3}\frac{v-x_2}{v-x_3}\right]^{m_{21}}\right\} =$$

$$= \left\{\left(\frac{w-x_1}{w-x_3}\right)^{m_{32}} \left[\frac{w-x_2}{w-x_3}\right]^{m_{21}}\right\} = \exp\left(V(\phi_u + \phi_v\right). \tag{3.12}$$

The problem is to find some rational function expressing w via the pair (u, v), i.e., the function $w = w(u, v)$ has to be a rational function.

Evidently, the method used in the previous section for the ordinary Riccati equation now is not applicable. According to Lemma 3.1, we are able to transform a product of ratios of n-order polynomials into the ratio of n-order polynomials if these polynomials are defined on roots of $n+1$-order polynomial. Thus, we have to seek another way of construction of a summation formula.

Let us present the integral (3.8) as a sum of two integrals by

$$\int^{w} \frac{dx}{x^3 - a_2x^2 + a_1x - a_0} = \int^{u} \frac{dx}{x^3 - a_2x^2 + a_1x - a_0} +$$

$$+ \int^{v} \frac{dx}{x^3 - a_2x^2 + a_1x - a_0} =$$

$$= \phi = V\log\left(\left(\frac{u-x_1}{u-x_3}\frac{v-x_1}{v-x_3}\right)^{m_{32}} \left(\frac{u-x_2}{u-x_3}\ \frac{v-x_2}{v-x_3}\right)^{m_{13}}\right). \tag{3.13}$$

In this way we arrive to the following algebraic equation:

$$\left(\frac{u-x_1}{u-x_3}\frac{v-x_1}{v-x_3}\right)^{m_{32}} \left(\frac{u-x_2}{u-x_3}\frac{v-x_2}{v-x_3}\right)^{m_{13}}\Bigg) =$$

$$= \exp\left(V\phi(u,v)\right) = \exp\left(V\phi_u\right)\exp\left(V\phi_v\right). \tag{3.14}$$

Let u, v be solutions of the quadratic equation

$$x^2 + tx + s = 0, \quad t = -(u+v), \quad s = uv. \tag{3.15}$$

Then, equation (3.14) is written as

$$\left[\frac{x_1^2 + tx_1 + s}{x_3^2 + tx_3 + s}\right]^{m_{32}} \left[\frac{x_2^2 + tx_2 + s}{x_3^2 + tx_3 + s}\right]^{m_{13}} = \exp\left(V\phi(t,s)\right). \tag{3.16}$$

Thus, from the pair of functions (u, v), we come to another pair (t, s). This pair of functions, in fact, admits a summation rule because the problem is reduced to the task of transformation four-degree polynomial into quadratic polynomial at the solutions of the cubic equation (3.5). Evidently, this task can be easily performed by simple algebraic operations.

Theorem 3.3. *The following summation formula for solutions of Riccati–Abel equation holds true:*

$$(t,s) \bigoplus (v,u) = (r,w),$$

where

$$\begin{aligned} r &= \frac{(a_0 - 2a_2a_1) - a_1(v+t) + (tu+sv)}{(3a_2^2 - a_1) + a_2(v+t) + (s+u+tv)}, \\ w &= \frac{a_2a_0 + (v+t)a_0 + su}{(3a_2^2 - a_1) + a_2(v+t) + (s+u+tv)}. \end{aligned} \tag{3.17}$$

Proof. Consider product of two monic polynomials

$$(x^2+tx+s)(x^2+vx+u) = x^4 + x^3(v+t) + x^2(s+u+tv) + x(tu+vs) + su,$$

where x is one of the roots of cubic equation

$$x^3 - a_2x^2 + a_1x - a_0 = 0. \tag{3.18}$$

From the cubic equation (3.18), we are able to express x^3 and x^4 as polynomials of the second order as follows:

$$x^3 = a_2x^2 - a_1x + a_0, \quad x^4 = (3a_2^2 - a_1)\, x^2 + (a_0 - a_1a_2)\, x + a_2a_0.$$

Then, the four-degree polynomial on roots of the cubic polynomial is reduced into a polynomial of the second order

$$x^4 + x^3(v+t) + x^2(s+u+tv) + x(tu+vs) + su = Ax^2 + Bx + C, \tag{3.19}$$

where A, B, C do not depend of x.

Since we deal with the ratios of polynomials the coefficients of the quadratic polynomial in (3.19) and polynomials in denominator and in numerator have the same leading coefficient, we are able to return to the ratio of monic polynomials. In this way we come to the relations

$$r = \frac{B}{A}, \quad w = \frac{C}{A}, \tag{3.20}$$

where

$$\begin{aligned} A &= (3a_2^2 - a_1) + a_2(v + t) + (s + u + tv), \\ B &= (a_0 - 2a_2a_1) - a_1(v + t) + (tu + sv), \\ C &= a_2a_0 + (v + t)a_0 + su. \end{aligned} \tag{3.21}$$

End of proof.

4. GENERALIZED COMPLEX ALGEBRA OF THE THIRD ORDER AND SOLUTIONS OF RICCATI–ABEL EQUATION

In this section, we will establish a relationship between characteristic functions of general complex algebra of the third order and solutions of Riccati–Abel equation.

The unique generator E of general complex algebra of the third order, CG_3, is defined by cubic equation [11]

$$E^3 - a_2E^2 + a_1E - a_0 = 0. \tag{4.1}$$

The companion matrix E of the cubic equation (4.1) is given by (3×3) matrix

$$E := \begin{pmatrix} 0 & 0 & a_0 \\ 1 & 0 & -a_1 \\ 0 & 1 & a_2 \end{pmatrix}. \tag{4.2}$$

Consider the expansion

$$\exp\left(E\phi_1 + E^2\phi_2\right) = g_0(\phi_1, \phi_2) + E\, g_1(\phi_1, \phi_2) + E^2\, g_2(\phi_1, \phi_2). \tag{4.3}$$

This is an analogue of the Euler formula for exponential function, the function $g_0(\phi_1, \phi_2)$ is an analogue of cosine function, and $g_k(\phi_1, \phi_2), k = 1, 2$ are extensions of the sine function. It is seen, the characteristic functions of GC_3 algebra depend on the pair of «angles». Correspondingly, for each of them we have the *formulae of differentiation*:

$$\frac{\partial}{\partial\phi_1} \begin{pmatrix} g_0 \\ g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a_0 \\ 1 & 0 & -a_1 \\ 0 & 1 & a_1 \end{pmatrix} \begin{pmatrix} g_0 \\ g_1 \\ g_2 \end{pmatrix}, \tag{4.4}$$

$$\frac{\partial}{\partial\phi_2} \begin{pmatrix} g_0 \\ g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} 0 & a_0 & a_0a_2 \\ 0 & -a_1 & a_0 - a_1a_2 \\ 1 & a_2 & -a_1 + a_2^2 \end{pmatrix} \begin{pmatrix} g_0 \\ g_1 \\ g_2 \end{pmatrix}. \tag{4.5}$$

The semigroup of multiplications of the exponential functions leads to the following *addition formulae for g-functions* [12]:

$$\begin{pmatrix} g_0 \\ g_1 \\ g_2 \end{pmatrix}_{(\psi_c=\psi_a+\psi_b)} =$$

$$= \begin{pmatrix} g_0 & g_2 a_0 & g_1 a_0 + g_2 a_0 a_2 \\ g_1 & g_0 - g_2 a_1 & -g_1 a_1 + g_2 (a_0 - a_1 a_2) \\ g_2 & g_1 + g_2 a_2 & g_0 + g_1 a_2 + g_2(-a_1 + a_2^2) \end{pmatrix}_{\psi_a} \begin{pmatrix} g_0 \\ g_1 \\ g_2 \end{pmatrix}_{\psi_b}, \quad (4.6)$$

where the sub-indices of the brackets indicate dependence of the g-functions of the pair of variables $\psi_i = (\phi_{1i}, \phi_{2i})$, $i = a, b, c$.

Introduce two fractions of g-functions by

$$tg = \frac{g_1}{g_2}, \quad sg = \frac{g_0}{g_2}. \quad (4.7)$$

It is seen, these functions are analogues of tangent–cotangent functions. From the addition formulae for g-functions (4.6), the following summation formulae for the general tangent functions are derived:

$$T_0 = \frac{t_0 r_0 + a_0(r_1 + t_1) + a_0 a_2}{r_0 + (t_1 + a_2) r_1 + t_0 + t_1 a_2 + (-a_1 + a_2^2)}, \quad (4.8)$$

$$T_1 = \frac{t_1 r_0 + t_0 r_1 - a_1(r_1 + t_1) + (a_0 - a_1 a_2))}{r_0 + t_0 + a_2(t_1 + r_1) + t_1 r_1 + (-a_1 + a_2^2)}. \quad (4.9)$$

Here the following notations are used:

$$\begin{aligned} T_0(\psi_c) &= \frac{g_0(\psi_c)}{g_2(\psi_c)}, \quad T_1(\psi_c) = \frac{g_1(\psi_c)}{g_2(\psi_c)}, \\ t_0(\psi_a) &= \frac{g_0(\psi_a)}{g_2(\psi_a)}, \quad r_0(\psi_b) = \frac{g_0(\psi_b)}{g_2(\psi_b)}, \\ t_1(\psi_a) &= \frac{g_1(\psi_a)}{g_2(\psi_a)}, \quad r_1(\psi_b) = \frac{g_1(\psi_b)}{g_2(\psi_b)}, \end{aligned} \quad (4.10)$$

and $\psi_i = (\phi_{1i}, \phi_{2i}), i = a, b$, $\psi_c = (\phi_{1c} = \phi_{1a} + \phi_{1b}, \phi_{2c} = \phi_{2a} + \phi_{2b})$.

Let $x_1, x_2, x_3 \in C$ be eigenvalues of E given by distinct values. Then, the matrix equation (4.3) is represented by three separated series ($k = 1, 2, 3$):

$$\exp(x_k \phi_1 + x_k^2 \phi_2) = g_0(\phi_1, \phi_2) + x_k\, g_1(\phi_1, \phi_2) + x_k^2\, g_2(\phi_1, \phi_2), \quad (4.11)$$

Form the following ratios for $i \neq k$:

$$\exp\left((x_i - x_k)\,\phi_1 + (x_i^2 - x_k^2)\,\phi_2\right) = \frac{g_0(\phi_1, \phi_2) + x_i\, g_1(\phi_1, \phi_2) + x_i^2\, g_2(\phi_1, \phi_2)}{g_0(\phi_1, \phi_2) + x_k\, g_1(\phi_1, \phi_2) + x_k^2\, g_2(\phi_1, \phi_2)}. \quad (4.12)$$

Consider two of these ratios, namely,

$$\exp\left(m_{13}\phi_1 + (x_1^2 - x_3^2)\,\phi_2\right) = \frac{g_2x_1^2 + g_1x_1 + g_0}{g_2x_3^2 + g_1x_3 + g_0}, \tag{4.13a}$$

$$\exp\left(m_{23}\phi_1 + (x_2^2 - x_3^2)\,\phi_2\right) = \frac{g_2x_2^2 + g_1x_2 + g_0}{g_2x_3^2 + g_1x_3 + g_0}, \tag{4.13b}$$

where $m_{ij} = x_i - x_j$. Both sides of equation (4.13a) raise to power m_{32} and both sides of equation (4.13b) raise to power m_{13} and multiply left and right sides of the obtained equations, correspondingly. And, by taking into account that $m_{13}m_{32} + m_{23}m_{13} = 0$, we arrive to the following equation:

$$\begin{aligned}\exp\left(m_{13}m_{32}\phi_1 + (x_1+x_3)m_{13}m_{32}\phi_2\right)\exp\left(m_{23}m_{13}\phi_1 + (x_2+x_3)m_{13}\phi_2\right) = \\ = \left[\frac{g_2x_2^2 + g_1x_2 + g_0}{g_2x_3^2 + g_1x_3 + g_0}\right]^{m_{13}}\left[\frac{g_2x_1^2 + g_1x_1 + g_0}{g_2x_3^2 + g_1x_3 + g_0}\right]^{m_{32}}.\end{aligned} \tag{4.14}$$

The left-hand side of this equation is equal to $\exp\left(V\phi_2\right)$, that is,

$$\exp\left(V\phi_2\right) = \left[\frac{g_2x_2^2 + g_1x_2 + g_0}{g_2x_3^2 + g_1x_3 + g_0}\right]^{m_{13}}\left[\frac{g_2x_1^2 + g_1x_1 + g_0}{g_2x_3^2 + g_1x_3 + g_0}\right]^{m_{32}}. \tag{4.15}$$

Let $g_2 \neq 0$, then by dividing numerator and denominator by g_2, we obtain

$$\exp\left(V\phi_2\right) = \left[\frac{x_2^2 + tg\,x_2 + sg}{x_3^2 + tg\,x_3 + sg}\right]^{m_{13}}\left[\frac{x_1^2 + tg\,x_1 + sg}{x_3^2 + tg\,x_3 + sg}\right]^{m_{32}}, \tag{4.16}$$

where

$$tg = \frac{g_1}{g_2}, \quad sg = \frac{g_0}{g_2}.$$

Let u, v be roots of the quadratic equation

$$g_0(\phi_1,\phi_2) + y\,g_1(\phi_1,\phi_2) + y^2\,g_2(\phi_1,\phi_2) = 0. \tag{4.17}$$

Then the ratios (4.13a,b) can be rewritten as follows:

$$\exp\left((x_k - x_l)\,\phi_1 + (x_k^2 - x_l^2)\,\phi_2\right) = \frac{(u - x_k)}{(u - x_l)}\frac{(v - x_k)}{(v - x_l)}. \tag{4.18}$$

This equation is true for any $k, l = 1, 2, 3,\ k \neq l$. This is to say, for any index we have the same ϕ_1, ϕ_2 and the same u, v. Here u, v depend on two parameters ϕ_1, ϕ_2.

Inversely, if functions $u = u(\varphi_u)$, $v = v(\varphi_v)$ are known, then we can find corresponding g by

$$\frac{g_0}{g_2} = uv, \quad \frac{g_1}{g_2} = u + v.$$

From these two equations we find ϕ_1 and ϕ_2. We expect that

$$\exp\left(V(\varphi_u + \varphi_v)\right) = \exp\left(V\phi_2\right),$$

or

$$\varphi_u + \varphi_v = \phi_2.$$

In this way we have established connection between the characteristics of general complex algebra CG_3 and solutions of the Riccati–Abel equation.

The next task is to prove that the ratio $u = -g_0/g_1|_{g_2=0}$, in fact, satisfies the Riccati–Abel equation.

With this purpose, let us calculate derivatives of g_1, g_0 under the condition

$$g_2(\phi_1, \phi_2) = 0. \tag{4.19}$$

From this equation, it follows that ϕ_1 is an implicit function of ϕ_2, *viz.*, $\phi_1 = \phi_1(\phi_2)$. Thus, we have to prove that the function

$$u(\phi_2) = -\frac{g_0(\phi_1(\phi_2), \phi_2)}{g_1(\phi_1(\phi_2), \phi_2)} \tag{4.20}$$

obeys the Riccati–Abel equation (3.1) with $\phi = \phi_2$.

Differentiating equation (4.19), we obtain

$$\frac{dg_2}{d\phi_2} + \frac{dg_2}{d\phi_1}\frac{d\phi_1}{d\phi_2} = 0, \quad \frac{d\phi_1}{d\phi_2} = -\frac{\left.\dfrac{dg_2}{d\phi_2}\right|_{g_2=0}}{\left.\dfrac{dg_2}{d\phi_1}\right|_{g_2=0}}. \tag{4.21}$$

The derivatives of g_2 with respect to ϕ_1, ϕ_2 under the constraint (4.19) we calculate by using (4.4), (4.5):

$$\begin{aligned} \left.\frac{dg_2}{d\phi_1}\right|_{g_2=0} &= (g_1 + a_1 g_2)|_{g_2=0} = g_1, \\ \left.\frac{dg_2}{d\phi_2}\right|_{g_2=0} &= (g_0 + a_1 g_1 + (a_2^2 - a_1)g_2)|_{g_2=0} = g_0 + a_1 g_1. \end{aligned} \tag{4.22}$$

By substituting this result into (4.21), we get

$$\frac{d\phi_1}{d\phi_2} = -\frac{1}{g_1}(g_0 + a_2 g_1). \tag{4.23}$$

Next, we have to calculate derivatives of g_0, g_1 with respect to ϕ_2 under the constraint (4.19). According to formulae (4.4), (4.5) and (4.23) we write:

$$\frac{dg_0}{d\phi_2} = \frac{\partial g_0}{\partial \phi_2}|_{g_2=0} + \frac{dg_0}{d\phi_1}|_{g_2=0}\frac{d\phi_1}{d\phi_2} = a_0 g_1, \tag{4.24}$$

$$\frac{dg_1}{d\phi_2} = \frac{\partial g_1}{\partial \phi_2}|_{g_2=0} + \frac{dg_1}{d\phi_1}|_{g_2=0}\frac{d\phi_1}{d\phi_2} = -a_1 g_1 - \frac{g_0}{g_1}(g_0 + a_2 g_1). \tag{4.25}$$

Now we are able to calculate the derivative of the function $u(\phi_2)$, which is defined by the fraction (4.20). Firstly, calculate derivative of the fraction:

$$\frac{d}{d\phi_2}\frac{g_0}{g_1} = \frac{1}{g_1^2}\left(g_1\frac{dg_0}{d\phi_2} - g_0\frac{dg_1}{d\phi_2}\right) = \frac{1}{g_1^3}(a_0 g_1^3 + a_1 g_1^2 g_0 + g_0^3 + a_2 g_1 g_0^2). \quad (4.26)$$

Now, rewrite this equation by taking into account the definition (4.20), where

$$u(\phi_2) = -\frac{g_0(\phi_1(\phi_2), \phi_2)}{g_1(\phi_1(\phi_2), \phi_2)}.$$

In this way we arrive to the Riccati–Abel equation:

$$\frac{du}{d\phi_2} = -a_0 + a_1 u^2 + u^3 - a_2 u^2. \quad (4.27)$$

5. THE RICCATI–ABEL EQUATION AS AN EVOLUTION EQUATION OF THE GENERALIZED DYNAMICS

In the relativistic mechanics the evolution of the energy p_0 and the momentum p are performed in such a way that the mass-shell equation remains invariant:

$$p_0^2 - p^2 = m^2, \quad (c = 1). \quad (5.1)$$

Let us consider a one-parametrical evolution of energy-momentum remaining invariant the (proper)mass m. Introduce a new variable X by $X = p_0 + m$ which obeys the following quadratic equation:

$$X^2 - 2p_0 X + p^2 = 0. \quad (5.2)$$

Let x_1, x_2 be two roots of this equation, that is to say

$$x_1 + x_2 = 2p_0, \quad x_1 x_2 = p^2, \quad x_1 - x_2 = 2m. \quad (5.3)$$

Now consider the evolution generated by the matrix solution of equation (5.2). The matrix obeying (5.2) is defined by (2.12):

$$E = \begin{pmatrix} 0 & -p^2 \\ 1 & 2p_0 \end{pmatrix}. \quad (5.4)$$

Thus the desired evolution equation is the Riccati equation

$$u^2 - 2p_0 u + p^2 = \frac{du}{d\phi}. \quad (5.5)$$

Consequently, from (2.6) we obtain

$$u(\phi, \phi_0) = m \coth(m\phi_0) - m \coth(m\phi) = p_0(\phi_0) - p_0(\phi). \qquad (5.6)$$

In the papers [13,14] (see, also, [15–17] and references therein), the classical dynamics of the third order has been suggested. The evolution in this dynamics is generated by the three order polynomial of the form

$$x^3 - 3P_1x^2 + 2P_2x - P^2 = 0. \qquad (5.7)$$

From this equation, it follows two algebraic equations connecting invariants with momentum P and energy P_1:

$$R_0 = P_1^3 - R_1P_1 - P^2, \quad R_1 = -2P_2 + 3P_1^2. \qquad (5.8)$$

The first of these equations is an analogue of the mass-shell equation (5.1).

The evolution generated by the polynomial (5.7) is the Riccati–Abel equation

$$\frac{du}{d\phi} = u^3 - 3P_1u^2 + 2P_2u - P^2. \qquad (5.9)$$

The solution of the Riccati–Abel equation is related with the evolution of energy P_1 as follows:

$$u(\phi, \phi_0) = P_1(\phi_0) - P_1(\phi). \qquad (5.10)$$

Concluding Remarks. As the ordinary Riccati equation, also the Riccati–Abel equation has a relationship with a linear differential equation. Seeking a summation formula for solutions of Riccati–Abel equation, we established a certain relationship of these solutions with multi-trigonometric functions of the third order. We have elaborated some rule according to which in order to build a summation formula for solutions of Riccati–Abel equations, it is necessary to consider a pair of solutions, which can be achieved by using an auxiliary variable. This idea can be successfully used for the solutions of generalized Riccati equations of any order with constant coefficients. By increasing the order of the nonlinearity, the number of auxiliary variables also will increase. For example, from solutions of generalized Riccati equations of the fourth order, we have to compose the triplet of solutions with two auxiliary variables, and for n-order generalized Riccati equations, it is necessary to compose a set of $(n-1)$ solutions with $(n-2)$ auxiliary variables.

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