

E2-2014-66

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REMARKS ON SIMPLE MODIFIED PERTURBATION  
THEORY

Submitted to «Письма в ЭЧАЯ»

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E2-2014-66

Заметки о простой модифицированной теории возмущений

Наша цель — предложить модификацию теории возмущений для квантовой хромодинамики (pQCD), которая регулярна в области низких энергий и может дать практический метод анализа данных ниже 1 ГэВ вплоть до инфракрасной границы. Недавно открытый рост 4-петлевых pQCD-рядов для формфактора Бьёркена в районе  $Q \lesssim 1$  ГэВ и частичное решение проблемы при помощи аналитической теории возмущений (АТВ) вплоть до  $Q \sim 0,6$  ГэВ послужили импульсом для такой попытки.

Рассматриваемая «массивная pQCD» имеет два основания. Первое состоит в том, что к pQCD добавляется только один новый параметр — эффективная «масса глобола»  $m_\rho \lesssim M_{\text{glb}} \lesssim 1$  ГэВ, которая является инфракрасным регулятором. Грубо говоря, мы вводим ее, заменяя ультрафиолетовый  $\ln Q^2$  массивным логарифмом,  $\ln(Q^2 + M_{\text{glb}}^2)$ , который регулярен в области низких энергий и является конечным в инфракрасном пределе. Второе основание связано со свободной от духов АТВ, которая включает нестепенные разложения. Последние делают теорию возмущений совместимой с линейными интегральными преобразованиями.

Работа выполнена в Лаборатории теоретической физики им. Н. Н. Боголюбова ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна, 2014

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E2-2014-66

Remarks on Simple Modified Perturbation Theory

The goal is to devise a pQCD modification that should be regular in the low-energy region and could serve practically for the data analysis below 1 GeV up to the IR limit. The recently observed “blow-up” of the 4-loop pQCD series for the Bjorken SR form factor around  $Q \lesssim 1$  GeV and partial resolving of the issue with the help of the Analytic Perturbation Theory (APT) until  $Q \sim 0.6$  GeV provided the impetus for this attempt.

The “massive pQCD” under construction has two grounds. The first is pQCD with only one parameter added, an effective “glueball mass”  $m_\rho \lesssim M_{\text{glb}} \lesssim 1$  GeV, serving as an IR regulator. Roughly, we introduce it by changing the UV  $\ln Q^2$  for a massive log,  $\ln(Q^2 + M_{\text{glb}}^2)$ , regular in the LE region and finite in the IR limit. The second stems from the ghost-free APT comprising non-power perturbative expansion that makes it compatible with linear integral transformations.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna, 2014

## 1. INTRODUCTION: MOTIVATION AND OUTLINE

The perturbative QCD (pQCD)\* is a firmly established part of the particle interaction theory. Starting with gauge-non-invariant quantization, it correlates several dozens of experiments at quite different scales from a few up to hundreds of GeV. At the same time, pQCD meets troubles in the low-energy (large-distance) domain, below a few GeV, at the scales marked by the QCD parameter  $\Lambda \lesssim 380$  MeV. This Achilles' heel is related to its UV origin essence.

To avoid the unwanted singularity in the LE region, several modifications [1–5] of the pQCD have been devised. Recently, one of them, the Analytic Perturbation Theory (APT) ([6, 7] and a later review paper [8]), has proved to be good [9] in describing the polarized  $\Gamma_1^{p-n}(Q^2) = \Gamma_1(Q^2)$  form factor of the Bjorken Sum Rules (BjSR) amplitude *down to a few hundred MeV*.

The difference of proton and neutron Bjorken moments is usually presented as a sum of PT and higher twist (HT) non-perturbative contributions

$$\Gamma_1(Q^2) = \frac{g_A}{6} [1 - \Delta_{\text{Bj}}^{\text{PT}}(Q^2)] + \Gamma_{\text{HT}}, \quad \Gamma_{\text{HT}} = \sum_{i=2}^{\infty} \frac{\mu_{2i}}{Q^{2i-2}}, \quad (1)$$

with  $\Delta_{\text{Bj}}^{\text{PT}}$ , including the  $N^3LO \sim (\alpha_s(Q^2))^4$  term. However, an attempt to fit rather precise JLab data by expression (1) with appropriate HT coefficients failed as the PT part exploded (Fig. 1) in the region 0.7–1 GeV and the extracted (via comparison with fitted JLab data)  $\mu_{2i}$  values turned out to be unstable w.r.t. higher loop terms in the first PT sum. This prevented data description below 1 GeV (Fig. 2). Along with Eq. (1), in [9] the PT sum was changed for the APT one\*\*

$$\Delta_{\text{Bj}}^{\text{PT}} = \sum_{k \leq 4} c_k (\alpha_s(Q^2))^k \Rightarrow \Delta_{\text{Bj}}^{\text{APT}}(Q^2) = \sum_{k \leq 4} c_k A_k(Q^2), \quad (2)$$

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\*The renormalization group improved QCD perturbation expansion *taken in the UV limit*.

\*\*In current paper, we change original notation  $A_k$  to  $\mathcal{A}_k$ . The notation  $A_k$  will be used only for results of previous articles just for comparison.

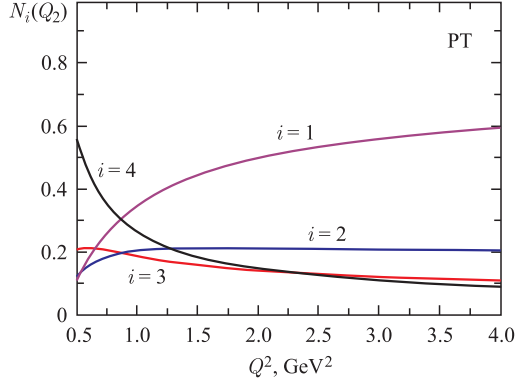


Fig. 1. Relative contributions  $N_i$  of PT terms to  $\Delta_{\text{Bj}}^{\text{PT}}(Q^2)$

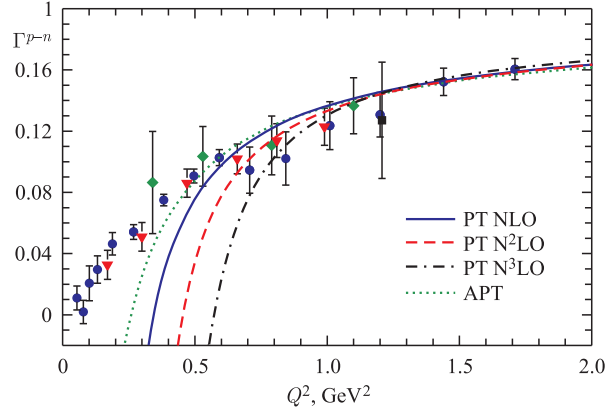


Fig. 2. The pQCD anti-progress below  $Q < 1$  GeV as  $2 \rightarrow 3 \rightarrow 4$ -loops

with  $A_k(Q^2)$ , the APT [6, 7] ghost-free expansion functions. The change resulted in good fitting of the JLab data down to a few hundred MeV (Fig. 2) with stable HT parameters.

This result begets hope for the global fitting down to the IR limit. Unfortunately, none of the above-mentioned ghost-free modifications [1–5] is suitable for this purpose. The common drawback is the use of UV logs in the IR region.

To approach the global fitting of data (like these for the BjSR form factor), one needs a modified perturbation theory (MPT) with two essential properties:

- Correspondence with common pQCD in UV (that is above a few GeV);
- Regularity and finiteness of the modified effective coupling  $\alpha_s^{\text{MPT}}(Q)$  and matrix elements in the low-energy domain.

As a primary launch pad for this construction, the above-mentioned APT seems good. It satisfies the first condition and, partially, the second one. To exempt the APT-like scheme from its last drawback — the singularity (infinite derivatives) in the IR limit, one has to disentangle it from the UV logs. To this end, in the first version of this research [10], the IR regulator has been introduced just by the shift of the  $Q^2$  scale,

$$Q^2 \rightarrow Q^2 + M_{\text{glb}}^2, \quad (3)$$

with the only fitting parameter added, an effective glueball mass,  $M_{\text{glb}}$ .

Here, we are going to address one more trick evoked by unitarity [11, 12] arguments (the  $J(x)$  differs from rapidity by factor  $\sqrt{(x+1)/x} = E/Q$ , which provides it with correct  $Q^2$ -analyticity; see, e. g., Eq. (24.4) in the text [13]):

$$\begin{aligned} \ln x \rightarrow J(x) &= 2\sqrt{\frac{x+1}{x}} \operatorname{arctanh}\left(\frac{x}{1+x}\right)^{1/2} = \\ &= 2\sqrt{\frac{x+1}{x}} \ln(\sqrt{x} + \sqrt{x+1}), \quad x = Q^2/M_{\text{glb}}^2 \end{aligned} \quad (4)$$

This “quasi-rapidity”  $J(x)$  tends to the usual UV log,  $\ln(Q^2/M_{\text{glb}}^2) + 2\ln 2 + (\ln x)/2x$ , at  $Q^2 \gg M_{\text{glb}}^2$  and behaves as  $J(x \ll 1) \sim 2 + 2x/3 - 4x^2/15$  in the IR.

## 2. GENERAL RELATIONS

**2.1. Two-Loop Massive RG Solution.** At LE (below  $m_\tau$ ) we use the two-loop massive RG solution in the denominator representation\* (see [14])

$$\mathcal{A}_1(x) = \mathfrak{a}(\phi) = \frac{\alpha_0}{1 + \alpha_0 \beta_0 \phi + \alpha_0 b \ln(1 + \alpha_0 \beta_0 \phi)}, \quad (5)$$

which corresponds to the PT expansion in the form

$$\alpha_s(x)_{\text{MPT}}^{[2]} = a_{\text{PT}}(\phi) = \alpha_0 - \alpha_0^2 \beta_0 \phi(x) + \alpha_0^3 [\beta_0 \phi(x)]^2 - \alpha_0^3 \beta_1 \phi(x) \quad (6)$$

that reflects our *ansatz on the functional equality* of one- and two-loop massive contributions which we take in a simple form

$$\phi = \ln(1 + px). \quad (7)$$

Below, we use the 3-flavor values  $\beta_0 = 0.716$ ,  $\beta_1 = 0.405$ ,  $b = \beta_1/\beta_0 = 0.566$ .

The 1-loop expression is also of interest:

$$\mathfrak{a}(\phi) \lesssim \mathfrak{a}_{[1]}(\phi) = \frac{\alpha_0}{1 + \alpha_0 \beta_0 \phi} \lesssim 1.16 \mathfrak{a}(\phi), \quad (8)$$

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\*For a more precise two-loop solution, see Subsection 2.5 below.

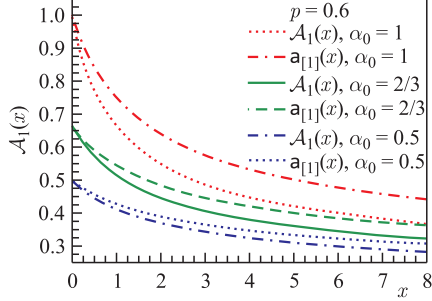


Fig. 3.  $a_{[1]}$  vs.  $\mathcal{A}_1(x) = a_{[2]}$

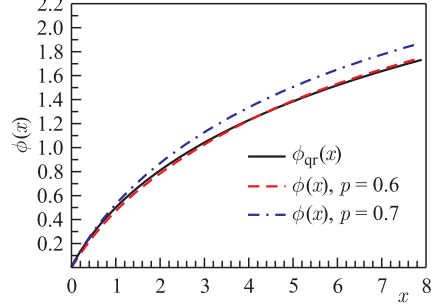


Fig. 4.  $\phi(x)$  vs.  $\phi_{qr}(x)$

with maximum *relative deviation* (see Fig. 3) from  $a_{[2]} = a(\phi)$  at  $px_{\max} = e - 1 = 1.72$ .

The  $p$  parameter can be estimated by comparing  $\phi$  (Fig. 4) with the “quasi-rapidity” model:  $\phi_{qr}(x) = 2\sqrt{\frac{x+1}{x}} \ln(\sqrt{x} + \sqrt{x+1}) - 2$ . This gives  $p \sim 0.60 \pm 0.05$ ,  $\sqrt{x_{\max}} \sim 1.7$ . For the  $\alpha_0$  *parameter assess*, one can use the BjSR threshold condition

$$\frac{\alpha_0}{\pi} + 0.363 \alpha_0^2 + 0.652 \alpha_0^3 + 1.804 \alpha_0^4 > 1 \Rightarrow \alpha_0^{\text{PT}} \gtrsim 0.69. \quad (9)$$

## 2.2. Condition on $\mathcal{A}_1(m_\tau)$ . The normalization (Conjunction) condition

$$\mathcal{A}_1(x_\tau) = 0.330 \pm 0.014 \quad (10)$$

combined with (5) yields a relation, at fixed  $p = 0.6$ , between two quantities  $\alpha_0$  and  $M_{\text{glb}}$ :

$$\begin{aligned} X_\tau + b \ln X_\tau &= 3.03 \pm 0.13 + b \ln \left( \frac{1}{\alpha_0} \right), \\ X_\tau = X_\tau(\alpha_0, M_{\text{glb}}) &= \frac{1}{\alpha_0} + b \ln \left( 1 + p \frac{m_\tau^2}{M_{\text{glb}}^2} \right). \end{aligned} \quad (11)$$

Under plausible assumption  $\alpha_0 \gtrsim 0.7$  ( $1/\alpha_0 \lesssim 1.43$ ), one gets  $X_\tau \gtrsim 2.70$ . In turn, this gives  $M_{\text{glb}} \gtrsim 450$  MeV. For more detailed information, see Figs. 5 and 6. The tendency is simple:  $\frac{dM_{\text{glb}}}{d\alpha_0} \sim -0.5$  GeV. However, one has to mind that estimate (9) in the MPT case is under suspicion as it does not account for non-powerness of the MPT expansion. Some more information on  $\mathcal{A}_1(x) = a(\phi(x))$  can be learnt from Figs. 7 and 8. From there we have

$$M_{\text{glb}}(a_0 = 1.0) \sim (512 \pm 43) \text{ MeV}, \quad M_{\text{glb}}(a_0 = 2/3) \sim (661 \pm 61) \text{ MeV}. \quad (12)$$

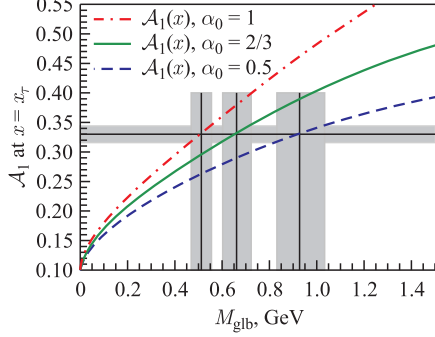


Fig. 5.  $M_{\text{glb}}$  estimate vs. three  $\alpha_0$  values

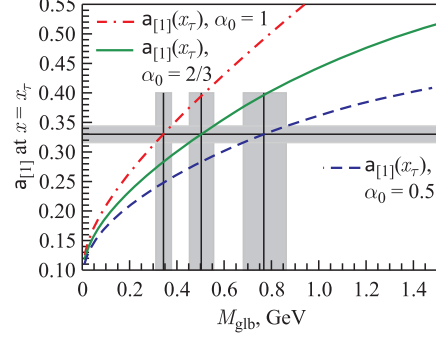


Fig. 6. The same as in Fig.5 but for the 1-loop case

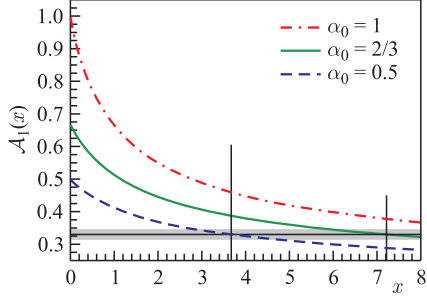


Fig. 7.  $\mathcal{A}_1(x)$  at three  $\alpha_0$  values

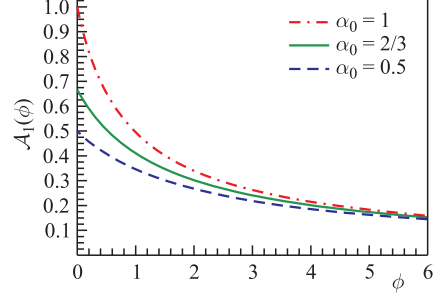


Fig. 8. The same for  $\mathcal{A}_1 = \alpha(\phi)$

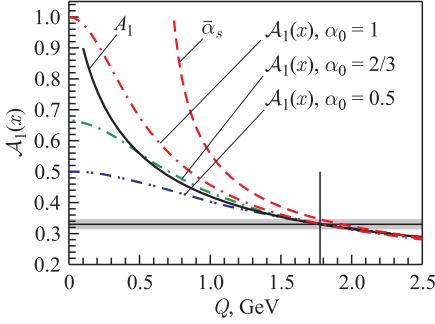


Fig. 9. Various couplings at LE domain

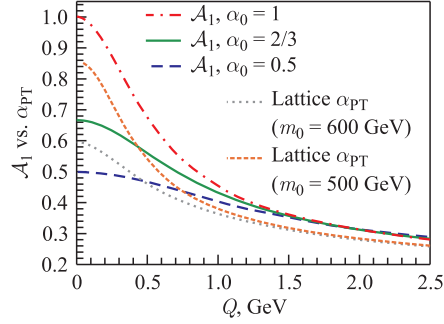


Fig. 10.  $\mathcal{A}_1(Q)$  vs. lattice results [15]

To conclude this part, we give the  $\mathcal{A}_1$  LE behavior vs. common  $\bar{\alpha}_s$  and APT  $A_1$  ones, on the one hand (Fig. 9), and vs. lattice simulation results [15] (Fig. 10), on the other hand. From Fig.8 one can get additional, process-independent, argument in favour of Eq. (9) numerical estimate — see Tables 2, 3 and the text afterwards.

**2.3. Recurrent Relation.** In the construction under devising, we intend to preserve an essential APT feature, namely, the non-polynomiality of “perturbative” MPT — expansion over a set (the same symbol  $\mathcal{A}$  as in the minimal APT is used) of functions ( $\mathcal{A}_k(Q^2)$ ).

2.3.1. *Differential Recurrency.* In the APT, higher functions are connected by the differential recurrent relation

$$-\frac{x}{k} \frac{\partial}{\partial x} \mathcal{A}_k(x) = \beta_0 \mathcal{A}_{k+1}(x) + \beta_1 \mathcal{A}_{k+2}(x) + \dots \quad (13)$$

To the arguments ascending to the 80s (see Ref. 11 in [16], and papers [17, 18]) and related to the  $\pi^2$ -term summation procedure in the s-channel (see also [1–5]), one can add a fresher reasoning [19, 20].

This differential recurrency ensures compatibility [21–23] with linear transformations involved in transition to the distance picture (Fourier-conjugated with the momentum-transfer one) and to the annihilation s-channel (reverse Adler transformation).

2.3.2. *Comment on Eq. (13).* This simple recurrent operator (the log derivative) is an *ansatz* mainly motivated by the structure of  $\pi^2$ -terms [17, 18] generated, in turn, by the UV log branching. Generally, it is an open question how to modify this log-derivative for the mass-dependent structures relevant to the LE domain. E.g., one can write instead

$$-\frac{1}{k} \bar{\partial}_\rho \mathcal{A}_k(x) := \beta_0 \mathcal{A}_{k+1}(x) + \beta_1 \mathcal{A}_{k+2}(x) + \dots, \quad \bar{\partial}_\rho = \frac{d}{d\rho(x)}. \quad (14)$$

Here,  $\rho(x)$  is an appropriate function, defined for a real positive argument, with log asymptotics.

For the recurrence function, we start with the *simplest* option without additional parameter

$$\begin{aligned} \rho(x) &= \phi(x) - \ln p = \ln(1 + px) - \ln p = \ln(x + x_*); \\ x &= Q^2/M_{\text{glb}}^2, \quad x_* = 1/p. \end{aligned} \quad (15)$$

Then, technically

$$\bar{\partial}_\rho = \bar{\partial}_\phi := \bar{\partial}, \quad \bar{\partial} \mathcal{A}_k(x) = (x + x_*) \mathcal{A}_k'(x), \quad (16)$$

and

$$\bar{\partial}^2 \mathcal{A}_k(x) = (x + x_*)^2 \mathcal{A}_k''(x) + (x + x_*) \mathcal{A}_k'. \quad (17)$$

Our recurrent ansatz

$$\boxed{\beta_0 \mathcal{A}_{k+1}(x) = -\frac{1}{k} \bar{\partial} \mathcal{A}_k(x) - \beta_1 \mathcal{A}_{k+2}, \quad \mathcal{A}_{k \geq 5} = 0} \quad (18)$$



leads to three equations for the 3 expansion functions  $\mathcal{A}_{2,3,4}$  in terms of the given  $\mathcal{A}_1(x) = a(\phi)$ ,

$$\begin{aligned} \Rightarrow \mathcal{A}_2 &= -\frac{1}{\beta_0} \bar{\partial} a(\phi) - b \mathcal{A}_3, \\ \mathcal{A}_3 &= -\frac{1}{2\beta_0} \bar{\partial} \mathcal{A}_2 - b \mathcal{A}_4, \quad \mathcal{A}_4 = -\frac{1}{3\beta_0} \bar{\partial} \mathcal{A}_3. \end{aligned} \quad (19)$$

**2.4. Higher SiMPT Functions.** Higher Simplest MPT (SiMPT) expansion functions  $\mathcal{A}_{2,3,4}$  are defined by Eqs. (19). Their combination yields linear DE for  $\mathcal{A}_3$  (in the  $\phi$  variable):

$$\mathcal{A}_3(x) = \frac{1}{2\beta_0^2} \bar{\partial}^2 a(\phi) + \frac{5b}{6\beta_0} \bar{\partial} \mathcal{A}_3(x), \quad \bar{\partial} \mathcal{A}_3 - k \mathcal{A}_3 = -f(\phi) \quad (20)$$

$$\left( k = \frac{6\beta_0}{5b} = 1.52, \quad q = \frac{3}{5\beta_1} = 1.48, \quad f(\phi) = q \bar{\partial}^2 a(\phi) \right). \quad (21)$$

Its solution with AF boundary condition

$$\begin{aligned} \mathcal{A}_3(x) = a_3(\phi) &= q e^{k\phi} \int_{\phi}^{\infty} \bar{\partial}^2 a(f) e^{-kf} df = \\ &= -q [\bar{\partial} a(\phi) + k a(\phi)] + k q I(\phi) \end{aligned} \quad (22)$$

contains integral

$$I(\phi) = k \int_{\phi}^{\infty} a(f) e^{k(\phi-f)} df, \quad (23)$$

which is calculated numerically — see Fig. 11 and Table 1. Below, in Figs. 12 and 13, all higher functions  $\mathcal{A}_{2,3,4}$  are given numerically via Eqs. (19).

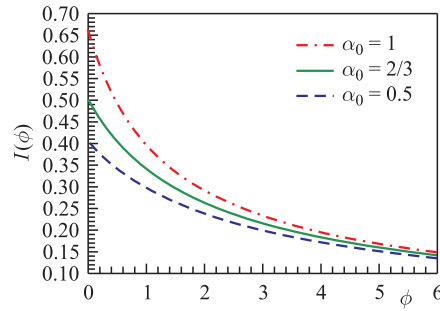


Fig. 11. Integral  $I(\phi)$

Table 1. The values of integral  $I(\phi)$  for definite  $\phi$  at fixed parameter  $\alpha_0$

$\alpha_0$	$\phi$												
	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0	5.5	6.0
0.5	0.667	0.492	0.397	0.335	0.292	0.259	0.233	0.213	0.196	0.181	0.169	0.158	0.149
1	0.502	0.404	0.342	0.297	0.263	0.237	0.216	0.198	0.183	0.171	0.160	0.150	0.142
2/3	0.402	0.340	0.296	0.263	0.237	0.218	0.198	0.184	0.171	0.160	0.151	0.142	0.135

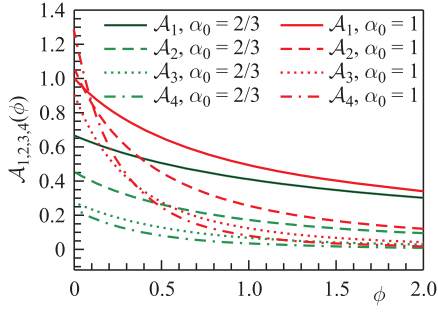


Fig. 12. Functions  $\mathcal{A}_{1,2,3,4}(\phi)$

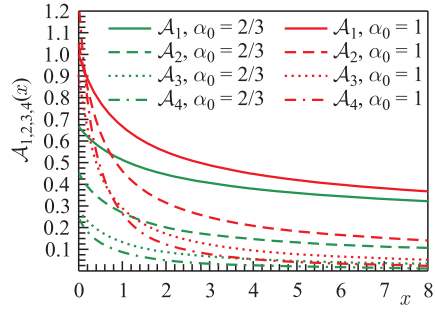


Fig. 13. Functions  $\mathcal{A}_{1,2,3,4}(x)$

Table 2. Quantities  $\mathcal{A}_{1,2,3,4}$  as functions of  $\phi$  and  $x$

$\phi$	$x$	$\alpha_0 = 1$				$\alpha_0 = 2/3$					
		$Q, \text{ GeV}$	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_4$	$Q, \text{ GeV}$	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_4$
0.00	0.00	0.00	1.000	1.050	0.911	1.289	0.00	0.667	0.454	0.279	0.240
0.15	0.27	0.27	0.858	0.774	0.600	0.717	0.34	0.607	0.377	0.215	0.166
0.25	0.47	0.35	0.786	0.649	0.470	0.509	0.46	0.574	0.337	0.183	0.133
0.50	1.08	0.53	0.653	0.447	0.278	0.245	0.69	0.505	0.261	0.127	0.080
1.00	2.86	0.87	0.495	0.255	0.125	0.080	1.12	0.410	0.172	0.069	0.035
1.25	4.15	1.04	0.443	0.204	0.090	0.051	1.35	0.376	0.144	0.054	0.024
1.50	5.80	1.23	0.402	0.168	0.068	0.034	1.59	0.347	0.123	0.042	0.018
1.70	7.46	1.40	0.375	0.145	0.055	0.026	1.81	0.328	0.109	0.036	0.014

Now, one can verify the preliminary estimate  $\alpha_0^{\text{PT}} \sim 0.686$ , Eq. (9). Towards this end, on the basis of Tables 2 and 3, we found the PT sum

$$\Delta_{\text{MPT}}(Q) = \sum_k c_k \mathcal{A}_k \frac{1}{\pi} \mathcal{A}_1 + 0.363 \mathcal{A}_2 + 0.652 \mathcal{A}_3 + 1.804 \mathcal{A}_4, \quad (24)$$

values at the IR limit:

$$\Delta_{\text{MPT}}(Q = 0; \alpha_0 = 1) = 3.619, \text{ and } \Delta_{\text{MPT}}(Q = 0; \alpha_0 = 2/3) = 0.991. \quad (25)$$

Linear extrapolation gives slightly more accurate bounds

$$\alpha_0^{\text{SiMPT}} \lesssim 0.67, \quad M_{\text{glb}} \gtrsim 650 \text{ MeV}. \quad (26)$$

Table 3. Quantities  $\mathcal{A}_{1,2,3,4}$  as functions of  $Q$ 

$Q$ , GeV	$\alpha_0 = 1$				$\alpha_0 = 2/3$				$\alpha_0 = 1/2$			
	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_4$	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_4$	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_4$
0.00	1.000	1.050	0.911	1.289	0.66(6)	0.454	0.279	0.239	0.500	0.253	0.120	0.073
0.20	0.911	0.873	0.707	0.904	0.644	0.424	0.253	0.209	0.494	0.247	0.116	0.070
0.40	0.747	0.586	0.408	0.418	0.591	0.357	0.198	0.149	0.477	0.230	0.105	0.061
0.60	0.612	0.392	0.231	0.189	0.530	0.288	0.146	0.097	0.454	0.209	0.091	0.050
0.80	0.519	0.280	0.143	0.097	0.476	0.232	0.107	0.063	0.428	0.186	0.077	0.040
1.00	0.454	0.214	0.097	0.056	0.432	0.191	0.081	0.043	0.404	0.165	0.065	0.031
1.40	0.374	0.145	0.055	0.026	0.369	0.139	0.051	0.023	0.361	0.132	0.047	0.020
1.78	0.330	0.112	0.038	0.015	0.330	0.111	0.037	0.014	0.330	0.110	0.036	0.014

The last limitation on  $M_{\text{glb}}$  does not contradict the current lattice estimate (see review [24]).

**2.5. On Exact Two-Loop Massive Solution.** The massive RG solution, more precise than (5), is expressible via a special Lambert function  $W(z)$ .

*In this connection, remind that the Lambert function naturally arises in the course of exact solving of the two-loop RG equation for the running coupling in the massless, pure log case. Under simple reservation, the same is true for the massive case.*

To this end, we analyze massive renormalization group (MRG) solution for the functional equality input. Omitting technical details (see [25, 26]), one gets the transcendental relation for  $\alpha_s(x)_{\text{MRG}}^{[2]} = a(\phi)$ :

$$\begin{aligned}
 F(\alpha_s(x)_{\text{MRG}}^{[2]}) &= F(\alpha_0) + \beta_0 \phi(x), \\
 F(a) &= -\beta_0 \int^a \frac{da}{\beta(a)}; \quad \beta(a) = \beta_0 a^2 + \beta_1 a^3.
 \end{aligned}
 \tag{27}$$

If, as commonly, the expansion  $\beta_0/\beta(a) \sim 1/a^2 - b/a$  is used, then  $F_{\text{ex}}(a) = 1/a + b \ln a = V_+(a)$ . Hence,

$$\alpha_s(x)_{\text{MRG},1}^{[2]} = F_1^{-1}(F_1(\alpha_0) + \beta_0 \phi(x))
 \tag{28}$$

At the same time, a straightforward calculation of the integral in (27) yields

$$F_2(a) = \frac{1}{a} - b \ln \left( \frac{1}{a} + b \right) := V_- \left( \frac{1}{a} + b \right) - b
 \tag{29}$$

and

$$\alpha_s(x)_{\text{MRG},2}^{[2]} = F_2^{-1}(F_2(\alpha_0) + \beta_0 \phi(x)).
 \tag{30}$$

The functions reverse to the just introduced  $V_{\pm}$  are simply related to the Lambert function.

### 3. LONGITUDINAL AND TRANSVERSE SPIN STRUCTURE AT LOW $Q^2$

For the purpose of a smooth continuation of  $\Gamma_1^{p,n}(Q^2)$  to the non-perturbative region  $0 \leq Q^2 \lesssim \Lambda_{\text{QCD}}^2$  [27, 28], it is convenient to consider the  $Q^2$ -evolution of the integral

$$I_1(Q^2) \equiv \frac{2M^2}{Q^2} \Gamma_1(Q^2) = \frac{2M^2}{Q^2} \int_0^1 dx g_1(x, Q^2), \quad (31)$$

which is equivalent to the integral over all energies of the spin-dependent photon–nucleon cross section, whose value at  $Q^2 = 0$  is defined by the GDH sum rule [29, 30]

$$I_1(0) = -\frac{\mu_A^2}{4}, \quad (32)$$

where  $\mu_A$  is the nucleon anomalous magnetic moment. Then, the function  $I_1(Q^2)$  can be written as a difference,

$$I_1(Q^2) = I_T(Q^2) - I_2(Q^2), \quad (33)$$

between the transverse and “longitudinal minus transverse” structures where

$$\begin{aligned} I_T(Q^2) &= \frac{2M^2}{Q^2} \Gamma_T(Q^2) \equiv \frac{2M^2}{Q^2} \int_0^1 dx g_T(x, Q^2), \\ I_2(Q^2) &= \frac{2M^2}{Q^2} \int_0^1 dx g_2(x, Q^2). \end{aligned} \quad (34)$$

The well-known Burkhardt–Cottingham (BC) sum rule [31] provides us with an exact expression for  $I_2(Q^2)$ , in terms of familiar electric  $G_E$  and magnetic  $G_M$  Sachs form factors as

$$I_2(Q^2) = \frac{1}{4} \mu G_M(Q^2) \frac{\mu G_M(Q^2) - G_E(Q^2)}{1 + Q^2/4M^2}, \quad (35)$$

where  $\mu$  is the nucleon magnetic moment. As a consequence of the strong  $Q^2$  behavior of the r.h.s. of Eq. (35), we get for large  $Q^2$

$$\int_0^1 g_2(x, Q^2) dx \Big|_{Q^2 \rightarrow \infty} = 0, \quad (36)$$

so  $I_2$  is much smaller than  $I_1$  for large  $Q^2$ . Now from the BC sum rule (35), it follows that

$$I_2(0) = \frac{\mu_A^2 + \mu_A e}{4}, \quad (37)$$

where  $e$  is the nucleon charge. The GDH value (32) is then reproduced with

$$I_T(0) = \frac{\mu_A e}{4}. \quad (38)$$

This slope is essentially larger than the one for  $I_1$ , explaining the observed excess of the slope of the latter over the GDH value. In practice, as there is currently no evidence that the BC sum rule has any perturbative or non-perturbative corrections, one may apply the SiMPT series to  $\Gamma_T(Q^2)$  instead of  $\Gamma_1(Q^2)$ , which will allow matching to the GDH value.

#### 4. LAMBERT FUNCTION

Generally, the multi-branch Lambert function  $W$  of the complex variable  $z$  is defined by the relation

$$W(z)e^{W(z)} = z, \quad W(z) + \ln W(z) = \ln z. \quad (39)$$

One of its asymptotics is

$$W(z)|_{z \rightarrow \infty} \rightarrow \ln z - \ln(\ln z). \quad (40)$$

Of our interest is a particular branch,  $W_{-1}(z)$ , real on a real negative semi-axis  $-z = t > 1/e$ . Farther in this text we shall omit lower index, always having in mind this branch. Then, in the massless case (see [32, 33] and references therein)

$$\alpha_s^{[2]}(Q^2) = -\frac{\beta_0}{\beta_1} \frac{1}{1 + \mathcal{W}(z)}, \quad z = -\exp\left(-\frac{L}{B} - 1\right); \quad (41)$$

$$L = \ln\left(\frac{Q^2}{\Lambda^2}\right), \quad B = \frac{\beta_1}{\beta_0^2}.$$

According to (33), its asymptotics can be presented as follows:

$$\alpha_s^{[2]}(Q^2) \sim \frac{1}{\beta_0 L + \beta_1 \ln L}. \quad (42)$$

In our massive SiMPT case, we can use this expression with the change of the log argument,  $L \rightarrow \phi(x)$ , that is  $\ln x$  to  $\ln(x + 1/p)$ . For a quantitative estimate, look at the numerical table of Lambert in spirit of Eqs. (4) and (15).

See page 31 in my P2-2008-107 (Russian-language preprint) and earlier papers in TMP (1981. V. 49. P. 291–297; Engl. transl.: P. 1039–1042); preprint E2-81-274. There, in the course of solving, one meets an equation like

$$\begin{aligned} \beta_0 \phi(x) &= J(A(x)) - J(a_0) = \\ &= -\beta_0 \int_{a_0}^{\mathcal{A}_1(x)} \frac{dg}{\beta_0 g^2 + \beta_1 g^3} = - \int_{a_0}^{\mathcal{A}_1(\phi(x))} \frac{dg}{g^2 + b g^3}, \quad b = \frac{\beta_1}{\beta_0} = 0.566. \end{aligned} \quad (43)$$

If the integrand is expanded  $\frac{1}{g^2 + b g^3} \simeq 1/g^2 - b/g$ , or exact  $\beta(g)$  is equal to a formally expanded expression (like in some 2-dim soluble models), then after integration

$$J(A) = \frac{1}{A} + b \ln A = b \left( \frac{1}{bA} - \ln \frac{1}{bA} \right) - b \ln b, \quad (44)$$

$$\frac{1}{b} \left( J(A) + b \ln b \right) = \frac{1}{bA} - \ln \frac{1}{bA}, \quad (45)$$

exponentiating the last relation, we get

$$e^{\phi(x)/B + J(a_0)/b + \ln b} = e^{J(A)/b + \ln b} = bA e^{1/bA}, \quad \text{where } B = \frac{\beta_1}{\beta_0^2} = 0.790, \quad (46)$$

where we used (43) to express  $J(A)$  in terms of  $J(a_0)$ . After rearrangement of (46) we get

$$bA e^{1/bA} = e^{\phi(x)/B + 1/b a_0 - \ln 1/b a_0} = b a_0 e^{\phi(x)/B + 1/b a_0}, \quad (47)$$

and, using the definition of the Lambert function  $W(z)$  (i. e.,  $W(z) e^{W(z)} = z$ ), one can write the result as

$$bA(x) = - \frac{1}{W \left( - \frac{1}{b a_0} e^{-\phi(x)/B - 1/b a_0} \right)} \quad (48)$$

If not expanded, it is still calculable and expressible (a bit differently!) via the  $W$  Lambert function

$$\frac{1}{g^2 + b g^3} = \frac{1}{g^2} - \frac{b}{g} + \frac{b^2}{1 + b g}, \quad J_2(A) = \frac{1}{A} - b \ln \left( \frac{1}{A} + b \right), \quad (49)$$

$$b \left( \frac{1}{bA} + 1 - \ln \left( \frac{1}{bA} + 1 \right) \right) - b(1 + \ln b). \quad (50)$$

Again after exponentiation we get

$$e^{\phi(x)/B + J_2(a_0)/b + 1 + \ln b} = e^{J_2(A)/b + 1 + \ln b} = (1/bA + 1)^{-1} e^{1/bA + 1}. \quad (51)$$

And the solution is the following:

$$bA(x) = - \frac{1}{W \left( - \frac{1 + b a_0}{b a_0} e^{-\phi(x)/B - 1/b a_0 - 1} \right) + 1}. \quad (52)$$

The difference  $\Delta$  between these two expressions

$$\begin{aligned}
 A_1(x) &= -\frac{1}{b W\left(-\frac{1}{b a_0} e^{-\phi(x)/B-1/b a_0}\right)}, \\
 A_2(x) &= -\frac{1}{b \left[ W\left(-\frac{1+b a_0}{b a_0} e^{-\phi(x)/B-1/b a_0-1}\right) + 1 \right]}
 \end{aligned} \tag{53}$$

could be essential when the cubical term in  $\beta(g)$  is a leading one. This can happen in the LE region at  $a_0 \sim 1$ . There, the difference  $A_2(x) - A_1(x)$  is positive and could reach several tenths.

#### 4.1. Auxiliary Relations

$$\begin{aligned}
 x \pm b \ln x &= b \left( \frac{1}{b a} \pm \ln b a \right) \mp b \ln b, \quad t + \ln t \rightarrow t e^t, \\
 t - \ln t &= -(-t + \ln t) \rightarrow \tau e^\tau, \quad \tau = -t.
 \end{aligned} \tag{54}$$

**4.2. Numerical Example.** Suppose  $a_0 = 0.8$ . Then,  $b \bar{a} = 0.566(1 - 0.522) = 0.566 \cdot 0.477855 = 0.270$ ,  $1/b \bar{a} = 3.697, \dots$

## 5. OUTLOOK: ON 3-LOOP LONG-LOG

One more way to realize the accuracy of SiMPT is to estimate the possible influence of three-loop effects. This can be done with a specific trick proposed in [34]:

$$L \rightarrow L^* = L + B \sqrt{L^2 + 2\pi^2}. \tag{55}$$

**Acknowledgements.** This work was supported in part by RFBR Grant 14-01-00647.

The manuscript was prepared for publication with the participation of P. Fiziev, S. V. Mikhailov and O. V. Teryaev.

## REFERENCES

1. *Shirkov D., Mikhailov S.* // Z. Phys. C. 1994. V. 63. P. 463; arXiv:hep-ph/9401270.
2. *Simonov Y.* // Phys. Atom. Nucl. 1995. V. 58. P. 107; arXiv:hep-ph/9311247.
3. *Badalian A., Simonov Y.* // Phys. Atom. Nucl. 1997. V. 60. P. 630.
4. *Shirkov D., Sidorov A., Mikhailov S.* arXiv:hep-ph/9707514. 1997.

5. *Nesterenko A., Papavassiliou J.* // J. Phys. G. 2006. V.32. P.1025; arXiv:hep-ph/0511215.
6. *Shirkov D., Solovtsov I.* // Phys. Rev. Lett. 1997. V.79. P.1209; arXiv:hep-ph/9704333.
7. *Solovtsov I., Shirkov D.* // Theor. Math. Phys. 1999. V.120. P.1220; arXiv:hep-ph/9909305.
8. *Shirkov D., Solovtsov I.* // Theor. Math. Phys. 2007. V.150. P.132; arXiv:hep-ph/0611229.
9. *Khandramai V. et al.* // Phys. Lett. B. 2012. V.706. P.340; arXiv:1106.6352.
10. *Shirkov D.* // Phys. Part. Nucl. Lett. 2013. V.10. P.186; arXiv:1208.2103.
11. *Lipatov L.* // Nucl. Phys. B. 1988. V.309. P.379.
12. *Shirkov D.* // Phys. Atom. Nucl. 1999. V.62. P.1928; arXiv:hep-ph/9903431.
13. *Bogolyubov N., Shirkov D.* Quantum Fields. Benjamin/Cummings, USA, 1983.
14. *Shirkov D.* // Theor. Math. Phys. 1982. V.49. P.1039.
15. *Aguilar A. et al.* // Phys. Rev. D. 2009. V.80. P.085018; arXiv:0906.2633.
16. *Radyushkin A.* // JINR Rapid Commun. 1996. No.78. P.96; arXiv:hep-ph/9907228.
17. *Krasnikov N., Pivovarov A.* // Phys. Lett. B. 1982. V.116. P.168.
18. *Jones H., Solovtsov I.* // Phys. Lett. B. 1995. V.349. P.519; arXiv:hep-ph/9501344.
19. *Shirkov D.* // Theor. Math. Phys. 1999. V.119. P.438; arXiv:hep-th/9810246.
20. *Shirkov D.* // Lett. Math. Phys. 1999. V.48. P.135.
21. *Shirkov D.* // Theor. Math. Phys. 2001. V.127. P.409; arXiv:hep-ph/0012283.
22. *Shirkov D.* // Theor. Math. Phys. 2003. V.136. P.893; arXiv:hep-th/0210013.
23. *Shirkov D. V.* // Nucl. Phys. Proc. Suppl. 2006. V.152. P.51; arXiv:hep-ph/0408272.
24. *Ochs W.* // J. Phys. G. 2013. V.40. P.043001; arXiv:1301.5183.
25. *Shirkov D.* // Nucl. Phys. B. 1992. V.371. P.467.
26. *Shirkov D.* // Theor. Math. Phys. 1992. V.93. P.1403.
27. *Soffer J., Teryaev O.* // Phys. Rev. Lett. 1993. V.70. P.3373.
28. *Soffer J., Teryaev O.* // Phys. Rev. D. 2004. V.70. P.116004; arXiv:hep-ph/0410228.
29. *Gerasimov S.* // Sov. J. Nucl. Phys. 1966. V.2. P.430.
30. *Drell S., Hearn A. C.* // Phys. Rev. Lett. 1966. V.16. P.908.
31. *Burkhardt H., Cottingham W.* // Annals Phys. 1970. V.56. P.453.
32. *Magradze B.* // Conf. Proc. C. 1999. V.980518. P.158; arXiv:hep-ph/9808247.
33. *Magradze B.* // Few Body Syst. 2006. V.40. P.71; arXiv:hep-ph/0512374.
34. *Shirkov D., Zayakin A.* // Phys. Atom. Nucl. 2007. V.70. P.775; arXiv:hep-ph/0512325.

Received on September 15, 2014.



Редактор *Е. И. Кравченко*

Подписано в печать 27.10.2014.

Формат 60 × 90/16. Бумага офсетная. Печать офсетная.

Усл. печ. л. 1,06. Уч.-изд. л. 1,49. Тираж 325 экз. Заказ № 58363.

Издательский отдел Объединенного института ядерных исследований  
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