Modification of Klauder's coherent states

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A modified version of Klauder's coherent state is presented. Klauder's state is a generalized coherent state that can be constructed in terms of the energy eigenstates of a given non-degenerate system without referring to any symmetry group. It can be formed for continuous as well as discrete dynamics. The proposed modification allows us to deal with degenerate systems and to treat discrete states and continuous states in a unified manner. Some examples are given for illustration.

1 Introduction

As is well-known, the standard coherent states $|z\rangle \in \mathcal{H}$ for the harmonic oscillator [1] are defined, for all $z \in \mathbf{C}$, by one of the following equivalent relations;

$$\hat{a}|z\rangle = z|z\rangle \tag{1}$$

$$|z\rangle = \hat{D}(z)|0\rangle \quad \text{with} \quad \hat{D}(z) = \mathrm{e}^{z\hat{a}^\dagger - z^*\hat{a}}\,, \tag{2}$$

$$|z\rangle = \mathrm{e}^{-|z|^2/2}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}\,|n\rangle\,, \tag{3}$$

where $[\hat{a}, \hat{a}^\dagger] = \hat{1}$, $\hat{a}^\dagger\hat{a}|n\rangle = n|n\rangle$ $(n = 0, 1, 2, \ldots)$, and $\hat{a}|0\rangle = 0$, as usual. The coherent states admit a resolution of unity,

$$\hat{1} = \int_{\mathbf{C}} |z\rangle\langle z|\,\frac{\mathrm{d}x\,\mathrm{d}y}{\pi}\,,$$

where $x = \mathrm{Re}\,z$ and $y = \mathrm{Im}\,z$. If the Hamiltonian is given by $\hat{H} = \omega\hat{a}^\dagger\hat{a}$, then the coherent states are temporally stable in the sense that

$$\mathrm{e}^{-\mathrm{i}\hat{H}t}|z\rangle = \left|\mathrm{e}^{-\mathrm{i}\omega t}z\right\rangle.$$

In generalizing the harmonic oscillator coherent states, we can start by choosing one of the above definitions. For instance, Barut and Girardello [2], noticing that the oscillator coherent states as defined by (1) are eigenstates of the non-compact operator $\hat{a}$ constructed a set of generalized coherent states for the group $\mathrm{SU}(1,1)$ as eigenstates of the non-compact operator of $\mathrm{SU}(1,1)$. Realizing that the displacement operator $\hat{D}(z)$ in relation (2) is a representation of the Heisenberg–Weyl group, Perelomov defined a set of generalized coherent states for any semi-simple group G with an isotropy subgroup H by $|x(g)\rangle = \hat{T}(g)|0\rangle$. Here $x(g)$ with $g \in G$ is a point in the coset space G/H, and $\hat{T}(g)$ is a representation of G acting in the Hilbert space $\mathcal{H}$. In most attempts, generalized coherent states are based on some group structures.

In [3], Klauder gave up the group, and generalized the relation (3) by stipulating that coherent states are (i) continuous in their parameters, (ii) admit a resolution of unity, and (iii) are temporally stable (i.e., evolution among themselves in time). Then he proposed a set of coherent states for a nondegenerate system with an energy spectrum $E_n = \omega e_n$ ($n = 0, 1, 2, \ldots; e_0 = 0$), which are labelled by two real parameters s ($0 \le s < \infty$) and γ ($-\infty < \gamma < \infty$) as

$$|s,\gamma\rangle = M(s^2)\sum_{n=0}^{\infty}\frac{s^n \mathrm{e}^{-\mathrm{i}\gamma e_n}}{\sqrt{\rho_n}}\,|n\rangle\,, \tag{4}$$

where $|n\rangle$ is the eigenstate belonging to E_n and ρ_n is the n-th moment of a probability distribution function $\rho(u) > 0$,

$$\rho_n = \int_0^\infty u^n\,\rho(u)\,\mathrm{d}u\,. \tag{5}$$

For Klauder's coherent states (4), the choice of $\rho(s^2)$ is arbitrary. The normalization factor $M(s^2)$ is determined so as to satisfy $\langle s,\gamma|s,\gamma\rangle = 1$, namely,

$$M(s^2)^{-2} = \sum_{n=0}^{\infty}\frac{s^{2n}}{\rho_n}\,.$$

With the Hamiltonian $\hat{H}$ such that $\hat{H}|n\rangle = \omega e_n|n\rangle$, it is apparent that

$$\mathrm{e}^{-\mathrm{i}\hat{H}t}|s,\gamma\rangle = |s,\gamma+\omega t\rangle\,,$$

which is taken in [3] as the exhibition of temporal stability of the coherent states. The states satisfy the resolution of unity,

$$\int \mathrm{d}\mu(s,\gamma)\,|s,\gamma\rangle\langle s,\gamma| = \hat{1}_{\mathrm{dis}}$$

with a measure $\mu(s,\gamma)$ defined by

$$\int \mathrm{d}\mu(s,\gamma)\,f(s,\gamma) = \lim_{\Gamma\to\infty}\frac{1}{2\Gamma}\int_0^\infty k(s^2)\,\mathrm{d}s^2\int_{-\Gamma}^{\Gamma}\mathrm{d}\gamma\,f(s,\gamma) \tag{6}$$

provided that

$$\lim_{\Gamma\to\infty}\frac{1}{2\Gamma}\int_{-\Gamma}^{\Gamma}\mathrm{d}\gamma\,\mathrm{e}^{\mathrm{i}\gamma(e_n-e_{n'})} = \delta_{n,n'}\,,$$

that is, that all e_n are distinct (no degeneracies). In (6),

$$k(s^2) = \frac{\rho(s^2)}{M(s^2)^2}\,,$$

which remains unspecified until the form of $\rho(s^2)$ in (5) is given. Later, Gazeau and Klauder [5], letting $s^2 = J$, imposed an additional condition, called the action identity [5],

$$\langle J,\gamma|\hat{H}|J,\gamma\rangle = \omega J\,, \tag{7}$$

which leads ρ_n to the form,

$$\rho_n = \prod_{j=1}^{n} e_j \,, \quad \rho_0 = 1 \,.$$

By this condition, they interpret the parameter J as the classical action variable conjugate to the angle variable ω.

Gazeau and Klauder [5] also proposed coherent states for continuum dynamics. For a Hamiltonian with a non-degenerate continuous spectrum $0 < \omega\varepsilon < \omega\bar{\varepsilon}$, the proposed coherent states take the form,

$$|s, \gamma\rangle = M(s^2) \int_0^{\bar{\varepsilon}} \frac{s^{\varepsilon} \mathrm{e}^{-\mathrm{i}\gamma\varepsilon}}{\sqrt{\rho(\varepsilon)}} \, |\varepsilon\rangle \mathrm{d}\varepsilon \,, \tag{8}$$

where

$$M(s^2)^{-2} = \int_0^{\bar{\varepsilon}} \frac{s^{2\varepsilon}}{\rho(\varepsilon)} \, \mathrm{d}\varepsilon$$

to meet $\langle s, \gamma | s, \gamma \rangle = 1$ for $0 \leq s < \bar{s}$. The function $\rho(\varepsilon)$ in (8) is determined with an appropriate non-negative weighting function $\sigma(s) \geq 0$ as

$$\rho(\varepsilon) = \int_0^{\bar{\varepsilon}} s^{2\varepsilon} \sigma(s) \, \mathrm{d}s \,. \tag{9}$$

These coherent states for a continuous spectrum evolve in time among themselves. With $\mathrm{d}\mu(s, \gamma) = (1/2\pi) M(s)^{-2} \sigma(s) \, \mathrm{d}s \, \mathrm{d}\gamma$, the resolution of unity,

$$\int \mathrm{d}\mu(s, \gamma) |s, \gamma\rangle\langle s, \gamma| = \hat{1}_{\mathrm{cont}} \,,$$

is fulfilled. In [5], the resolution of unity is set up independently for the discrete and the continuous case.

The major merit of Klauder's coherent states is that these states can be constructed by means of energy eigenstates for any physical system. However, there are some shortcomings; Klauder's states cannot be used for degenerate systems and the way (à la Gazeau–Klauder) of constructing continuous coherent states is somewhat unnatural. Furthermore, it is questionable that the action identity necessarily leads to the classical action–angle variable interpretation. Indeed the parameter $J = s^2$ plays a role of the action variable for the harmonic oscillator. If $\langle J, \gamma | \hat{H} | J, \gamma \rangle$ corresponds to the classical energy E_{cl}, then $J = E_{\mathrm{cl}}/\omega$ is the adiabatic invariant. Application of the Bohr–Ishiwara–Sommerfeld–Wilson quantization $J = n$ $(n = 0, 1, 2, \ldots)$ results in $E_{\mathrm{cl}} \to e_n = n$, which is true only for the harmonic oscillator.

In the present paper, we modify Klauder's coherent states so as to accommodate degenerate systems and to treat the discrete and continuous states together in a unified manner. We give up the action identity. We follow closely to the formal structures of the harmonic oscillator coherent states, replacing the action identity by the condition that the distribution function has the universal form $\rho(u) = \mathrm{e}^{-u}$ not only for the harmonic oscillator but also for all other relevant systems. For simplicity, we abbreviate now on the term *coherent states* by CS.

2 Generalized coherent states

We begin by considering a system with the Hamiltonian $\hat{H}$. Let $\hat{H}|n\rangle = E_n|n\rangle$. As in [3], let the energy spectrum be expressed in the form $E_n = \omega[n]$ ($n = 0, 1, 2, \ldots$); $[0] = 0$), where $[n]$ is a dimensionless energy spectrum identical with e_n in [3]. The main reason, why e_n is replaced by $[n]$ is to emphasize the close affinity of the present generalized CS to the standard harmonic oscillator CS. In this sense, we treat $[n]$ as a generalized number. However, the integral number n only labels the energy levels and is not necessarily referring to a particular quantum number.

By utilizing the generalized number $[n]$, we propose a set of generalized coherent states,

$$|Z\rangle = N(Z) \sum_{n=0}^{\infty} \frac{Z^{[n]}}{\sqrt{\rho_n}} |n\rangle .$$

Here Z is a complex number defined over a covering space of the complex plane (i.e., the multiple Riemann sheets with the branch point at the origin) denoted by ${}^*\mathbf{C}$, and ρ_n is the n-th generalized moment of a distribution function $\rho(u) > 0$,

$$\rho_n = \int_0^{\infty} \mathrm{d}u\, u^{[n]} \rho(u) .$$

Here we *choose* $\rho(u) = \mathrm{e}^{-u}$ ($u > 0$) for all relevant systems, so that

$$\rho_n = \Gamma\big([n] + 1\big) = [n]! .$$

This factorization condition replaces the action identity of Gazeau and Klauder.

The normalization factor $N(Z)$ can be determined by

$$\langle Z|Z\rangle = |N(Z)|^2 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{Z^{*\,[m]} Z^{[n]}}{\sqrt{[m]!\,[n]!}} \langle m|n\rangle = |N(Z)|^2 \sum_{n} \frac{|Z|^{2[n]}}{[n]!} = 1 .$$

Introducing the generalized exponential function,

$$[\mathrm{e}]^x \equiv \sum_{n=0}^{\infty} \frac{x^{[n]}}{[n]!} \quad (x \in \mathbf{R}) ,$$

we express the normalization factor as

$$N(Z) = [\mathrm{e}]^{-|Z|^2/2} .$$

Consequently, the proposed CS takes the form,

$$|Z\rangle = [\mathrm{e}]^{-|Z|^2/2} \sum_{n=0}^{\infty} \frac{Z^{[n]}}{\sqrt{[n]!}} |n\rangle .$$

It is apparent that the structure of this CS is very similar to that of the harmonic oscillator CS. In particular, for the harmonic oscillator, $[n] = n$ and $[\mathrm{e}]^x = \mathrm{e}^x$; furthermore the general eigenstate $|n\rangle$ becomes Fock's number state. Hence the generalized CS reduces to the standard CS.

3 Properties

If we let $Z = s\,\mathrm{e}^{-\mathrm{i}\gamma}$ $(0 < s < \infty, -\infty < \gamma < \infty)$, then the proposed CS coincides with Klauder's CS. Accordingly, the three properties Klauder's states possess are to be shared with the present CS. Evidently the CS is continuous in the two parameters, s and γ. It is also easy to show that

$$\mathrm{e}^{-\mathrm{i}\hat{H}t}|Z\rangle = |Z\mathrm{e}^{-\mathrm{i}\omega t}\rangle\,.$$

Hence the CS is temporally stable.

To derive the resolution of unity, we have to exercise caution. Before carrying out the integration of $|Z\rangle\langle Z|$, we write the CS as

$$|Z\rangle = \lim_{\epsilon\to 0}\sum_{n=0}^{\infty}\frac{Z^{[n]+\epsilon n}}{\sqrt{[n]!}}\,|n\rangle$$

and modify Klauder's measure (6) as

$$\int \mathrm{d}\mu(Z)\,f(Z) = \lim_{\epsilon\to 0}\lim_{\Gamma\to\infty}\frac{1}{2\Gamma}\int_{-\Gamma}^{\Gamma}\mathrm{d}\gamma\int \mathrm{d}s^2\,N^2(s^2)\,\mathrm{e}^{-s^2}\,f(s,\gamma) \tag{10}$$

with

$$\lim_{\epsilon\to 0}\lim_{\Gamma\to\infty}\frac{1}{2\Gamma}\int_{-\Gamma}^{\Gamma}\mathrm{d}\gamma\mathrm{e}^{\mathrm{i}\gamma([n]+\epsilon n-[n']-\epsilon n')} = \lim_{\epsilon\to 0}\delta([n]+\epsilon n-[n']-\epsilon n')\,. \tag{11}$$

For non-degenerate systems, by letting $\epsilon\to 0$, we see that the present CS admits the resolution of unity as Klauder's CS does. In the case of degeneracy, we first calculate the integration and take the limit $\epsilon\to 0$. For either case, we obtain the resolution of unity,

$$\int \mathrm{d}^2 Z\,|Z\rangle\langle Z| = \mathbf{1}_{\mathrm{disc}}\,. \tag{12}$$

4 Extension to the continuous states

Next we construct the coherent states for a system with a continuous spectrum by the limiting procedure $[n]\to\varepsilon$. Replacing $[n]!$ by

$$\rho(\varepsilon) = \int_0^\infty \mathrm{d}u\,u^{\varepsilon}\,\mathrm{e}^{-u} = \Gamma(\varepsilon+1) = \varepsilon!\,,$$

we can go naturally over to the continuous limit,

$$|\zeta\rangle = \nu^{-2}(|Z|^2)\int_0^\infty \mathrm{d}\varepsilon\,\frac{Z^{\varepsilon}}{\sqrt{\varepsilon!}}\,|\varepsilon\rangle\,,$$

where $\nu(x)$ is the ν-function [6] defined by

$$\nu(x) = \int_0^\infty \frac{x^t}{\Gamma(t+1)}\,\mathrm{d}t\,.$$

In what follows we shall discuss two examples.

4.1 Particle in a one-dimensional box

The first example is a free particle of mass m confined to move in an infinite square well potential,

$$V(x) = \begin{cases} 0 & \text{for } -a/2 < x < a/2\,, \\ \infty & \text{otherwise}\,. \end{cases}$$

The solutions of this elementary example are well known. Since we are interested in the limiting case where the two walls are moved to infinity in the separate directions, that is, $a \to \infty$, we express the solutions satisfying the boundary conditions $\psi(\pm a/2) = 0$ in the form,

$$\psi_n(x) = A_n \left\{ \mathrm{e}^{\mathrm{i}k_n x} - \mathrm{e}^{\mathrm{i}k_n a}\, \mathrm{e}^{-\mathrm{i}k_n x} \right\}, \tag{13}$$

where

$$k_n = \frac{\pi n}{a}\,, \quad (n \in \mathbf{N})\,.$$

The energy spectrum is

$$E_n = \frac{k_n^2}{2m} = \frac{\pi^2 n^2}{2ma^2}\,.$$

Before moving the walls to infinity, we apply the adiabatic trick by assuming that k has a small positive imaginary part ϵ, and rewrite (13) as

$$\psi_n(x) = \lim_{\epsilon \to 0} A_n \left\{ \mathrm{e}^{\mathrm{i}k_n x - \epsilon x} - \mathrm{e}^{\mathrm{i}k_n a - \epsilon a}\, \mathrm{e}^{-\mathrm{i}k_n x - \epsilon x} \right\}. \tag{14}$$

Then we let $\alpha \to \infty$ before letting $\epsilon \to 0$. In this limit, the last term damps down, so that the bound state functions tend to the free particle wave function

$$\psi(x) = A\, \mathrm{e}^{\mathrm{i}kx}\,, \tag{15}$$

as the discrete spectrum k_n approaches the continuous one k,

$$\lim_{a,n \to \infty} \frac{\pi n}{a} = k\,.$$

Now we turn ourselves to the coherent states. The generalized number for this system is obtained as

$$[n] = \frac{E_n - E_0}{\omega} = n^2$$

with

$$\omega = (2ma^2)^{-1}\,.$$

From this follows

$$[n]! = \sum_{j=1}^{n} j^2 = (n!)^2\,.$$

As a result the coherent states for the particle in the well are given by

$$|Z\rangle_{\mathrm{disc}} = N(z)_{\mathrm{disc}} \sum_{n=0}^{\infty} \frac{Z^{n^2}}{n!}\, |n\rangle\,, \tag{16}$$

where the eigenstates $\langle x|n\rangle$ correspond to the solutions (13). The inverse square of the normalization factor is

$$[N(Z)_{\rm disc}]^{-2} = \sum_{n=0}^{\infty} \frac{|Z|^{2n^2}}{(n!)^2}\,,$$

which cannot be given in closed form.

Next we consider the limit $a \to \infty$ of the coherent states (16). As k_n tends to the continuous value k,

$$[n] \to \varepsilon = \frac{k^2}{2m}\,,$$

and

$$[n]! \to \varepsilon! = \Gamma(\varepsilon+1)\,.$$

Hence we obtain the CS for the free particle,

$$|Z\rangle_{\rm cont} = N(Z)_{\rm cont}\int_0^\infty {\rm d}\varepsilon\, \frac{Z^\varepsilon}{\sqrt{\Gamma(\varepsilon+1)}}\,|\varepsilon\rangle.$$

The continuous energy eigenstate states $|\varepsilon\rangle$ correspond to the free particle wave function (15). This time the normalization factor is given in a closed form expression via

$$[N(Z)_{\rm cont}]^{-2} = \int_0 \infty {\rm d}\varepsilon\, \frac{|Z|^{2\varepsilon}}{\Gamma(\varepsilon+1)} = \nu(|Z|^2)\,,$$

namely

$$N(Z) = \frac{1}{\sqrt{\nu(|Z|^2)}}\,,$$

where $\nu(x)$ is the ν-function as has been defined earlier. By this limiting procedure, the discrete CS is naturally converted to the continuous CS.

4.2 A compactified hydrogen atom

The next example is a radial Coulomb problem defined on a three dimensional sphere of radius R. In [3], Klauder constructed the CS for the the hydrogen atom in flat space but only for the bound states. Since the energy spectrum of the hydrogen atom in flat space has the continuous part (for scattering states) as well as the discrete part (for bound states), Klauder's formulation is incomplete. However, if the hydrogen atom is placed on the sphere, the continuous part of the energy spectrum disappears. In other words, the confinement on the sphere compactifies the hydrogen atom. Then we can construct the discrete CS for the compactified hydrogen atom according to Klauder's recipe. It is shown in [7] that the flat space limit $R \to \infty$ splits the discrete Klauder CS into the discrete and continuous portions. Here we deal with this example to demonstrate that our CS can accommodate both the discrete and the continuous portion in a natural manner.

The energy spectrum for the compactified hydrogen atom is given by [7] − [14]

$$E_n = \frac{(n+l)(n+l+2)}{2R^2} - \frac{\omega}{(n+l+1)^2}\,,$$

where $n = 0, 1, 2, \ldots$ and $l = 0, 1, 2, \ldots, n$ are the radial quantum number and the angular momentum quantum number, respectively, and $\omega = e^4/2$. The corresponding radial wave function has the form [12, 7],

$$w_{n,l}(\chi) = C_{nl} \sin^l \chi\, \mathrm{e}^{-\mathrm{i}\chi(n+\mathrm{i}\lambda_n}\ {}_2F_1\big(-n, l+1-\mathrm{i}\lambda_n; 2l+2; 1-\mathrm{e}^{2\mathrm{i}\chi}\big) \tag{17}$$

with the normalization factor

$$C_{n,l} = \mathrm{e}^{\mathrm{i}\pi(2n+l+1)/2} \frac{2^{l+1}}{\Gamma(2l+2)} \left[\frac{\mathrm{i}\{(n+l+1)^2+\lambda_n^2\}\Gamma(l+1+i\lambda\mathrm{i}_n)\Gamma(n+2l+2)}{R^3\kappa_n\Gamma(\mathrm{i}\lambda_n - l)\Gamma(n+1)}\right]^{1/2}$$

where $\chi = \arcsin(r/R)$,

$$\lambda_n = -\frac{e^2 R}{n+l+1}\,,$$

and

$$\kappa_n = \min\{|n+l+1|, |\lambda_n|\}\,.$$

In writing down the coherent states for this system, we must notice that the accidental degeneracy of the hydrogen atom does not disappear even on the sphere. For the present demonstration purpose, we confine ourselves to the non-degenerate case by considering only the radial portion for the S-wave sector with $l = 0$. The generalized number $[n]_R$ defined for the radial quantum number $n = 0, 1, 2, \ldots$ and $l = 0$ is obtained in the form,

$$[n]_R = \frac{1}{2R^2}\, n(n+2) - \frac{e^4 n(n+2)}{2(n+1)^2}\,.$$

The factorial of $[n]$ is given by

$$[n]_R! = \prod_{j=1}^{n} \left[\frac{j(j+2)}{(j+1)^2}\left(1 + \frac{(j+1)^2}{2\omega R^2}\right)\right], \tag{18}$$

where $\omega = e^4/2$. Although $[n]!$ is not in closed form, we can formally express the CS for the S-wave sector of the radial Coulomb problem on the sphere as

$$|Z\rangle_R = N(Z)_R \sum_{n=0}^{\infty} \frac{Z^{[n]_R}}{\sqrt{[n]_R!}}\, |n\rangle_R\,, \tag{19}$$

where $\langle\chi|n\rangle = w_{n,0}(\chi)$. Since we have

$$[N(z)_R]^{-2} = \sum_{n=0}^{\infty} \frac{|Z|^{2[n]_R}}{[n]_R!}\,,$$

the normalization factor cannot be given in closed form either. In this regard, the above CS have no practical values. However, it is our purpose to show that this CS will generate a continuous as well as a discrete portion in the flat space limit.

Before taking the flat space limit $R \to \infty$, we introduce the critical number n_c such that $E_{n_c} = 0$, i.e.,

$$n_c(n_c + 2)(n_c + 1)^2 = 2\omega R^2$$

and separately consider the two cases (a) $n < n_c$ and (b) $n \geq n_c$. Note that n_c goes to infinity as fast as $\sqrt{R}$ as R tends to infinity.

Case (a): For $n < n_c$

$$\left| E_n + \frac{e^4}{2(n+1)^2} \right| < \frac{n_c(n_c+2)}{2R^2} \sim \frac{1}{R} \to 0\,.$$

Hence, we obtain

$$E_n = -\frac{e^4}{2(n+1)^2}\,,$$

which is the bound state spectrum in flat space. In the same limit, the S-wave radial functions (17) become

$$u_n(r) = C_n \mathrm{e}^{-re^2/(n+1)}\,{}_1F_1\left(-n-1; 2; 2re^2/(n+1)\right)$$

with the normalization constant,

$$C_n = \left[\left(\frac{2e^2}{n+1} \right)^3 \frac{1}{2(n+1)} \right]^{1/2}.$$

These are as expected the S-wave radial wave functions of the usual hydrogen atom bound states.

Case (b): For $n \geq n_c$, we approximate $\Delta n/R$ by $\mathrm{d}k$ with a continuous parameter $k > 0$, so that $n - n_c = kR$. In the limit $R \to \infty$,

$$E_n = \frac{kR + n_c)(kR + n_c + 2)}{2R^2} - \frac{\omega}{(kR + n_c + 1)^2} \to \frac{k^2}{2}\,.$$

This part of the spectrum turns into a continuous spectrum,

$$E_k = \tfrac{1}{2}\, k^2 \geq 0\,.$$

In the limit $R \to \infty$ for $n \geq n_c$, the radial functions $w_{n,l}(\chi)$ for $l = 0$ become

$$v_k(r) = \left(\frac{2}{\pi e^2} \right)^{1/2} k^2 \left| \Gamma\left(1 - \frac{\mathrm{i}e^2}{2k}\right) \right| \mathrm{e}^{\mathrm{i}kr}\,{}_1F_1\left(1 + \frac{\mathrm{i}e^4}{2k}; 2; 2\mathrm{i}kr\right),$$

which are the S-wave functions for the scattering states of the hydrogen atom in flat space.

Now we consider the flat space limit of the formal result (19),

$$|Z\rangle = \lim_{R\to\infty} N(Z)\left\{\sum_{n<n_c} + \sum_{n\geq n_c}\right\} \frac{Z^{[n]_R}}{\sqrt{[n]_R!}}\,|n\rangle_R\,. \tag{20}$$

For $n < n_c$,

$$\lim_{R\to\infty}[n]_R = [n] = -\frac{\omega}{(n+1)^2}\,, \quad n = 0, 1, 2, \ldots\,,$$

and

$$\lim_{R\to\infty}[n]_R! = [n]! = \prod_{j=1}^{n}\frac{j(j+2)}{(j+1)^2} = \frac{n!(n+2)!}{[(n+1)!]^2}\,.$$

Thus the limit of the first term of (20) for $n < n_c$ becomes

$$|Z\rangle_{\rm disc} = \mathcal{N}(Z)\sum_{n=0}^{\infty}\frac{Z^{[n]}}{\sqrt{[n]!}}\,|n\rangle\,,$$

where $\langle r|n\rangle = u_n(r)$.

For $n \geq n_c$,

$$\lim_{R\to\infty}[n]_R = \tfrac{1}{2}\,k^2 = \varepsilon$$

and

$$\lim_{R\to\infty}[n]_R = \lim_{R\to\infty}\int_0^\infty {\rm d}u\, u^{[n]_R}{\rm e}^{-u} = \Gamma(\varepsilon+1) = \varepsilon!\,.$$

Here use of $\varepsilon!$ for $\Gamma(\varepsilon+1)$ is to emphasize that $\varepsilon!$ is a natural continuum counterpart of $[n]!$.

The second term of (20) for $n \geq n_c$ reduces to

$$|Z\rangle_{\rm cont} = \mathcal{N}(Z)\int_0^\infty {\rm d}\varepsilon\,\frac{Z^\varepsilon}{\sqrt{\varepsilon!}}\,|\varepsilon\rangle\,,$$

where $\langle r|\varepsilon\rangle = v_n(r)$.

Consequently, as the flat space limit of (20), we obtain CS for the S-wave radial Coulomb problem, which consists of the discrete and continuous portions:

$$|Z\rangle = \mathcal{N}(Z)\left\{\sum_{n=0}^{\infty}\frac{Z^{n(n+2)/(n+1)^2}}{\sqrt{(n+2)/(2n+2)}}\,|n\rangle + \int_0^\infty {\rm d}\varepsilon\,\frac{Z^\varepsilon}{\sqrt{\varepsilon!}}\,|\varepsilon\rangle\right\}.$$

The normalization factor $\mathcal{N}(Z)$ common to the discrete and continuous portions is determined by

$$[\mathcal{N}(Z)]^{-2} = \sum_{n=0}^{\infty}\frac{|Z|^{2n(n+2)/(n+1)^2}}{(n+2)/(2n+2)} + \int_0^\infty {\rm d}\varepsilon\,\frac{|Z|^\varepsilon}{\varepsilon!}\,.$$

In this way we are able to demonstrate that the continuous portion of CS for the radial Coulomb problem can be obtained very naturally.

5 Propagator

Here we consider the propagator with the present CS,

$$K(Z'',t'';Z',t') = \langle Z''|\mathrm{e}^{-\mathrm{i}\hat{H}(t''-t')}|Z'\rangle\,. \tag{21}$$

5.1 Path integral representation

As usual, with the aid of the resolution of unity (12), we discretize the propagator (21) as

$$\langle Z''|\mathrm{e}^{-\mathrm{i}\hat{H}(t''-t')}|Z'\rangle \lim_{N\to\infty}\int\prod_{j=1}^{N}\langle Z_j|\mathrm{e}^{-\mathrm{i}\epsilon\hat{H}}|Z_{j-1}\rangle\prod_{j=1}^{N-1}\mathrm{d}\mu(Z_j)\,,$$

where $\mathrm{d}\mu(Z)$ is the modified Klauder measure (10). The propagator for a short time $\epsilon=(t''-t')/N$ can be calculated by taking only the first order approximation in ϵ;

$$\langle Z_j|\mathrm{e}^{-\mathrm{i}\epsilon\hat{H}}|Z_{j-1}\rangle = \langle Z_j|1-\mathrm{i}\epsilon\hat{H}|Z_{j-1}\rangle = \langle Z_j|Z_{j-1}\rangle(1-\mathrm{i}\epsilon h_j)\langle Z_j|Z_{j-1}\rangle\mathrm{e}^{-\mathrm{i}\epsilon}\,,$$

where

$$h_j = \frac{\langle Z_j|\hat{H}|Z_{j-1}\rangle}{\langle Z_j|Z_{j-1}\rangle}\,.$$

Hence we have

$$\langle Z_j|\mathrm{e}^{-\mathrm{i}\epsilon\hat{H}}|Z_{j-1}\rangle = \exp\left\{\mathrm{i}\epsilon\left(-\frac{\mathrm{i}}{\epsilon}\,\ln\langle Z_j|Z_{j-1}\rangle - h_j\right)\right\}.$$

Let $\Delta|Z_{j-1}\rangle = |Z_j\rangle - |Z_{j-1}\rangle$. Then

$$\langle Z_j|\Delta|Z_{j-1}\rangle = 1-\langle Z_j|Z_{j-1}\rangle = -\ln\langle Z_j|Z_{j-1}\rangle\,.$$

Thus we arrive at

$$K(Z'',t'';Z',t') = \lim_{N\to\infty}\int\prod_{j=1}^{N}\mathrm{d}\mu(Z_j)\,\exp\left\{\mathrm{i}\epsilon\sum_{j=0}^{N}\left(\frac{\mathrm{i}}{\epsilon}\,\langle Z_j|\Delta|Z_{j-1}\rangle - h_j\right)\right\}.$$

In the path integral representation, the propagator is formally given by

$$K(Z'',t'';Z',t') = \int\mathcal{D}\mu(Z)\,\exp\left\{\mathrm{i}\epsilon\int\mathrm{d}t\left(\mathrm{i}\langle Z|\,\frac{\mathrm{d}}{\mathrm{d}t}\,|Z\rangle - \langle Z|\hat{H}|Z\rangle\right)\right\},$$

which is the same in structure as that of the harmonic oscillator.

It has been pointed out by Kuratsuji [15] that the first term in the action integral is geometrical while the second term (the Hamiltonian term) is dynamical. In classical limit, the first term corresponds to Hamilton's characteristic function,

$$W(E) = \oint_{H=E}\langle Z|\mathrm{i}\,\frac{\mathrm{d}}{\mathrm{d}t}|Z\rangle\mathrm{d}t\,,$$

where the integral is taken along periodic orbits on the energy surface. The poles of the energy Green function defined by

$$G(E) = \mathrm{i} \int_0^\infty \mathrm{e}^{\mathrm{i}E\tau} K(\tau)\, \mathrm{d}\tau$$

with

$$K(\tau) = \int \langle Z|\mathrm{e}^{-\mathrm{i}\tau \hat{H}}|Z\rangle \mathrm{d}^2 Z$$

yields a quantization rule,

$$\oint \langle Z|\mathrm{i}\,\frac{\mathrm{d}}{\mathrm{d}t}|Z\rangle \mathrm{d}t = n\pi\,, \quad n = 0, 1, 2, \ldots\,,$$

details of which will be discussed elsewhere. For instance, the integral along a periodic orbit for a classical particle of momentum k confined in the one-dimensional box of length a has a value ak. Hence the quantization rule yields $k = \pi n/a$. The energy $E_{\mathrm{cl}} = k^2/2m$ is quantized as $E_n = \pi^2 n^2/(2ma^2)$. This example shows that the integral along a periodic orbit is not identifiable with the classical energy contrary to what the Gazeau–Klauder action identity claims.

5.2 Series representation

The propagator (21) can also be put in the form,

$$K(Z'', t''; Z', t') = \langle Z''|\mathrm{e}^{-\mathrm{i}\hat{H}(t''-t')}|Z'\rangle = \langle Z''\mathrm{e}^{-\mathrm{i}\omega t''}|Z'\mathrm{e}^{-\mathrm{i}\omega t'}\rangle\,. \tag{22}$$

From this immediately follows

$$K(Z'', t''; Z', t') = N(Z'')N(Z') \sum_{n=0}^{\infty} \frac{(Z''^{*} Z' \mathrm{e}^{\mathrm{i}\omega(t''-t')})^{[n]}}{[n]!}\,,$$

where

$$N^{-2}(Z) = \sum_{n=0}^{\infty} \frac{|z|^{2[n]}}{[n]!} = [\mathrm{e}]^{|Z|^2}\,,.$$

In terms of the generalized exponential function, the propagator reads

$$K(Z'', t''; Z', t') = [\mathrm{e}]^{-|Z''|^2/2}\, [\mathrm{e}]^{-|Z'|^2/2}\, [\mathrm{e}]^{Z' Z''^{*} \exp\left(\mathrm{i}\omega(t''-t')\right)}\,.$$

In particular, if $[n] = n$, it reduces to

$$K(z'', t''; z', t')_{\mathrm{osc}} = \exp\left\{-\frac{|z''|^2}{2} - \frac{|z'|^2}{2} + z' z''^{*} \mathrm{e}^{\mathrm{i}\omega(t''-t')}\right\},$$

which is the propagator for the harmonic oscillator.

6 Concluding remarks

By modifying Klauder's coherent states, we have proposed a set of generalized coherent states which are applicable to some degenerate systems and appropriate for defining discrete and continuous coherent states in a natural and unified manner. However, more often than not, the discrete portion of the normalization factor cannot be obtained in closed form. Therefore, the Gazeau–Klauder version are more practical than ours insofar as non-degenerate systems are concerned.

We would like to thank John R. Klauder for his useful comments.

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