The Titius−Bode law and a quantum-like description of the planetary systems

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The Titius–Bode law for planetary distances is reviewed. A model describing the basic features of this law in the "quantum-like" language of a wave equation is proposed. Some considerations about the 't Hooft idea on the quantum behaviour of deterministic systems with dissipation are discussed.

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1 Introduction

In the last years many papers, in particular those of 't Hooft [1], but also of several other authors [2–4], have described a possible way out from some of the unpleasant (or, to cite Feynman, "peculiar") features of Quantum Mechanics.

The idea proposed by 't Hooft is, in extreme synthesis, that a classical deterministic theory (at the Planck scale) supplemented with a dissipation mechanism (information loss) should produce at larger scales the observed quantum mechanical behaviour of our world. The emergent model, after that the dissipation mechanism has been implemented, can be described with the usual tools of Quantum Mechanics, namely, states which evolve in Hilbert space, unitary evolution matrix, wave equation, etc..

Originally motivated (also) by the challenge of Quantum Gravity, the accent on the Planck scale collocation of the deterministic underlying theory has been more recently weakened in favour of the description of several classes of deterministic classical systems, where quantum behaviours emerge from classical dynamics constrained with dissipation mechanism(s) (see [2–4]).

Inspired by this use of the dissipation to obtain *"the apparent quantization of the orbits which resembles the structures seen in the real world"* [5], the present paper aims to describe in a quantum-like language some enigmatic aspects of the known planetary systems (including our Solar System), which, however, can be regarded as deterministic systems *par excellence*. In particular, a phenomenological law that seems to resemble closely a quantum feature, and brings to many reflections in this direction, is the so called Titius–Bode law for planetary distances.

This work is driven by the analogy between the 't Hooft idea of the quantization of a classical deterministic system via the dissipation at the Planck scale, and the dissipation occurred during the formation period of a planetary system, which has produced the "quantization" of the orbits described by the Titius–Bode rule. In the present paper, the dissipation mechanism which should have brought the

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proto-planetary nebula to the stable, "quantized" orbits of today (in the sense of the Titius–Bode law), is not yet explicitly constructed. Nevertheless, an "effective" quantum-like model (in the form of a wave equation) is developed and applied to the description of planetary systems, and a dissipation mechanism is realistically devised as responsible of the discrete nature of planetary orbits.

2 Titius−Bode law

The Titius–Bode law is an empirical rule which gives, in the hypothesis of circular orbits, the distances of the planets from the Sun as a function of a single parameter, an integer n. There are several versions of the law. The eldest one is perhaps the following

$$r(n) = 0.4 + 0.3 \cdot 2^n \,,$$

where $r(n)$ is given in Astronomical Unit (1 A.U. $\simeq 150 \cdot 10^6$ km). For $n = -\infty, 0, 1, 2, \ldots$ this law gives the distance respectively of Mercury, Venus, Earth, Mars, etc., including the asteroids belt (actually, Cerere was discovered following the indications of this law) and Uranus, which at the moment of the first formulation of the law (1766 − 1772) had not yet been discovered.

In this original formulation, the law was not able to account for the distance of Neptune and Pluto. More recent versions of the law have been elaborated during the XX sec., as for example the Blagg law (1913) and the Richardson law (around 1943) (see the book of Nieto [6] for history, explicit formulations, theories and extensive comments). In these last versions the law is able to describe not only the planetary distances within the solar system, including planets like Neptune and Pluto, but also can be successfully applied to the systems of satellites orbiting Jupiter, Saturn and Uranus. The agreement between the predicted and the observed distances of the various satellites from the central body is really astonishing, of the order of a few percents, as can be checked in the tables of Nieto [6].
The main feature shared by these modern versions of the Titius–Bode law is that the rule can be expressed, if we neglect second order corrections, by an exponential relation as

$$r = a\mathrm{e}^{2\lambda n} \,, \tag{1}$$

where the factor 2 is introduced for convenience reasons and $n = 1, 2, 3, \ldots$.

For the Solar System we have

$$\begin{aligned} 2\lambda &= 0.53707 \,, \qquad\qquad \mathrm{e}^{2\lambda} \simeq 1.7110 \,, \\ a &= 0.21363 \text{ A.U.} \end{aligned}$$

The amazing thing found by Blagg was that the geometric progression ratio $\mathrm{e}^{2\lambda}$ is roughly the same both for the Solar System, and also for the satellite systems of Jupiter ($\mathrm{e}^{2\lambda} \simeq 1.7277$), Saturn ($\mathrm{e}^{2\lambda} \simeq 1.5967$), and Uranus ($\mathrm{e}^{2\lambda} \simeq 1.4662$). Of course the parameter a, which is linked to the radius of the first orbit, will take case by case the opportune values (see for more comments the Nieto book [6]).

A plenty of theories have been developed during the last 240 years in order to explain the Titius–Bode law.

There have been dynamical models connected with the theory of the origin of the solar system, electromagnetic theories, gravitational theories, nebular theories. All of them can be found in literature (see, for example, [7]) and they have been excellently reviewed in the book [6]. Therefore they will not be described in details here. We remind that also the idea of using a Schrödinger-type equation in order to give account of the law (1) is not new (see for example the papers [8]) [1]).

The aim of the present paper is to develop a model able to describe, in the language of a Schrödinger-like equation, the observed law of planetary distance, as an eigenvalue problem.

3 Bohr model of the hydrogen atom

In this section we remind sketchly the Bohr model for the hydrogen atom. The electron orbits are supposed circular (this will be held for planetary orbits also). The two main equations are the equation for the force (i.e. the equation of motion)

$$m\,\frac{v^2}{r} = \frac{e^2}{r^2}\,,$$

where m and e are the mass and charge of the electron; and the quantization condition on the (z-component) of the angular momentum

$$mvr = n\hbar\,, \quad n = 1, 2, 3, \ldots\,. \tag{2}$$

In the Bohr model, all the orbits belong to the same plane, and this is also taken for true in the planetary models. From the two equations above, one easily derives

$$\begin{cases} r = \dfrac{\hbar^2}{me^2}\,n^2 \\ v = \dfrac{\hbar n}{mr}\,. \end{cases}$$

The first equation is the law of electron distance from the nucleus in the Bohr model. With this law, from the classical expression for the total energy we get the energy spectrum of the bound orbits

$$E = \frac{1}{2}\,mv^2 - \frac{e^2}{r} = -\frac{e^2}{2r} = -\frac{me^4}{2\hbar^2}\,\frac{1}{n^2}\,.$$

Now we shall try to apply analog ideas to the solar and planetary systems.

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[1]) However, in some of Ref. [8], a Schrödinger equation has been used mainly to calculate distances of a *set* of planets (e.g. terrestrial planets, as Mercury, Venus, etc., or gigantic planets, as Jupiter, Saturn, etc.), but not of the whole planetary system. This because in [8] the newtonian potential $1/r$ is usually assumed together with a Bohr-like quantization condition, and these two things together yield (as it is well known) a law for the orbital radii as $r \sim n^2$ (not as $r \sim \mathrm{e}^{2\lambda n}$).

4 Model *a la Bohr* for a planetary system

In this section we introduce a model for the "quantization" of a planetary system. The model acquires its discrete, or "quantum", properties from a modification of the Bohr quantization rule for the angular momentum. The equations here proposed, for a generic planet of mass m, orbiting a central body of mass M, are

$$\begin{cases} m\dfrac{v^2}{r} = \dfrac{GMm}{r^2}\,, \\ \dfrac{J}{m} = vr = s\mathrm{e}^{\lambda n} \end{cases} \tag{3}$$

where $n = 1, 2, 3, \ldots$ and s is a constant. Some comments are immediately required:

• Because of the principle of equivalence the masses m on the LHS and on the RHS of eq. (3) cancel out each other.

• The constant s in the RHS of the second of eqs. (3) has the dimensions of an action per unit mass. It plays the role of $\hbar$ and it must be understood as an action typical of the planetary system under consideration. It is not possible to use $\hbar$ itself, because this would fix the wrong initial radius in the Titius–Bode law, that is the constant a in $r = a\mathrm{e}^{2\lambda n}$.

• The constant λ is the one obtained from the observation ($2\lambda = 0.53707$ for the Sun, $2\lambda = 0.54677$ for Jupiter, $2\lambda = 0.46794$ for Saturn, $2\lambda = 0.38271$ for Uranus).

• In the second of the eqs. (3), we quantize the angular momentum *per unit mass*. This is somewhat a consequence of the principle of equivalence. If we did not do so, we would obtain a law for $r(n)$ where the scale of distance changes from a planet to another, as the planetary masses change. We should in fact remind that not all the planets have the same mass, as instead the electrons have.

From the eqs. (3) one immediately gets

$$v = \frac{s\mathrm{e}^{\lambda n}}{r} \;\Rightarrow\; r(n) = \frac{s^2}{GM}\,\mathrm{e}^{2\lambda n}\,, \tag{4}$$

which is the Titius–Bode law if we identify $a = s^2/(GM)$.

We can also compute the energy spectrum for the i-th planet from the eq. (4)

$$E_{(i)} = \frac{1}{2}\,m_i v^2 + U(r) = \frac{1}{2}\,m_i v^2 - \frac{GMm_i}{r} = -\frac{GMm_i}{2r} = -\left(\frac{GM}{s}\right)^2 \frac{m_i}{2\mathrm{e}^{2\lambda n}}\,,$$

where $n = 1, 2, 3, \ldots$. As we see, the energy of the i-th planet is not properly quantized by itself. This is because the mass m_i changes in general with the planet, and this would imply different sets of energy levels for different planets. Instead, the energy per unit mass

$$\mathcal{E} := \frac{E_{(i)}}{m_i} = -\left(\frac{GM}{s}\right)^2 \frac{1}{2\mathrm{e}^{2\lambda n}} \tag{5}$$

is exactly quantized, i.e. it is a quantity which depends on n only (apart from the general constants G, M, s). Therefore, the energy levels per unit mass are valid for the whole set of planetary orbits.

Also here some comments are needed, in order to complete the explanation given before.

• The constant s can be computed in terms of the mass of the central body and of the parameter a (remind that the radius of the first orbit is $a\mathrm{e}^{2\lambda}$)

$$s = \sqrt{GMa}\,.$$

• This constant is *not* the same for all the planetary systems (Sun, Jupiter, Saturn, Uranus). In fact, if it were so, this would imply that the parameter $a = s^2/(GM)$ should be in inverse proportion to the mass M of the central body, which is not true. Therefore the constant s is not universal, like $\hbar$, but it depends on the planetary system under consideration.

• If the quantization rule (3) had been written with the mass of the planet, namely

$$mvr = \tilde{s}\mathrm{e}^{\lambda n}\,,$$

this would have implied for $r(n)$

$$r(n) = \frac{\tilde{s}^2}{GMm_i^2}\,\mathrm{e}^{2\lambda n}$$

that is, the parameter $a = \tilde{s}^2/(GMm_i^2)$ would change from planet to planet, contrary to the generality of the Titius–Bode law, which maintains the same parameters within the same planetary system.

• The quantization rule (3) does not allow us to compute some known experimental constant, as instead it happens in the case of the Bohr model of the hydrogen atom, where the Rydberg constant was computed from the model. Nevertheless, a semiclassical quantum language is introduced.

• It should be noted also that a condition like $vr = s\mathrm{e}^{\lambda n}$ presents some difficulties for a wave interpretation. In fact, the Bohr quantization condition for the H-atom can be easily interpreted in terms of de Broglie's stationary matter waves

$$mvr = n\hbar \;\Rightarrow\; pr = n\hbar \;\Rightarrow\; r = n\,\frac{\hbar}{p} \;\Rightarrow\; 2\pi r = n\left(\frac{h}{p}\right)\,,$$

where $\ell_B = h/p$ and n is an integer. The quantity h/p can be interpreted as a wavelength of a stationary wave just because n is an integer. The analog condition in our model yields (for a given planet of mass m)

$$vr = s\mathrm{e}^{\lambda n} \;\Rightarrow\; r = \mathrm{e}^{\lambda n}\,\frac{sm}{p} \;\Rightarrow\; 2\pi r = \mathrm{e}^{\lambda n}\left(\frac{2\pi sm}{p}\right)\,.$$

The number $e^{\lambda n}$ is not an integer, in general. Hence is difficult to interpret $(2\pi sm)/p$ as a wavelength of a stationary wave. Moreover, even using a de Broglie-like relation ($n\ell_B = 2\pi r$), the wavelength of the matter wave associated to the planet has to be of the same order of the parameter a. In fact

$$\ell_B = 2\pi \, \frac{r}{n} = 2\pi \, \frac{a e^{2\lambda n}}{n} \, .$$

In principle, this could create interference phenomena in the probability amplitudes, but these phenomena are not observed at classical level in planetary systems. We must therefore *postulate* a unknown mechanism which suppresses these interferences of probability waves. From this last observation, it appears clearly that the model we are building is not actually a quantum model, in the sense of ordinary quantum theory. Rather, it resembles some *quantum-like* properties, mainly the quantization of the orbital radii.

In spite of all these difficulties, we shall see that a wave equation can still be written in coherence with the condition $vr = s e^{\lambda n}$, and this wave equation will be able to describe the main features of planetary systems.

5 Permitted orbits, dissipation and gravity

One of the main objections that it is possible to rise against the existence of permitted discrete orbits in a planetary system is the following: It is a common experience, in this era of space travels, that a satellite can be put in any orbit we wish around the Earth, the Sun, or any other planet. Why therefore there should exist stable *permitted* orbits? In what sense they are "permitted"? How they are reached? To answer to these questions, it is fundamental to remind the concepts of *dissipation* and *limit cycles* emphasized by 't Hooft in his seminal papers [1] (see also further references therein). Although in the proto-planetary nebula dust, particles and other bodies could be found at any distance from the central body, after a huge amount of time, friction and mutual gravitational actions produced a dissipation of the total energy and brought matter to stabilize in several orbits, the limit cycles, where particles and dust aggregated to form planets. It is in this way that we can speak about "permitted orbits": they are the "limit cycles" of the dissipative processes started in the primitive nebula.

Of course, it is in principle possible to put, today, a body in any orbit we wish. But if we wait for a time of the order of $5 \cdot 10^9$ years, and if the body has a sufficient mass, it is likely that we finally find it in one of the permitted limit cycles.

It is interesting to compute and compare the dissipation time taken by an electron to fall on the first permitted orbit in the hydrogen atom, and the time taken by a planet, say Jupiter, to fall from the infinite to its own orbit. We suppose that the loss of energy occurs by emission of electromagnetic or gravitational waves, respectively. For the electron, we use the classical dipole emission formula

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{2e^2}{3c^3} \, \ddot{x}^2 \, .$$

The acceleration can be roughly valued with the Bohr model

$$a = \frac{v^2}{r} = \frac{e^2}{mr^2} \, .$$

Therefore

$$P = \frac{\mathrm{d}E}{\mathrm{d}t} = \frac{2e^6}{3m^2c^3r^4} \, .$$

The total energy of an electron in a orbit of radius r is $E = e^2/(2r)$. Hence, the decay time is

$$\Delta t = \frac{E}{P} = \frac{3m^2c^3r^3}{4e^4} = 0.4 \cdot 10^{-10} \,\mathrm{sec} \, .$$

For Jupiter, if we suppose that the energy dissipation is completely due to gravitational waves, we compute for the radiated power (see for example [9])

$$P = \frac{32G\Omega^6m^2r^4}{5c^5} = 5.5 \cdot 10^3 \,\mathrm{watt} \, ,$$

where Ω is the revolution frequency. Given a total energy of $E_J = GMm/(2r)$, we can write for the decay time

$$\Delta t = \frac{E_J}{P} = \frac{5}{64}\frac{Mc^5}{\Omega^6mr^5} = 3 \cdot 10^{31} \,\mathrm{sec} \simeq 10^{24} \,\mathrm{years} \, ,$$

a time much longer than the life of the Universe. Evidently the dissipation mechanisms at work in the solar (and planetary) system are much more efficient than the simply energy loss via gravitational radiation. The more efficient mechanisms (friction, viscosity, etc.) bring the planets on stable orbits (limits cycles) in less than $5 \cdot 10^9$ years.

On the contrary, the dissipation of energy via electromagnetic radiation brings the electron on stable orbitals in less than 10^{-10} sec.. Electrons dissipate very rapidly and collapse on limit cycles in very short times. We can perhaps say that a dissipation mechanism is at work in the "quantization" of the solar system as well as of the atom (see also [2–4]).

6 Wave equation for a planetary system

The relative success of the Bohr-like model introduced in section 4, at least in reproducing the exponential nature of the Titius–Bode law, induces us to look for the corresponding Schrödinger-like equation. This could be quite at odd with the observations made at the end of section 4. There we noted that it seems difficult even to define a wavelength for a stationary "matter" (or probability) wave, which closes around to the classical orbit. And even if a wavelength could be defined, since it would be of the same order of the orbit radius, unwanted interference phenomena would occur among planets along their orbits. In other words, the de Broglie matter waves interpretation, which is the intermediate step between a Bohr-like and a Schrödinger-like model, seems to be missed.

Despite all that, we shall now see how a wave equation can be correctly constructed, and how this equation predicts the energy spectrum and the exponential positions of the orbits, when the usual probabilistic interpretation of the wave function $\psi(\vec{x},t)$ is adopted.

We suppose, hence, to associate to a planetary system a scalar field $\psi(\vec{x},t)$, the so called wave function. We shall see that the wave function does not give us information on the behaviour of the single planet, but rather on the structure (orbits, energy levels, etc.) of the whole planetary system.

On comparing the Bohr quantization condition $mvr = n\hbar$ with the condition given in (3), $vr = s\mathrm{e}^{\lambda n}$, we see that the most straightforward correspondence is

$$s \longleftrightarrow \frac{\hbar}{m} \quad \text{or} \quad \hbar \longleftrightarrow sm\,.$$

This correspondence allows us to write immediately the wave equation for stationary states. In fact, from the Schrödinger eigenvalue equation

$$\left[-\frac{\hbar^2}{2m}\,\nabla^2 + U(r)\right]\psi = E\psi$$

we can write

$$\left[-\frac{s^2 m}{2}\,\nabla^2 + U(r)\right]\psi = E\psi\,.$$

Defining

$$\mathcal{E} := \frac{E}{m} = \text{energy per unit mass,}$$
$$V(r) := \frac{U}{m} = \text{potential energy per unit mass,}$$

we get

$$\left[-\frac{s^2}{2}\,\nabla^2 + V(r)\right]\psi = \mathcal{E}\psi\,, \tag{6}$$

which we adopt as the fundamental wave equation of our planetary system.

We note again that the quantity correctly quantized is $\mathcal{E}$, the energy per unit mass, and not the energy itself. As we already said for the problem *a la Bohr*, this is a consequence of the fact that the masses of the planets change from planet to planet, contrary to what happens for the electrons.

7 Wave equation for the Titius−Bode problem

In this section we look for a wave equation corresponding to the model (3). Such equation should implement the particular quantization condition on the angular momentum, namely $vr = s\mathrm{e}^{\lambda n}$. This condition seems to go to touch the delicate structures of the angular momentum algebra and of the spherical harmonic functions, and indeed it does. We have also another requirement: we wish to describe the observed fact that all the planetary orbits lie (more or less) in the same plane. From this elementary observation, we are pushed to consider a 2-dimensional Schrödinger-like equation, that is to write down the previous wave equation in a plane. This choice will simplify and clarify very much the problem on how to modify the standard Schrödinger equation in order to accommodate for the condition (3).

The equation (6) can be written in operatorial form

$$\hat{H}_{\mathrm{M}}\psi = \mathcal{E}\psi\,,$$

where $\hat{H}_{\mathrm{M}}$ is the *hamiltonian per unit mass*

$$\hat{H}_{\mathrm{M}} = \frac{\vec{p}^{\,2}}{2} + V(r) \tag{7}$$

and the association is made

$$\vec{p} \longleftrightarrow -\mathrm{i}s\,\vec{\nabla}\,.$$

In fact $\vec{p}^{\,2} = \vec{p}\cdot\vec{p} = -s^2\nabla^2$.

The equation (7) written in planar polar coordinates reads

$$\hat{H}_{\mathrm{M}} = \frac{1}{2}\left(\hat{p}_r^2 + \frac{\hat{p}_\phi^2}{r^2}\right) + V(r)\,. \tag{8}$$

With the usual associations of ordinary quantum mechanics, in planar polar coordinates,

$$\hat{p}_r^2 \longrightarrow -s^2\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)\,,$$

$$\hat{p}_\phi \longrightarrow -\mathrm{i}s\frac{\partial}{\partial\phi} \implies \hat{p}_\phi^2 \longrightarrow -s^2\frac{\partial^2}{\partial\phi^2}\,,$$

the hamiltonian (8) reads

$$\hat{H}_{\mathrm{M}} = -\frac{s^2}{2r^2}\left[r\frac{\partial}{\partial r}\left(r\,\frac{\partial}{\partial r}\right) + \frac{\partial^2}{\partial\phi^2}\right] + V(r)\,.$$

The operator which accounts for the angular momentum is $\hat{p}_\phi$. The motion takes place in the plane (x, y), that is, the only non zero component of the angular momentum is J_z. When we solve the usual Schrödinger equation we proceed in the construction of the spherical harmonic functions $Y_{lm}(\theta,\phi) = A_{lm}P_l^m(\cos\theta)\mathrm{e}^{\mathrm{i}m\phi}$ and the operator $\hat{p}_\phi$ acts on the eigenfunctions $\mathrm{e}^{\mathrm{i}m\phi}$. These belong to the Hilbert space $\mathcal{L}^2([0,2\pi])$ of the squared integrable functions on the interval $[0,2\pi]$. On such space, $\hat{p}_\phi$ is hermitian (self-adjoint) and its eigenvalues are ms, $m \in Z$. In fact

$$\hat{p}_\phi\,\mathrm{e}^{\mathrm{i}m\phi} = -\mathrm{i}s\frac{\partial}{\partial\phi}\,\mathrm{e}^{\mathrm{i}m\phi} = ms\,\mathrm{e}^{\mathrm{i}m\phi}\,.$$

This corresponds to the quantum condition (*a la Bohr*) $vr = ms$ (see eq. (2)). But we want to have eigenvalues different from ms, we want $se^{\lambda m}$ (see eq.(3)). And, if possible, we do not want to spoil or loose the eigenfunctions $\mathrm{e}^{\mathrm{i}m\phi}$ and their Hilbert space $\mathcal{L}^2([0, 2\pi])$. This aim can be reached if we define the operator

$$\hat{\mathcal{P}}_\varphi \, \mathrm{e}^{\mathrm{i}m\phi} := \mathrm{i}\, \mathrm{e}^{m\lambda}\, \mathrm{e}^{\mathrm{i}m\phi} \tag{9}$$

for every $\mathrm{e}^{\mathrm{i}m\phi} \in \mathcal{L}^2([0, 2\pi])$, $m \in Z$, where λ is the phenomenological parameter given in (1) ($2\lambda \simeq 0.53707$ for the Solar System).

The operator $\hat{\mathcal{P}}_\varphi$ can be defined to be linear, and being defined on an orthonormal basis of $\mathcal{L}^2([0, 2\pi])$, is therefore well defined on all $\mathcal{L}^2([0, 2\pi])$. In the Appendix we shall show that the operator $\hat{\mathcal{P}}_\varphi$ is nothing else than the exponential of ∂_ϕ, that is

$$\hat{\mathcal{P}}_\varphi = \mathrm{i}\mathrm{e}^{-\mathrm{i}\lambda\partial_\phi} \,.$$

Therefore we substitute the usual association

$$\hat{p}_\phi \;\longrightarrow\; -\mathrm{i}s\frac{\partial}{\partial\phi}$$

with the new one

$$\hat{p}_\phi \;\longrightarrow\; -\mathrm{i}s\hat{\mathcal{P}}_\varphi \,.$$

In this way we have

$$\hat{p}_\phi \mathrm{e}^{\mathrm{i}m\phi} = -\mathrm{i}s\hat{\mathcal{P}}_\varphi \mathrm{e}^{\mathrm{i}m\phi} = -\mathrm{i}s(\mathrm{i}\mathrm{e}^{m\lambda})\mathrm{e}^{\mathrm{i}m\phi} = s\mathrm{e}^{\lambda m}\mathrm{e}^{\mathrm{i}m\phi} \,.$$

The operator $\hat{p}_\phi$ can be proved to be self-adjoint (see Appendix). Of course, the operator $\hat{\mathcal{P}}_\varphi$ is here introduced by hand. Further studies will be devoted to a fully explanation of its physical meaning and of its link with ∂_ϕ.

The new hamiltonian per unit mass reads

$$\hat{H}_{\mathrm{M}} = -\frac{s^2}{2r^2}\left[r\frac{\partial}{\partial r}\left(r\,\frac{\partial}{\partial r}\right) + \hat{\mathcal{P}}_\varphi^2\right] + V(r) \,.$$

The wave function in these *polar planar coordinates* can be written as

$$\psi(r, \phi) = R(r)\Phi(\phi) \,,$$

where the separation of variables was supposed, in order to proceed towards a solution of the Schrödinger-like equation. The normalization condition is

$$\int_{R^2} |\psi(r, \phi)|^2 \mathrm{d}^2 x = \int_0^\infty R^2(r) r \,\mathrm{d}r \cdot \int_0^{2\pi} |\Phi(\phi)|^2 \mathrm{d}\phi = 1 \,. \tag{10}$$

The time-independent Schrödinger-like equation is

$$\hat{H}_{\mathrm{M}}\psi = \mathcal{E}\psi \,,$$

where $V(r) = -GM/r$. This can be written as

$$r\,\frac{\partial}{\partial r}\left(r\,\frac{\partial \psi}{\partial r}\right) + \hat{\mathcal{P}}_\varphi^2\psi + \frac{2r^2}{s^2}\,(\mathcal{E} - V(r))\psi = 0 \tag{11}$$

which, with the position $\psi(r,\phi) = R(r)\Phi(\phi)$, becomes

$$\frac{r}{R}\,\frac{\partial}{\partial r}\left(r\,\frac{\partial R}{\partial r}\right) + \frac{1}{\Phi}\,\hat{\mathcal{P}}_\varphi^2\psi + \frac{2r^2}{s^2}\,(\mathcal{E} - V(r)) = 0\,.$$

Separating the variables we get the two equations

$$\begin{cases} \dfrac{1}{r}\,\dfrac{\partial}{\partial r}\left(r\,\dfrac{\partial R}{\partial r}\right) - \dfrac{\mu}{r^2}R + \dfrac{2}{s^2}(\mathcal{E} - V(r))R = 0\,, \\ \hat{\mathcal{P}}_\varphi^2\Phi = -\mu\Phi\,, \end{cases}$$

where we suppose $\mu \in R$.

8 The radial equation

The radial equation is quite similar to the standard one of the hydrogen atom theory, apart of course for the radial part of the laplacian, which here is 2-dimensional. For its solution we shall therefore use standard techniques (see e.g. [11]).

Let's now look for the asymptotic behaviour of $R(r)$. We ask $R(r)$ to be finite everywhere including $r = 0$. Under the hypothesis

$$\lim_{r\to 0} V(r)r^2 = 0\,,$$

which is fulfilled by $V(r) = -GM/r$, the radial equation for $r \to 0$ becomes

$$r\,\frac{\partial}{\partial r}\left(r\,\frac{\partial R}{\partial r}\right) - \mu R = 0\,. \tag{12}$$

We seek $R(r)$ in the form of a power series and we retain only the first term for small r. That is, we put $R(r) = kr^t$ for $r \to 0$. Substituting this in the equation (12) we find

$$t^2 = \mu\,.$$

We want $R(r)$ real, therefore t must be real, and $\mu \geq 0$. So we have two roots

$$\begin{aligned} t_1 &= -\sqrt{\mu}\,, \\ t_2 &= +\sqrt{\mu}\,. \end{aligned}$$

But $t_1 \leq 0$, hence $r^{t_1} \to \infty$ for $r \to 0$. So t_1 does not yields a $R(r)$ finite near the origin, and must be discarded. The only acceptable solution is $t_2 = +\sqrt{\mu} \geq 0$.

Therefore we put $R(r) \sim kr^{t_2}$ for $r \to 0$. For the Newtonian potential $V(r) = -GM/r$ the radial equation reads

$$\frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\frac{\partial R}{\partial r} - \frac{\mu}{r^2}R + \frac{2}{s^2}\left(\mathcal{E} + \frac{GM}{r}\right)R = 0\,.$$

We choose as natural units for mass, length, and energy, respectively,

$$M\,, \quad \frac{s^2}{GM}\,, \quad \frac{G^2M^2}{s^2}$$

so that the radial equation can be rewritten as

$$\frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\frac{\partial R}{\partial r} - \frac{\mu}{r^2}R + 2\left(\mathcal{E} + \frac{1}{r}\right)R = 0\,. \tag{13}$$

To study the discrete spectrum (bound orbits, $\mathcal{E} < 0$), we introduce in place of $\mathcal{E}$, r, the variables

$$n = \frac{1}{\sqrt{-2\mathcal{E}}} \quad \text{and} \quad \rho = \frac{2}{n}r = 2\sqrt{-2\mathcal{E}}\,r$$

with $\mathcal{E} < 0$, $n > 0$, and their inverse relations

$$\mathcal{E} = -\frac{1}{2n^2}\,, \quad r = \frac{n}{2}\rho\,.$$

The equation (13) then becomes

$$\frac{\partial^2 R}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial R}{\partial \rho} + \left(-\frac{1}{4} + \frac{n}{\rho} - \frac{\mu}{\rho^2}\right)R = 0\,. \tag{14}$$

We already know that $R(\rho) \sim \rho^{t_2}$ for $\rho \to 0$. If now we take $\rho \to \infty$, then eq. (14) reads

$$\frac{\partial^2 R}{\partial \rho^2} - \frac{1}{4}R = 0\,,$$

whose solutions are $\mathrm{e}^{\pm\rho/2}$. We want $R(\rho) \to 0$ for $\rho \to \infty$, therefore we must choose the second, $R(\rho) \sim \mathrm{e}^{-\rho/2}$ for $\rho \to \infty$.

Now we make the substitution

$$R(\rho) = \rho^{t_2}\mathrm{e}^{-\rho/2}w(\rho)$$

and the eq. (14) becomes

$$\rho\, w'' + (2t_2 + 1 - \rho)\, w' + (n - t_2 - \tfrac{1}{2})\, w = 0\,.$$

We look for a solution of this equation which diverges at infinity no more rapidly than a finite power of ρ, while for $\rho \to 0$ we should have $w \to w_0$ finite. Such a solution is the *confluent hypergeometric function* (see Appendix)

$$w(\rho) = F(\alpha, \gamma, \rho) = F(t_2 + \tfrac{1}{2} - n, 2t_2 + 1, \rho)\,. \tag{15}$$

In particular, it behaves as a polynomial ($w \to \rho^p$ for $\rho \to \infty$) only if $\alpha = -N$ with $N \geq 0$ integer. Thus

$$t_2 + \tfrac{1}{2} - n = -N \implies n = t_2 + \tfrac{1}{2} + N\,, \quad N = 0, 1, 2, 3, \ldots\,.$$

9 The angular equation and the spectrum of $\mathcal{E}$

Now we need to know what is $t_2 = \sqrt{\mu}$. To this aim, we solve the angular equation

$$\hat{\mathcal{P}}_\varphi^2 \Phi = -\mu \Phi \,, \quad \mu \geq 0 \,. \tag{16}$$

We take $\Phi \in \mathcal{L}^2([0, 2\pi])$ and we know that an orthonormal basis in $\mathcal{L}^2([0, 2\pi])$ is

$$\frac{1}{\sqrt{2\pi}} \, \mathrm{e}^{\mathrm{i}m\phi} \,, \quad m = 0, \pm 1, \pm 2, \ldots \,.$$

From eq. (16), we see that we are looking for eigenvectors and eigenvalues of the operator $\hat{\mathcal{P}}_\varphi^2$. From the definition (9) we immediately get

$$\hat{\mathcal{P}}_\varphi^2 \mathrm{e}^{\mathrm{i}m\phi} = -\mathrm{e}^{2\lambda m} \mathrm{e}^{\mathrm{i}m\phi} \,,$$

which means that

- the eigenvectors of $\hat{\mathcal{P}}_\varphi^2$ are $u_m = \mathrm{e}^{\mathrm{i}m\phi}/\sqrt{2\pi}$;

- the eigenvalues of $\hat{\mathcal{P}}_\varphi^2$ are $\{-\mathrm{e}^{2\lambda m}\}_{m \in Z}$, and therefore $\mu_m = \mathrm{e}^{2\lambda m}$, $m = 0, \pm 1, \pm 2, \ldots$.

Hence we have

$$t_2 = \sqrt{\mu} = \mathrm{e}^{\lambda m} \,, \quad m = 0, \pm 1, \pm 2, \ldots \,,$$

then

$$n = t_2 + \tfrac{1}{2} + N = \mathrm{e}^{\lambda m} + \tfrac{1}{2} + N$$

with $N = 0, 1, 2, 3, \ldots$, $m = 0, \pm 1, \pm 2, \ldots$.

Finally, we are able to write down the spectrum of the energy

$$\mathcal{E} = -\frac{1}{2n^2} = -\frac{1}{2(\mathrm{e}^{\lambda m} + \frac{1}{2} + N)^2}$$

which in ordinary units reads

$$\mathcal{E} = -\frac{1}{2} \left(\frac{GM}{s} \right)^2 \frac{1}{(\mathrm{e}^{\lambda m} + \frac{1}{2} + N)^2} \tag{17}$$

with $N = 0, 1, 2, 3, \ldots$, $m = 0, \pm 1, \pm 2, \ldots$.

Some considerations are now in order:

- We see that for N fixed and large positive m, the levels behave like

$$\mathcal{E} \sim -\frac{1}{2} \left(\frac{GM}{s} \right)^2 \frac{1}{\mathrm{e}^{2\lambda m}} \,,$$

which is the formula (5) obtained from the "Bohr" model.

• Clearly, the closest agreement between the relations (17) and (5) is reached for

$$N = 0 \quad \text{and} \quad m = 1, 2, 3, \ldots .$$

We call this sequence the *principal sequence*. The formula (17) seems to suggest the existence of other sequences, like, for example,

$$m = 1 , \quad N = 0, 1, 2, \ldots ,$$
$$m = 2 , \quad N = 0, 1, 2, \ldots ,$$

or also

$$N = 0 \quad \text{and} \quad m = 0, -1, -2, -3, \ldots .$$

These sequences can give rise, in principle, to possible resonances (whose properties and connections with the observational data are going to be explored in future works), or to system of rings (about this see section 11).

10 Mean value of r

In order to complete our analysis of the solution of eq. (11), we want to compute the mean values taken by the variable r in various eigenstates. We are particularly interested in the eigenstates of the principal sequence, $N = 0$, $m = 1, 2, 3, \ldots$, which is the one matching the observational data in the closest way. For $\psi = R(r)\Phi(\phi)$, the normalization condition (10) holds, and the $\Phi(\phi) = \mathrm{e}^{im\phi}/\sqrt{2\pi}$ are already normalized to unity. Therefore we are left with the condition on $R(r)$ only

$$1 = \int_0^\infty \mathrm{d}r \, r R(r)^2 .$$

The function R is given in term of the variable ρ by

$$R(\rho) \sim \rho^{t_2} \, \mathrm{e}^{-\rho/2} \, w(\rho)$$

with $w(\rho)$ given by eq. (15) and $\rho = 2r/n$. First, we find the correct normalization constant for $R(\rho)$ by translating the normalization condition for $R(r)$ into the one for $R(\rho)$. We write

$$R(\rho) = A\rho^{t_2} \, \mathrm{e}^{-\rho/2} \, w(\rho) ,$$

so we have

$$1 = \int_0^\infty \mathrm{d}r \, r R(r)^2 = \frac{n^2}{4} \int_0^\infty \mathrm{d}\rho \, \rho R(\rho)^2 = \frac{n^2}{4} A^2 \int_0^\infty \mathrm{d}\rho \, \rho^{2t_2+1} \, \mathrm{e}^{-\rho} [w(\rho)]^2 .$$

Since we are mainly interested in the mean values of r for the eigenstates belonging to the principal sequence, ($N = 0$, $m = 1, 2, 3, \ldots$), we set $N = n - t_2 - \frac{1}{2} = 0$. This means $n = t_2 + \frac{1}{2}$ and

$$w(\rho) = F(0, 2t_2 + 1, \rho) = 1 .$$

This simplify very much the calculation of the integral. In fact we have

$$1 = \frac{n^2}{4}\, A^2 \int_0^\infty \mathrm{d}\rho\, \rho^{2t_2+1} \mathrm{e}^{-\rho} = \frac{n^2}{4}\, A^2 \int_0^\infty \mathrm{d}\rho\, \rho^{(2t_2+2)-1} \mathrm{e}^{-\rho} = \frac{n^2}{4}\, A^2 \Gamma(2t_2+2)\,,$$

where Γ is the Euler' Γ-function. Thus we have

$$A^2 = \frac{4}{n^2\,\Gamma(2t_2+2)} = \frac{2}{n^3\,\Gamma 2t_2+1)}\,, \tag{18}$$

where we used $2n = 2t_2 + 1$ and $\Gamma(x+1) = x\Gamma(x)$.

We can now compute the mean value of r:

$$\bar{r} = \int_0^\infty \mathrm{d}r\, r^2 R(r)^2 = \frac{n^3}{8} \int_0^\infty \mathrm{d}\rho\, \rho^2 R(\rho)^2 = \frac{n^3}{8}\, A^2 \int_0^\infty \mathrm{d}\rho\, \rho^{2t_2+2} \mathrm{e}^{-\rho}\, [w(\rho)]^2\,.$$

We are interested in the principal sequence. So, $N = 0$ and $w(\rho) = 1$, and we use for A the value just obtained in eq. (18). Hence

$$\begin{aligned}\bar{r} &= \frac{n^3}{8}\, A^2 \int_0^\infty \mathrm{d}\rho\, \rho^{2t_2+2} \mathrm{e}^{-\rho} = \frac{n^3}{8}\, A^2 \int_0^\infty \mathrm{d}\rho\, \rho^{(2t_2+3)-1} \mathrm{e}^{-\rho} = \frac{n^3}{8}\, A^2 \Gamma(2t_2+3) = \\ &= \frac{1}{4}\, \frac{\Gamma(2t_2+3)}{\Gamma(2t_2+1)} = \frac{1}{4}\, (2t_2+2)(2t_2+1) = \frac{1}{2}\, n(2n+1) = n^2 + \frac{n}{2}\,.\end{aligned}$$

Since $N = 0$, then $n = t_2 + \frac{1}{2} = \mathrm{e}^{\lambda m} + \frac{1}{2}$. Therefore

$$\bar{r} = n^2 + \tfrac{1}{2}\, n = \left(\mathrm{e}^{\lambda m} + \tfrac{1}{2}\right)^2 + \tfrac{1}{2}\left(\mathrm{e}^{\lambda m} + \tfrac{1}{2}\right) \sim \mathrm{e}^{2\lambda m}\,, \tag{19}$$

where the last holds for large and positive m.

Restoring the ordinary units, we have for the mean value of r, for large and positive m

$$\bar{r} \sim \frac{s^2}{GM}\, \mathrm{e}^{2\lambda m}\,,$$

which agrees with the Bohr model developed in section 4.

11 Prediction of the rings

It is tempting to speculate on the other possible sequences, in particular the one $N = 0$ and $m = 0, -1, -2, -3, \ldots$. First, we note that for $N = 0$ the mean value $\bar{r}$ is still given by the relation (19), or in ordinary units

$$\bar{r} = a\left[(\mathrm{e}^{\lambda m} + \tfrac{1}{2})^2 + \tfrac{1}{2}\,(\mathrm{e}^{\lambda m} + \tfrac{1}{2})\right], \tag{20}$$

where $a = s^2/(GM)$. Clearly, the formula (20) matches the phenomenological formula (1) only for large, positive m. This could be expected as due to the "quantum" origin (i.e. from a wave equation) of the formula (20). But the interesting feature of the equation (20) is that it can be considered also for $m = 0, -1, -2, -3, \ldots$. In the limit $m \to -\infty$ we have for (20)

$$\bar{r} \longrightarrow \tfrac{1}{2}\, a \tag{21}$$

contrary to the limit given by phenomenological rule (1), which predicts $\bar{r} \to 0$ for $m \to -\infty$. The sequence $m = 0, -1, -2, -3, \ldots$ would in such a way correspond to a system of permitted concentric orbits, accumulating on the limit orbit $\bar{r} = a/2$: clearly, a system of rings. We can check the predictive ability of the relation (21) using the planets equipped with a system of rings: Jupiter, Saturn, Uranus.

The procedure is the following:

— Calculate the parameter a from the phenomenological Titius–Bode rule, using the radius r_1 of the first satellite orbit:

$$a = \frac{r_1}{e^{2\lambda}}\,.$$

— Calculate the radius that the inner ring should have:

$$R_{\mathrm{in-ring}} = \frac{a}{2} = \frac{r_1}{2e^{2\lambda}}\,.$$

We can check the above relation also in the reverse form

$$2e^{2\lambda}\, R_{\mathrm{in-ring}} = r_1\,.$$

Using the observational data given, for example, in [10] we can write, for:

Jupiter:

$$2e^{2\lambda}\, R_{\mathrm{in-ring}} = 2 \cdot 1.7277 \cdot 122\,500 = 423\,280\ \mathrm{km}\,.$$

This value agrees almost perfectly (within an error of less than 1 %) with the radius of the orbit of Io, which is 421 600 km. To be precise, we should say that we discarded the ring "Halo" because it has a thickness of 20 000 km, and therefore does not seem a "real" ring, but rather just a halo. As first ring we use the ring "Main". Besides, we considered Io as the first satellite (as regard the distance from Jupiter), instead of Metis, Adrastea, Amalthea or Thebe. This because the latter satellites have sizes of, at most, 100 km and masses which are $10^{-4}-10^{-6}$ the mass of Io. This choice takes into account the well known fact that Titius–Bode rule works well for quite large and quite massive objects. For example, it does not work for comets or light asteroids. This criterium will be adopted also in the forthcoming considerations about the Saturn and Uranus systems.

Saturn:

$$2e^{2\lambda}\, R_{\mathrm{in-ring}} = 2 \cdot 1.5967 \cdot 66\,000 = 210\,700\ \mathrm{km}\,. \tag{22}$$

Here we have used the radius of the ring "D", the inner one. We see that the number (22) lies half a way in between the orbits of Mimas (185 520 km) and Enceladus (238 020 km), which are the first two "big" satellites. The error is in the range of

11.5 % − 13.5 %. If we use the radius of the ring "C", namely $R_{\rm in-ring} = 74\,500$ km, we get for the radius $r_1 = 237\,900$ km, which agrees almost perfectly with orbital radius of Enceladus (less than 1 % error). However, Enceladus is not the first satellite but only the second. Here also, as already done for Jupiter, we have discarded the too light bodies, and the satellites discovered only with spacecrafts (therefore, very small). The reasons are the same as in the above.

Uranus:

$$2e^{2\lambda}\,R_{\rm in-ring} = 2\cdot 1.4662\cdot 41\,840 = 122\,690\ \text{km}\,.$$

This value agrees with the radius of the orbit of Miranda (129 780 km), the first "big" moon considered, within an error of 5.5 %. Here we considered as first inner ring the ring "6". Miranda is the first satellite with relevant mass and size. In fact it was discovered from the Earth by Kuiper in 1948. If we use the ring "Alpha" as inner ring, we get for the radius of the first satellite orbit $r_1 = 131\,140$ km, in agreement within 1 % with the Miranda orbital radius.

Even if the agreement between the predicted inner radius of the rings and the observational data is not perfect (however with errors around 10 %), and the statistics of only three cases is really poor, nevertheless this "prediction" seems to corroborate the quantum-like model presented in this paper. On the other hand, it should be noted that these errors are of the same order of those affecting the phenomenological Titius–Bode rule, therefore acceptable.

Moreover, noting that the Sun, for example, does not have rings, we must also say that the model does not predict a compulsory presence of the rings. However, it allows us to describe the existing rings.

12 Conclusions

In this paper we have shown that a wave equation is able, under certain restrictive hypothesis, to describe some basic properties of the planetary systems, namely the law of the distances of the planets (or satellites) from the central body. The wave equation adopted for this scope is a deformation of the Schrödinger equation.

We have been pushed to the choice of a deformed Schrödinger equation by the analogy between the mechanism devised by 't Hooft to produce quantization at atomic level via dissipation, and the dissipation occurred during the history of the proto-planetary nebula. An analogue of such a dissipation mechanism could have been at work (of course, on much larger time scales) during the evolution of the planetary systems. From the primitive nebula, where all the orbits were filled by dust and rubble, we arrive, after 5 billion years of evolution, to the "quantized" orbits of today.

Of course, having marked the analogies, also the evident differences must be underlined. We do not have quantum jumps in the Solar System, we do not have quantum interferences between planets, neither quantum superpositions nor zero-point energy. A planetary system *is not* a quantum system. On the contrary, we have shown that a deformation of the Schrödinger equation (one of the basic tools of Quantum Mechanics) seems to be able to play a role also in the description of some quantum-like features of planetary systems. The descriptive power (somehow "mysterious") of eigenvalue wave equations seems to be confirmed. Indirectly, this quantum-like description of the planetary systems seems also to strengthen the 't Hooft ideas on the origin of quantization from dissipation.

Appendix

Linearity, self-adjointness and explicit form of $\hat{p}_\phi = -\mathrm{i}s\hat{\mathcal{P}}_\varphi$

$\hat{\mathcal{P}}_\varphi$ is defined on the orthonormal basis $u_m = \mathrm{e}^{\mathrm{i}m\phi}/\sqrt{2\pi}$ of $\mathcal{L}^2([0,2\pi])$ as

$$\hat{\mathcal{P}}_\varphi \mathrm{e}^{\mathrm{i}m\phi} := \mathrm{i}\, \mathrm{e}^{m\lambda}\mathrm{e}^{\mathrm{i}m\phi}\,.$$

Defining $\hat{p}_\phi = -\mathrm{i}s\hat{\mathcal{P}}_\varphi$ we have

$$\hat{p}_\phi \mathrm{e}^{\mathrm{i}m\phi} = s\mathrm{e}^{m\lambda}\mathrm{e}^{\mathrm{i}m\phi}\,.$$

Hence $u_m = \mathrm{e}^{\mathrm{i}m\phi}/\sqrt{2\pi}$ are the eigenvectors of $\hat{p}_\phi$ with the eigenvalues $\mu_m = s\mathrm{e}^{m\lambda}$.

Moreover we define $\hat{\mathcal{P}}_\varphi$ to be linear by stating

$$\begin{cases} \hat{\mathcal{P}}_\varphi(\mathrm{e}^{\mathrm{i}n\phi} + \mathrm{e}^{\mathrm{i}m\phi}) := \mathrm{i}\,\mathrm{e}^{n\lambda}\mathrm{e}^{\mathrm{i}n\phi} + \mathrm{i}\,\mathrm{e}^{m\lambda}\mathrm{e}^{\mathrm{i}m\phi} = \hat{\mathcal{P}}_\varphi(\mathrm{e}^{\mathrm{i}n\phi}) + \hat{\mathcal{P}}_\varphi(\mathrm{e}^{\mathrm{i}m\phi})\,, \\ \hat{\mathcal{P}}_\varphi(\alpha\mathrm{e}^{\mathrm{i}n\phi}) := \mathrm{i}\alpha\mathrm{e}^{n\lambda}\mathrm{e}^{\mathrm{i}n\phi} = \alpha\hat{\mathcal{P}}_\varphi(\mathrm{e}^{\mathrm{i}n\phi})\,. \end{cases}$$

Being $\hat{\mathcal{P}}_\varphi$ linear on an orthonormal basis, then $\hat{\mathcal{P}}_\varphi$ is linear all over $\mathcal{L}^2([0,2\pi])$. We remind also the

Theorem: *If a linear operator is self-adjoint on an orthonormal basis of a Hilbert space $\mathcal{H}$, then it is self-adjoint over all $\mathcal{H}$.*

Therefore we have simply to show that $\hat{p}_\phi$ is self-adjoint on the basis u_m. In fact we have

$$\begin{aligned} \langle \hat{p}_\phi u_m | u_n \rangle &= \int_0^{2\pi} \mathrm{d}\phi\, (\hat{p}_\phi u_m)^* u_n = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi\, (\hat{p}_\phi \mathrm{e}^{\mathrm{i}m\phi})^* \mathrm{e}^{\mathrm{i}n\phi} = \\ &= \frac{s\mathrm{e}^{m\lambda}}{2\pi}\int_0^{2\pi} \mathrm{d}\phi\, \mathrm{e}^{-\mathrm{i}m\phi}\mathrm{e}^{\mathrm{i}n\phi} = s\mathrm{e}^{m\lambda}\delta_{mn} \end{aligned}$$

and

$$\langle u_m | \hat{p}_\phi u_n \rangle = \int_0^{2\pi} \mathrm{d}\phi\, u_m^* (\hat{p}_\phi u_n) = \frac{s\mathrm{e}^{n\lambda}}{2\pi}\int_0^{2\pi} \mathrm{d}\phi\, \mathrm{e}^{-\mathrm{i}m\phi}\mathrm{e}^{\mathrm{i}n\phi} = s\mathrm{e}^{n\lambda}\delta_{mn}\,.$$

Besides, we check the identity

$$\hat{\mathcal{P}}_\varphi = \mathrm{i}\,\mathrm{e}^{-\mathrm{i}\lambda\partial_\phi}\,.$$

In fact, since

$$-\mathrm{i}\lambda\partial_\phi \mathrm{e}^{\mathrm{i}m\phi} = \lambda m \mathrm{e}^{\mathrm{i}m\phi}\,,$$

we have

$$
\begin{aligned}
\mathrm{e}^{-\mathrm{i}\lambda\partial_\phi}\mathrm{e}^{\mathrm{i}m\phi} &= \left(1+(-\mathrm{i}\lambda)\partial_\phi+\frac{1}{2}(-\mathrm{i}\lambda)^2\partial_\phi^2+\dots+\frac{1}{n!}(-\mathrm{i}\lambda)^n\partial_\phi^n+\cdots\right)\cdot\mathrm{e}^{\mathrm{i}m\phi} = \\
&= \left(1+\lambda m+\frac{\lambda^2 m^2}{2!}+\dots+\frac{\lambda^n m^n}{n!}+\cdots\right)\cdot\mathrm{e}^{\mathrm{i}m\phi} = \mathrm{e}^{m\lambda}\mathrm{e}^{\mathrm{i}m\phi}\,.
\end{aligned}
$$

Hence, the thesis.

Confluent hypergeometric function

The *confluent hypergeometric function* is defined via the series

$$
F(\alpha,\gamma,z) = 1+\frac{\alpha}{\gamma}\frac{z}{1!}+\frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}\frac{z^2}{2!}+\dots\,, \tag{23}
$$

which converges for all finite z; the parameter α is any number in C, and the parameter γ must be different from zero and from any negative integer. If α is a negative integer (or zero), the function $F(\alpha,\gamma,z)$ becomes a polynomy of degree $|\alpha|$.

The function $F(\alpha,\gamma,z)$ is a solution of the differential equation

$$
zu''+(\gamma-z)u'-\alpha u = 0\,, \tag{24}
$$

as can be directly checked.

With the substitution $u = z^{1-\gamma}u_1$ the eq. (24) is transformed in

$$
zu_1''+(2-\gamma-z)u_1'-(\alpha-\gamma+1)u_1 = 0\,.
$$

From here we see that, for a non integer γ, eq. (24) admits also the integral

$$
z^{1-\gamma}F(\alpha-\gamma+1,2-\gamma,z)\,,
$$

which is linearly independent from (23), so that the general solution of eq. (24) has the form

$$
u = c_1 F(\alpha,\gamma,z)+c_2 z^{1-\gamma}F(\alpha-\gamma+1,2-\gamma,z)\,.
$$

The second term, contrary to the first, has a singularity in $z = 0$.

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