Nonlinear spinor field in Bianchi type-I Universe filled with perfect fluid: Exact self-consistent solutions

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Self-consistent solutions to nonlinear spinor field equations in general relativity have been studied for the case of Bianchi type-I space–time filled with perfect fluid. The initial and the asymptotic behaviors of the field functions and the metric one have been thoroughly studied. The absence of initial singularity for some types of solutions and also the isotropic mode of space–time expansion in some special cases should be emphasized.
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I. INTRODUCTION

The quantum field theory in curved space–time has been a matter of great interest in recent years because of its applications to cosmology and astrophysics. The evidence of existence of strong gravitational fields in our Universe led to the study of the quantum effects of material fields in external classical gravitational field. After the appearance of Parker's paper on scalar fields[1] and spin-$\frac{1}{2}$ fields,[2] several authors have studied this subject. Although the Universe seems homogenous and isotropic at present, there are no observational data guaranteeing the isotropy in the era prior to the recombination. In fact, there are theoretical arguments that sustain the existence of an anisotropic phase that approaches an isotropic one.[3] Interest in studying Klein–Gordon and Dirac equations in anisotropic models has increased since Hu and Parker[4] have shown that the creation of scalar particles in anisotropic backgrounds can dissipate the anisotropy as the Universe expands.

A Bianchi type-I (B-I) Universe, being the straightforward generalization of the flat Robertson–Walker (RW) Universe, is one of the simplest models of an anisotropic Universe that describes a homogenous and spatially flat Universe. Unlike the RW Universe which has the same scale factor for each of the three spatial directions, a B-I Universe has a different scale factor in each direction, thereby introducing an anisotropy to the system. It moreover has the agreeable property that near the singularity it behaves like a Kasner Universe, even in the presence of matter, and consequently falls within the general analysis of the singularity given by Belinskii *et al.*[5] Also in a Universe filled with matter for $p=\gamma\varepsilon,\ \gamma<1$, it has been shown that any initial anisotropy in a B-I Universe quickly dies away and a B-I Universe eventually evolves into a RW Universe.[6] Since the present-day Universe is surprisingly isotropic, this feature of the B-I Universe makes it a prime candidate for studying the possible effects of an anisotropy in the early Universe on present-day observations. In light of the importance of mentioned above, several authors have studied linear spinor field equations[7,8] and the behavior of gravitational waves (GWs)[9–11] in a B-I Universe. Nonlinear spinor field (NLSF) in external cosmological gravitational field was first studied by G. N. Shikin in 1991.[12] This study was extended by us for the more general case where we consider the nonlinear term as an arbitrary function of all possible invariants generated from

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spinor bilinear forms. In that paper we also studied the possibility of elimination of initial singularity especially for the Kasner Universe.[13] In a recent paper[14] we studied the behavior of self-consistent NLSF in a B-I Universe that was followed by Refs. 15 and 16 where we studied the self-consistent system of interacting spinor and scalar fields. The purpose of the paper is to extend our study for more general NLSF in the presence of perfect fluid. In Sec. II we derive fundamental equations corresponding to the Lagrangian for the self-consistent system of spinor and gravitational fields in the presence of a perfect fluid and seek their general solutions. In Sec. III we give a detailed analysis of the solutions obtained for different kinds of nonlinearity. In Sec. IV we study the role of a perfect fluid and in Sec. V we sum up the results obtained.

II. FUNDAMENTAL EQUATIONS AND GENERAL SOLUTIONS

The Lagrangian for the self-consistent system of spinor and gravitational fields in the presence of a perfect fluid is

$$L=\frac{R}{2\kappa}+\frac{i}{2}[\bar{\psi}\gamma^{\mu}\nabla_{\mu}\psi-\nabla_{\mu}\bar{\psi}\gamma^{\mu}\psi]-m\bar{\psi}\psi+L_N+L_m, \tag{2.1}$$

with R being the scalar curvature and κ being the Einstein gravitational constant. The nonlinear term L_N describes the self-interaction of a spinor field and can be presented as some arbitrary functions of invariants generated from the real bilinear forms of a spinor field having the form

$$S=\bar{\psi}\psi,\quad P=i\bar{\psi}\gamma^5\psi,\quad v^{\mu}=(\bar{\psi}\gamma^{\mu}\psi),\quad A^{\mu}=(\bar{\psi}\gamma^5\gamma^{\mu}\psi),\quad T^{\mu\nu}=(\bar{\psi}\sigma^{\mu\nu}\psi),$$

where $\sigma^{\mu\nu}=(i/2)[\gamma^{\mu}\gamma^{\nu}-\gamma^{\nu}\gamma^{\mu}]$. Invariants, corresponding to the bilnear forms, look like

$$I=S^2,\quad J=P^2,\quad I_v=v_{\mu}v^{\mu}=(\bar{\psi}\gamma^{\mu}\psi)g_{\mu\nu}(\bar{\psi}\gamma^{\nu}\psi),$$

$$I_A=A_{\mu}A^{\mu}=(\bar{\psi}\gamma^5\gamma^{\mu}\psi)g_{\mu\nu}(\bar{\psi}\gamma^5\gamma^{\nu}\psi),\quad I_T=T_{\mu\nu}T^{\mu\nu}=(\bar{\psi}\sigma^{\mu\nu}\psi)g_{\mu\alpha}g_{\nu\beta}(\bar{\psi}\sigma^{\alpha\beta}\psi).$$

According to the Pauli–Fierz theorem,[17] among the five invariants only I and J are independent as all others can be expressed by them: $I_v=-I_A=I+J$ and $I_T=I-J$. Therefore we choose the nonlinear term $L_N=F(I,J)$, thus claiming that it describes the nonlinearity in the most general of its form. L_m is the Lagrangian of perfect fluid.

We choose B-I space–time metric in the form

$$ds^2=dt^2-\gamma_{ij}(t)dx^i dx^j. \tag{2.2}$$

As it admits no rotational matter, the spatial metric $\gamma_{ij}(t)$ can be put into diagonal form. Now we can rewrite the B-I space–time metric in the form[18]

$$ds^2=dt^2-a^2(t)dx^2-b^2(t)dy^2-c^2(t)dz^2, \tag{2.3}$$

where the velocity of light is taken to be unity. Einstein equations for $a(t)$, $b(t)$, and $c(t)$ corresponding to the metric (2.3) and Lagrangian (2.1) read[18]

$$\frac{\ddot{a}}{a}+\frac{\dot{a}}{a}\left(\frac{\dot{b}}{b}+\frac{\dot{c}}{c}\right)=-\kappa\left(T_1^1-\frac{1}{2}\,T\right), \tag{2.4}$$

$$\frac{\ddot{b}}{b}+\frac{\dot{b}}{b}\left(\frac{\dot{a}}{a}+\frac{\dot{c}}{c}\right)=-\kappa\left(T_2^2-\frac{1}{2}\,T\right), \tag{2.5}$$

$$\frac{\ddot{c}}{c}+\frac{\dot{c}}{c}\left(\frac{\dot{a}}{a}+\frac{\dot{b}}{b}\right)=-\kappa\left(T_3^3-\frac{1}{2}T\right), \tag{2.6}$$

$$\frac{\ddot{a}}{a}+\frac{\ddot{b}}{b}+\frac{\ddot{c}}{c}=-\kappa\left(T_0^0-\frac{1}{2}T\right), \tag{2.7}$$

where points denote differentiation with respect to t, and $T=T_\mu^\mu$.

The NLSF equations and components of the energy-momentum tensor for the spinor field and perfect fluid corresponding to (2.1) are

$$\begin{aligned} &i\gamma^\mu\nabla_\mu\psi-m\psi+F_I 2S\psi+F_J 2Pi\gamma^5\psi=0,\\ &i\nabla_\mu\bar{\psi}\gamma^\mu+m\bar{\psi}-F_I 2S\bar{\psi}-F_J 2Pi\bar{\psi}\gamma^5=0, \end{aligned} \tag{2.8}$$

where $F_I:=\partial F/\partial I$ and $F_J:=\partial F/\partial J$. Here

$$T_\mu^\rho=\frac{i}{4}g^{\rho\nu}(\bar{\psi}\gamma_\mu\nabla_\nu\psi+\bar{\psi}\gamma_\nu\nabla_\mu\psi-\nabla_\mu\bar{\psi}\gamma_\nu\psi-\nabla_\nu\bar{\psi}\gamma_\mu\psi)-\delta_\mu^\rho L_{sp}+T_{\mu(m)}^\rho, \tag{2.9}$$

while L_{sp}, on account of spinor field equations, takes the form

$$L_{sp}=-\left[\frac{1}{2}\left(\bar{\psi}\frac{\partial L_N}{\partial\bar{\psi}}+\frac{\partial L_N}{\partial\psi}\psi\right)-L_N\right]=-[2IF_I+2JF_J-L_N].$$

Here $T_{\mu(m)}^\rho$ is the energy-momentum tensor of a perfect fluid. For a Universe filled with perfect fluid, in the concomitant system of reference ($u^0=1$, $u^i=0$, $i=1,2,3$) we have

$$T_{\mu(m)}^\nu=(p+\varepsilon)u_\mu u^\nu-\delta_\mu^\nu p=(\varepsilon,-p,-p,-p), \tag{2.10}$$

where energy ε is related to the pressure p by the equation of state $p=\gamma\varepsilon$. The general solution has been derived by Jacobs.[6] Here γ varies between the interval $0\leqslant\gamma\leqslant 1$, whereas $\gamma=0$ describes the dust Universe, $\gamma=\frac{1}{3}$ presents radiation Universe, $\frac{1}{3}<\gamma<1$ ascribes hard Universe, and $\gamma=1$ corresponds to the stiff matter. In (2.8) and (2.9) ∇_μ denotes the covariant derivative of spinor, having the form[19]

$$\nabla_\mu\psi=\frac{\partial\psi}{\partial x^\mu}-\Gamma_\mu\psi, \tag{2.11}$$

where $\Gamma_\mu(x)$ are spinor affine connection matrices. The $\gamma^\mu(x)$ matrices are defined for the metric (2.3) as follows. Using the equalities[20,21]

$$g_{\mu\nu}(x)=e_\mu^a(x)e_\nu^b(x)\eta_{ab}, \quad \gamma_\mu(x)=e_\mu^a(x)\bar{\gamma}^a,$$

where $\eta_{ab}=\text{diag}\,(1,-1,-1,-1)$, $\bar{\gamma}_\alpha$ are the Dirac matrices of Minkowski space and $e_\mu^a(x)$ are the set of tetradic four-vectors, we obtain the Dirac matrices $\gamma^\mu(x)$ of curved space-time:

$$\gamma^0=\bar{\gamma}^0, \quad \gamma^1=\bar{\gamma}^1/a(t), \quad \gamma^2=\bar{\gamma}^2/b(t), \quad \gamma^3=\bar{\gamma}^3/c(t),$$

$$\gamma_0=\bar{\gamma}_0, \quad \gamma_1=\bar{\gamma}_1 a(t), \quad \gamma_2=\bar{\gamma}_2 b(t), \quad \gamma_3=\bar{\gamma}_3 c(t).$$

The $\Gamma_\mu(x)$ matrices are defined by the equality

$$\Gamma_\mu(x)=\tfrac{1}{4}g_{\rho\sigma}(x)(\partial_\mu e_\delta^b e_b^\rho-\Gamma_{\mu\delta}^\rho)\gamma^\sigma\gamma^\delta,$$

which gives

$$\Gamma_0=0, \quad \Gamma_1=\tfrac{1}{2}\dot{a}(t)\overline{\gamma^1}\,\overline{\gamma^0}, \quad \Gamma_2=\tfrac{1}{2}\dot{b}(t)\overline{\gamma^2}\,\overline{\gamma^0}, \quad \Gamma_3=\tfrac{1}{2}\dot{c}(t)\overline{\gamma^3}\,\overline{\gamma^0}. \tag{2.12}$$

Flat space–time matrices we choose in the form given in Ref. 22.

$$\overline{\gamma^0}=\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \overline{\gamma^1}=\begin{pmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & -1 & 0 & 0\\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\overline{\gamma^2}=\begin{pmatrix} 0 & 0 & 0 & -i\\ 0 & 0 & i & 0\\ 0 & i & 0 & 0\\ -i & 0 & 0 & 0 \end{pmatrix}, \quad \overline{\gamma^3}=\begin{pmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\\ -1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Defining γ^5 as follows,

$$\gamma^5=-\frac{i}{4}E_{\mu\nu\sigma\rho}\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho, \quad E_{\mu\nu\sigma\rho}=\sqrt{-g}\varepsilon_{\mu\nu\sigma\rho}, \quad \varepsilon_{0123}=1,$$

$$\gamma^5=-i\sqrt{-g}\gamma^0\gamma^1\gamma^2\gamma^3=-i\overline{\gamma^0}\,\overline{\gamma^1}\,\overline{\gamma^2}\,\overline{\gamma^3}=\overline{\gamma^5},$$

we obtain

$$\overline{\gamma^5}=\begin{pmatrix} 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\\ -1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

We study the space-independent solutions to NLSF equation (2.8). In this case the first equation of the system (2.8) together with (2.11) and (2.12) is

$$i\overline{\gamma^0}\left(\frac{\partial}{\partial t}+\frac{\dot{\tau}}{2\tau}\right)\psi-m\psi+\mathscr{D}\psi+i\mathscr{G}\gamma^5\psi=0, \quad \tau(t)=a(t)b(t)c(t)=\sqrt{-g}, \tag{2.13}$$

where we denote $\mathscr{D}:=2SF_I$ and $\mathscr{G}:=2PF_J$. For the components $\psi_\rho=V_\rho(t)$, where ρ=1,2,3,4, from (2.13) one deduces the following system of equations:

$$
\begin{aligned}
&\dot V_1+\frac{\dot\tau}{2\tau}V_1+i(m-\mathscr{D})V_1-\mathscr{G}V_3=0,\\
&\dot V_2+\frac{\dot\tau}{2\tau}V_2+i(m-\mathscr{D})V_2-\mathscr{G}V_4=0,\\
&\dot V_3+\frac{\dot\tau}{2\tau}V_3-i(m-\mathscr{D})V_3+\mathscr{G}V_1=0,\\
&\dot V_4+\frac{\dot\tau}{2\tau}V_4-i(m-\mathscr{D})V_4+\mathscr{G}V_2=0.
\end{aligned}
\tag{2.14}
$$

Let us now define the equations for

$$
\begin{aligned}
&P=i(V_1V_3^*-V_1^*V_3+V_2V_4^*-V_2^*V_4),\\
&R=(V_1V_3^*+V_1^*V_3+V_2V_4^*+V_2^*V_4),\\
&S=(V_1^*V_1+V_2^*V_2-V_3^*V_3-V_4^*V_4).
\end{aligned}
\tag{2.15}
$$

After a little manipulation one finds

$$
\begin{aligned}
&\frac{dS_0}{dt}-2\mathscr{G}R_0=0,\\
&\frac{dR_0}{dt}+2(m-\mathscr{D})P_0+2\mathscr{G}S_0=0,\\
&\frac{dP_0}{dt}-2(m-\mathscr{D})R_0=0,
\end{aligned}
\tag{2.16}
$$

where $S_0=\tau S$, $P_0=\tau P$, and $R_0=\tau R$. From this system we obtain

$$S_0\dot S_0+R_0\dot R_0+P_0\dot P_0=0,$$

which gives

$$S^2+R^2+P^2=C^2/\tau^2,\quad C^2=\text{const.} \tag{2.17}$$

Let us go back to the system of equations (2.14). It can be written as follows if one defines $W_\alpha=\sqrt{\tau}V_\alpha$:

$$
\begin{aligned}
&\dot W_1+i\Phi W_1-\mathscr{G}W_3=0,\quad \dot W_2+i\Phi W_2-\mathscr{G}W_4=0,\\
&\dot W_3-i\Phi W_3+\mathscr{G}W_1=0,\quad \dot W_4-i\Phi W_4+\mathscr{G}W_2=0,
\end{aligned}
\tag{2.18}
$$

where $\Phi=m-\mathscr{D}$. Defining $U(\sigma)=W(t)$, where $\sigma=\int\mathscr{G}\,dt$, we rewrite the foregoing system as

$$
\begin{aligned}
&U_1'+i(\Phi/\mathscr{G})U_1-U_3=0,\quad U_2'+i(\Phi/\mathscr{G})U_2-U_4=0,\\
&U_3'-i(\Phi/\mathscr{G})U_3+U_1=0,\quad U_4'-i(\Phi/\mathscr{G})U_4+U_2=0,
\end{aligned}
\tag{2.19}
$$

where prime $(')$ denotes differentiation with respect to σ.

Let us now solve the Einstein equations. To do it we first write the expressions for the components of the energy-momentum tensor explicitly. Using the property of flat space–time Dirac matrices and the explicit form of covariant derivative ∇_μ, one can easily find

$$T_0^0 = mS - F(I,J) + \varepsilon, \quad T_1^1 = T_2^2 = T_3^3 = 2IF_I + 2JF_J - F(I,J) - p. \tag{2.20}$$

Summation of Einstein equations (2.4), (2.5), and (2.6) leads to the equation

$$\frac{\ddot{\tau}}{\tau} = -\kappa\left(T_1^1 + T_2^2 + T_3^3 - \frac{3}{2}\,T\right) = \frac{3\kappa}{2}\,(mS + 2IF_I + 2JF_J - 2F(I,J) + \varepsilon - p). \tag{2.21}$$

In the case of the right-hand side of (2.21) being the function of $\tau(t) = a(t)b(t)c(t)$, this equation takes the form

$$\ddot{\tau} + \Phi(\tau) = 0. \tag{2.22}$$

As is known, this equation possesses exact solutions for arbitrary function $\Phi(\tau)$. Giving the explicit form of $L_N = F(I,J)$, from (2.21) one can find concrete function $\tau(t) = abc$. Once the value of τ is obtained, one can get expressions for components $V_\alpha(t)$, $\alpha = 1,2,3,4$. Let us express a, b, c through τ. For this we notice that subtraction of Einstein equations (2.4) and (2.5) leads to the equation

$$\frac{\ddot{a}}{a} - \frac{\ddot{b}}{b} + \frac{\dot{a}\dot{c}}{ac} - \frac{\dot{b}\dot{c}}{bc} = \frac{d}{dt}\left(\frac{\dot{a}}{a} - \frac{\dot{b}}{b}\right) + \left(\frac{\dot{a}}{a} - \frac{\dot{b}}{b}\right)\left(\frac{\dot{a}}{a} + \frac{\dot{b}}{b} + \frac{\dot{c}}{c}\right) = 0. \tag{2.23}$$

Equation (2.23) possesses the solution

$$\frac{a}{b} = D_1 \exp\left(X_1 \int \frac{dt}{\tau}\right), \quad D_1 = \text{const}, \quad X_1 = \text{const}. \tag{2.24}$$

Subtracting Eqs. (2.4)−(2.6) and (2.5)−(2.6), one finds the equations similar to (2.23), having solutions

$$\frac{a}{c} = D_2 \exp\left(X_2 \int \frac{dt}{\tau}\right), \quad \frac{b}{c} = D_3 \exp\left(X_3 \int \frac{dt}{\tau}\right), \tag{2.25}$$

where D_2, D_3, X_2, X_3 are integration constants. There is a functional dependence between the constants D_1, D_2, D_3, X_1, X_2, X_3:

$$D_2 = D_1 D_3, \quad X_2 = X_1 + X_3.$$

Using the equations (2.24) and (2.25), we rewrite $a(t)$, $b(t)$, and $c(t)$ in the explicit form

$$\begin{aligned} a(t) &= (D_1^2 D_3)^{1/3}\tau^{1/3} \exp\left[\frac{2X_1 + X_3}{3}\int \frac{dt}{\tau(t)}\right], \\ b(t) &= (D_1^{-1} D_3)^{1/3}\tau^{1/3} \exp\left[-\frac{X_1 - X_3}{3}\int \frac{dt}{\tau(t)}\right], \\ c(t) &= (D_1 D_3^2)^{-1/3}\tau^{1/3} \exp\left[-\frac{X_1 + 2X_3}{3}\int \frac{dt}{\tau(t)}\right]. \end{aligned} \tag{2.26}$$

Thus the previous system of Einstein equations is completely integrated. In this process of integration only the first three of the complete system of Einstein equations have been used. General solutions to these three second-order equations have been obtained. The solutions contain six arbitrary constants, D_1, D_3, X_1, X_3, and two others, that were obtained while solving Eq. (2.22). Equation (2.7) is the consequence of the first three Einstein equations. To verify the correctness of obtained solutions, it is necessary to put a, b, c into (2.7). It should lead either to the identity or to some additional constraint between the constants. Putting a, b, c from (2.26) into (2.7) one can get the following equality:

$$\frac{1}{3\tau}\left[3\ddot{\tau}-2\frac{\dot{\tau}^2}{\tau}+\frac{2}{3\tau}(X_1^2+X_1X_3+X_3^2)\right]=-\kappa\left(T_0^0-\frac{1}{2}T\right), \tag{2.27}$$

which guarantees the correctness of the solutions obtained. In fact we can rewrite (2.21) and (2.27) as

$$\frac{\ddot{\tau}}{\tau}=\frac{3\kappa}{2}(T_0^0+T_1^1) \tag{2.28}$$

and

$$\frac{\ddot{\tau}}{\tau}-\frac{2}{3}\frac{\dot{\tau}^2}{\tau^2}+\frac{2}{9\tau^2}\mathscr{X}=-\frac{\kappa}{2}(T_0^0-3T_1^1), \tag{2.29}$$

where $\mathscr{X}:=X_1^2+X_1X_3+X_3^2$. Combining (2.28) and (2.29) together one gets the solution for τ in quadrature:

$$\int\frac{d\tau}{\sqrt{3\kappa\tau^2T_0^0+\mathscr{X}/3}}=t. \tag{2.30}$$

Let us note that in our further study we exploit the equations (2.21) to obtain τ and (2.27) to estimate integration constants.

It should be emphasized that we are dealing with a cosmological problem and our main goal is to investigate the initial and the asymptotic behavior of the field functions and the metric ones. As one sees, all of these functions are in some functional dependence with τ: $\psi\sim 1/\sqrt{\tau}$ and $a_i\sim\tau^{1/3}e^{\pm\int dt/\tau}$. Therefore in our further investigation we mainly look for τ, though in some particular cases we write down field and metric functions explicitly.

III. ANALYSIS OF THE SOLUTIONS OBTAINED FOR SOME SPECIAL CHOICE OF NONLINEARITY

Let us now study the system for some special choice of L_N. First we analyze the system only for the NLSF which will be followed by the study when the Universe is filled with perfect fluid. However, first of all we study the linear case. The reason for getting the solution to the self-consistent system of equations for the linear spinor and gravitational fields is the necessity of comparing this solution with that for the system of equations for the nonlinear spinor and gravitational fields that permits clarification of the role of nonlinear spinor terms in the evolution of the cosmological model in question. Using the equation (2.21) one gets

$$\tau(t)=(1/2)Mt^2+y_1t+y_0, \tag{3.1}$$

where $M=\frac{3}{2}\kappa mC_0$, $C_0=C_1^2+C_2^2-C_3^2-C_4^2$, and y_1, y_0 are the constants. In this case we get explicit expressions for the components of spinor field functions and metric functions:

$$V_r(t)=(C_r/\sqrt{\tau})e^{-imt}, \quad r=1,2; \quad V_l(t)=(C_l/\sqrt{\tau})e^{imt}, \quad l=3,4. \tag{3.2}$$

$$a(t)=(D_1^2D_3)^{1/3}(\tfrac{1}{2}Mt^2+y_1t+y_0)^{1/3}Z^{2(2X_1+X_3)/3B},$$

$$b(t)=(D_1^{-1}D_3)^{1/3}(\tfrac{1}{2}Mt^2+y_1t+y_0)^{1/3}Z^{-2(X_1-X_3)/3B}, \tag{3.3}$$

$$c(t)=(D_1D_3^2)^{-1/3}(\tfrac{1}{2}Mt^2+y_1t+y_0)^{1/3}Z^{-2(X_1+2X_3)/3B},$$

where $Z=(t-t_1)/(t-t_2)$, $B=M(t_1-t_2)$, and $t_{1,2}=-y_1/M\pm\sqrt{(y_1/M)^2-2y_0/M}$ are the roots of the quadratic equation $Mt^2+2y_1t+2y_0=0$. Substituting $\tau(t)$ into (2.27), one gets

$$y_1^2-2My_0=(X_1^2+X_1X_3+X_3^2)/3=\mathscr{X}/3>0. \tag{3.4}$$

This means that the quadratic polynomial in (3.1) possesses real roots, i.e., $\tau(t)$ in (3.1) turns into zero at $t=t_{1,2}$ and the solution obtained is the singular one. Let us now study the solutions (3.1)–(3.3) at $t\to\infty$. In this case we have

$$\tau(t)\approx\frac{3}{4}\,\kappa mC_0t^2, \quad a(t)\approx b(t)\approx c(t)\approx t^{2/3},$$

which leads to the conclusion about the asymptotical isotropization of the expansion process for the initially anisotropic B-I space. Thus the solution to the self-consistent system of equations for the linear spinor and gravitational fields is the singular one at the initial time. In the initial state of evolution of the field system the expansion process of space is anisotropic, but at $t\to\infty$ there happens isotropization of the expansion process.

Once the solutions to the linear spinor field equations and those corresponding to the metric functions are obtained, let us study the nonlinear case.

I. Let us consider the case when $L_N=F(I)$. It is clear that in this case $\mathscr{G}=0$. From (2.16) we find

$$S=C_0/\tau, \quad C_0=\text{const}. \tag{3.5}$$

As in the considered case where $L_N=F$ depends only on S, from (3.5) it follows that $F(I)$ and $F_I(I)$ are functions of $\tau=abc$. Taking this fact into account, integration of the system of equations (2.14) leads to the expressions

$$V_r(t)=(C_r/\sqrt{\tau})e^{-i\Omega}, \quad r=1,2, \quad V_l(t)=(C_l/\sqrt{\tau})e^{i\Omega}, \quad l=3,4, \tag{3.6}$$

where C_r and C_l are integration constants and $\Omega=\int\Phi dt$. Putting (3.6) into (2.15) one gets

$$S=(C_1^2+C_2^2-C_3^2-C_4^2)/\tau. \tag{3.7}$$

Comparison of (3.5) with (3.7) gives $C_0=C_1^2+C_2^2-C_3^2-C_4^2$.

Let us consider the concrete type of NLSF equation with $F(I)=\lambda I^{(n/2)}=\lambda S^n$ where λ is the coupling constant, $n>1$. In this case for τ one gets

$$\ddot{\tau}=(3/2)\kappa C_0[m+\lambda(n-2)C_0^{n-1}/\tau^{n-1}]. \tag{3.8}$$

The first integral of the foregoing equation takes form

$$\dot{\tau}^2=3\kappa C_0[m\tau-\lambda C_0^{n-1}/\tau^{n-2}+g^2], \tag{3.9}$$

where from (2.27) one determines $g^2 = \mathscr{X}/9\kappa C_0$. The sign C_0 is determined by the positivity of the energy-density T_0^0 of linear spinor field:

$$T_0^0 = mC_0/\tau > 0. \tag{3.10}$$

It is obvious from (3.10) that $C_0 > 0$. Now one can write the solution to the equation (3.9) in quadratures:

$$\int \frac{\tau^{(n-2)/2} d\tau}{\sqrt{m\tau^{n-1} + g^2\tau^{n-2} - \lambda C_0^{n-1}}} = \sqrt{3\kappa C_0} t. \tag{3.11}$$

The constant of integration in (3.11) has been taken to be zero, as it only gives the shift of the initial time. Let us study the properties of solution to Eq. (3.8) for $n > 2$. From (3.11) one gets

$$\tau(t)|_{t\to\infty} \approx (3/4)\kappa m C_0 t^2, \tag{3.12}$$

which coincides with the asymptotic solution to the equation (3.3). It leads to the conclusion about isotropization of the expansion process of the B-I space. It should be remarked that the isotropization takes place if and only if the spinor field equation contains the massive term [cf. the parameter m in (3.12)]. If $m = 0$, the isotropization does not take place. In this case from (3.11) we get

$$\tau(t)|_{t\to\infty} \approx \sqrt{3\kappa C_0 g^2}\, t. \tag{3.13}$$

Substituting (3.13) into (2.26) one comes to the conclusion that the functions $a(t)$, $b(t)$, and $c(t)$ are different. Let us consider the properties of solutions to Eq. (3.8) when $t \to 0$. For $\lambda < 0$ from (3.11) we get

$$\tau(t) = [(3/4)n^2\kappa|\lambda|C_0^n]^{1/n} t^{2/n} \to 0, \tag{3.14}$$

i.e., solutions are singular. For $\lambda > 0$, from (3.11) it follows that $\tau = 0$ cannot be reached for any value of t as in this case when the denominator of the integrand in (3.11) becomes imaginary. It means that for $\lambda > 0$ there exist regular solutions to the previous system of equations.[14] The absence of the initial singularity in the considered cosmological solution appears to be consistent with the violation for $\lambda > 0$ of the dominant energy condition in the Hawking–Penrose theorem.[18]

Let us consider the Heisenberg–Ivanenko equation when in (3.8) $n = 2$.[23] In this case the equation for $\tau(t)$ does not contain the nonlinear term and its solution coincides with that of the linear equation (3.3). With such n chosen the metric functions a, b, c are given by the equality (3.2), and the spinor field functions are written as follows:

$$V_\tau = (C_\tau/\sqrt{\tau}) e^{-imt} Z^{4i\lambda C_0/B}, \quad V_l = (C_l/\sqrt{\tau}) e^{imt} Z^{-4i\lambda C_0/B}. \tag{3.15}$$

As in the linear case, the obtained solution is singular at initial time and asymptotically isotropic as $t \to \infty$.

We now study the properties of solutions to Eq. (3.8) for $1 < n < 2$. In this case it is convenient to present the solution (3.11) in the form

$$\int \frac{d\tau}{\sqrt{m\tau - \lambda\tau^{2-n} C_0^{n-1} + g^2}} = \sqrt{3\kappa C_0} t. \tag{3.16}$$

As $t \to \infty$, from (3.16) we get the equality (3.12), leading to the isotropization of the expansion process. If $m = 0$ and $\lambda > 0$, $\tau(t)$ lies on the interval

$$0 \leqslant \tau(t) \leqslant (g^2/\lambda C_0^{n-1})^{1/(2-n)}.$$

If $m=0$ and $\lambda<0$, the relation (3.16) at $t\to\infty$ leads to the equality

$$\tau(t) \approx [(3/4)n^2\kappa|\lambda|C_0^n]^{1/n} t^{2/n}. \tag{3.17}$$

Substituting (3.17) into (2.26) and taking into account that at $t\to\infty$

$$\int \frac{dt}{\tau} \approx \frac{n(3\kappa|\lambda|n^2 C_0^n)^{1/n}}{(n-2)2^{2/n}}\, t^{-2/n+1} \to 0$$

due to $-2/n+1<0$, we obtain

$$a(t) \sim b(t) \sim c(t) \sim [\tau(t)]^{1/3} \sim t^{2/3n} \to \infty. \tag{3.18}$$

This means that the solution obtained tends to the isotropic one. In this case the isotropization is provided not by the massive parameter, but by the degree n in the term $L_N=\lambda S^n$. Equation (3.16) implies

$$\tau(t)|_{t\to 0} \approx \sqrt{3\kappa C_0 g^2}\, t \to 0, \tag{3.19}$$

which means the solution obtained is initially singular. Thus for $1<n<2$ there exist only singular solutions at initial time. At $t\to\infty$ the isotropization of the expansion process of the B-I space takes place both for $m\neq 0$ and for $m=0$.

Finally, let us study the properties of the solution to the equation (3.8) for $0<n<1$. In this case we use the solution in the form (3.16). Since now $2-n>1$, then with the increasing of $\tau(t)$ in the denominator of the integrand in (3.16) the second term $\lambda\tau^{2-n}C_0^{n-1}$ increases faster than the first one. Therefore the solution describing the space expansion can be possible only for $\lambda<0$. In this case at $t\to\infty$, for $m=0$ as well as for $m\neq 0$, one can get the asymptotic representation (3.17) of the solution. This solution, as for the choice $1<n<2$, provides asymptotically isotropic expansion of the B-I space. For $t\to 0$ in this case we shall get only the singular solution of the form (3.19).

II. We study the system when $L_N=F(J)$, which means in the case considered $\mathscr{D}=0$. Let us note that, in the unified nonlinear spinor theory of Heisenberg, the massive term remains absent, and, according to Heisenberg, the particle mass should be obtained as a result of quantization of spinor prematter[24]. In the nonlinear generalization of classical field equations, the massive term does not possess the significance that it possesses in the linear one, as it by no means defines total energy (or mass) of the nonlinear field system. Thus without losing the generality we can consider massless spinor field putting $m=0$ that leads to $\Phi=0$. This assumption metamorphoses (2.16) to get

$$P(t)=D_0/\tau, \quad D_0=\text{const}. \tag{3.20}$$

The system of equations (2.19) in this case reads

$$\begin{aligned} U_1' - U_3 = 0, \quad U_2' - U_4 = 0, \\ U_3' + U_1 = 0, \quad U_4' + U_2 = 0. \end{aligned} \tag{3.21}$$

Differentiating the first equation of system (3.21) and taking into account the third one we get

$$U_1'' + U_1 = 0, \tag{3.22}$$

which leads to the solution

$$U_1 = D_1 e^{i\sigma} + iD_3 e^{-i\sigma}, \quad U_3 = iD_1 e^{i\sigma} + D_3 e^{-i\sigma}. \tag{3.23}$$

Analogically for U_2 and U_4 one gets

$$U_2 = D_2 e^{i\sigma} + iD_4 e^{-i\sigma}, \quad U_4 = iD_2 e^{i\sigma} + D_4 e^{-i\sigma}, \tag{3.24}$$

where D_i are the constants of integration. Finally, we can write

$$\begin{aligned} V_1 &= (1/\sqrt{\tau})(D_1 e^{i\sigma} + iD_3 e^{-i\sigma}), \quad V_2 = (1/\sqrt{\tau})(D_2 e^{i\sigma} + iD_4 e^{-i\sigma}), \\ V_3 &= (1/\sqrt{\tau})(iD_1 e^{i\sigma} + D_3 e^{-i\sigma}), \quad V_4 = (1/\sqrt{\tau})(iD_2 e^{i\sigma} + D_4 e^{-i\sigma}). \end{aligned} \tag{3.25}$$

Putting (3.25) into the expressions (2.15) one finds

$$P = 2(D_1^2 + D_2^2 - D_3^2 - D_4^2)/\tau. \tag{3.26}$$

Comparison of (3.20) with (3.26) gives $D_0 = 2(D_1^2 + D_2^2 - D_3^2 - D_4^2)$.

Let us now estimate τ using the equation

$$\ddot{\tau}/\tau = 3k\lambda(n-1)P^{2n}, \tag{3.27}$$

where we chose $L_N = \lambda P^{2n}$. Putting the value of P into (3.20) and integrating one gets

$$\dot{\tau}^2 = -3k\lambda D_0^{2n}\tau^{2-2n} + y^2, \tag{3.28}$$

where y^2 is the integration constant and can be defined from (2.27): $y^2 = \mathscr{X}/3 > 0$. The solution to the equation (3.28) in quadrature reads

$$\int \frac{d\tau}{\sqrt{-3k\lambda D_0^{2n}\tau^{2-2n} + y^2}} = t. \tag{3.29}$$

Let us now analyze the solution obtained here. As one can see the case $n=1$ is the linear one. In case of $\lambda<0$ for $n>1$, i.e., $2-2n<0$, we get

$$\tau(t)\big|_{t\to 0} \approx [(\sqrt{3k|\lambda|}D_0^n n)t]^{1/n}$$

and

$$\tau\big|_{t\to\infty} \approx \sqrt{3ky^2}t.$$

This means that for the term L_N considered with $\lambda<0$ and $n>1$, the solution is initially singular and the space–time is anisotropic at $t\to\infty$. Let us now study it for $n<1$. In this case we obtain

$$\tau\big|_{t\to 0} \approx \sqrt{3ky^2}t$$

and

$$\tau\big|_{t\to\infty} \approx [(\sqrt{3k|\lambda|}D_0^n n)t]^{1/n}.$$

The solution is initially singular as in the previous case, but as far as $1/n>1$, it provides an asymptotically isotropic expansion of the B-I space–time.

III. In this case we study $L_N = F(I,J)$. Choosing

$$L_N = F(K_\pm), \quad K_+ = I + J = I_v = -I_A, \quad K_- = I - J = I_T, \tag{3.30}$$

in the case of massless NLSF we find

$$\mathscr{D}=2SF_{K_\pm}, \quad \mathscr{G}=\pm 2PF_{K_\pm}, \quad F_{K_\pm}=dF/dK_\pm .$$

Putting them into (2.16) we find

$$S_0^2 \pm P_0^2 = D_\pm . \tag{3.31}$$

Choosing $F=\lambda K_\pm^n$ from (2.21) we get

$$\ddot{\tau}=3k\lambda(n-1)D_\pm^n \tau^{1-2n}, \tag{3.32}$$

with the solution

$$\int \frac{\tau^{n-1}\, d\tau}{\sqrt{g^2\tau^{2n-2}-3k\lambda D_\pm^n}}=t, \tag{3.33}$$

where $g^2=\mathscr{X}/3$. Let us study the case with $\lambda<0$. For $n<1$ from (3.33) one gets

$$\tau(t)|_{t\to 0}\approx gt \to 0, \tag{3.34}$$

i.e., the solutions are initially singular, and

$$\tau(t)|_{t\to\infty}\approx[\sqrt{(3\kappa|\lambda|D_\pm^n)}t]^{1/n}, \tag{3.35}$$

which means that the anisotropy disappears as the Universe expands. In the case of $n>1$ we get

$$\tau(t)|_{t\to 0}\approx t^{1/n}\to 0$$

and

$$\tau(t)|_{t\to\infty}\approx gt,$$

i.e., the solutions are initially singular and the metric functions $a(t)$, $b(t)$, and $c(t)$ are different at $t\to\infty$, i.e., the isotropization process remains absent. For $\lambda>0$ we get that the solutions are initially regular, but it violates the dominant energy condition in the Hawking–Penrose theorem[18]. Note that one comes to the analogical conclusion choosing $L_N=\lambda S^{2n}P^{2n}$.

IV. ANALYSIS OF THE RESULTS OBTAINED WHEN THE B-I UNIVERSE IS FILLED WITH PERFECT FLUID

Let us now analyze the system filled with perfect fluid. Let us recall that the energy-momentum tensor of perfect fluid is

$$T_{\mu(m)}^{\nu}=(p+\varepsilon)u_\mu u^\nu-\delta_\mu^\nu p=(\varepsilon,-p,-p,-p). \tag{4.1}$$

As we saw earlier the introduction of perfect fluid does not change the field equations, thus leaving the solutions to the NLSF equations externally unchanged. Changes in the solutions performed by perfect fluid are carried out through Einstein equations, namely through τ. So, let us first see how the quantities ε and p connected with τ. In doing this we use the well-known equality $T_{\mu;\nu}^{\nu}=0$, which leads to

$$\frac{d}{dt}(\tau\varepsilon)+\dot{\tau}p=0, \tag{4.2}$$

with the solution

$$\ln \tau = -\int \frac{d\varepsilon}{(\varepsilon + p)}. \tag{4.3}$$

Recalling the equation of state $p = \xi\varepsilon$, $0 \leqslant \xi \leqslant 1$, finally we get

$$T^0_{0(m)} = \varepsilon = \frac{\varepsilon_0}{\tau^{1+\xi}}, \quad T^1_{1(m)} = T^2_{2(m)} = T^3_{3(m)} = -p = -\frac{\varepsilon_0 \xi}{\tau^{1+\xi}}, \tag{4.4}$$

where ε_0 is the integration constant. Putting them into (2.21) we get

$$\frac{\ddot{\tau}}{\tau} = \frac{3\kappa}{2}\frac{(\xi - 1)\varepsilon_0}{\tau^{(\xi+1)}}, \tag{4.5}$$

which shows that for stiff matter (ξ=1) the contribution of fluid to the solution is missing. Let us now study the system with nonlinearity type **I**. In this case we get

$$\int \frac{d\tau}{\sqrt{mC_0\tau - \lambda C_0^n/\tau^{(n-2)} + \varepsilon_0 \tau^{(1-\xi)} + g^2}} = \pm\sqrt{3\kappa}t. \tag{4.6}$$

As one can see in the case of dust (ξ=0), the fluid term can be combined with the massive one, whereas in the case of stiff matter (ξ=1) it mixes up with the constant. Analyzing the equation (4.6) one comes to the conclusion that the presence of perfect fluid does not influence the result obtained earlier for the nonlinear term type **I**. One comes to the same conclusion analyzing the system with perfect fluid for the other types of nonlinear terms considered here. At least both at $t \to 0$ and at $t \to \infty$ the key role is played by the other terms rather than the term presenting fluid.

V. CONCLUSIONS

Exact solutions to the NLSF equations have been obtained for the nonlinear terms being arbitrary functions of the invariant $I = S^2$ and $J = P^2$, where $S = \bar{\psi}\psi$ and $P = i\bar{\psi}\gamma^5\psi$ are the real bilinear forms of spinor field, for B-I space–time. Equations with power nonlinearity in spinor field Lagrangian $L_N = \lambda S^n$, where λ is the coupling constant, have been thoroughly studied. In this case it is shown that the equations mentioned possess solutions both regular and singular at the initial moment of time for $n > 2$. Singularity remains absent for the case of a field system with broken dominant energy condition. It is also shown that if in the NLSF equation the massive parameter $m \neq 0$ and $n \geqslant 2$, then at $t \to \infty$ isotropization of the B-I space–time expansion takes place, while for $m = 0$ the expansion is anisotropic. Properties of the solutions to the spinor field equation for $1 < n < 2$ and $0 < n < 1$ were also studied. It was found that in these cases there does not exist a solution that is initially regular. At $t \to \infty$ the isotropization process of the B-I space–time takes place both for $m \neq 0$ and for $m = 0$. In the case of the nonlinear term $L_N = \lambda P^{2n}$, we found the solutions are initially singular and the isotropization process of the B-I space–time depends on the choice of n. For $L_N = \lambda(I \pm J)^n$ we obtained the solutions that may be initially singular or regular, depending on the sign of coupling constant λ, but the isotropization process depends on the value of power n. It is also shown that the results remain unchanged even in the case when the B-I space–time is filled with perfect fluid.

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