

# Interacting spinor and scalar fields in a Bianchi type-I Universe: Oscillatory solutions

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**Abstract.** Self-consistent system of spinor, scalar and BI gravitational fields is considered. Exact solutions to the field equations in terms of volume scale of the BI metric are obtained. Einstein field equations in account of the cosmological constant  $\Lambda$  and perfect fluid are studied. Oscillatory mode of expansion of the universe is obtained. It is shown that for the interaction term being a power function of the invariants of bilinear spinor forms and  $\Lambda > 0$  and given other parameters, *e.g.*, coupling constant, spinor mass, *etc.*, there exists a finite range of integration constant which generates oscillatory mode of evolution.

## 1. Introduction

The discovery of the cosmic microwave radiation has stimulated a growing interest in anisotropic, general-relativistic cosmological models of the universe. The choice of anisotropic cosmological models in the system of Einstein field equation enable us to study the early day universe, which had an anisotropic phase that approaches an isotropic one [1]. BI universe is the simplest model of an anisotropic universe and eventually evolves into a Friedmann-Robertson-Walker (FRW) universe [2], if filled with matter obeying  $p = \zeta \varepsilon$ ,  $\zeta < 1$ , where  $\varepsilon$  and  $p$  are the energy and pressure of the material field, respectively. Since the present-day universe is surprisingly isotropic, this feature of the BI Universe makes it a prime candidate for studying the possible effects of an anisotropy in the early universe on present-day observations.

In this paper we study the self-consistent system of spinor, scalar and BI gravitational fields in presence of perfect fluid. Solutions of Einstein equations coupled to a spinor and a scalar fields in BI spaces have been extensively studied by Saha and Shikin [3]-[6]. In those papers, the field equations were solved qualitatively. In this report, we consider some key equations occurred in those papers. Initial value problem has been posed for those equations and solved numerically. Application of numerical methods enables us to view this problem from totally different angle and gives rise to some interesting results previously unknown.

## 2. Fundamental Equations and General Solutions

The action of the nonlinear spinor, scalar and gravitational fields can be written as

$$\mathcal{S}(g; \psi, \bar{\psi}, \varphi) = \int (R + L)\sqrt{-g}d\Omega, \tag{1}$$

where  $R$  is the Ricci scalar and  $L$  is the spinor and scalar field Lagrangian density chosen in the form [3]

$$L = \frac{i}{2} \left[ \bar{\psi}\gamma^\mu\nabla_\mu\psi - \nabla_\mu\bar{\psi}\gamma^\mu\psi \right] - m\bar{\psi}\psi + \frac{1}{2}\varphi_{,\alpha}\varphi^{,\alpha}(1 + \lambda F). \tag{2}$$

Here  $\lambda$  is the coupling constant and  $F$  is some arbitrary functions of invariants generated from the real bilinear forms of a spinor field. We choose  $F$  to be the function of  $I = S^2 = (\bar{\psi}\psi)^2$  and  $J = P^2 = (i\bar{\psi}\gamma^5\psi)^2$ , i.e.,  $F = F(I, J)$ , that describes the nonlinearity in the most general of its form [5]. As one sees, for  $\lambda = 0$  we have the system with minimal coupling.

The gravitational field in our case is given by a Bianchi type I (BI) metric in the form

$$ds^2 = a_0^2(dx^0)^2 - a_1^2(dx^1)^2 - a_2^2(dx^2)^2 - a_3^2(dx^3)^2, \tag{3}$$

with  $a_0 = 1$ ,  $x^0 = ct$  and  $c = 1$ . The metric functions  $a_i$  ( $i = 1, 2, 3$ ) are the functions of time  $t$  only.

Variation of (1) with respect to spinor field  $\psi$  ( $\bar{\psi}$ ) gives nonlinear spinor field equations

$$i\gamma^\mu\nabla_\mu\psi - m\psi + \mathcal{D}\psi + \mathcal{G}i\gamma^5\psi = 0, \tag{4a}$$

$$i\nabla_\mu\bar{\psi}\gamma^\mu + m\bar{\psi} - \mathcal{D}\bar{\psi} - \mathcal{G}i\bar{\psi}\gamma^5 = 0, \tag{4b}$$

where we denote

$$\mathcal{D} = \lambda S\varphi_{,\alpha}\varphi^{,\alpha}\partial F/\partial I, \quad \mathcal{G} = \lambda P\varphi_{,\alpha}\varphi^{,\alpha}\partial F/\partial J,$$

whereas, variation of (1) with respect to scalar field yields the following scalar field equation

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\nu}\left(\sqrt{-g}g^{\nu\mu}(1 + \lambda F)\varphi_{,\mu}\right) = 0. \tag{5}$$

Varying (1) with respect to metric tensor  $g_{\mu\nu}$  one finds the gravitational field equation which in account of cosmological constant  $\Lambda$  has the form

$$\frac{\ddot{a}_2}{a_2} + \frac{\ddot{a}_3}{a_3} + \frac{\dot{a}_2}{a_2}\frac{\dot{a}_3}{a_3} = \kappa T_1^1 - \Lambda, \tag{6a}$$

$$\frac{\ddot{a}_3}{a_3} + \frac{\ddot{a}_1}{a_1} + \frac{\dot{a}_3}{a_3}\frac{\dot{a}_1}{a_1} = \kappa T_2^2 - \Lambda, \tag{6b}$$

$$\frac{\ddot{a}_1}{a_1} + \frac{\ddot{a}_2}{a_2} + \frac{\dot{a}_1}{a_1}\frac{\dot{a}_2}{a_2} = \kappa T_3^3 - \Lambda, \tag{6c}$$

$$\frac{\dot{a}_1}{a_1}\frac{\dot{a}_2}{a_2} + \frac{\dot{a}_2}{a_2}\frac{\dot{a}_3}{a_3} + \frac{\dot{a}_3}{a_3}\frac{\dot{a}_1}{a_1} = \kappa T_0^0 - \Lambda. \tag{6d}$$

Here  $\kappa$  is the Einstein gravitational constant and over-dot means differentiation with respect to  $t$ . The energy-momentum tensor of the material field is given by

$$T_{\mu}^{\rho} = \frac{i}{4} g^{\rho\nu} \left( \bar{\psi} \gamma_{\mu} \nabla_{\nu} \psi + \bar{\psi} \gamma_{\nu} \nabla_{\mu} \psi - \nabla_{\mu} \bar{\psi} \gamma_{\nu} \psi - \nabla_{\nu} \bar{\psi} \gamma_{\mu} \psi \right) + (1 - \lambda F) \varphi_{,\mu} \varphi^{,\rho} - \delta_{\mu}^{\rho} L + T_{\text{m}\mu}^{\nu}. \quad (7)$$

Here  $T_{\mu}^{\nu(m)} = (\varepsilon, -p, -p, -p)$  is the energy-momentum tensor of a perfect fluid. Energy  $\varepsilon$  is related to the pressure  $p$  by the equation of state  $p = \zeta \varepsilon$ . Here  $\zeta$  varies between the interval  $0 \leq \zeta \leq 1$ , whereas  $\zeta = 0$  describes the dust Universe,  $\zeta = \frac{1}{3}$  presents radiation Universe,  $\frac{1}{3} < \zeta < 1$  ascribes hard Universe and  $\zeta = 1$  corresponds to the stiff matter. The Dirac matrices  $\gamma_{\mu}(x)$  of curve space-time are connected with those of Minkowski space as

$$\gamma^{\mu} = \bar{\gamma}^{\mu} / a_{\mu}, \quad \gamma_{\mu} = \bar{\gamma} a_{\mu}, \quad \mu = 0, 1, 2, 3. \quad (8)$$

The explicit form of the covariant derivative of spinor is [7]

$$\nabla_{\mu} \psi = \partial_{\mu} \psi - \Gamma_{\mu} \psi, \quad \nabla_{\mu} \bar{\psi} = \partial_{\mu} \bar{\psi} + \bar{\psi} \Gamma_{\mu}, \quad \mu = 0, 1, 2, 3, \quad (9)$$

where  $\Gamma_{\mu}(x)$  are spinor affine connection matrices. For the metric (3) one has the following components of the affine spinor connections

$$\Gamma_{\mu} = (1/2) \dot{a}_{\mu} \bar{\gamma}^{\mu} \bar{\gamma}^0. \quad (10)$$

We study the space-independent solutions to the spinor and scalar field Eqs.(4) and (5) so that  $\psi = \psi(t)$  and  $\varphi = \varphi(t)$ . Defining

$$\tau = a_0 a_1 a_2 a_3 = \sqrt{-g} \quad (11)$$

from (5) for the scalar field we have

$$\varphi = C \int [\tau(1 + \lambda F)]^{-1} dt. \quad (12)$$

Setting  $V_j(t) = \sqrt{\tau} \psi_j(t)$ ,  $j = 1, 2, 3, 4$ , in view of (9) and (10) from (4a) one deduces the following system of equations:

$$\dot{V}_1 + i(m - \mathcal{D})V_1 - \mathcal{G}V_3 = 0, \quad (13a)$$

$$\dot{V}_2 + i(m - \mathcal{D})V_2 - \mathcal{G}V_4 = 0, \quad (13b)$$

$$\dot{V}_3 - i(m - \mathcal{D})V_3 + \mathcal{G}V_1 = 0, \quad (13c)$$

$$\dot{V}_4 - i(m - \mathcal{D})V_4 + \mathcal{G}V_2 = 0. \quad (13d)$$

From (4a) we also write the equations for the bilinear spinor forms  $S$ ,  $P$  and  $A^0 = \bar{\psi} \bar{\gamma}^5 \bar{\gamma}^0 \psi$

$$\dot{S}_0 - 2\mathcal{G} A_0^0 = 0, \quad (14a)$$

$$\dot{P}_0 - 2(m - \mathcal{D}) A_0^0 = 0, \quad (14b)$$

$$\dot{A}_0^0 + 2(m - \mathcal{D}) P_0 + 2\mathcal{G} S_0 = 0, \quad (14c)$$

where  $Q_0 = \tau Q$ , leading to the relation  $S^2 + P^2 + (A^0)^2 = C^2/\tau^2$ ,  $C^2 = \text{const}$ . As one sees, for  $F = F(I)$  (14a) gives  $S = C_0/\tau$ , whereas for the massless spinor field with  $F = F(J)$  (14b) yields  $P = D_0/\tau$ . In view of it for  $F = F(I)$  we obtain the following expression for the components of spinor field

$$\begin{aligned}\psi_1(t) &= C_1\tau^{-1/2}e^{-i\beta}, & \psi_2(t) &= C_2\tau^{-1/2}e^{-i\beta}, \\ \psi_3(t) &= C_3\tau^{-1/2}e^{i\beta}, & \psi_4(t) &= C_4\tau^{-1/2}e^{i\beta},\end{aligned}\quad (15)$$

with  $C_i$  being the integration constants and are related to  $C_0$  as  $C_0 = C_1^2 + C_2^2 - C_3^2 - C_4^2$ . Here  $\beta = \int(m - \mathcal{D})dt$ . In case of  $F = F(J)$  for the massless spinor field we get

$$\begin{aligned}\psi_1 &= \tau^{-1/2}(D_1e^{i\sigma} + iD_3e^{-i\sigma}), & \psi_2 &= \tau^{-1/2}(D_2e^{i\sigma} + iD_4e^{-i\sigma}), \\ \psi_3 &= \tau^{-1/2}(iD_1e^{i\sigma} + D_3e^{-i\sigma}), & \psi_4 &= \tau^{-1/2}(iD_2e^{i\sigma} + D_4e^{-i\sigma}).\end{aligned}\quad (16)$$

The integration constants  $D_i$  are connected to  $D_0$  by  $D_0 = 2(D_1^2 + D_2^2 - D_3^2 - D_4^2)$ . Here we set  $\sigma = \int \mathcal{G}dt$ .

Once the spinor functions are known explicitly, one can write the components of spinor current  $j^\mu = \bar{\psi}\gamma^\mu\psi$ , the charge density of spinor field  $\varrho = (j_0 \cdot j^0)^{1/2}$ , the total charge of spinor field  $Q = \int_{-\infty}^{\infty} \varrho\sqrt{-^3g}dx dy dz$ , the components of spin tensor  $S^{\mu\nu,\epsilon} = \frac{1}{4}\bar{\psi}\{\gamma^\epsilon\sigma^{\mu\nu} + \sigma^{\mu\nu}\gamma^\epsilon\}\psi$  and other physical quantities.

Let us now solve the Einstein equations. In doing so we first write the expressions for the components of the energy-momentum tensor explicitly:

$$\begin{aligned}T_0^0 &= mS + C^2/2\tau^2(1 + \lambda F) + \varepsilon, \\ T_1^1 &= T_2^2 = T_3^3 = \mathcal{D}S + \mathcal{G}P - C^2/2\tau^2(1 + \lambda F) - p.\end{aligned}\quad (17)$$

In account of (17) from (6a), (6b), (6c) we find the metric functions [5]

$$a_i(t) = D_i\tau^{1/3}\exp\left[X_i\int[\tau(t)]^{-1}dt\right], \quad i = 1, 2, 3, \quad (18)$$

with the constants of integration  $D_i$  and  $X_i$  obeying

$$\prod_{i=1}^3 D_i = 1, \quad \sum_{i=1}^3 X_i = 0.$$

As one sees from (18) for  $\tau \sim t^n$  with  $n > 1$  the exponent tends to unity at large  $t$ , and the anisotropic model becomes isotropic one. Let us also write the invariants of gravitational field. They are the Ricci scalar  $I_1 = R \approx 1/\tau$ ,  $I_2 = R_{\mu\nu}R^{\mu\nu} \equiv R_\mu^\nu R_\nu^\mu \approx 1/\tau^3$  and the Kretschmann scalar  $I_3 = R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} \approx 1/\tau^6$ . As we see, the space-time becomes singular at a point where  $\tau = 0$ , as well as the scalar and spinor fields. Thus we see, all the functions in question are expressed via  $\tau$ . In what follows, we write the equation for  $\tau$  and study it in details.

Summation of Einstein Eqs. (6a), (6b), (6c) and (6d) multiplied by 3 gives

$$\frac{\ddot{\tau}}{\tau} = \frac{3}{2}\kappa(mS + \mathcal{D}S + \mathcal{G}P + \varepsilon - p) - 3\Lambda. \quad (19)$$

From energy-momentum conservation law  $T_{\mu;\nu}^\nu = 0$ , in account of the equation of state  $p = \zeta\varepsilon$ , we obtain

$$\varepsilon = \varepsilon_0/\tau^{1+\zeta}, \quad p = \zeta\varepsilon_0/\tau^{1+\zeta}. \quad (20)$$

In our consideration of  $F$  as a function of  $I$ ,  $J$  or  $I \pm J$  we get these arguments, as well as  $\mathcal{D}$  and  $\mathcal{G}$  as functions of  $\tau$ . Hence the right-hand-side of (19) is a function of  $\tau$  only. In what follows we consider the case with  $F = F(I)$ . Recalling the definition of  $\mathcal{D}$ , in view of (12) and (20) Eq. (19) can be written as

$$\ddot{\tau} = \mathcal{F}(\tau, p), \quad (21)$$

where we denote

$$\mathcal{F} \equiv \frac{3}{2}\kappa \left( mC_0 + \lambda C_0^2 C^2 F_I(\tau)/\tau^3 (1 + \lambda F(\tau))^2 + \varepsilon_0(1 - \zeta)/\tau^\zeta \right) - 3\Lambda\tau, \quad (22)$$

and  $p \equiv \{\kappa, \lambda, m, C_0, C, \varepsilon_0, \zeta, \Lambda\}$  is the set of the parameters. Here we take into account that  $S = C_0/\tau$ . From mechanical point of view Eq.(21) can be interpreted as an equation of motion of a single particle with unit mass under the force  $\mathcal{F}(\tau, p)$ . Then the following first integral exists [8]

$$\dot{\tau} = \sqrt{2[E - \mathcal{U}(\tau, p)]}. \quad (23)$$

Here  $E$  is the integration constant and

$$\mathcal{U} \equiv -\frac{3}{2} \left[ \kappa \left( mC_0\tau + C^2/2(1 + \lambda F) + \varepsilon_0\tau^{-\zeta} \right) - \Lambda\tau^2 \right],$$

is the potential of the force  $\mathcal{F}$ . We note that the radical expression must be non-negative. The zeroes of this expression, which depend on all the problem parameters  $p$  define the boundaries of the possible rates of changes of  $\tau(t)$ . Note that setting  $m = 0$  in (22) we come to the case of massless spinor field with  $F = F(J)$  or  $F = F(I \pm J)$ .

We formulate the initial value problem for the Eq. (21) with initial condition

$$\tau(0) = \tau_0 > 0,$$

which we solve numerically.

Since  $\tau$  is the volume-scale, it cannot be negative for every  $t \geq 0$ . On the other-hand, BI space-time models a non-static universe, *i.e.*, the derivative  $\dot{\tau}(t)$  should be nontrivial at the initial moment  $t = 0$ . This leads to fact that for fixed  $p$ , the constant  $E$  and the initial value of  $\tau$  are inter-related in the sense that for a given  $E$  the value of  $\tau_0$  should belong to some interval.

In what follows, we numerically solve (21) for some concrete form of  $F$ .

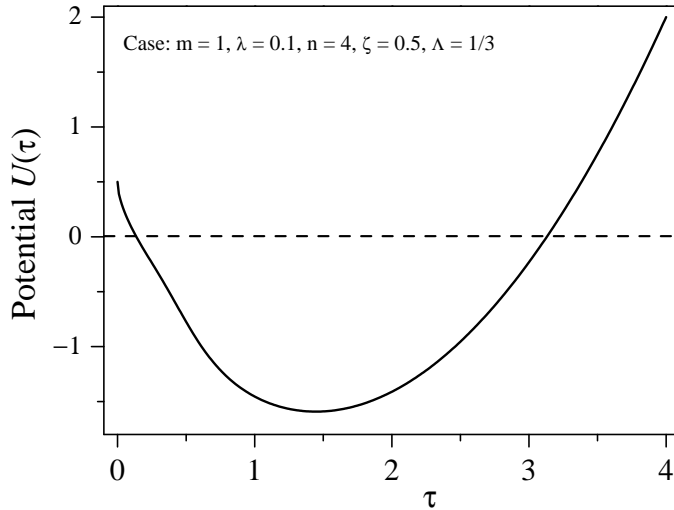
Let us choose  $F$  as a power function of  $S$ , namely,  $F = S^n$ . In this case setting  $C_0 = 1$  and  $C = 1$  we obtain

$$\mathcal{F} = \frac{3\kappa}{2} \left( m + \frac{\lambda n \tau^{n-1}}{2(\lambda + \tau^n)} + \varepsilon_0 \frac{(1 - \zeta)}{\tau^\zeta} \right) - 3\Lambda\tau, \quad (24)$$

with the potential

$$\mathcal{U} = -\frac{3}{2} \left\{ \kappa \left[ m\tau - \frac{\lambda}{2(\lambda + \tau^n)} \right] - \Lambda\tau^2 + \varepsilon_0\tau^{1-\zeta} \right\}. \quad (25)$$

Note that the nonnegativity of the radical in (23) in view of (25) imposes restriction on  $\tau$  from above in case of  $\Lambda > 0$ . It means that in case of  $\Lambda > 0$  the value of  $\tau$  runs between 0 and some  $\tau_{\max}$ , where  $\tau_{\max}$  is the maximum value of  $\tau$  for the given value of  $p$ . This equation has been studied for different values of parameters  $p$ . Here we demonstrate the evolution of  $\tau$  for different choice of  $\tau_0$  for fixed “energy”  $E$  and *vice versa*.



**Figure 1:** Perspective view of the potential  $U(\tau)$

As the first example we consider massive spinor field with  $m = 1$ . Other parameters are chosen in the following way: coupling constant  $\lambda = 0.1$ , power of nonlinearity  $n = 4$ , and cosmological constant  $\Lambda = 1/3$ . We also choose  $\zeta = 0.5$  describing a hard universe.

In Figure 1 we plot corresponding potential  $U(\tau)$  multiplied by the factor  $2/3$ . As is seen from Figure 1 and Figure 2, choosing the integration constant  $E$  we may obtain two different types of solutions. For  $E > 0.5$  solutions are non-periodic, whereas for  $E_{\min} < E \leq 0.5$  the evolution of the universe is oscillatory.

As a second example we consider the massless spinor field. Other parameters of the problem are left unaltered with the exception of  $\zeta$ . Here we choose  $\zeta = 1$  describing stiff matter. It should be noted that this particular choice of  $\zeta$  gives rise to a local maximum. This results in two types of solutions for a single choice of  $E$ .

As one sees from Figure 3, for  $E > M$  there exists only non-periodic solutions, whereas, for  $E_{\min} < E < -0.5$  the solutions are always oscillatory. For  $E \in (-0.5, M)$  there exists two types of solutions depending on the choice of  $\tau_0$ . In Figure 4 we plot the evolution of  $\tau$  for  $E \in (-0.5, M)$ . As is seen, for  $\tau_0 \in (0, A)$  we have periodical solution, but due to the fact that  $\tau$  is non-negative, the physical solutions happen to be semi-periodic. For  $\tau_0 \in (B, C)$  we again have oscillatory mode of the evolution of  $\tau$ . This two region is separated by a no-solution zone  $(A, B)$ .

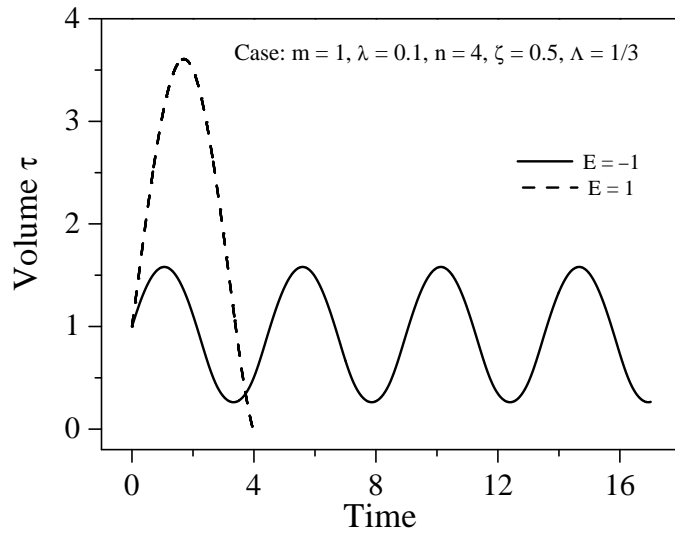


Figure 2: Perspective view of  $\tau$  for different choice of  $E$

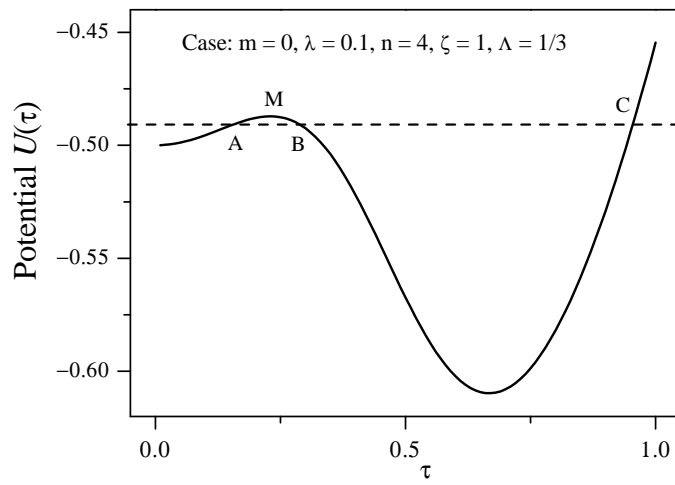
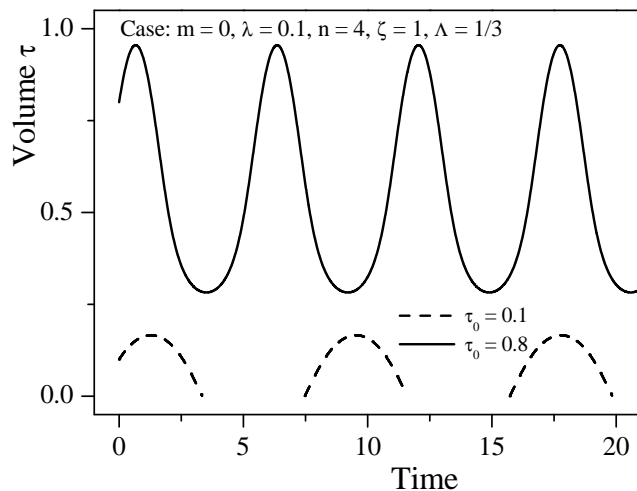


Figure 3: Perspective view of the potential  $\mathcal{U}(\tau)$  with BI universe being filled with stiff matter



**Figure 4:** Perspective view of  $\tau$  for different choice of  $\tau_0$  with  $E \in (-0.5, M)$

### 3. Conclusions

A self-consistent system of spinor, scalar and gravitation fields has been studied in presence of perfect fluid and cosmological term  $\Lambda$ . Oscillatory mode of evolution of the universe is obtained. It is shown that for the interaction term being a power function of the invariants of bilinear spinor forms, the oscillatory solution is possible if  $\Lambda$ -term is positive. It is also shown that only for a finite range the integration constant  $E$  there exists oscillatory mode of evolution. It should be emphasized that a third type of solution is possible, if the BI universe is filled with stiff matter.

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