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BEL–ROBINSON TENSOR AND DOMINANT ENERGY PROPERTY IN THE BIANCHI TYPE I UNIVERSE

BIJAN SAHA* and VICTOR RIKHVITSKY
Laboratory of Information Technologies, Joint Institute for Nuclear Research, 141980 Dubna, Moscow region, Russia
**bijan@jinr.ru*

MIHAI VISINESCU
Department of Theoretical Physics, National Institute for Physics and Nuclear Engineering, Magurele, P. O. Box MG-6, RO-077125, Bucharest, Romania
mvisin@theory.nipne.ro

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Within the framework of Bianchi type-I spacetime we study the Bel–Robinson tensor and its impact on the evolution of the Universe. We use different definitions of the Bel–Robinson tensor existing in the literature and compare the results. Finally we investigate the so called "dominant super-energy property" for the Bel–Robinson tensor as a generalization of the usual dominant energy condition for energy momentum tensors.

Keywords: Bianchi type-I model; super-energy tensors.

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1. Introduction

The lack of a well-posed definition of *local energy–momentum tensor* is the consequence of the Principle of Equivalence,[1] which lies at the heart of Einstein's theory of general relativity. Nevertheless, quest for the local tensors describing the strength of gravitational field has long been going on. One of the first successful attempt to address this problem was taken by Bel[2–4] and independently Robinson.[5,6] In their works, in analogy with the electromagnetic energy–momentum tensor, they constructed a four-index tensor for the gravitational field in vacuum. The properties of the now famous Bel–Robinson (BR) tensor are similar to the traditional energy–momentum tensor and following Senovilla[7,8] can be formulated as follows: (i) it possesses a positive-definite time-like component and a "causal" momentum vector; (ii) its divergence vanishes (in vacuum); (iii) the tensor is zero if and only if the curvature of the spacetime vanishes; (iv) it has positivity property similar to the electromagnetic one; and some others. Construction of BR and the study of its properties were widely considered by a number of authors, e.g., Desher *et al.*,[9,10] Teyssandier,[11,12] Senovilla,[7] Bergqvist,[13] Andersson,[14] Wingbrant,[15] Choquet-Bruhat *et al.*[16] etc.

It should be noted that the authors of the papers mentioned above considered the BR and established its properties in general. On the other hand, in general relativity there exists a number of interesting and widely studied models of spacetime. Therefore, in our view it is interesting to consider the BR within the scope of some concrete metric. In a recent paper[17] we studied the BR within the framework of Bianchi type I (BI) universe using two different definitions. The purpose of this paper is to extend that study for some other definitions and analyze the dominant energy property (DEP) and dominant super-energy property (DSEP) within this model.

2. Bianchi I Universe: A Brief Description

A Bianchi type-I (BI) universe is the straightforward generalization of the flat Robertson–Walker (RW) universe and is one of the simplest models of an anisotropic universe that describes a homogeneous and spatially flat universe. It has the agreeable property that near the singularity it behaves like a Kasner universe, even in the presence of matter, and consequently falls within the general analysis of the singularity given by Belinskii *et al.*[18] Also in a universe filled with matter for $p = \zeta\varepsilon$, $\zeta < 1$, it has been shown that any initial anisotropy in a BI universe quickly dies away and a BI universe eventually evolves into a Friedmann–RW (FRW) universe.[19] Since the present-day universe is surprisingly isotropic, this feature of the BI universe makes it a prime candidate for studying the possible effects of an anisotropy in the early universe on present-day observations. In light of the importance mentioned above, several authors have studied BI universe from different aspects.

A diagonlal BI spacetime is a spatially homogeneous spacetime, which admits an Abelian group G_3, acting on spacelike hypersurfaces, generated by the spacelike Killing vectors $\mathbf{x}_1 = \partial_1$, $\mathbf{x}_2 = \partial_2$ and $\mathbf{x}_3 = \partial_3$. In synchronous coordinates, the metric is[20,21]:

$$ds^2 = dt^2 - \sum_{i=1}^{3} a_i^2(t) dx_i^2 \,. \tag{2.1}$$

If the three scale factors are equal (i.e. $a_1 = a_2 = a_3$), Eq. (2.1) describes an isotropic and spatially flat FRW universe. The BI universe has a different scale factor in each direction, thereby introducing an anisotropy to the system. Thus, a BI universe, being the straightforward generalization of the flat FRW universe, is one of the simplest models of an anisotropic universe that describes a homogeneous and spatially flat universe. When two of the metric functions are equal (e.g., $a_2 = a_3$) the BI spacetime is reduced to the important class of plane symmetric spacetime (a special class of the locally rotational symmetric spacetimes[22,23]), which admits a G_4 group of isometries acting multiply transitively on the spacelike hypersurfaces of homogeneity generated by the Killing vectors $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ and $\mathbf{x}_4 = x_2\partial_3 - x_3\partial_2$. The BI has the agreeable property that near the singularity it behaves like a Kasner universe, given by

$$a_1(t) = a_1^0 t^{p_1}\,, \qquad a_2(t) = a_2^0 t^{p_2}\,, \qquad a_3(t) = a_3^0 t^{p_3}\,, \tag{2.2}$$

with p_j being the parameters of the BI spacetime which measure the relative anisotropy between any two asymmetry axes and satisfy the constraints

$$p_1 + p_2 + p_3 = 1\,, \qquad p_1^2 + p_2^2 + p_3^2 = 1\,. \tag{2.3}$$

As one sees, p_1, p_2 and p_3 cannot be equal. Only two of them can be equal, and only in two special cases, namely, $(0,0,1)$ and $(-1/3, 2/3, 2/3)$. In all other cases p_1, p_2 and p_3 are different, moreover, one of them is negative, while the other two are positive. If it is supposed that $p_1 < p_2 < p_3$, then their values are confined to the following intervals:

$$-1/3 \le p_1 \le 0\,, \qquad 0 \le p_2 \le 2/3\,, \qquad 2/3 \le p_3 \le 1\,.$$

The solutions of the algebraic equations (2.3) can be presented as

$$p_1 = \frac{-p}{p^2+p+1}\,, \qquad p_2 = \frac{p(p+1)}{p^2+p+1}\,, \qquad p_3 = \frac{p+1}{p^2+p+1}\,. \tag{2.4}$$

Thus instead of three, we have now one parameter p, which lies in the interval $0 \le p \le 1$.

Another particular parametrization can be given using an angle on the unit circle, since Eq. (2.3) describes the intersection of a sphere with a plane in the parameter space (p_1, p_2, p_3):

$$p_1 = \frac{1+\cos\vartheta + \sqrt{3}\sin\vartheta}{3}\,, \qquad p_2 = \frac{1+\cos\vartheta - \sqrt{3}\sin\vartheta}{3}\,, \qquad p_3 = \frac{1-2\cos\vartheta}{3}\,.$$

Although ϑ ranges over the unit circle, the labeling of each p_j is quite arbitrary. Thus the unit circle can be divided into six equal parts, each of which span 60°, and the choice of p_j is unique within each section separately. For $\vartheta = 0$, $p_1 = p_2 = \frac{2}{3}$ and $p_3 = -\frac{1}{3}$ while for $\vartheta = \pi/3$, $p_1 = 1$ and $p_2 = p_3 = 0$.

For later convenience we list the Christoffel symbol, scalar curvature, Ricci, Riemann and Weyl tensors for the BI spacetime. The nontrivial Christoffel symbols are the following:

$$\Gamma^i_{i0} = \frac{\dot a_i}{a_i}\,, \qquad \Gamma^0_{ii} = a_i \dot a_i\,, \qquad i = 1,2,3\,,$$

and the nontrivial components of Riemann tensors are

$$R^{0i}{}_{0i} = -\frac{\ddot a_i}{a_i}\,, \qquad R^{ij}{}_{ij} = -\frac{\dot a_i}{a_i}\frac{\dot a_j}{a_j}\,, \qquad i,j = 1,2,3, \quad i \ne j\,. \tag{2.5}$$

Finally the nontrivial components of the Ricci tensors for the BI metric are

$$R_0^0 = -\sum_{i=1}^3 \frac{\ddot a_i}{a_i}\,, \qquad R_i^i = -\left[\frac{\ddot a_i}{a_i} + \frac{\dot a_i}{a_i}\left(\frac{\dot a_j}{a_j} + \frac{\ddot a_k}{a_k}\right)\right]\,,$$

$$i,j,k = 1,2,3, \quad i \ne j \ne k\,,$$

and the scalar curvature is

$$R = -2\left(\sum_{i=1}^{3}\frac{\ddot{a}_i}{a_i} + \frac{\dot{a}_1}{a_1}\frac{\dot{a}_2}{a_2} + \frac{\dot{a}_2}{a_2}\frac{\dot{a}_3}{a_3} + \frac{\dot{a}_3}{a_3}\frac{\dot{a}_1}{a_1}\right). \tag{2.6}$$

It is convenient to separate the Riemann tensor into a trace-free part and a "Ricci" part. This gives the Weyl tensor

$$C_{ijkl} = R_{ijkl} - \frac{1}{(n-2)}(g_{ik}R_{jl} + g_{jl}R_{ik} - g_{jk}R_{il} - g_{il}R_{jk}) + \frac{1}{(n-1)(n-2)}(g_{ik}g_{jl} - g_{il}g_{jk})R\,.$$

This tensor has manifestly all the symmetries of the Riemann tensor; but contrary to the Riemann tensor while it gives rise to Ricci tensor, the Weyl tensor gives

$$g^{ik}C_{ijkl} \equiv 0\,. \tag{2.7}$$

A further distinction is that while the Riemann tensor can be defined in a manifold endowed only with a connection, the Weyl tensor can be defined only when a metric is also defined. In four dimensions the Riemann tensor has 20 distinct components, while the Weyl and the Ricci have ten components each. The nontrivial components of the Weyl tensor for the BI spacetime are

$$\begin{aligned} C_{0i0i} &= \frac{a_i}{6a_ja_k}\{2\ddot{a}_ia_ja_k - \ddot{a}_ja_ka_i - \ddot{a}_ka_ia_j - \dot{a}_i\dot{a}_ja_k - \dot{a}_k\dot{a}_ia_j + 2\dot{a}_j\dot{a}_ka_i\}\,,\\ C_{jkjk} &= -\frac{a_ja_k}{6a_i}\{2\ddot{a}_ia_ja_k - \ddot{a}_ja_ka_i - \ddot{a}_ka_ia_j - \dot{a}_i\dot{a}_ja_k - \dot{a}_k\dot{a}_ia_j + 2\dot{a}_j\dot{a}_ka_i\}\,,\\ &\qquad i,j,k = 1,2,3, \quad i \neq j \neq k\,. \end{aligned} \tag{2.8}$$

From (2.8) one easily finds the following relation:

$$C_{0i0i} = -\frac{a_i^2}{a_j^2a_k^2}C_{jkjk}\,, \qquad i,j,k = 1,2,3, \quad i \neq j \neq k\,. \tag{2.9}$$

Now having all the nontrivial components of Ricci and Riemann tensors, one can easily write the invariants of gravitational field which we need to study the spacetime singularity. Moreover, we can now construct the BR tensor that is defined differently by different authors.

3. Einstein Equations and Their Solutions

In this section we study the Einstein equation. In doing so let us first write the Einstein equation for the BI metric governing the evolution of the Universe. In the presence of a cosmological constant Λ, the Einstein equation has the form

$$\frac{\ddot{a}_2}{a_2} + \frac{\ddot{a}_3}{a_3} + \frac{\dot{a}_2}{a_2}\frac{\dot{a}_3}{a_3} = \kappa T_1^1 + \Lambda\,, \tag{3.1a}$$

$$\frac{\ddot{a}_3}{a_3} + \frac{\ddot{a}_1}{a_1} + \frac{\dot{a}_3}{a_3}\frac{\dot{a}_1}{a_1} = \kappa T_2^2 + \Lambda\,, \tag{3.1b}$$

$$\frac{\ddot{a}_1}{a_1} + \frac{\ddot{a}_2}{a_2} + \frac{\dot{a}_1}{a_1}\frac{\dot{a}_2}{a_2} = \kappa T_3^3 + \Lambda\,, \tag{3.1c}$$

$$\frac{\dot{a}_1}{a_1}\frac{\dot{a}_2}{a_2} + \frac{\dot{a}_2}{a_2}\frac{\dot{a}_3}{a_3} + \frac{\dot{a}_3}{a_3}\frac{\dot{a}_1}{a_1} = \kappa T_0^0 + \Lambda\,. \tag{3.1d}$$

Here over-dot means differentiation with respect to t and T_μ^ν is the energy–momentum tensor of the matter field which we choose in the form:

$$T_\mu^\nu = (\varepsilon + p)u_\mu u^\nu - p\delta_\mu^\nu\,, \tag{3.2}$$

where u^μ is the flow vector satisfying

$$g_{\mu\nu}u^\mu u^\nu = 1\,. \tag{3.3}$$

Here ε is the total energy density of a perfect fluid and/or dark energy density, while p is the corresponding pressure. p and ε are related by an equation of state which will be studied below in detail. In a co-moving system of coordinates from (3.2) one finds

$$T_0^0 = \varepsilon\,, \qquad T_1^1 = T_2^2 = T_3^3 = -p\,. \tag{3.4}$$

In view of (3.4) from (3.1a)–(3.1d) one immediately obtains[24]

$$a_i(t) = D_i[\tau(t)]^{1/3}\exp\left[X_i\int[\tau(t')]^{-1}\,dt'\right]\,, \qquad i = 1, 2, 3\,.$$

Here D_i and X_i are some arbitrary constants obeying

$$D_1D_2D_3 = 1\,, \qquad X_1 + X_2 + X_3 = 0\,,$$

and τ is a function of t defined to be

$$\tau = a_1a_2a_3\,. \tag{3.5}$$

From (3.1a)–(3.1d) for τ one finds

$$\frac{\ddot{\tau}}{\tau} = \frac{3\kappa}{2}(\varepsilon - p) + 3\Lambda\,. \tag{3.6}$$

On the other hand, the conservation law for the energy–momentum tensor gives

$$\dot{\varepsilon} = -\frac{\dot{\tau}}{\tau}(\varepsilon + p)\,. \tag{3.7}$$

After a little manipulation from (3.6) and (3.7) we find

$$\dot{\tau}^2 = 3(\kappa\varepsilon + \Lambda)\tau^2 + C_1\,, \tag{3.8}$$

with C_1 being an arbitrary constant. Let us now, in analogy with Hubble constant, define

$$\frac{\dot{\tau}}{\tau} = \frac{\dot{a}_1}{a_1} + \frac{\dot{a}_2}{a_2} + \frac{\dot{a}_3}{a_3} = 3H\,. \tag{3.9}$$

On account of (3.9) from (3.8) one derives

$$\kappa\varepsilon = 3H^2 - \Lambda - C_1/(3\tau^2)\,. \tag{3.10}$$

It should be noted that the energy density of the Universe is a positive quantity. It is believed that at the early stage of evolution when the volume scale τ was close to zero, the energy density of the Universe was infinitely large. On the other hand, with the expansion of the Universe, i.e. with the increase of τ, the energy density ε decreases and an infinitely large τ corresponds to a ε close to zero. Now if we consider the case when $\tau = \tau_{\rm L}$ is big enough for the T_0^0 and $1/\tau_{\rm L}^2$ be ignored, from (3.10) then follows

$$3H^2 - \Lambda \to 0. \tag{3.11}$$

Then on account of (3.9) from (3.11) for τ one finds

$$\tau = \tau_{\rm L} \exp\,[\sqrt{3\Lambda}\, t]. \tag{3.12}$$

As it is seen from (3.12) in this case Λ is essentially non-negative. In case of $\Lambda = 0$ we find that beginning from some value of τ the rate of expansion of the Universe becomes trivial, that is the universe does not expand with time. Whereas, for $\Lambda > 0$ the expansion process continues forever. Moreover, it is believed that the presence of the dark energy (which can be explained with a positive Λ as well) results in the accelerated expansion of the Universe. As far as negative Λ is concerned, its presence imposes some restriction on ε, namely, ε can never be small enough to be ignored. In case of the perfect fluid given by $p = \zeta\varepsilon$ there exists some upper limit for τ as well (note that τ is essentially non-negative, i.e. bound from below). In our previous papers we came to the same conclusion[24,25] (with a positive Λ which in the present paper appears to be negative). A suitable choice of parameters in this case may give rise to an oscillatory mode of expansion, whereas in case of a van der Waals fluid the highly nonlinear equation of state may result in an exponential expansion as well.

Inserting (3.9) and (3.10) into (3.6) one now finds

$$\dot H = -\frac{1}{2}\left(3H^2 - \Lambda + \frac{C_1}{3\tau^2} + \kappa p\right) = -\frac{\kappa}{2}(\varepsilon + p) - \frac{C_1}{3\tau^2}\,. \tag{3.13}$$

In view of (3.10), from (3.13), it follows that if the perfect fluid is given by a stiff matter where $p = \varepsilon$, the corresponding solution does not depend on the constant C_1.

Let us now go back to Eq. (3.8). It is in fact the first integral of (3.6) and can be written as

$$\dot\tau = \pm\sqrt{C_1 + 3(\kappa\varepsilon + \Lambda)\tau^2}\,. \tag{3.14}$$

On the other hand, rewriting (3.7) in the form

$$\frac{\dot\varepsilon}{\varepsilon + p} = \frac{\dot\tau}{\tau}\,, \tag{3.15}$$

and taking into account that p is a function of ε, one concludes that the right-hand side of Eq. (3.6) is a function of τ only, i.e.

$$\ddot{\tau} = \frac{3\kappa}{2}(\varepsilon - p)\tau + 3\Lambda\tau = \mathcal{F}(\tau)\,. \tag{3.16}$$

From a mechanical point of view Eq. (3.16) can be interpreted as an equation of motion of a single particle with unit mass under the force $\mathcal{F}(\tau)$. Then the following first integral exists[25]:

$$\dot{\tau} = \sqrt{2[\mathcal{E} - \mathcal{U}(\tau)]}\,. \tag{3.17}$$

Here $\mathcal{E}$ can be viewed as energy and $\mathcal{U}(\tau)$ is the potential of the force $\mathcal{F}$. Comparing Eqs. (3.14) and (3.17) one finds $\mathcal{E} = C_1/2$ and

$$\mathcal{U}(\tau) = -\frac{3}{2}(\kappa\varepsilon + \Lambda)\tau^2\,. \tag{3.18}$$

Let us finally write the solution to Eq. (3.6) in quadrature:

$$\int \frac{d\tau}{\sqrt{C_1 + 3(\kappa\varepsilon + \Lambda)\tau^2}} = t + t_0\,, \tag{3.19}$$

where the integration constant t_0 can be taken to be zero, since it only gives a shift in time. Equations (3.6) and (3.7) for perfect fluid obeying different equations of state has been thoroughly studied by us.[24,25]

4. Bel–Robinson Tensors

BR tensor first appeared in the endless search for a covariant version of gravitational energy. In general relativity, the energetic content of an electromagnetic field propagating in a region free of charge is described by the well-known symmetric traceless tensor

$$T_{\rm el}^{\alpha\beta} = -\frac{1}{4\pi}\left(F^{\alpha\lambda}F_{\lambda}^{\beta} - \frac{1}{4}g^{\alpha\beta}F^{\mu\nu}F_{\mu\nu}\right), \tag{4.1}$$

where $F^{\alpha\beta}$ is the electromagnetic field tensor. This tensor satisfies:

$$T_{{\rm el}\,;\alpha}^{\alpha\beta} = 0 \tag{4.2}$$

as a consequence of Maxwell equations with $j^\mu = 0$. The tensor $T_{\rm el}^{\alpha\beta}$ enables us to define a local density of electromagnetic energy as measured by an observer moving with the unit 4-velocity u:

$$w_{\rm el}(u) = T_{\rm el}^{\alpha\beta}u_\alpha u_\beta\,. \tag{4.3}$$

It follows from (4.1) that the energy density is positive definite for any time-like vector u.

Within the scope of general relativity, however, it is well known that the concept of local energy density is meaningless for a gravitational field. To overcome this difficulty one is led to introduce the notion of super-energy tensor constructed with the curvature tensor $R_{\mu\nu\alpha\beta}$. The first example of such a tensor was exhibited by Bel,[2] that was further generalized to the case of an arbitrary gravitational field.[3] Note that a similar tensor was also introduced by Robinson.[5] This tensor is now commonly known as the BR tensor as well. Since we are going to compare some distinct definitions of BR in this paper, before defining them let us see what kind of properties they should have. In general, the BR tensor has the following symmetry properties:

$$B_{\mu\nu\alpha\beta} = B_{\nu\mu\alpha\beta}\,, \tag{4.4a}$$

$$B_{\mu\nu\alpha\beta} = B_{\mu\nu\beta\alpha}\,, \tag{4.4b}$$

$$B_{\mu\nu\alpha\beta} = B_{\alpha\beta\mu\nu}\,. \tag{4.4c}$$

The symmetry property leads to the fact that in n-dimensional case there are $n(n+1)[n(n+1)+2]/8$ independent components of the BR tensor. In case of $n = 4$ out of 256 components only 55 are linearly independent.

In literature there are a few definitions of BR. Here we mention only three.

I. By analogy with the tensor (4.1) which may be written as

$$T_{\mu\nu} = F_{\mu\alpha}F_\nu^\alpha + *F_{\mu\alpha} * F_\nu^\alpha\,, \tag{4.5}$$

the BR tensor is defined as[9]:

$$B_{\mu\nu\alpha\beta} = R^\rho{}_\mu{}^\sigma{}_\alpha R_{\rho\nu\sigma\beta} + *R^\rho{}_\mu{}^\sigma{}_\alpha * R_{\rho\nu\sigma\beta}\,. \tag{4.6}$$

Here the dual curvature is $*R^{\mu\nu}{}_{\lambda\sigma} \equiv (1/2)\epsilon^{\mu\nu}{}_{\alpha\beta}R^{\alpha\beta}{}_{\lambda\sigma}$. It should be noted that this definition is adequate only in 4 dimensions and in vacuum. Otherwise this tensor cannot satisfy the DEP[26] and therefore this expression should not be used in other dimensions or in non-Ricci-flat spacetimes.

Using the definition of dual curvature, from (4.6) we find

$$B_{\mu\nu\alpha\beta} = R^\rho{}_\mu{}^\sigma{}_\alpha R_{\rho\nu\sigma\beta} + R^\rho{}_\mu{}^\sigma{}_\beta R_{\rho\nu\sigma\alpha} - \frac{1}{2}g_{\mu\nu}R_\alpha{}^{\rho\sigma\tau}R_{\beta\rho\sigma\tau}\,. \tag{4.7}$$

The properties (4.4a) and (4.4b) follow immediately from (4.6) thanks to the symmetry property of Riemann tensor. The property (4.4c) is straightforward from (4.6), but for (4.7) it requires

$$g_{\mu\nu}R_\alpha{}^{\rho\sigma\tau}R_{\beta\rho\sigma\tau} = g_{\alpha\beta}R_\mu{}^{\rho\sigma\tau}R_{\nu\rho\sigma\tau}\,. \tag{4.8}$$

In view of (2.5) for the BR tensor in this case we obtain the following nontrivial components:

$$B_{0000} = \sum_{i=1}^{3}\frac{\ddot a_i^2}{a_i^2}\,,$$

$$B_{iiii} = \ddot a_i^2 + \dot a_i^2\left\{\frac{\dot a_j^2}{a_j^2} + \frac{\dot a_k^2}{a_k^2}\right\},$$

$$B_{0i0i} = a_i\ddot{a}_i\left\{\frac{\ddot{a}_j}{a_j} + \frac{\ddot{a}_k}{a_k}\right\},$$

$$B_{ijij} = \ddot{a}_i\ddot{a}_j + a_i\dot{a}_i a_j\dot{a}_j\frac{\dot{a}_k^2}{a_k^2},$$

$$B_{00ii} = \ddot{a}_i^2 - \dot{a}_i^2\left\{\frac{\dot{a}_j^2}{a_j^2} + \frac{\dot{a}_k^2}{a_k^2}\right\},$$

$$B_{iijj} = \dot{a}_i^2\dot{a}_j^2 - a_i^2\ddot{a}_j^2 - a_i^2\dot{a}_j^2\frac{\dot{a}_k^2}{a_k^2}, \qquad i,j,k = 1,2,3 \quad i \neq j \neq k\,.$$

Inserting (2.5) into (4.8) we obtain the following additional relations:

$$\left(\frac{\ddot{a}_i}{a_i}\right)^2 \pm \left(\frac{\ddot{a}_j}{a_j}\right)^2 = \left(\frac{\dot{a}_k}{a_k}\right)^2\left[\left(\frac{\dot{a}_i}{a_i}\right)^2 \pm \left(\frac{\dot{a}_j}{a_j}\right)^2\right], \qquad i,j,k = 1,2,3, \quad i \neq j \neq k\,. \tag{4.9}$$

Among the six constrains in (4.9) only three are linearly independent. After a little manipulations with them finally obtains the following relations between the metric functions:

$$\frac{\ddot{a}_i}{a_i} = \frac{\dot{a}_j}{a_j}\frac{\dot{a}_k}{a_k}, \qquad i,j,k = 1,2,3, \quad i \neq j \neq k\,. \tag{4.10}$$

As one sees, in account of (4.10) the Einstein equation (3.1a)–(3.1d) leads to $T_0^0 = T_1^1 = T_2^2 = T_3^3$, which can be realized only when the source field satisfies the following equation of state:

$$p = -\varepsilon\,. \tag{4.11}$$

It is well known that only vacuum satisfies the state of equation given by (4.11). Thus we see that if we are to define BR tensor given by (4.6) or (4.7) we should deal with the Einstein equations with the source field given by a vacuum.

II. The restriction that arises above is due to the fact that in defining the BR tensor we used the dual term with the duality operator acting on the left pair only. To avoid this restrictions the BR tensor can be defined by[11,12,27]

$$\begin{aligned} 2B_{\mu\nu\alpha\beta} &= R^{\rho\ \ \sigma}_{\ \mu\ \ \alpha}R_{\rho\nu\sigma\beta} + {*R}^{\rho\ \ \sigma}_{\ \mu\ \ \alpha} * R_{\rho\nu\sigma\beta} \\ &\quad + R*^{\rho\ \ \sigma}_{\ \mu\ \ \alpha}\, R*_{\rho\nu\sigma\beta} + *\,R*^{\rho\ \ \sigma}_{\ \mu\ \ \alpha}\, {*R*}_{\rho\nu\sigma\beta}\,, \end{aligned} \tag{4.12}$$

where the duality operator acts on the left or on the right pair of indices according to its position. Nowadays this is known as the Bel tensor and was introduced by Bel[3] in a slightly different form.

From (4.12) one easily finds

$$\begin{aligned} B_{\mu\nu\alpha\beta} &= R^{\rho\ \ \sigma}_{\ \mu\ \ \alpha}R_{\rho\nu\sigma\beta} + R^{\rho\ \ \sigma}_{\ \mu\ \ \beta}R_{\rho\nu\sigma\alpha} - \frac{1}{2}g_{\mu\nu}R_{\alpha}^{\ \ \rho\sigma\tau}R_{\beta\rho\sigma\tau} \\ &\quad - \frac{1}{2}g_{\alpha\beta}R_{\mu}^{\ \ \rho\sigma\tau}R_{\nu\rho\sigma\tau} + \frac{1}{8}g_{\mu\nu}g_{\alpha\beta}R^{\rho\sigma\tau\eta}R_{\rho\sigma\tau\eta}\,. \end{aligned}$$

Under the new definition the symmetry properties (4.4a), (4.4b) and (4.4c) follow immediately, without any restriction to the metric functions.

Let us now write the nontrivial components of the BR tensor for the BI metric. In view of (2.5) we now find

$$\begin{aligned}
B_{0000} &= \frac{1}{2}\left\{\sum_{i=1}^{3}\frac{\ddot a_i^2}{a_i^2} + \frac{\dot a_1^2}{a_1^2}\frac{\dot a_2^2}{a_2^2} + \frac{\dot a_2^2}{a_2^2}\frac{\dot a_3^2}{a_3^2} + \frac{\dot a_3^2}{a_3^2}\frac{\dot a_1^2}{a_1^2}\right\},\\
B_{iiii} &= a_i^4 B_{0000}\,,\\
B_{0i0i} &= -a_i\dot a_i\left\{\frac{\dot a_j}{a_j}\frac{\ddot a_j}{a_j} + \frac{\dot a_k}{a_k}\frac{\ddot a_k}{a_k}\right\},\\
B_{ijij} &= a_i a_j\left\{\ddot a_i\ddot a_j + \dot a_i\dot a_j\frac{\dot a_k^2}{a_k^2}\right\},\\
B_{00ii} &= \frac{1}{2}\left\{-\ddot a_i^2 + \dot a_i^2\left(\frac{\dot a_j^2}{a_j^2} + \frac{\dot a_k^2}{a_k^2}\right) + a_i^2\left(\frac{\ddot a_j^2}{a_j^2} + \frac{\ddot a_k^2}{a_k^2} - \frac{\dot a_j^2}{a_j^2}\frac{\dot a_k^2}{a_k^2}\right)\right\},\\
B_{iijj} &= \frac{1}{2}\left\{\dot a_i^2\dot a_j^2 - \ddot a_j^2 a_j^2 - a_i^2\ddot a_j^2 - \frac{\dot a_k^2}{a_k^2}(\dot a_i^2 a_j^2 + a_i^2\dot a_j^2) + a_i^2 a_j^2\frac{\dot a_k^2}{a_k^2}\right\},\\
&\qquad\qquad i,j,k = 1,2,3,\quad i \neq j \neq k\,.
\end{aligned}$$

But the BR tensor defined in this way is not trace-free and is not completely symmetric. It is achieved if and only if the manifold is Ricci flat, i.e. $R_{ij} = 0$. Since for the BI universe we have nontrivial components of Ricci tensor, we give an alternative definition of BR where it is totally symmetric and trace-free.

III. Here we give another definition that gives rise to BR tensor, that is traceless and totally symmetric. It can be achieved by constructing BR by means of Weyl tensor[13,28].

$$B_{\mu\nu\alpha\beta} = C^{\rho}{}_{\mu}{}^{\sigma}{}_{\alpha}C_{\rho\nu\sigma\beta} + *C^{\rho}{}_{\mu}{}^{\sigma}{}_{\alpha} * C_{\rho\nu\sigma\beta}\,. \tag{4.13}$$

It can be shown that this BR is totally symmetric, i.e.

$$B_{ijkl} = B_{(ijkl)}\,. \tag{4.14}$$

Moreover, as one can easily find from (2.7), the BR defined through Weyl tensor is trace-free, i.e.

$$g^{jl}B_{ijkl} \equiv 0\,. \tag{4.15}$$

Let us study this case in detail. Using the properties of Levi–Cività tensor, we first rewrite (4.13) in the form

$$B_{\mu\nu\alpha\beta} = C^{\rho}{}_{\mu}{}^{\sigma}{}_{\alpha}C_{\rho\nu\sigma\beta} + C^{\rho}{}_{\mu}{}^{\sigma}{}_{\beta}C_{\rho\nu\sigma\alpha} - \frac{1}{2}g_{\mu\nu}C_{\alpha}{}^{\rho\sigma\tau}C_{\beta\rho\sigma\tau}\,. \tag{4.16}$$

After a little manipulation one easily finds the following nontrivial components of the BR tensor.

$$B_{0000} = (g^{11}C_{1010})^2 + (g^{22}C_{2020})^2 + (g^{33}C_{3030})^2 \,, \tag{4.17}$$

which on account of (2.9) gives

$$B_{0000} = (\frac{1}{a_2^2 a_3^2} C_{2323})^2 + \left(\frac{1}{a_3^2 a_1^2} C_{3131}\right)^2 + \left(\frac{1}{a_1^2 a_2^2} C_{1212}\right)^2 . \tag{4.18}$$

On the other hand for we find

$$B_{1111} = (C_{1010})^2 + (g^{22}C_{1212})^2 + (g^{33}C_{1313})^2 \,, \tag{4.19}$$

which in view of (2.9) can be rewritten as

$$\begin{aligned} B_{1111} &= \left(\frac{a_1^2}{a_2^2 a_3^2} C_{2323}\right)^2 + \left(\frac{1}{a_3^2} C_{3131}\right)^2 + \left(\frac{1}{a_2^2} C_{1212}\right)^2 , \\ &= a_1^4 \left[\left(\frac{1}{a_2^2 a_3^2} C_{2323}\right)^2 + \left(\frac{1}{a_3^2 a_1^2} C_{3131}\right)^2 + \left(\frac{1}{a_1^2 a_2^2} C_{1212}\right)^2\right] = a_1^4 B_{0000} \,. \end{aligned} \tag{4.20}$$

In view of (4.18) and (4.20) symbolically we can write

$$\begin{aligned} B_{0000} &= \left(\frac{1}{a_j^2 a_k^2} C_{jkjk}\right)^2 + \left(\frac{1}{a_k^2 a_i^2} C_{kiki}\right)^2 + \left(\frac{1}{a_i^2 a_j^2} C_{ijij}\right)^2 \\ &= \frac{1}{6}\Bigg\{ \sum_{i=1}^{3} \frac{\ddot{a}_i^2}{a_i^2} + \frac{\dot{a}_1^2}{a_1^2}\frac{\dot{a}_2^2}{a_2^2} + \frac{\dot{a}_2^2}{a_2^2}\frac{\dot{a}_3^2}{a_3^2} + \frac{\dot{a}_3^2}{a_3^2}\frac{\dot{a}_1^2}{a_1^2} \\ &\quad - \left(\frac{\ddot{a}_1}{a_1}\frac{\ddot{a}_2}{a_2} + \frac{\ddot{a}_2}{a_2}\frac{\ddot{a}_3}{a_3} + \frac{\ddot{a}_3}{a_3}\frac{\ddot{a}_1}{a_1}\right) - \frac{\dot{a}_1}{a_1}\frac{\dot{a}_2}{a_2}\left(\frac{\ddot{a}_1}{a_1} + \frac{\ddot{a}_2}{a_2} - 2\frac{\ddot{a}_3}{a_3} + \frac{\dot{a}_3^2}{a_3^2}\right) \\ &\quad - \frac{\dot{a}_2}{a_2}\frac{\ddot{a}_3}{a_3}\left(\frac{\ddot{a}_2}{a_2} + \frac{\ddot{a}_3}{a_3} - 2\frac{\ddot{a}_1}{a_1} + \frac{\dot{a}_1^2}{a_3^1}\right) - \frac{\dot{a}_3}{a_3}\frac{\ddot{a}_1}{a_1}\left(\frac{\ddot{a}_3}{a_3} + \frac{\ddot{a}_1}{a_1} - 2\frac{\ddot{a}_2}{a_2} + \frac{\dot{a}_2^2}{a_2^2}\right)\Bigg\} , \end{aligned} \tag{4.21a}$$

$$B_{iiii} = a_i^4 B_{0000} \,, \tag{4.21b}$$

$$\begin{aligned} B_{ijij} &= 2C_{0i0i}C_{0j0j} \\ &= \frac{1}{18a_k^2}(2\dot{a}_i a_j \dot{a}_k - a_i \dot{a}_j \dot{a}_k - \dot{a}_i \dot{a}_j a_k - a_i a_j \ddot{a}_k - \ddot{a}_i a_j a_k + 2a_i \ddot{a}_j a_k) \\ &\quad \times (2a_i \dot{a}_j \dot{a}_k - \dot{a}_i a_j \dot{a}_k - \dot{a}_i \dot{a}_j a_k - a_i a_j \ddot{a}_k - a_i \ddot{a}_j a_k + 2\ddot{a}_i a_j a_k) \,, \end{aligned} \tag{4.21c}$$

$$B_{0k0k} = -\frac{a_k^2}{a_i^2 a_j^2} B_{ijij} \,, \qquad i,j,k = 1,2,3, \quad i \neq j \neq k \,. \tag{4.21d}$$

Thus we have used three different definitions of BR. The first one defined in Ref. 9 imposes some restriction on the metric functions, namely for the BI spacetime, it coincides with vacuum solution of Einstein equations. The second definition removes this restriction, but since BI metric admits nontrivial Ricci tensor, the BR in this case is not totally symmetric. Finally we gave the definition used by Bergqvist and Senovilla. It satisfies all the properties of BR and it is totally symmetric. In what follows we study the DEP and DSEP for BR in BI Universe.

5. Cosmological Singularity and the Dominant Energy Condition

Recalling that a timelike geodesic is a world line for a particle moving without acceleration in the proper reference system we define the following:

A spacetime is nonsingular if any timelike geodesics, or null geodesics, can be continued into the past and the future without bound, i.e. to infinite proper length for the timelike geodesics and to an infinite value of an affine parameter for the null geodesics. Such a spacetime is termed "causally, geodesically complete". The requirements on the completeness are the minimum necessary so that the spacetime does not contain a singularity. It is necessary to point out that a spacetime not satisfying these requirements, however, one with a singularity, does not necessarily contain points with infinite curvature or with small hole.

From physical point of view, of course, one ought to take as singular any spacetime in which the geodesic world line of a particle cannot be continued without bound with respect to the proper time of this particle, for such a singular spacetime would lead to a violation of conservation laws.

As applied to the cosmological problem, the Hawking-Penrose theorem reads as follows[29]:

Theorem. *A spacetime $\mathcal{M}$ cannot satisfy causal geodesic completeness if the GTR* (*General Theory of Relativity*) *equations hold and if the following conditions are fulfilled*:

(i) *The spacetime $\mathcal{M}$ does not contain closed time-like curves.*

(ii) *The energy condition* (*DEP*) *is satisfied at every point.*

The energy condition may be expressed as

$$t^\alpha t_\alpha = 1 \quad \text{implies} \quad R_{\alpha\beta} t^\alpha t^\beta \leq 0 \,. \tag{5.1}$$

With Einstein's equations

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = -\kappa T_{\alpha\beta} \,, \tag{5.2}$$

(5.1) becomes

$$t^\alpha t_\alpha = 1 \quad \text{implies} \quad T_{\alpha\beta} t^\alpha t^\beta \geq \frac{1}{2} T_\mu^\mu \,. \tag{5.3}$$

If, in an eigentetrad of $T_{\mu\nu}$, ε denotes the energy density and p_1, p_2, p_3 denote the three principal pressures, then (5.3) can be written as

$$\begin{aligned} &\varepsilon + \textstyle\sum_\alpha p_\alpha \geq 0\,; \\ &\quad \varepsilon + p_\alpha \geq 0\,, \qquad \alpha = 1, 2, 3\,. \end{aligned} \tag{5.4}$$

The weak energy condition is

$$t^\alpha t_\alpha = 0 \quad \text{implies} \quad R_{\alpha\beta} t^\alpha t^\beta \leq 0\,,$$

which is a consequence of (5.1).

(iii) *On each time-like or null geodesic* γ, *there is at least one point for which*

$$K_{[a} R_{b]cd[e} K_{f]} K^c K^d \neq 0\,, \tag{5.5}$$

where K_a is the tangent to the curve γ at the given point and where the brackets on the subscripts imply antisymmetrization. If γ is timelike, we can rewrite (5.5) as

$$R_{abcd} K^c K^d \neq 0\,. \tag{5.6}$$

(iv) *The spacetime* $\mathcal{M}$ *contains either* (a) *a trapped surface*, (b) *a point* P *for which the convergence of all the null geodesics through* P *changes sign somewhere to the past of* P, *or* (c) *a compact spacelike hypersurface.*

The DEP for the BI metric can be written in the form:

$$T_0^0 \geq T_1^1 a_1^2 + T_2^2 a_2^2 + T_3^3 a_3^2\,, \tag{5.7a}$$

$$T_0^0 \geq T_1^1 a_1^2\,, \tag{5.7b}$$

$$T_0^0 \geq T_2^2 a_2^2\,, \tag{5.7c}$$

$$T_0^0 \geq T_3^3 a_3^2\,. \tag{5.7d}$$

In analogy with the Hawking–Penrose theorem stated above, Senovilla and others introduced DEP for the higher dimensional tensors. A detailed description of singularity theorems and their consequences can be found in the review paper by Senovilla[30].

The DSEP as defined by Senovilla reads[7]:

Theorem. *A rank-s tensor* $T_{\mu_1\cdots\mu_s}$, *is said to satisfy the DSEP if*

$$T_{\mu_1\cdots\mu_s} k_1^{\mu_1} \cdots k_s^{\mu_s} \geq 0 \tag{5.8}$$

for any future-pointing causal vectors $k_1^{\mu_1} \cdots k_s^{\mu_s}$. *To justify its name the dominant DSEP obeys the following lemma.*

Lemma. *If a tensor* $T_{\mu_1\cdots\mu_s}$ *satisfies the DSEP*, *then*

$$T_{0\cdots 0} \geq |T_{\mu_1\cdots\mu_s}|\,, \quad \forall\ \mu_1, \ldots, \mu_s = 0, \ldots, n-1 \tag{5.9}$$

in any orthonormal basis $\{\mathbf{e}_\nu\}$.

It was also established in Ref. 31, that any tensor satisfying the DEP possesses the following property:

Property. $T_{\mu_1\cdots\mu_s}$ satisfies DEP if and only if

$$T_{\mu_1\cdots\mu_s} l_1^{\mu_1}\cdots l_s^{\mu_s} \geq 0 \tag{5.10}$$

for any set $l_1^{\mu_1}\cdots l_s^{\mu_s}$ *of future-pointing null vectors.*

Let us now back to the 4-rank BR tensor and to check the DSEP for it. Since BR defined as (4.13) is a completely symmetric, trace-free 4-rank tensor, then it satisfy the DEP.[32] Therefore from the foregoing theorem, lemma and property for B_{ijkl} we can write:

$$B_{abcd} k_1^a k_2^b k_3^c k_4^d \geq 0\,, \tag{5.11}$$

$$B_{0\cdots 0} \geq |B_{abcd}|\,, \quad \forall\ a,b,c,d = 0,1,2,3 \tag{5.12}$$

and

$$B_{abcd} l_1^a l_2^b l_3^c l_4^d \geq 0\,. \tag{5.13}$$

Apparently, in view of (4.21b), Eq. (5.12) imposes some restriction on the metric functions, e.g., $a_i^4 \leq 1$. But this is not the case, since we have used a coordinate basis to compute the BR tensor components (4.21a)–(4.21d) and Eq. (5.9) refers to an orthonormal basis. We recall that in an orthonormal basis, the components of the Weyl tensor obey the following relation:

$$C_{0i0i} = -C_{jkjk}\,, \qquad i,j,k = 1,2,3 \quad \text{and} \quad i \neq j \neq k\,. \tag{5.14}$$

In view of (5.14) the expressions (4.21a)–(4.21d) now read

$$B_{0000} = (C_{jkjk})^2 + (C_{kiki})^2 + (C_{ijij})^2\,, \tag{5.15a}$$

$$B_{iiii} = B_{0000}\,, \tag{5.15b}$$

$$B_{ijij} = 2C_{0i0i}C_{0j0j}\,, \tag{5.15c}$$

$$B_{0k0k} = -B_{ijij}\,, \quad i,j,k = 1,2,3, \quad i \neq j \neq k\,. \tag{5.15d}$$

Thus in connection with the above Lemma relative to DSEP, Eq. (5.12) is fulfilled without any restrictions on the metric functions.

6. Conclusions

In view of the importance of the BI model in the study of the present day Universe, we considered the most simple model with a perfect fluid as a source field. The corresponding solutions to the Einstein equations have been obtained. Three alternative definitions of Bel–Robinson tensor are considered. It is shown that the definition used by Deser *et al.* is consistent with the Einstein equations when the source field is given by a vacuum only. The second definition used by Teyssandier is free from this restriction, but BR defined in this way is not totally symmetric. Definition used by Senovilla and Bergqvist does not suffer from this shortcomings, i.e., it has all the symmetries and the DSEP is satisfied.

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References

1. C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation* (W.H. Freeman and Co., 1973).
2. L. Bel, Paris *C. R. Acad. Sci. Paris* **247**, 1094 (1958).
3. L. Bel, Paris *C. R. Acad. Sci. Paris* **248**, 1297 (1959).
4. L. Bel, *Chahiers Phys.* **138**, 59 (1962).
5. I. Robinson, unpublished (1959).
6. I. Robinson, *Class. Quantum Grav.* **14**, 331 (1997).
7. J. M. M. Senovilla, *Class. Quantum Grav.* **17**, 2799 (2000).
8. R. Lazkoz, J. M. M. Senovilla and R. Vera, *Class. Quantum Grav.* **20**, 4135 (2003).
9. S. Deser, J. S. Franklin and D. Seminara, *Class. Quantum Grav.* **16**, 2815 (1999).
10. S. Deser, in *Gravitation and Relativity in General*, eds. A. Molina, J. Martin, E. Ruiz and F. Atrio (World Scientific, 1999).
11. P. Teyssandier, *Ann. Found. L. de Broglie* **26**, 459 (2001).
12. P. Teyssandier, gr-qc/9905080.
13. G. Bergqvist, *J. Math. Phys.* **39**, 2141 (1998).
14. L. Andersson, *Ann. Henri Poincare* **5**, 235 (2004).
15. O. Wingbrant, gr-qc/0310120.
16. Y. Choquet-Bruhat and J. W. York, Jr., in *General Relativity and Gravitation*, ed. A. Held (Plenum, 1980), Vol. I.
17. B. Saha, V. Rikhvitsky and M. Visinescu, *Romanian Rep. Phys.* **57**, 518 (2005).
18. V. A. Belinskii, I. M. Khalatnikov and E. M. Lifshitz, *Adv. Phys.* **19**, 525 (1970).
19. K. C. Jacobs, *Astrophys. J.* **153**, 661 (1968).
20. L. Bianchi, *Lezioni Sulla Teoria dei Gruppi Continui Finiti Transformazioni* (Spoerri, 1918).
21. M. Tsamparlis and P. S. Apostolopoulos, *J. Math. Phys.* **41**, 7573 (2000).
22. G. F. R. Ellis, *J. Math. Phys.* **8**, 1171 (1967).
23. J. M. Stewart and G. F. R. Ellis, *J. Math. Phys.* **9**, 1072 (1968).
24. B. Saha, *Phys. Rev. D* **64**, 123501 (2001).
25. B. Saha and T. Boyadjiev, *Phys. Rev. D* **69**, 124010 (2004).
26. J. M. M. Senovilla, private communication.
27. M. Chevreton, *Nuovo Cimento* **34**, 901 (1964).
28. J. M. M. Senovilla, in *Gravitation and Relativity in General* (World Scientific, 1999).
29. S. W. Hawking and R. Penrose, *Proc. R. Soc. London A* **314**, 528 (1970).
30. J. M. M. Senovilla, *Gen. Rel. Grav.* **30**, 701 (1998).
31. G. Bergqvist and J. M. M. Senovilla, *Class. Quantum Grav.* **18**, 5299 (2001).
32. G. Bergqvist and P. Lankinen, *Class. Quantum Grav.* **21**, 3499 (2004).