ISSN 1063-7796, Physics of Particles and Nuclei, 2006, Vol. 37, Suppl. 1, pp. S13–S44. © Pleiades Publishing, Inc., 2006.

Spinor Fields in Bianchi Type-I Universe*

B. Saha

Laboratory of Information Technologies, Joint Institute for Nuclear Research, 141980 Dubna, Moscow oblast, Russia;
e-mail: saha@thsun1.jinr.ru; bijan@jinr.ru
URL: http://thsun1.jinr.ru/~saha

Abstract—A system of minimally coupled nonlinear spinor and scalar fields within the scope of a Bianchi type-I (BI) cosmological model in the presence of a perfect fluid and a cosmological constant (Λ term) is studied, and solutions to the corresponding field equations are obtained. The problem of initial singularity and the asymptotical isotropization process of the Universe are thoroughly studied. The effect of the Λ term on the character of evolution is analyzed. It is shown that some special choice of spinor field nonlinearity generates a regular solution, but the absence of singularity results in violating the dominant energy condition in the Hawking–Penrose theorem. It is also shown that a positive Λ, which denotes an additional gravitational force in our case, gives rise to an oscillatory or a non-periodic mode of expansion of the Universe depending on the choice of problem parameter. The regular oscillatory mode of expansion violets the dominant energy condition if the spinor field nonlinearity occurs as a result of self-action, whereas, in the case of a linear spinor field or nonlinear one that occurs due to interaction with a scalar field, the dominant condition remains unbroken. A system with time-varying gravitational (G) and cosmological (Λ) constants is also studied to some extent. The introduction of magneto-fluid in the system generates nonhomogeneity in the energy-momentum tensor and can be exactly solved only under some additional condition. Though in this case, we indeed deal with all four known fields, i.e., spinor, scalar, electromagnetic, and gravitational, the over-all picture of evolution remains unchanged.

PACS numbers: 04.20.Ha, 03.65.Pm, 04.20.Jb, 98.80.Cq

DOI: 10.1134/S1063779606070021

1. INTRODUCTION

Nonlinear phenomena have been one of the most popular topics during last years. Nevertheless, it must be admitted that nonlinear classical fields have not received general consideration. This is probably due to the mathematical difficulties which arise because of the nonrenormalizability of the Fermi and other nonlinear couplings [1]. Nonlinear self-coupling of the spinor fields may arise as a consequence of the geometrical structure of space-time and, more precisely, because of the existence of torsion. Ivanenko [2, 3] and Rodichev [4] showed that a relativistic theory imposes in some cases a fourth-order self-coupling. In 1950, Weyl [5] proved that if the affine and the metric properties of space-time are taken as independent, the spinor field obeys either a linear equation in a space with torsion or a nonlinear one in a Reimannian space. As the self-action is of the spin–spin type, it allows the assignment of a dynamical role to the spin and offers a clue about the origin of the nonlinearities. This question was further clarified in some important papers by Utiyama, Kibble, and Sciama [6–8]. In the simplest scheme, the self-action is of the pseudovector type, but it can be shown that one can also get a scalar coupling [9]. An excellent review of the problem may be found in [10]. Nonlinear quantum Dirac fields were used by Heisenberg [11, 12] in his ambitious unified theory of elementary particles. They have been the object of renewed interest since the publication of the widely known paper by Gross and Neveu [13]. A nonlinear spinor field, suggested by the symmetric coupling between nucleons, muons, and leptons, was investigated by Finkelstein et al. [14] in the classical approximation. Although the existence of a spin-1/2 fermion is both theoretically and experimentally undisputed, these are described by *quantum* spinor fields. Possible justifications for the existence of classical spinors were addressed in [15].

The quantum field theory in curved space-time has been a matter of great interest in recent years because of its applications to cosmology and astrophysics. The evidence for the existence of strong gravitational fields in our Universe led to the study of the quantum effects of material fields in external classical gravitational field. In [16, 17], Parker considers spin-0 and spin-1/2 fields respectively, in a FRW space-time that is not quantized. The equations governing the spin-0 field are the covariant generalization of the special-relativistic free-field equations, whereas the spin-1/2 field satisfies the fully covariant generalization of the Dirac equation. In [16], the author showed that massless particles of arbitrary nonzero spin, such as photons or gravitons,

* The text was submitted by the author in English.

are not created by expansion, regardless of its form. Consideration of the special-relativistic limit in [17] provides new proof of the connection between spin and statistics. It is shown that the expansion of the universe in general results the production of spin-1/2 particles, which is not the case in the limits of zero and infinite mass. He also considered the Friedmann expansion of a radiation-filled universe, emphasizing the effect of the initial stage of the expansion. Since the appearance of these papers on scalar fields [16] and spin-1/2 fields [17], several authors have studied this subject. Present-day cosmology is based largely on Friedmann's solutions of the Einstein equations which describe the completely uniform and isotropic universe ("closed" and "open" models, i.e., bounded or unbounded universe). The main feature of these solutions is their non-stationarity. The idea of an expanding universe, following from this property, is confirmed by the astronomical observations and it now safe to assume that the isotropic model provides, in its general features, an adequate description of the present state of the universe.

Though spatially homogeneous and isotropic Fiedman–Robertson–Walker (FRW) models are widely considered as a good approximation of the present and early stages of the universe, the large-scale matter distribution in the observable universe, largely manifested in the form of discrete structures, does not exhibit homogeneity of a higher order. In contrast, the cosmic background radiation, which is significant in the microwave region, is extremely homogeneous, however, recent space investigations detect anisotropy in the cosmic microwave background. The observations from Cosmic Background Explorer's differential radiometer have detected and measured cosmic microwave background anisotropies in different angular scales. These anisotropies are believed to hide in their fold the entire history of cosmic evolution dating back to the recombination era and are being considered as indicative of the geometry and the content of the Universe. More about cosmic microwave background anisotropy is expected to be uncovered by the investigations of the microwave anisotropy probe. There is widespread consensus among cosmologists that cosmic microwave background anisotropies in small angular scales hold the key to the formation of discrete structures. The theoretical arguments [18] and recent experimental data that support the existence of an anisotropic phase that approaches an isotropic one leads us to consider the models of universe with anisotropic background. Zel'dovich was the first to assume that the early izotropization of the cosmological expanding process could have taken place as a result of the quantum effect of particle creation near singularity [19]. This assumption was further supported by several authors [20–22]. Interest in studying the Klein–Gordon and Dirac equations in anisotropic models has increased since Hu and Parker [22] showed that the creation of scalar particles in anisotropic backgrounds can dissipate the anisotropy as the Universe expands.

A Bianchi type-I (BI) universe, being the straightforward generalization of the flat Friedmann–Robertson–Walker (FRW) Universe, is one of the simplest models of an anisotropic Universe that describes a homogenous and spatially flat Universe. Unlike the FRW Universe which has the same scale factor for each of the three spatial directions, a BI universe has a different scale factor in each direction, thereby introducing an anisotropy to the system. Moreover, it has the agreeable property that near the singularity it behaves like a Kasner Universe, even in the presence of matter, and consequently falls within the general analysis of the singularity given by Belinskii et al. [23]. Also, in a Universe filled with matter for $p = \zeta\varepsilon$, $\zeta < 1$, it has been shown that any initial anisotropy in a BI universe quickly dies away and a BI universe eventually evolves into a FRW Universe [24]. Since the present-day Universe is surprisingly isotropic, this feature of the BI universe makes it a prime candidate for studying the possible effects of an anisotropy in the early Universe on present-day observations. In light of the importance of the above-mentioned information, several authors have studied the BI universe from different aspects.

In [25], Chimento and Mollerach studied the Dirac equations in a BI universe and obtained their classical solutions. They also claimed that for each value of the momentum, only two independent solutions exist and showed that it is not possible to obtain the solutions from those of a FRW universe only by perturbation. One of the solutions obtained would describe a particle with a given helicity, while the other one would represent antiparticles with the opposite helicity. This fact posed a very interesting problem—spin-1/2 particles cannot live in a BI, at least if they keep their well-known properties of flat space-time. This problem was handled by Castagnino et al. [26], where it was shown that if the Dirac equation is separable, the number of independent solutions is four, contrary to the claim made in [25]. The spinor field in a BI universe was also studied by Belinskii and Khalatnikov. In this paper, they solved the Einstein–Dirac equations when both the cosmological constant and the mass of the spinor field vanish (neutrinos). They also noticed that for BI models filled with neutrinos, the principal directions of expansion vary with time. Using Hamiltonian techniques, M. Henneaux studied class-A Bianchi universes generated by a spinor source [27, 28]. In [27], he derived the general solution to the massive Dirac equation in Bianchi type-I space-time with a cosmological constant [27] which was further extended for the Bianchi type-II model [28].

In a number of papers [29–31], several authors studied the behavior of gravitational waves (GW's) in a BI universe. In [30], the evolution equations for small perturbations in the metric, energy density, and material velocity were derived for an anisotropic viscous BI universe. It was shown that the results were independent of the equation of state of the cosmic fluid and its viscosity. It was also shown that the GWs need not necessarily be transversal in an anisotropically expanding BI universe and the longitudinal components of the gravitational waves have no physical significance. In [31], Cho and Speliotopoulos studied the propagation of classical gravitational waves in a BI universe. They found that GWs in a BI universe are not equivalent to two minimally coupled massless scalar fields as in a FRW universe. Because of its tensorial nature, the GW is much more sensitive to the anisotropy in space-time than the scalar field, and it gains an effective mass term. Moreover, they found a coupling between the two polarization states of the GW, which is not present in a FRW universe.

A nonlinear spinor field (NLSF) in an external FRW cosmological gravitational field was first studied by G.N. Shikin in 1991 [32]. Here we would like to note that the presence of a singular point to time in its space-time meric is another important property of the isotropic model. The presence of such a singular point means that the time is restricted. The motivation for the introduction a nonlinear term in the spinor field Lagrangian was to answer the natural question that arises in connection with the presence of a singular point, i.e., to what extent the presence of a singular point is an inherent property of the relativistic cosmological models or whether it is only a consequence of the specific simplifying assumptions underlying these models. The study by Shikin [32] shows that the presence of spinor filed nonlinearity is not enough to remove singularity in FRW space-time. The natural choice was to introduce anisotropy into the model and analyze the nonlinear spinor field equations in an external BI universe; this was carried out in [33]. In that paper, we considered the nonlinear term in the spinor field Lagrangian as an arbitrary function of all possible invariants generated from spinor bilinear forms. We also studied the possibility of elimination of initial singularity, especially for the Kasner Universe. For a few years, we studied the behavior of self-consistent NLSF in a BI universe [34, 35] both in the presence of perfect fluid and without it; this was followed by [36–38], where we studied the self-consistent system of interacting spinor and scalar fields. Recently, we have been studying [39, 40] the role of the cosmological constant (Λ) in the Lagrangian which together with Newton's gravitational constant (G) is considered as one of the fundamental constants in Einstein's theory of gravity [41, 42].

2. REVIEW OF BI COSMOLOGY

A diagonal Bianchi type-I space-time (hereafter BI) is a spatially homogeneous space-time which admits an abelian group G_3, acting on spacelike hyper-surfaces, generated by the spacelike Killing vectors $\mathbf{x}_1 = \partial_1$, $\mathbf{x}_2 = \partial_2$, and $\mathbf{x}_3 = \partial_3$. In synchronous coordinates, the metric is [43]:

$$ds^2 = dt^2 - \sum_{i=1}^{3} a_i^2(t) dx_i^2. \quad (2.1)$$

If the three scale factors are equal (i.e., $a_1 = a_2 = a_3$), (2.1) describes an isotropic and spatially flat Friedmann–Robertson–Walker (FRW) universe. The BI universe has a different scale factor in each direction, thereby introducing an anisotropy to the system. Thus, a Bianchi type-I (BI) universe, being the straightforward generalization of the flat Friedmann–Robertson–Walker (FRW) universe, is one of the simplest models of an anisotropic universe, describing a homogenous and spatially flat universe. When the two metric functions are equal (e.g., $a_2 = a_3$), the BI space-time is reduced to the important class of plane symmetric space-time (a special class of the Locally Rotational Symmetric space-times [44, 45] which admits a G_4 group of isometries acting multiply transitively on the space-like hyper-surfaces of homogeneity generated by the vectors $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, and $\mathbf{x}_4 = x^2\partial_3 - x^3\partial_2$. The BI has the agreeable property that near the singularity it behaves like a Kasner universe, given by

$$a_1(t) = a_1^0 t^{p_1}, \quad a_2(t) = a_2^0 t^{p_2}, \quad a_3(t) = a_3^0 t^{p_3}, \quad (2.2)$$

with p_j being the parameters of the BI space-time which measure the relative anisotropy between any two asymmetry axes and satisfy the constraints

$$p_1 + p_2 + p_3 = 1 \quad (2.3a)$$

$$p_1^2 + p_2^2 + p_3^2 = 1. \quad (2.3b)$$

Thus, out of three parameters, only one is arbitrary. One particular choice of parametrization is

$$p_1 = \frac{-p}{p^2 + p + 1} \quad (2.4a)$$

$$p_2 = \frac{p(p+1)}{p^2 + p + 1} \quad (2.4b)$$

$$p_3 = \frac{p+1}{p^2 + p + 1}. \quad (2.4c)$$

The condition $0 \le p \le 1$ on p then yields the condition $-1/3 \le p_1 \le 0$, $0 \le p_2 \le 2/3$, $2/3 \le p_3 \le 1$. Another particular parametrization can be given using an angle on the unit circle, since (2.3a) and (2.3b) describe the intersection of a sphere with a plane in the parameter space (p_1, p_2, p_3):

$$p_1 = \frac{1}{3}(1 + \cos\vartheta + \sqrt{3}\sin\vartheta), \quad (2.5a)$$

$$p_2 = \frac{1}{3}(1 + \cos\vartheta - \sqrt{3}\sin\vartheta), \quad (2.5b)$$

$$p_3 = \frac{1}{3}(1 - 2\cos\vartheta). \quad (2.5c)$$

Although ϑ ranges over the unit circle, the labeling of each p_j is quite arbitrary. Thus, the unit circle can be divided into six equal parts, each of which span 60°, and the choice of p_j is unique within each section separately. For $\vartheta = 0$, $p_1 = p_2 = 2/3$ and $p_3 = -1/3$, while for $\vartheta = \pi/3$, $p_1 = 1$ and $p_2 = p_3 = 0$.

Sometimes it proves convenient to introduce a new time parameter η by

$$\eta = \int^{t} a^{-1}(\tilde{t})d\tilde{t}, \tag{2.6}$$

where we define

$$[a(t)]^2 = C(t) \equiv (a_1 a_2 a_3)^{2/3} = (C_1 C_2 C_3)^{1/3}, \tag{2.7}$$

with $C_i \equiv a_i^2$. Note that in the isotropic limit, i.e., $a_1 = a_2 = a_3$, η reduces to conformal time. Further, defining

$$d_i = \frac{C_i'}{C_i}, \quad D \equiv \frac{1}{3}\sum_{i=1}^{3} d_i = \frac{C'}{C}, \quad Q \equiv \frac{1}{72}\sum_{i<j}(d_i - d_j)^2, \tag{2.8}$$

where prime denotes differentiation with respect to η, we get the following nonzero Christoffel symbols for the metric (2.1)

$$\Gamma^{\eta}_{\eta\eta} = \frac{1}{2}D, \quad \Gamma^{\eta}_{ii} = \frac{1}{2}\frac{d_i C_i}{C}, \quad \Gamma^{i}_{i\eta} = \Gamma^{i}_{\eta i} = \frac{1}{2}d_i. \tag{2.9}$$

The nonzero components of the Ricci tensor now read

$$R_{\eta\eta} = \frac{3}{2}D' + 6Q, \quad R_{ii} = -\frac{C_i}{2C}(d_i' + d_i D), \tag{2.10}$$

and the Ricci scalar

$$R = C^{-1}\left[3D' + \frac{3}{2}D^2 + 6Q\right]. \tag{2.11}$$

Note that in the sections to follow, we work with the usual time t.

3. BASIC EQUATIONS AND THEIR GENERAL SOLUTIONS

Using the variational principle, in this section we derive the basic equations for the corresponding spinor, scalar, and gravitational fields from the action (3.1).

We consider a system of the nonlinear spinor, scalar, and BI gravitational fields given by the action

$$\mathcal{S}(g;\psi,\bar{\psi},\varphi) = \int \mathcal{L}\sqrt{-g}d\Omega \tag{3.1}$$

with

$$\mathcal{L} = \mathcal{L}_{\rm g} + \mathcal{L}_{\rm sp} + \mathcal{L}_{\rm sc} + \mathcal{L}_{\rm int} + \mathcal{L}_{\rm pf}. \tag{3.2}$$

The gravitational part of the Lagrangian (3.2) $\mathcal{L}_{\rm g}$ is given by a Bianchi type-I metric, whereas the terms $\mathcal{L}_{\rm sp}$ and $\mathcal{L}_{\rm sc}$ describe the spinor and the scalar fields, respectively. The term $\mathcal{L}_{\rm int}$ stands for the interaction between the spinor and the scalar fields. Finally, $\mathcal{L}_{\rm pf}$ describes the perfect fluid.

3.1. Matter Field Lagrangian

3.1.1. Spinor field, its invariants and covariant derivatives. For a spinor field ψ, the symmetry between ψ and $\bar{\psi}$ appears to demand that we choose the symmetrized Lagrangian [7]. Keeping this in mind, we choose the spinor field Lagrangian as

$$\mathcal{L}_{\rm sp} = \frac{i}{2}[\bar{\psi}\gamma^{\mu}\nabla_{\mu}\psi - \nabla_{\mu}\bar{\psi}\gamma^{\mu}\psi] - m\bar{\psi}\psi + F, \tag{3.3}$$

where the term F describes the self-interaction of a spinor field and can be presented as some arbitrary functions of invariants generated from the real bilinear forms of a spinor field.

Let us now construct the invariants of spinor field. Since ψ and ψ^* (complex conjugate of ψ) have four components each, one can construct $4 \times 4 = 16$ independent bilinear combinations. They are

$$S = \bar{\psi}\psi \quad \text{(scalar)}, \tag{3.4a}$$

$$P = i\bar{\psi}\gamma^5\psi \quad \text{(pseudoscalar)}, \tag{3.4b}$$

$$v^{\mu} = (\bar{\psi}\gamma^{\mu}\psi) \quad \text{(vector)}, \tag{3.4c}$$

$$A^{\mu} = (\bar{\psi}\gamma^5\gamma^{\mu}\psi) \quad \text{(pseudovector)}, \tag{3.4d}$$

$$T^{\mu\nu} = (\bar{\psi}\sigma^{\mu\nu}\psi) \quad \text{(antisymmetric tensor)}, \tag{3.4e}$$

where $\sigma^{\mu\nu} = (i/2)[\gamma^{\mu}\gamma^{\nu} - \gamma^{\nu}\gamma^{\mu}]$. Invariants, corresponding to the bilinear forms are

$$I = S^2, \tag{3.5a}$$

$$J = P^2, \tag{3.5b}$$

$$I_v = v_{\mu}v^{\mu} = (\bar{\psi}\gamma^{\mu}\psi)g_{\mu\nu}(\bar{\psi}\gamma^{\nu}\psi), \tag{3.5c}$$

$$I_A = A_{\mu}A^{\mu} = (\bar{\psi}\gamma^5\gamma^{\mu}\psi)g_{\mu\nu}(\bar{\psi}\gamma^5\gamma^{\nu}\psi), \tag{3.5d}$$

$$I_T = T_{\mu\nu}T^{\mu\nu} = (\bar{\psi}\sigma^{\mu\nu}\psi)g_{\mu\alpha}g_{\nu\beta}(\bar{\psi}\sigma^{\alpha\beta}\psi). \tag{3.5e}$$

According to the Pauli–Fierz theorem [46], among the five invariants only I and J are independent as all others can be expressed by them: $I_v = -I_A = I + J$ and $I_T = I - J$. In view of the theorem mentioned above, we choose the nonlinear term F to be the function of I and J only, i.e., $F = F(I, J)$, thus claiming that it describes the nonlinearity in the most general form.

In (3.3), ∇_{μ} denotes the covariant differentiation; its explicit form depends on the quantity it acts on. This covariant differentiation has the standard properties

$$\nabla_\mu(AB) = (\nabla_\mu A)B + A(\nabla_\mu B), \tag{3.6a}$$

$$\nabla_\mu(A^*) = (\nabla_\mu A)^*, \tag{3.6b}$$

$$\nabla_\mu \gamma_\nu = 0, \tag{3.6c}$$

where the symbol * means the Hermitian adjoint (the transpose of the complex conjugate). The explicit form of the covariant derivative of spinor is [47, 48]

$$\nabla_\mu \psi = \frac{\partial \psi}{\partial x^\mu} - \Gamma_\mu \psi, \tag{3.7a}$$

$$\nabla_\mu \bar{\psi} = \frac{\partial \bar{\psi}}{\partial x^\mu} + \bar{\psi}\Gamma_\mu, \tag{3.7b}$$

where $\Gamma_\mu(x)$ are the Fock–Ivanenko spinor affine connection matrices. γ matrices in the above equations obey the following algebra

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} \tag{3.8}$$

and are connected with the flat space-time Dirac matrices $\bar{\gamma}$ in the following way

$$g_{\mu\nu}(x) = e^a_\mu(x) e^b_\nu(x)\eta_{ab}, \quad \gamma_\mu(x) = e^a_\mu(x)\bar{\gamma}_a, \tag{3.9}$$

where $\eta_{ab} = \mathrm{diag}(1, -1, -1, -1)$ and e^a_μ is a set of tetrad 4-vectors. The spinor affine connection matrices $\Gamma_\mu(x)$ are uniquely determined up to an additive multiple of the unit matrix by the equation

$$\nabla_\mu \gamma_\nu = \frac{\partial \gamma_\nu}{\partial x^\mu} - \Gamma^\rho_{\nu\mu}\gamma_\rho - \Gamma_\mu\gamma_\nu + \gamma_\nu\Gamma_\mu = 0, \tag{3.10}$$

with the solution

$$\Gamma_\mu(x) = \frac{1}{4} g_{\rho\sigma}(x)(\partial_\mu e^b_\delta e^\rho_b - \Gamma^\rho_{\mu\delta})\gamma^\sigma\gamma^\delta. \tag{3.11}$$

Let us now write the γ's and Γ_μ's explicitly for the BI metric (3.18). For the metric (3.18) from (3.9), one finds

$$\begin{gathered} \gamma_0 = \bar{\gamma}_0, \quad \gamma_1 = a(t)\bar{\gamma}_1, \quad \gamma_2 = b(t)\bar{\gamma}_2, \\ \gamma_3 = c(t)\bar{\gamma}_3, \\ \gamma^0 = \bar{\gamma}^0, \quad \gamma^1 = \bar{\gamma}^1/a(t), \quad \gamma^2 = \bar{\gamma}^2/b(t), \\ \gamma^3 = \bar{\gamma}^3/c(t). \end{gathered} \tag{3.12}$$

For the affine spinor connections from (3.11), we find

$$\begin{gathered} \Gamma_0 = 0, \quad \Gamma_1 = \frac{1}{2}\dot{a}(t)\bar{\gamma}^1\bar{\gamma}^0, \\ \Gamma_2 = \frac{1}{2}\dot{b}(t)\bar{\gamma}^2\bar{\gamma}^0, \quad \Gamma_3 = \frac{1}{2}\dot{c}(t)\bar{\gamma}^3\bar{\gamma}^0. \end{gathered} \tag{3.13}$$

Flat space-time matrices $\bar{\gamma}$ we will choose in the form given in [49]:

$$\bar{\gamma}^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \bar{\gamma}^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\bar{\gamma}^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \quad \bar{\gamma}^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Defining γ^5 as follows,

$$\gamma^5 = -\frac{i}{4}E_{\mu\nu\sigma\rho}\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho, \quad E_{\mu\nu\sigma\rho} = \sqrt{-g}\varepsilon_{\mu\nu\sigma\rho},$$

$$\varepsilon_{0123} = 1,$$

$$\gamma^5 = -i\sqrt{-g}\gamma^0\gamma^1\gamma^2\gamma^3 = -i\bar{\gamma}^0\bar{\gamma}^1\bar{\gamma}^2\bar{\gamma}^3 = \bar{\gamma}^5,$$

we obtain

$$\bar{\gamma}^5 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

3.1.2. Scalar field Lagrangian. The massless scalar field Lagrangian is chosen to be an arbitrary function of invariant $\Upsilon = \varphi_{,\alpha}\varphi^{,\alpha}$:

$$\mathcal{L}_{\rm sc} = \Psi(\Upsilon), \quad \Upsilon = \varphi_{,\alpha}\varphi^{,\alpha}. \tag{3.14}$$

The scalar field Lagrangian (3.14) becomes linear at $\Upsilon \longrightarrow 0$, i.e.,

$$\lim_{\Upsilon \to 0}\Psi(\Upsilon) = \frac{1}{2}\Upsilon + \ldots. \tag{3.15}$$

As a massless nonlinear scalar field Lagrangian, one can choose, e.g., the Born–Infeld Lagrangian (4.56) that becomes linear at the weak limit.

3.1.3. Interacting term. The interacting term in the Lagrangian is chosen in the form of derivative coupling [34, 37, 38], i.e.,

$$\mathcal{L}_{\rm int} = \lambda_1\varphi_{,\alpha}\varphi^{,\alpha}F_1 = \lambda_1\Upsilon F_1, \tag{3.16}$$

with λ_1 being the coupling constant and F_1 some arbitrary function of I and J, i.e., $F_1 = F_1(I, J)$. In this paper, $F_1(I, J)$ is taken to be either a power law or a trigonometric function of its arguments.

The contribution of the perfect fluid to the system is performed by means of its energy-momentum tensor, which acts as one of the sources of the corresponding gravitational field equations. So here we do not need to write the Lagrangian density $\mathcal{L}_{\rm pf}$ explicitly. The reason for writing $\mathcal{L}_{\rm pf}$ in Eqs. (3.1) and (3.2) action and lag is to underline that we are dealing with a self-consistent system. An interesting discussion of the action and Lagrangian for a perfect fluid can be found in [50, 51].

3.2. Gravitational Field

As a gravitational field, we consider the homogeneous anisotropic Bianchi type-I space-time. We chose the gravitational part of the Lagrangian (3.2) in the form

$$\mathcal{L}_{\rm g} = \frac{R+2\Lambda}{2\kappa}, \qquad (3.17)$$

where R is the scalar curvature, $\kappa = 8\pi G$ with G being Einstein's gravitational constant, and Λ is the cosmological constant.

We chose the B–I metric in the form [52]

$$ds^2 = dt^2 - a^2(t)dx^2 - b^2(t)dy^2 - c^2(t)dz^2. \qquad (3.18)$$

Here, the metric functions a, b and c are the functions of time t only. The inequality $a \neq b \neq c$ represents the anisotropy in space-time, whereas their space independence shows its homogeneity.

The metric (3.18) has the following non-trivial Christoffel symbols

$$\Gamma^1_{10} = \frac{\dot a}{a}, \quad \Gamma^2_{20} = \frac{\dot b}{b}, \quad \Gamma^3_{30} = \frac{\dot c}{c},$$
$$\Gamma^0_{11} = a\dot a, \quad \Gamma^0_{22} = b\dot b, \quad \Gamma^0_{33} = c\dot c. \qquad (3.19)$$

The nontrivial components of the Ricci tensors are

$$R^0_0 = -\left(\frac{\ddot a}{a} + \frac{\ddot b}{b} + \frac{\ddot c}{c}\right), \qquad (3.20a)$$

$$R^1_1 = -\left[\frac{\ddot a}{a} + \frac{\dot a}{a}\left(\frac{\dot b}{b} + \frac{\dot c}{c}\right)\right], \qquad (3.20b)$$

$$R^2_2 = -\left[\frac{\ddot b}{b} + \frac{\dot b}{b}\left(\frac{\dot c}{c} + \frac{\dot a}{a}\right)\right], \qquad (3.20c)$$

$$R^3_3 = -\left[\frac{\ddot c}{c} + \frac{\dot c}{c}\left(\frac{\dot a}{a} + \frac{\dot b}{b}\right)\right]. \qquad (3.20d)$$

From (3.20), one finds the following Ricci scalar for the BI universe

$$R = -2\left(\frac{\ddot a}{a} + \frac{\ddot b}{b} + \frac{\ddot c}{c} + \frac{\dot a\dot b}{ab} + \frac{\dot b\dot c}{bc} + \frac{\dot c\dot a}{ca}\right). \qquad (3.21)$$

The non-trivial components of Riemann tensors in this case read

$$R^{01}_{01} = -\frac{\ddot a}{a}, \quad R^{02}_{02} = -\frac{\ddot b}{b}, \quad R^{03}_{03} = -\frac{\ddot c}{c},$$
$$R^{12}_{12} = -\frac{\dot a\dot b}{ab}, \quad R^{23}_{23} = -\frac{\dot b\dot c}{bc}, \quad R^{31}_{31} = -\frac{\dot c\dot a}{ca}. \qquad (3.22)$$

Now having all the non-trivial components of Ricci and Riemann tensors, one can easily write the invariants of gravitational field which we need to study the space-time singularity. We return to this study later.

3.3. Field Equations

Using the variational principle, let us now write the field equations corresponding to the action (3.1).

Variation of (3.1) action with respect to spinor field $\psi(\bar\psi)$ gives nonlinear spinor field equations

$$i\gamma^\mu\nabla_\mu\psi - m\psi + \mathcal{D}\psi + \mathcal{G}i\gamma^5\psi = 0, \qquad (3.23a)$$

$$i\nabla_\mu\bar\psi\gamma^\mu + m\bar\psi - \mathcal{D}\bar\psi - \mathcal{G}i\bar\psi\gamma^5 = 0, \qquad (3.23b)$$

where we denote

$$\mathcal{D} = \mathcal{D}_1 + \lambda_1\mathcal{D}_2 = 2S\left(\frac{\partial F}{\partial I} + \lambda_1\Upsilon\frac{\partial F_1}{\partial I}\right),$$

$$\mathcal{G} = \mathcal{G}_1 + \lambda_1\mathcal{G}_2 = 2P\left(\frac{\partial F}{\partial J} + \lambda_1\Upsilon\frac{\partial F_1}{\partial J}\right),$$

whereas variation of (3.1) with respect to scalar field yields the following scalar field equation

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\nu}\left[\sqrt{-g}g^{\nu\mu}\left(\frac{\partial\Psi}{\partial\Upsilon} + \lambda_1 F_1\right)\varphi_{,\mu}\right] = 0. \qquad (3.24)$$

Varying (3.1) with respect to metric tensor $g_{\mu\nu}$, one finds the Einstein's field equation

$$R^\mu_\nu - \frac{1}{2}\delta^\mu_\nu R = -\kappa T^\mu_\nu + \Lambda\delta^\mu_\nu, \qquad (3.25)$$

where R^μ_ν is the Ricci tensor; $R = g^{\mu\nu}R_{\mu\nu}$ is the Ricci scalar; and T^μ_ν is the energy-momentum tensor of the material field given by

$$T^\nu_\mu = T^\nu_{{\rm sp}\mu} + T^\nu_{{\rm sc}\mu} + T^\nu_{{\rm int}\mu} + T^\nu_{{\rm pf}\mu}. \qquad (3.26)$$

Here $T^\nu_{{\rm sp}\mu}$ is the energy-momentum tensor of the spinor field having the form

$$T^\rho_{{\rm sp}\mu} = \frac{i}{4}g^{\rho\nu}(\bar\psi\gamma_\mu\nabla_\nu\psi + \bar\psi\gamma_\nu\nabla_\mu\psi$$
$$- \nabla_\mu\bar\psi\gamma_\nu\psi - \nabla_\nu\bar\psi\gamma_\mu\psi) - \delta^\rho_\mu\mathcal{L}_{\rm sp}, \qquad (3.27)$$

where $\mathcal{L}_{\rm sp}$ with respect to (3.23a) and (3.23b) takes the form

$$\mathcal{L}_{\rm sp} = -(\mathcal{D}S + \mathcal{G}P) + F(I, J). \qquad (3.28)$$

The energy-momentum tensor of the scalar field $T^\nu_{{\rm sc}\mu}$ is given by

$$T^\nu_{{\rm sc}\mu} = 2\frac{d\Psi}{d\Upsilon}\varphi_{,\mu}\varphi^{,\nu} - \delta^\nu_\mu\Psi. \qquad (3.29)$$

For the interaction field, we find

$$T^{\nu}_{\text{int}\mu} = 2\lambda_1 F_1 \varphi_{,\mu}\varphi^{,\nu} - \delta^{\nu}_{\mu}\mathcal{L}_{\text{int}}. \quad (3.30)$$

$T^{\nu}_{\text{pf}\mu}$ is the energy-momentum tensor of a perfect fluid. For a Universe filled with perfect fluid, in the comoving system of reference ($u^0 = 1$, $u^i = 0$, $i = 1, 2, 3$), we have

$$T^{\nu}_{\text{pf}\mu} = (p+\varepsilon)u_{\mu}u^{\nu} - \delta^{\nu}_{\mu}p = (\varepsilon, -p, -p, -p). \quad (3.31)$$

The energy density ε of the perfect fluid is related to the corresponding pressure p by the equation of state

$$p_{\text{pf}} = \zeta\varepsilon_{\text{pf}}, \quad (3.32)$$

where ζ is a constant and lies in the interval $\zeta \in [0, 1]$. Depending on its numerical value, ζ describes the following types of Universes [24]

$$\zeta = 0, \quad \text{(dust Universe)}, \quad (3.33a)$$

$$\zeta = 1/3, \quad \text{(radiation Universe)}, \quad (3.33b)$$

$$\zeta \in (1/3, 1), \quad \text{(hard Universes)}, \quad (3.33c)$$

$$\zeta = 1, \quad \text{(Zel'dovich Universe or stiff matter)}. \quad (3.33d)$$

It was also shown by Jacobs [24] that if filled with matter obeying (3.32) and (3.33), any initial anisotropy in a BI universe quickly dies away and a BI universe eventually evolves into a FRW Universe.

3.4. Solutions to the Field Equations

In this subsection, we solve the matter and gravitational field equations. We will study the space-independent solutions to the spinor and scalar field equations (3.23a), (3.23b), and (3.24) so that $\psi = \psi(t)$ and $\varphi = \varphi(t)$. We also define a time-dependent function $\tau(t)$:

$$\tau = abc = \sqrt{-g}, \quad (3.34)$$

which is indeed the volume scale of the BI space-time. The spinor field equation (3.23a) then can be rewritten as

$$i\bar{\gamma}^0\left(\frac{\partial}{\partial t} + \frac{\dot{\tau}}{2\tau}\right)\psi - m\psi + \mathcal{D}\psi + \mathcal{G}i\gamma^5\psi = 0. \quad (3.35)$$

Introducing $V_j(t) = \sqrt{\tau}\psi_j(t)$, $j = 1, 2, 3, 4$, from (3.35), one deduces the following system of equations:

$$\dot{V}_1 + i(m-\mathcal{D})V_1 - \mathcal{G}V_3 = 0, \quad (3.36a)$$

$$\dot{V}_2 + i(m-\mathcal{D})V_2 - \mathcal{G}V_4 = 0, \quad (3.36b)$$

$$\dot{V}_3 - i(m-\mathcal{D})V_3 + \mathcal{G}V_1 = 0, \quad (3.36c)$$

$$\dot{V}_4 - i(m-\mathcal{D})V_4 + \mathcal{G}V_2 = 0. \quad (3.36d)$$

From (3.23a) and (3.23b), we also write the equations for the invariants $S = \bar{\psi}\psi$, $P = i\bar{\psi}\gamma^5\psi$, and $A = \bar{\psi}\bar{\gamma}^5\bar{\gamma}^0\psi$

$$\dot{S}_0 - 2\mathcal{G}A_0 = 0, \quad (3.37a)$$

$$\dot{P}_0 - 2(m-\mathcal{D})A_0 = 0, \quad (3.37b)$$

$$\dot{A}_0 + 2(m-\mathcal{D})P_0 + 2\mathcal{G}S_0 = 0, \quad (3.37c)$$

where $S_0 = \tau S$, $P_0 = \tau P$, and $A_0 = \tau A$, leading to the following relation

$$S^2 + P^2 + A^2 = C^2/\tau^2, \quad C^2 = \text{const}. \quad (3.38)$$

From the scalar field equation (3.24), in this case we have

$$\left(\frac{d\Psi}{d\Upsilon} + \lambda_1 F_1\right)\dot{\varphi} = \frac{C_s}{\tau}, \quad C_s = \text{const}. \quad (3.39)$$

Let us now solve the Einstein equations. For this purpose we rewrite the Einstein equations (3.25) corresponding to the metric (3.18) explicitly,

$$\frac{\ddot{b}}{b} + \frac{\ddot{c}}{c} + \frac{\dot{b}\dot{c}}{bc} = \kappa T_1^1 - \Lambda, \quad (3.40a)$$

$$\frac{\ddot{c}}{c} + \frac{\ddot{a}}{a} + \frac{\dot{c}\dot{a}}{ca} = \kappa T_2^2 - \Lambda, \quad (3.40b)$$

$$\frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\dot{a}\dot{b}}{ab} = \kappa T_3^3 - \Lambda, \quad (3.40c)$$

$$\frac{\dot{a}\dot{b}}{ab} + \frac{\dot{b}\dot{c}}{bc} + \frac{\dot{c}\dot{a}}{ca} = \kappa T_0^0 - \Lambda, \quad (3.40d)$$

where overdots denote a differentiation with respect to t.

Using the property of flat space-time Dirac matrices and the explicit form of covariant derivative ∇_μ, one finds the expressions for the components of the energy-momentum tensor:

$$\begin{aligned} T_0^0 &= mS - F(I, J) + 2\frac{d\Psi}{d\Upsilon}\dot{\varphi}^2 - \Psi(\Upsilon) \\ &\quad + 2\lambda_1 F_1(I, J)\dot{\varphi}^2 - \lambda_1 F_1(I, J)\Upsilon + \varepsilon \\ T_1^1 &= T_2^2 = T_3^3 \\ &= \mathcal{D}S + \mathcal{G}P - F(I, J) - \Psi(\Upsilon) - \lambda_1 F_1\Upsilon - p. \end{aligned} \quad (3.41)$$

Let us express a, b, c in terms of τ. For this we notice that subtraction of Einstein equations (3.40b) and (3.40a) leads to the equation

$$\begin{aligned} &\frac{\ddot{a}}{a} - \frac{\ddot{b}}{b} + \frac{\dot{a}\dot{c}}{ac} - \frac{\dot{b}\dot{c}}{bc} \\ &= \frac{d}{dt}\left(\frac{\dot{a}}{a} - \frac{\dot{b}}{b}\right) + \left(\frac{\dot{a}}{a} - \frac{\dot{b}}{b}\right)\left(\frac{\dot{a}}{a} + \frac{\dot{b}}{b} + \frac{\dot{c}}{c}\right) = 0. \end{aligned} \quad (3.42)$$

From (3.42), one finds

$$\left(\frac{\dot{a}}{a} - \frac{\dot{b}}{b}\right)\tau = X_1, \quad (3.43)$$

where X_1 is the integration constant. This means that as $\tau \longrightarrow \infty$, the expansion rate becomes isotropic in the (x, y)-plane, i.e.,

$$\frac{\dot{a}}{a} = \frac{\dot{b}}{b} \Rightarrow a = qb, \tag{3.44}$$

with q being some integration constant.

In general, the Eq. (3.43) gives the following relation between the metric functions a and b:

$$\frac{a}{b} = D_1 \exp\left(X_1 \int \frac{dt}{\tau}\right), \tag{3.45}$$

with D_1 being the constant of integration. Analogously, one also finds

$$\frac{b}{c} = D_2 \exp\left(X_2 \int \frac{dt}{\tau}\right), \tag{3.46}$$

$$\frac{c}{a} = D_3 \exp\left(X_3 \int \frac{dt}{\tau}\right), \tag{3.47}$$

where D_2, D_3, X_2, X_3 are integration constants. Note that, as in case of (3.43), one can easily prove that the expansion rate becomes isotropic in all directions as $\tau \longrightarrow \infty$.

After a little manipulation with the Eqs. (3.45), (3.46) and (3.47), we find the following functional dependence between the constants X_i's and D_i's:

$$D_1 D_2 D_3 = 1, \quad X_1 + X_2 + X_3 = 0 \tag{3.48}$$

Finally, on account of (3.34) from (3.45), (3.46), and (3.47), we write the metric functions $a(t)$, $b(t)$, and $c(t)$ explicitly as [53]

$$a(t) = A_1 \tau^{1/3} \exp[(B_1/3) \int \tau^{-1} dt], \tag{3.49a}$$

$$b(t) = A_2 \tau^{1/3} \exp[(B_2/3) \int \tau^{-1} dt], \tag{3.49b}$$

$$c(t) = A_3 \tau^{1/3} \exp[(B_3/3) \int \tau^{-1} dt], \tag{3.49c}$$

where

$$A_1 = \sqrt[3]{(D_1/D_3)}, \quad A_2 = \sqrt[3]{1/(D_1^2 D_3)},$$
$$A_3 = \sqrt[3]{(D_1 D_3^2)},$$
$$B_1 = X_1 - X_3, \quad B_2 = -(2X_1 + X_3), \quad B_3 = X_1 + 2X_3.$$

Thus, the system of Einstein's equations is completely integrated. In this process of integration, only the first three of the complete system of Einstein equations have been used. General solutions to these three second order equations have been obtained. The solutions contain four arbitrary constants: D_1, D_3, X_1, X_3. To verify the correctness of the obtained solutions, it is necessary to use the fourth equation. This can be done either by putting a, b, c into Eq. (3.40d), or by solving all four equations of the system (3.40) together.

Insertion of a, b, c into (3.40d) should lead either to identity or to some additional constraint between the constants. Indeed, putting a, b, c from (3.49) into (3.40d) one come to the equality

$$\begin{gathered} \dot{\tau}^2 - \mathcal{X} = 3\tau^2(\kappa T_0^0 - \Lambda), \\ \mathcal{X} := X_1^2 + X_1 X_3 + X_3^2, \end{gathered} \tag{3.50}$$

with the solution in quadrature

$$\int \frac{d\tau}{\sqrt{\mathcal{X} + 3\tau^2(\kappa T_0^0 - \Lambda)}} = t + t_0, \quad t_0 = \text{const}. \tag{3.51}$$

In our further investigation, we consider a second possibility of using all four equations of the system (3.40). For this, we take the sum of Einstein equations (3.40a), (3.40b), (3.40c), and (3.40d) multiplied by 3, i.e. ((3.40a) + (3.40b) + (3.40c) + 3 × (3.40d)). This leads to the second order differential equation for defining $\tau(t)$:

$$\frac{\ddot{\tau}}{\tau} = \frac{3}{2}\kappa(T_1^1 + T_0^0) - 3\Lambda. \tag{3.52}$$

For the right-hand-side of (3.52) to be a function of τ only, the solution to this equation is well-known [54]. As we see in the next section, the right-hand-side of (3.52) is indeed a function of τ. Given the explicit form of F and $\mathcal{L}_{\text{int}}$ from (3.52), one finds the concrete solution for τ in quadrature.

For purposes of our further study, we rewrite the metric functions and their derivatives in a compact form setting a_1, a_2, and a_3 for a, b, and c, respectively. From (3.49), we then have

$$a_i = A_i \tau^{1/3} \exp[(B_i/3) \int \tau^{-1} dt], \tag{3.53a}$$

$$\dot{a}_i = \frac{\dot{\tau} + B_i}{3} \frac{a_i}{\tau}, \quad (i = 1, 2, 3) \tag{3.53b}$$

$$\ddot{a}_i = \frac{3\tau\ddot{\tau} - 2(\dot{\tau})^2 - B_i\dot{\tau} + B_i^2)}{9} \frac{a_i}{\tau^2}. \tag{3.53c}$$

Defining the Hubble constant in analogy with a FRW universe from (3.53), we obtain

$$H_j = \frac{\dot{a}_j}{a_j} = \frac{\dot{\tau} + B_j}{3\tau}, \quad j = 1, 2, 3, \tag{3.54}$$

or a generalized one

$$H = (H_1 + H_2 + H_3)/3 = \dot{\tau}/3\tau. \tag{3.55}$$

The deceleration parameter given by

$$q = -\frac{\ddot{R}R}{\dot{R}^2} \tag{3.56}$$

for a FRW universe with R being the scale factor can also be generalized for the BI space-time to obtain

$$q_i = -\frac{\ddot{a}_i a_i}{\dot{a}_i^2} = -\left[\left(\frac{\ddot{a}_i}{a_i}\right)/\left(\frac{\dot{a}_i}{a_i}\right)^2\right] \\ = -\left[1 + \left(\frac{\dot{a}_i}{a_i}\right)^{\cdot}/\left(\frac{\dot{a}_i}{a_i}\right)^2\right]. \quad (3.57)$$

Inserting (3.53) into (3.57), one obtains

$$q_i = -\frac{3\tau\ddot{\tau} - 2(\dot{\tau})^2 - B_i\dot{\tau} + B_i^2}{(\dot{\tau})^2 + 2B_i\dot{\tau} + B_i^2}, \quad i = 1, 2, 3. \quad (3.58)$$

Let us now go back to the Einstein equation (3.25). Taking the divergence of the Einstein equation, we obtain

$$T^{\nu}_{\mu;\,\nu} = T^{\nu}_{\mu,\,\nu} + \Gamma^{\nu}_{\rho\nu}T^{\rho}_{\mu} - \Gamma^{\rho}_{\mu\nu}T^{\nu}_{\rho} = 0, \quad (3.59)$$

which in our case reads

$$\dot{T}^0_0 + \frac{\dot{\tau}}{\tau}(T^0_0 - T^1_1) = 0. \quad (3.60)$$

We have

$$T^0_0 = mS - F(I, J) + 2\left(\frac{d\Psi}{d\Upsilon} + \lambda_1 F_1\right)\dot{\varphi}^2 \\ - \psi(\Upsilon) - \lambda_1 F_1(I, J)\Upsilon + \varepsilon \\ = mS - F(I, J) + 2\frac{C_s}{\tau}\dot{\varphi} - \psi(\Upsilon)\lambda_1 F_1(I, J)\Upsilon + \varepsilon,$$

which gives

$$\dot{T}^0_0 = m\dot{S} - (\mathcal{D}\dot{S} + \mathcal{G}\dot{P}) - 2\frac{C_s\dot{\varphi}\dot{\tau}}{\tau^2} + \dot{\varepsilon}. \quad (3.61)$$

On the other hand,

$$T^0_0 - T^1_1 = mS - (\mathcal{D}S + \mathcal{G}P) + 2\frac{C_s}{\tau}\dot{\varphi} + \varepsilon + p. \quad (3.62)$$

Putting (3.61) and (3.62) into (3.60), one finds

$$(m - \mathcal{D}\dot{S}_0 - \mathcal{G}\dot{P}_0 + (\varepsilon + p)\dot{\tau} + \dot{\tau}\dot{\varepsilon} = 0, \quad (3.63)$$

where $S_0 = \tau S$ and $P_0 = \tau P$. From (3.37a) and (3.37b), we have $(m - \mathcal{D})\dot{S}_0 - \mathcal{G}\dot{P}_0 = 0$. Further taking into account the equation of state, i.e., $p = \zeta\varepsilon$, we find

$$\frac{d\varepsilon}{(1+\zeta)\varepsilon} + \frac{d\tau}{\tau} = 0, \quad (3.64)$$

with the solutions

$$\varepsilon = \frac{\varepsilon_0}{\tau^{1+\zeta}}, \quad p = \frac{\zeta\varepsilon_0}{\tau^{1+\zeta}}, \quad (3.65)$$

where ε_0 is the integration constant. Note that the relation (3.65) holds for any combination of the material field Lagrangian, e.g., spinor or scalar, or interacting spinor and scalar fields. Thus we see that the right-hand side of (3.52) is a function of τ only. Then (3.52), multiplied by $2\dot{\tau}$, can be written as

$$2\dot{\tau}\ddot{\tau} = [3(\kappa(T^1_1 + T^0_0 - 2\Lambda)\tau]\dot{\tau} = \Psi(\tau)\dot{\tau}. \quad (3.66)$$

We write the solution to Eq. (3.66) in quadrature:

$$\int\frac{d\tau}{\sqrt{\int\Psi(\tau)d\tau}} = t + t_0. \quad (3.67)$$

From here on, we set $t_0 = 0$, as this gives only the shift of the initial time. Given the explicit form of $F(I, J)$, from (3.67), one finds concrete function $\tau(t)$. Once the value of τ is obtained, one can get expressions for components $\psi_j(t)$, $j = 1, 2, 3, 4$. Thus the initial systems of Einstein and Dirac equations have been completely integrated.

3.5. Invariants of Space-Time

To investigate the existence of singularity (singular point) of the gravitational case, one has to study the invariant characteristics of the space-time. In general relativity, these invariants are composed from the curvature tensor and the metric one. Contrary to the electrodynamics, where there are two invariants only ($J_1 = F_{\mu\nu}F^{\mu\nu}$ and $J_2 = *F_{\mu\nu}F^{\mu\nu}$), in 4-D Riemann space-time, there are 14 independent invariants. They are [55]

$$I_1 = R, \quad (3.68a)$$

$$I_2 = R_{\mu\nu}R^{\mu\nu}, \quad (3.68b)$$

$$I_3 = R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}, \quad (3.68c)$$

$$I_4 = *R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}, \quad (3.68d)$$

$$I_5 = R^{\alpha}_{\beta}R^{\beta}_{\mu}R^{\mu}_{\alpha}, \quad (3.68e)$$

$$I_6 = R^{\alpha\beta}R^{\mu\nu}R_{\alpha\mu\beta\nu}, \quad (3.68f)$$

$$I_7 = R^{\alpha\beta}R^{\mu\nu}*R_{\alpha\mu\beta\nu}, \quad (3.68g)$$

$$I_8 = R^{\alpha\beta\mu\nu}R_{\alpha\beta\sigma\rho}R^{\sigma\rho}_{\mu\nu}, \quad (3.68h)$$

$$I_9 = *R^{\alpha\beta\mu\nu}R_{\alpha\beta\sigma\rho}R^{\sigma\rho}_{\mu\nu}, \quad (3.68i)$$

$$I_{10} = R^{\beta}_{\alpha}R^{\alpha\mu}R_{\mu\nu}R^{\nu}_{\beta}, \quad (3.68j)$$

$$I_{11} = R^{\mu}_{\nu}R^{\sigma\alpha}_{\rho\mu}R^{\beta[\nu}_{\sigma\alpha}R^{\rho]}_{\beta}, \quad (3.68k)$$

$$I_{12} = R^{\mu}_{\nu}*R^{\sigma\alpha}_{\rho\mu}R^{\beta[\nu}_{\sigma\alpha}R^{\rho]}_{\beta}, \quad (3.68l)$$

$$I_{13} = R^{\mu\nu}_{\alpha\beta}(A^{\alpha\beta}_{\mu\nu} + R^{\alpha}_{\rho}R^{\rho}_{\alpha}R^{\sigma}_{\eta}R^{\eta}_{\mu}\delta^{\beta}_{\nu}), \quad (3.68m)$$

$$I_{14} = *R^{\mu\nu}_{\alpha\beta}A^{\alpha\beta}_{\mu\nu}, \quad (3.68n)$$

where $A^{\alpha\beta}_{\mu\nu} = 4R^{\alpha}_{\rho}R^{\rho}_{\sigma}R^{\sigma}_{\mu}R^{\beta}_{\nu} + 3R^{\alpha}_{\rho}R^{\rho}_{\mu}R^{\beta}_{\sigma}R^{\sigma}_{\nu}$ and $*R_{\alpha\beta\mu\nu} = \frac{1}{2}E_{\alpha\beta\sigma\rho}R^{\sigma\rho}_{\mu\nu} = \frac{1}{2}E_{\sigma\rho\mu\nu}R^{\sigma\rho}_{\alpha\beta}$, $*R^{\mu\nu}_{\alpha\beta} = \frac{1}{2}E_{\alpha\beta\sigma\rho}R^{\sigma\rho\mu\nu}$ with $E_{\alpha\beta\mu\nu} = \sqrt{-g}\varepsilon_{\alpha\beta\mu\nu}$ and $E^{\alpha\beta\mu\nu} = \frac{-1}{\sqrt{-g}}\varepsilon^{\alpha\beta\mu\nu}$. Here, $\varepsilon_{\alpha\beta\mu\nu}$ is the totally antisymmetric Levi–Civita tensor with $\varepsilon_{0123} = 1$. Instead of analyzing all 14 invariants mentioned above, one can confine this study to only three, namely, the scalar curvature $I_1 = R$, $I_2 = R_{\mu\nu}R^{\mu\nu}$, and the Kretschmann scalar $I_3 = R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$ [56, 57]. At any regular space-time point, these three invariants, $I - 1$, I_2, and I_3 should be finite. Let us rewrite these invariants in detail.

For the Bianchi I metric, one finds the scalar curvature

$$I_1 = R = -2\frac{\ddot{\tau} - \dot{a}\dot{b}c - \dot{b}\dot{c}a - \dot{c}\dot{a}b}{\tau}. \tag{3.69}$$

Since the Ricci tensor for the Bianchi I metric is diagonal, the invariant $I_2 = R_{\mu\nu}R^{\mu\nu} \equiv R^{\nu}_{\mu}R^{\mu}_{\nu}$ is a sum of squares of diagonal components of the Ricci tensor, i.e.,

$$I_2 = [(R^0_0)^2 + (R^1_1)^2 + (R^2_2)^2 + (R^3_3)^2], \tag{3.70}$$

with

$$R^0_0 = -\frac{\ddot{a}bc + a\ddot{b}c + ab\ddot{c}}{\tau}, \tag{3.71a}$$

$$R^1_1 = -\frac{\ddot{a}bc + \dot{a}\dot{b}c + \dot{a}b\dot{c}}{\tau}, \tag{3.71b}$$

$$R^2_2 = -\frac{a\ddot{b}c + \dot{a}\dot{b}c + a\dot{b}\dot{c}}{\tau}, \tag{3.71c}$$

$$R^3_3 = -\frac{ab\ddot{c} + a\dot{b}\dot{c} + \dot{a}b\dot{c}}{\tau}. \tag{3.71d}$$

Analogously, for the Kretschmann scalar in this case we have $I_3 = R^{\mu\nu}_{\alpha\beta}R^{\alpha\beta}_{\mu\nu}$, a sum of squared components of all nontrivial $R^{\mu\nu}_{\alpha\beta}$:

$$\begin{aligned} I_3 &= 4[(R^{01}_{01})^2 + (R^{02}_{02})^2 + (R^{03}_{03})^2 \\ &+ (R^{12}_{12})^2 + (R^{23}_{23})^2 + (R^{31}_{31})^2] = \frac{4}{\tau^2}[(\ddot{a}bc)^2 \\ &+ (a\ddot{b}c)^2 + (ab\ddot{c})^2 + (\dot{a}\dot{b}c)^2 + (\dot{a}b\dot{c})^2 + (a\dot{b}\dot{c})^2], \\ \tau &= abc. \end{aligned} \tag{3.72}$$

It was established earlier that the metric functions a, b, c and their derivatives are in functional dependence with τ. Therefore, in view of (3.53), we see that at any space-time point where $\tau = 0$, the invariants I_1, I_2, I_3 become infinity, hence the space-time becomes singular at this point.

3.6. Physical Observable Values

To build a descriptive picture of any physical theory, we need to express the results through real physical values that can be measured experimentally. In General Relativity, where we are dealing with objects in 4**D** space-time, the problem of defining the physical observable values is not a trivial one. A mathematical apparatus to calculate the physical observable values in 4**D** pseudo-Riemannian space was first introduced by Zelmanov and is referred to as the theory of *chronometric invariants* [58–60]. By chronometric invariant values, it is understood that physical observable values in the accompanying frame should be invariant with respect to the transformation of time.

Let us study this point in detail. Consider two systems of coordinates x^i and x'^i. These two systems are said to be related to the same system of reference if they obey the following relations:

$$\frac{\partial x'^i}{\partial x^0} = 0, \quad \frac{\partial x^i}{\partial x'^0} = 0. \tag{3.73}$$

The above conditions mean that the systems of coordinates belonging to the same system of reference are stationary with respect to each other. On the other hand, if two system of coordinates move relative to each other, they belong to different systems of references. Then the coordinate transformation that leaves the two systems of coordinates in the same system of reference can be written as a system of two transformations realized together: *chronometric transformations* and 3**D** *transformation*

$$x'^0 = x'^0(x^0, x^1, x^2, x^3), \tag{3.74a}$$

$$x'^i = x'^i(x^1, x^2, x^3). \tag{3.74b}$$

Quantities invariant under the group of transformations (3.74) are considered to be chronometric invariant quantities. According to the Zelmanov theorem, chronometrically invariant (physical observable) projections of 4**D** vector Q^{α} are [60]

$$u^{\alpha}Q_{\alpha} = \frac{Q_0}{\sqrt{g_{00}}}, \quad P^i_{\alpha}Q^{\alpha} = Q^i, \tag{3.75}$$

where u^{α} is the 4**D** velocity with $u_{\alpha}u^{\alpha} = 1$ and $P_{\alpha\beta}$ is the projection operator:

$$P_{\alpha\beta} = g_{\alpha\beta} - u_{\alpha}u_{\beta} \tag{3.76a}$$

$$P^{\alpha\beta} = g^{\alpha\beta} - u^{\alpha}u^{\beta} \tag{3.76b}$$

$$P^{\alpha}_{\beta} = \delta^{\alpha}_{\beta} - u^{\alpha}u_{\beta} \tag{3.76c}$$

$$P^{\alpha}_{\lambda}P^{\lambda}_{\beta} = P^{\alpha}_{\beta}. \tag{3.76d}$$

To study the role of the nonlinear spinor and scalar fields in the formation of configurations with localized energy density and limited total energy, spin, and charge of the spinor field, we first define the spinor current and spin tensor.

Using the solutions obtained, one can write the components of spinor current:

$$j^{\mu} = \bar{\psi}\gamma^{\mu}\psi. \tag{3.77}$$

Taking into account that $\bar{\psi} = \psi^{\dagger}\bar{\gamma}^{0}$, where $\psi^{\dagger} = (\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*)$ and $\psi_j = V_j/\sqrt{\tau}$, $j = 1, 2, 3, 4$ for the components of spin current, we write

$$j^0 = \frac{1}{\tau}[V_1^*V_1 + V_2^*V_2 + V_3^*V_3 + V_4^*V_4], \tag{3.78a}$$

$$j^1 = \frac{1}{a\tau}[V_1^*V_4 + V_2^*V_3 + V_3^*V_2 + V_4^*V_1], \tag{3.78b}$$

$$j^2 = \frac{1}{b\tau}[V_1^*V_4 - V_2^*V_3 + V_3^*V_2 - V_4^*V_1], \tag{3.78c}$$

$$j^3 = \frac{1}{c\tau}[V_1^*V_3 - V_2^*V_4 + V_3^*V_1 - V_4^*V_2]. \tag{3.78d}$$

The component j^0 defines the charge density of spinor field that has the following chronometric-invariant form

$$\rho = (j_0 j^0)^{1/2}. \tag{3.79}$$

Note that the definition of chronometric invariant values adopted here differs from the one suggested by Zelmanov. In our case, we simply underline the fact that the experimentalist measures ρ, not j^0. So, being the physical observable value, ρ can be termed as chronometrically invariant. The total charge of the spinor field is defined as

$$Q = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\rho\sqrt{-{}^3g}\,dxdydz. \tag{3.80}$$

Since $\rho = \rho(t)$ and $\sqrt{-{}^3g} = \tau(t)$, for the charge Q to make any sense, we should integrate it for any finite range by x, y, and z, and then normalize it to unity.

Let us consider the spin tensor [49]

$$S^{\mu\nu,\varepsilon} = \frac{1}{4}\bar{\psi}\{\gamma^{\varepsilon}\sigma^{\mu\nu} + \sigma^{\mu\nu}\gamma^{\varepsilon}\}\psi. \tag{3.81}$$

We write the components $S^{ik,0}$ ($i, k = 1, 2, 3$), defining the spatial density of spin vector explicitly. From (3.81) we have

$$S^{ij,0} = \frac{1}{4}\bar{\psi}\{\gamma^0\sigma^{ij} + \sigma^{ij}\gamma^0\}\psi = \frac{1}{2}\bar{\psi}\gamma^0\sigma^{ij}\psi, \tag{3.82}$$

which defines the projection of spin vector on the k axis. Here, i, j, k takes the value 1, 2, 3, and $i \neq j \neq k$. Thus, for the projection of spin vectors on the X, Y, and Z axes, we find

$$S^{23,0} = \frac{1}{2bc\tau}[V_1^*V_2 + V_2^*V_1 + V_3^*V_4 + V_4^*V_3], \tag{3.83a}$$

$$S^{31,0} = \frac{-i}{2ca\tau}[V_1^*V_2 - V_2^*V_1 + V_3^*V_4 - V_4^*V_3], \tag{3.83b}$$

$$S^{12,0} = \frac{1}{2ab\tau}[V_1^*V_1 - V_2^*V_2 + V_3^*V_3 - V_4^*V_4]. \tag{3.83c}$$

The chronometric invariant spin tensor takes the form

$$S_{\text{ch}}^{ij,0} = (S_{ij,0}S^{ij,0})^{1/2}, \tag{3.84}$$

and the projection of the spin vector on the k axis is defined by

$$S_k = \int_{-\infty}^{\infty} S_{\text{ch}}^{ij,0}\sqrt{-{}^3g}\,dxdydz. \tag{3.85}$$

Once we solve the spinor field equations, inserting them into (3.78)–(3.80), (3.83), and (3.85), we find the corresponding physical observable quantities. We return to these definitions in the next section, where the exact solutions of the spinor field will be given for some concrete choice of nonlinearity.

3.7. Λ Term and Its Role in the Evolution of the Universe

To allow a steady-state cosmological solution to the gravitational field equations, Einstein [41, 42] introduced a fundamental constant, known as the cosmological constant or Λ term, into the system. Soon after E. Hubble had experimentally established that the Universe is expanding, Einstein returned to the original form of his equations citing his temporary modification of them as the biggest blunder of his life. The Λ term made a temporary comeback in the late 60's. Finally, after the pioneering paper by A. Guth [61] on inflationary cosmology, researchers began to study the models with the Λ term with growing interest (an excellent review on the cosmological constant can be found in [62]). In this paper, a negative Λ corresponds to the universal repulsive force, while a positive one gives an additional gravitational force. Note that a negative Λ is often considered to be a form of dark energy. To see the role of a Λ term in general, let us study the system of Einstein equation once again. In doing so, we first rewrite the equation for τ as

$$\ddot{\tau} = \frac{3}{2}\kappa(T_1^1 + T_0^0)\tau - 3\Lambda\tau. \tag{3.86}$$

On the other hand, from the Bianchi identity $G^{\nu}_{\mu;\nu} = 0$, we find

$$\dot{T}_0^0 + \frac{\dot{\tau}}{\tau}(T_0^0 - T_1^1) = 0. \quad (3.87)$$

After a little manipulation from (3.86) and (3.87), one finds the following expression for T_0^0:

$$\kappa T_0^0 = 3H^2 + \Lambda - C_{00}/\tau^2, \quad (3.88)$$

where the definition of the generalized Hubble constant $\dot{\tau}/\tau = 3H$ has been used. Let us now study the relation (3.88) in detail. Consider the case where $\Lambda = 0$. At the moment when the expansion rate is zero (this might be at a time prior to the "Big Bang," or a time in the far future when the universe has ceased to expand), we have $H = 0$. Then the nonnegativity of T_0^0 suggests that $C_{00} \le 0$. Let us now consider another case when τ is large enough for the term $1/\tau^2$ to be omitted. As we know, T_0^0 (note that T_0^0 is the energy density of usual matter) decreases with the increase of τ. If τ is big enough for T_0^0 to be neglected, from (3.88) we find

$$3H^2 + \Lambda \longrightarrow 0.$$

This means that for τ to be infinitely large, λ should be non-positive. In case of $\Lambda = 0$, we find that beginning from some value of τ, the rate of expansion of the Universe becomes trivial, i.e., the universe does not expand with time, whereas for $\Lambda < 0$ the expansion process continues forever. As far as positive Λ is concerned, its presence imposes some restriction on the energy density T_0^0, namely, T_0^0 can never be small enough to be ignored. For the material field in question, as will be shown later, there exists some upper limit for τ as well (note that τ is essentially nonnegative, i.e., bounded from below). Thus we see that a positive Λ, depending on the choice of parameters, can give rise to an oscillatory mode of expansion. Thus we come to the following conclusion:

Let T_μ^ν be the source of the Einstein field equation; T_0^0 is the energy density and T_1^1, T_2^2, T_3^3 are the principal pressure, and $T_1^1 = T_2^2 = T_3^3$. An ever-expanding BI Universe may be obtained if and only if the Λ term is negative (describes a repulsive force and can be viewed as a form of dark energy) and is introduced into the system as in (3.25).

3.8. Cosmological Singularity and the Dominant Energy Condition

Recalling that a timelike geodesic is a world line for a particle moving without acceleration in the proper reference system, we define the following:

A spacetime is nonsingular if any timelike geodesics, or null geodesics, can be continued into the past and the future without bound, i.e., to infinite proper length for the timelike geodesics and to an infinite value of an affine parameter for the null geodesics. Such a spacetime is termed "causally, geodesically complete." The requirements on the completeness are the minimum necessary so that the spacetime does not contain a singularity. It is necessary to point out that a spacetime not satisfying these requirements, but with a singularity, does not necessarily contain points with infinite curvature or with a small hole.

From the physical point of view, of course, one ought to take as singular any spacetime in which the geodesic world line of a particle cannot be continued without bound with respect to the proper time of this particle, for such a singular spacetime would lead to a violation of conservation laws.

As applied to the cosmological problem, the Hawking–Penrose theorem reads as follows [63]:

Theorem. A space-time $\mathcal{M}$ cannot satisfy causal geodesic completeness if the GTR equations hold and if the following conditions are fulfilled:

(i) The space-time $\mathcal{M}$ does not contain closed timelike curves.

(ii) The energy condition (dominant energy condition) is satisfied at every point. The energy condition may be expressed as

$$t^\alpha t_\alpha = 1 \quad \text{implies} \quad R_{\alpha\beta} t^\alpha t^\beta \le 0. \quad (3.89)$$

With Einstein's equations

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = -\kappa T_{\alpha\beta}, \quad (3.90)$$

(3.89) becomes

$$t^\alpha t_\alpha = 1 \quad \text{implies} \quad T_{\alpha\beta} t^\alpha t^\beta \ge \frac{1}{2} T_\mu^\mu. \quad (3.91)$$

If, in an eigentetrad of $T_{\mu\nu}$, ε denotes the energy density and p_1, p_2, p_3 denote the three principal pressures, then (3.91) can be written as

$$\varepsilon + \sum_\alpha p_\alpha \ge 0; \quad (3.92a)$$

$$\varepsilon + p_\alpha \ge 0, \quad \alpha = 1, 2, 3. \quad (3.92b)$$

The weak energy condition is

$$\ell^\alpha \ell_\alpha = 0 \quad \text{implies} \quad R_{\alpha\beta} t^\alpha t^\beta \le 0, \quad (3.93)$$

which is a consequence of (3.89).

(iii) On each timelike or null geodesic γ, there is at least one point for which

$$K_{[a} R_{b]cd[e} K_{f]} K^c K^d \ne 0, \quad (3.94)$$

where K_a is the tangent to the curve γ at the given point and where the brackets on the subscripts imply antisymmetrization. If γ is timelike, we can rewrite (3.94) as

$$R_{abcd}K^cK^d \neq 0. \qquad (3.95)$$

(iv) The space-time $\mathcal{M}$ contains either (*a*) a trapped surface, (*b*) a point *P* for which the convergence of all the null geodesics through *P* changes sign somewhere to the past of *P*, or (*c*) a compact spacelike hypersurface.

In our further analysis of the results, consider specifically the dominant energy condition. As will be shown later, the regular solutions obtained by means of the nonlinear term do not always satisfy the dominant energy condition.

4. QUALITATIVE ANALYSIS OF THE RESULTS

In this section, we shall analyze the general results obtained in the previous section. In the subsections that follow, we will study the system with nonlinear spinor and nonlinear scalar fields respectively, as well with interacting spinor and scalar fields in the absence of perfect fluid and Λ term. Further, we will introduce the Λ term and then perfect fluid to determine their role in the evolution of the universe.

4.1. Nonlinear Spinor Field in BI Universe

In this subsection we study the nonlinear spinor field in BI universe. In doing so, we first consider the linear case. The reason for getting the solution to the self-consistent system of equations for the linear spinor and gravitational fields is the necessity of comparing this solution with that for the system of equations for the nonlinear spinor and gravitational fields, which permits clarification of the role of nonlinear spinor terms in the evolution of the cosmological model in question.

In this case we get explicit expressions for the components of spinor field functions and metric functions:

$$\psi_1(t) = (C_1/\sqrt{\tau}\exp[-imt], \qquad (4.1a)$$

$$\psi_2(t) = (C_2/\sqrt{\tau}\exp[-imt], \qquad (4.1b)$$

$$\psi_3(t) = (C_3/\sqrt{\tau}\exp[imt], \qquad (4.1c)$$

$$\psi_4(t) = (C_4/\sqrt{\tau}\exp[imt], \qquad (4.1d)$$

with C_1, C_2, C_3, C_4 being the integration constants. On the other hand, from (3.37a) we find

$$S = \frac{C_0}{\tau}, \qquad (4.2)$$

where C_0 is an integration constant and related to the previous ones as $C_0 = C_1^2 + C_2^2 - C_3^2 - C_4^2$. For the components of the spin current from (3.78a)–(3.78d), we find

$$j^0 = \frac{1}{\tau}[C_1^2 + C_2^2 + C_3^2 + C_4^2], \qquad (4.3a)$$

$$j^1 = \frac{2}{a\tau}[C_1C_4 + C_2C_3]\cos(2mt), \qquad (4.3b)$$

$$j^2 = \frac{2}{b\tau}[C_1C_4 - C_2C_3]\sin(2mt), \qquad (4.3c)$$

$$j^3 = \frac{2}{c\tau}[C_1C_3 - C_2C_4]\cos(2mt), \qquad (4.3d)$$

whereas, for the projection of spin vectors on the *X*, *Y* and *Z* axes, we find

$$S^{23,0} = \frac{1}{bc\tau}[C_1C_2 + C_3C_4], \qquad (4.4a)$$

$$S^{31,0} = 0, \qquad (4.4b)$$

$$S^{12,0} = \frac{1}{2ab\tau}[C_1^2 - C_2^2 + C_3^2 - C_4^2]. \qquad (4.4c)$$

From (3.80), we find the charge of the system in a volume $\mathcal{V}$

$$Q = [C_1^2 + C_2^2 + C_3^2 + C_4^2]\mathcal{V}. \qquad (4.5)$$

Thus, we see that the total charge of the system in a finite volume is always finite.

Let us now determine the function τ. In absence of perfect fluid, for the linear spinor field, we have

$$T_0^0 = mS, \quad T_1^1 = T_2^2 = T_3^3 = 0. \qquad (4.6)$$

Taking (4.6) into account, for τ we write

$$\ddot{\tau} = M - 3\Lambda\tau, \qquad (4.7)$$

with the solutions

$$\tau = \begin{cases} (1/3\Lambda)[M - q_1\sinh(\sqrt{-3\Lambda}t)], & \Lambda < 0, \\ (1/2)Mt^2 + y_1t + y_0, & \Lambda = 0, \\ (1/3\Lambda)[M - q_2\sin(\sqrt{3\Lambda}t)], & \Lambda > 0, \end{cases} \qquad (4.8)$$

where $M = \frac{3}{2}\kappa mC_0$ and y_1, y_0, q_1, q_2 are the constants. Let us now analyze the solutions obtained.

First, we study the case when $\Lambda = 0$. It can be shown that [35]

$$y_1^2 - 2My_0 = (X_1^2 + X_1X_3 + X_3^2/3 > 0. \qquad (4.9)$$

This means that the quadratic polynomial $(1/2)Mt^2 + y_1t + y_0 = 0$ possesses real roots, i.e., $\tau(t)$ in case of $\Lambda = 0$ turns into zero at $t = t_{1,2} = -y_1/M \pm \sqrt{(y_1/M)^2 - 2y_0/M}$ and the solution obtained is the singular one. At $t \longrightarrow \infty$, in this case we have

$$\tau(t) \approx \frac{3}{4}\kappa mC_0t^2, \quad a(t) \approx b(t) \approx c(t) \approx t^{2/3},$$

which leads to the conclusion about the asymptotical isotropization of the expansion process for the initially anisotropic BI space. Thus, the solution to the self-consistent system of equations for the linear spinor and gravitational fields is the singular one at the initial time. In the initial state of evolution of the field system, the expansion process of space is anisotropic, but at $t \longrightarrow \infty$, the isotropization of the expansion process takes place. As one sees, the components of spin current and projections of spin vector are singular at space-time points $t_{1,2}$, where τ vanishes.

For $\Lambda < 0$, we see that the solution is singular at $t = t_0 = (1/\sqrt{-3\Lambda})\mathrm{arcsinh}(M/q_1)$, and the isotropization of the expansion process takes place as $t \longrightarrow \infty$. Note that the isotropization process in this case is rather rapid.

For $\Lambda > 0$, we have the oscillatory solutions. Taking into account that τ is a non-negative quantity, it can be shown that the model has singular solutions at $t = (4k + 1)\pi/2\sqrt{3\Lambda}$, $k = 0, 1, 2, 3, \ldots$, with $M = q_2$. For $M > q_2$, we have τ that is always positive definite, i.e., the solutions obtained are regular at each space-time point.

Let us now go back to the nonlinear case. We consider the following forms of nonlinear term: (i) $F = F(I)$; (ii) $F = F(J)$; (iii) $F = F(K_\pm)$ with $K_\pm = I \pm J$.

(i) Let us consider the case when $F = F(I)$. From (3.37a), we find

$$S = \frac{C_0}{\tau}, \quad C_0 = \mathrm{const}. \tag{4.10}$$

Note that in this case we denote the constants in the same way as we did for the linear case, but the constants in these cases are not necessarily identical. Spinor field equations in this case read

$$\dot{V}_1 + i(m - \mathcal{D}_1)V_1 = 0, \tag{4.11a}$$

$$\dot{V}_2 + i(m - \mathcal{D}_1)V_2 = 0, \tag{4.11b}$$

$$\dot{V}_3 - i(m - \mathcal{D}_1)V_3 = 0, \tag{4.11c}$$

$$\dot{V}_4 - i(m - \mathcal{D}_1)V_4 = 0. \tag{4.11d}$$

As in the considered case $F = F(S)$, from (4.10) it follows that $F(I)$ and $\mathcal{D}_1$ are functions of τ only. Taking this fact into account, we get explicit expressions for the components of spinor field functions

$$\psi_1(t) = (C_1/\sqrt{\tau}\exp[-i\int(m - \mathcal{D}_1)dt], \tag{4.12a}$$

$$\psi_2(t) = (C_2/\sqrt{\tau}\exp[-i\int(m - \mathcal{D}_1)dt], \tag{4.12b}$$

$$\psi_3(t) = (C_3/\sqrt{\tau}\exp[i\int(m - \mathcal{D}_1)dt], \tag{4.12c}$$

$$\psi_4(t) = (C_4/\sqrt{\tau}\exp[i\int(m - \mathcal{D}_1)dt], \tag{4.12d}$$

with C_1, C_2, C_3, C_4 being the integration constants, and related to C_0 as $C_0 = C_1^2 + C_2^2 - C_3^2 - C_4^2$. For the components of the spin current from (3.78a)–(3.78d), we find

$$j^0 = \frac{1}{\tau}[C_1^2 + C_2^2 + C_3^2 + C_4^2], \tag{4.13a}$$

$$j^1 = \frac{2}{a\tau}[C_1C_4 + C_2C_3]\cos[2\int(m - \mathcal{D}_1)dt], \tag{4.13b}$$

$$j^2 = \frac{2}{b\tau}[C_1C_4 - C_2C_3]\sin[2\int(m - \mathcal{D}_1)dt], \tag{4.13c}$$

$$j^3 = \frac{2}{c\tau}[C_1C_3 - C_2C_4]\cos[2\int(m - \mathcal{D}_1)dt], \tag{4.13d}$$

whereas, for the projection of spin vectors on the X, Y, and Z axes, we find

$$S^{23,0} = \frac{1}{bc\tau}[C_1C_2 + C_3C_4], \tag{4.14a}$$

$$S^{31,0} = 0, \tag{4.14b}$$

$$S^{12,0} = \frac{1}{2ab\tau}[C_1^2 - C_2^2 + C_3^2 - C_4^2]. \tag{4.14c}$$

We now study the equation for τ in detail, choosing the nonlinear spinor term as $F = \lambda I^{(n/2)} = \lambda S^n$, with λ being the coupling constant and $n > 1$. In this case, for τ, one gets

$$\ddot{\tau} = (3/2)\kappa[mC_0 + \lambda(n-2)C_0^n/\tau^{n-1}] - 3\Lambda\tau. \tag{4.15}$$

The first integral of the foregoing equation takes the form

$$\dot{\tau}^2 = 3\kappa[mC_0\tau - \lambda C_0^n/\tau^{n-2} + y_1^2] - 3\Lambda\tau^2. \tag{4.16}$$

Here, y_1^2 is the integration constant that is positively defined and connected with the constants X_i as $y_1^2 = (X_1^2 + X_1X_3 + X_3^2)/9\kappa C_0$ [35]. The sign C_0 is determined by the positivity of the energy-density T_0^0 of the linear spinor field, i.e.,

$$T_0^0 = mC_0/\tau > 0. \tag{4.17}$$

It is obvious from (4.17) that $C_0 > 0$. Now one can write the solution to the equation (4.16) in quadratures:

$$\int\frac{\tau^{(n-2)/2}d\tau}{\sqrt{\kappa[mC_0\tau^{n-1} + y_1^2\tau^{n-2} - \lambda C_0^n] - \Lambda\tau^n}} = \sqrt{3}t. \tag{4.18}$$

The constant of integration in (4.18) has been taken to be zero, as it only gives the shift of the initial time. Let us study the properties of solution obtained for different choice of n, λ, and Λ. First we study the case with $\Lambda = 0$.

For $n > 2$ from (4.18), one gets

$$\tau(t)|_{t\to\infty} \approx (3/4)\kappa mC_0t^2. \tag{4.19}$$

This leads to the conclusion about isotropization of the expansion process of the BI space-time. It should be remarked that the isotropization takes place if and only if the spinor field equation contains the massive term (cf. the parameter m in (4.18)). This is not the case for a massless spinor field, since from (4.18) we get

$$\tau(t)|_{t \to \infty} \approx \sqrt{3\kappa C_0 y_1^2}\, t. \tag{4.20}$$

Inserting (4.20) into (3.49a)–(3.49c), one comes to the conclusion that the functions $a(t)$, $b(t)$, and $c(t)$ are different.

Let us consider the properties of solutions to Eq. (4.15) when $t \longrightarrow 0$. For $\lambda < 0$, from (4.18) we get

$$\tau(t) = [(3/4)n^2\kappa|\lambda|C_0^n]^{1/n} t^{2/n} \longrightarrow 0, \tag{4.21}$$

i.e., solutions are singular. For $\lambda > 0$, from (4.18) it follows that $\tau = 0$ cannot be reached for any value of t, as in this case the denominator of the integrand in (4.18) becomes imaginary. This means that for $\lambda > 0$, there exist regular solutions to the previous system of equations [34]. The absence of the initial singularity in the considered cosmological solution appears to be consistent with the violation for $\lambda > 0$ of the dominant energy condition in the Hawking–Penrose theorem.

A particular choice of spinor field nonlinearity (generated by self-action) gives rise to a singularity-free BI cosmological model even in absence of the Λ *term but at the expense of a broken dominant energy condition in the Hawking–Penrose theorem.*

Proof. To prove that in the case considered the dominant energy condition is violated, we rewrite it in the following form:

$$T_{00} + T_{11} + T_{22} + T_{33} \geq 0, \tag{4.22a}$$

$$T_{00} + T_{11} \geq 0, \tag{4.22b}$$

$$T_{00} + T_{22} \geq 0, \tag{4.22c}$$

$$T_{00} + T_{33} \geq 0, \tag{4.22d}$$

which for the BI metric read

$$T_0^0 \geq T_1^1 a^2 + T_2^2 b^2 + T_3^3 c^2, \tag{4.23a}$$

$$T_0^0 \geq T_1^1 a^2, \tag{4.23b}$$

$$T_0^0 \geq T_2^2 b^2, \tag{4.23c}$$

$$T_0^0 \geq T_3^3 c^2. \tag{4.23d}$$

Let us go back to the energy density of spinor field, which in this case coincides with the total energy density. From

$$T_0^0 = \frac{mC_0}{\tau} - \frac{\lambda C_0^n}{\tau^n}, \tag{4.24}$$

it follows that at

$$\tau^{n-1} < \frac{\lambda C_0^{n-1}}{m}, \tag{4.25}$$

the energy density of spinor field becomes negative. On the other hand, we have

$$T_1^1 = T_2^2 = T_3^3 = \frac{\lambda(n-1)C_0^n}{\tau^n} > 0 \tag{4.26}$$

for any non-negative value of τ. Thus, we see all four conditions in (4.23) violated, i.e., the absence of initial singularity in the considered cosmological solution appears to be consistent with the violation of the dominant energy condition in the Hawking–Penrose theorem.

Let us consider the **Heisenberg–Ivanenko** equation [64] setting $n = 2$ in (4.15). In this case, the equation for $\tau(t)$ does not contain the nonlinear term and its solution coincides with that of the linear one. The spinor field functions in this case are written as follows:

$$V_1 = \frac{C_1}{\sqrt{\tau}} \exp[-imt] Z^{4i\lambda C_0/B}, \tag{4.27a}$$

$$V_2 = \frac{C_2}{\sqrt{\tau}} \exp[-imt] Z^{4i\lambda C_0/B}, \tag{4.27b}$$

$$V_3 = \frac{C_3}{\sqrt{\tau}} \exp[imt] Z^{-4i\lambda C_0/B}, \tag{4.27c}$$

$$V_4 = \frac{C_4}{\sqrt{\tau}} \exp[imt] Z^{-4i\lambda C_0/B}, \tag{4.27d}$$

where $Z = \frac{(t-t_1)}{(t-t_2)}$, $B = M(t_1 - t_2)$, and $t_{1,2} = -y_1/M \pm \sqrt{(y_1/M)^2 - 2y_0/M}$ are the roots of the quadratic equation $Mt^2 + 2y_1 t + 2y_0 = 0$. As in the linear case, the obtained solution is singular at initial time and asymptotically isotropic as $t \longrightarrow \infty$.

We now study the properties of solutions to Eq. (4.15) for $1 < n < 2$. In this case, it is convenient to present the solution (4.18) in the form

$$\int \frac{d\tau}{\sqrt{m\tau - \lambda\tau^{2-n} C_0^{n-1} + y_1^2}} = \sqrt{3\kappa C_0}\, t. \tag{4.28}$$

As $t \longrightarrow \infty$, from (4.28) we get the equality (4.19), leading to the isotropization of the expansion process. If $m = 0$ and $\lambda > 0$, $\tau(t)$ lies on the interval

$$0 \leq \tau(t) \leq (y_1^2/\lambda C_0^{n-1})^{1/(2-n)}.$$

If $m = 0$ and $\lambda < 0$, the relation (4.28) at $t \longrightarrow \infty$ leads to the equality

$$\tau(t) \approx [(3/4)n^2\kappa|\lambda|C_0^n]^{1/n} t^{2/n}. \tag{4.29}$$

Substituting (4.29) into (3.49) and taking into account that at $t \longrightarrow \infty$,

$$\int \frac{dt}{\tau} \approx \frac{n(3\kappa|\lambda|n^2 C_0^n)^{1/n}}{(n-2)2^{2/n}} t^{-2/n+1} \longrightarrow 0,$$

due to $-2/n + 1 < 0$, we obtain

$$a(t) \sim b(t) \sim c(t) \sim [\tau(t)]^{1/3} \sim t^{2/3n} \longrightarrow \infty. \quad (4.30)$$

This means that the solution obtained tends to the isotropic one. In this case, the isotropization is provided not by the massive parameter, but by the degree n in the term $F = \lambda S^n$. Equation (4.28) implies

$$\tau(t)|_{t \to 0} \approx \sqrt{3\kappa C_0 y_1^2} t \longrightarrow 0, \quad (4.31)$$

which means the solution obtained is initially singular. Thus, for $1 < n < 2$, there exist only singular solutions at the initial time. At $t \longrightarrow \infty$, the isotropization of the expansion process of the BI space takes place both for $m \neq 0$ and for $m = 0$.

Finally, let us study the properties of the solution to the equation (4.15) for $0 < n < 1$. In this case, we use the solution in the form (4.28). Since now $2 - n > 1$, then with the increasing of $\tau(t)$ in the denominator of the integrand in (4.28), the second term $\lambda\tau^{2-n}C_0^{n-1}$ increases faster than the first one. Therefore the solution describing the space expansion can be possible only for $\lambda < 0$. In this case at $t \longrightarrow \infty$, for $m = 0$ as well as for $m \neq 0$, one can get the asymptotic representation (4.29) of the solution. This solution, as for the choice $1 < n < 2$, provides asymptotically isotropic expansion of the BI space-time. For $t \longrightarrow 0$, in this case we shall get only the singular solution of the form (4.31).

For a nonzero Λ term, we study the following situations depending on the sign of Λ and λ.

case (i). $\Lambda = -\varepsilon^2 < 0$, $\lambda > 0$. In this case for $n > 2$ and $t \longrightarrow \infty$, we find

$$\tau(t) \approx e^{\sqrt{3}\varepsilon t}. \quad (4.32)$$

Thus we see that the asymptotic behavior of τ does not depend on n and is defined by the Λ term. From (3.49), it is obvious that asymptotic isotropization takes place.

From (4.18), it also follows that τ cannot be zero at any moment, since the integrand turns out to be imaginary as τ approaches zero. Thus, the solution obtained is a nonsingular one, thanks to spinor field nonlinearity, and is asymptotically isotropic. As has been noted earlier, the absence of initial singularity in the considered cosmological model results in the violation of the dominant energy condition.

case (ii). $\Lambda > 0$ and $\lambda > 0$. For $n > 2$, (4.18) admits only nonsingular oscillating solutions, since $\tau > 0$ and is bounded from above. Consider the case with $n = 4$ and for simplicity set $m = 0$. Then, from (4.18), one gets

$$\tau(t) = \frac{1}{\sqrt{2\Lambda}}[\kappa C_0 \tau_0 + \sqrt{\kappa^2 C_0^2 \tau_0^2 + 4\Lambda\lambda C_0^4} \sin 2\sqrt{3\Lambda} t]^{1/2}. \quad (4.33)$$

case (iii). $\Lambda < 0$ and $\lambda < 0$. In this case we find

$$\lim_{t \to 0} \tau \approx [-3\lambda n^2 C_0^n t^2/4]^{1/n}, \quad (4.34)$$

and

$$\lim_{t \to \infty} \tau \approx e^{\sqrt{3\Lambda} t}, \quad (4.35)$$

which means that the solutions obtained are initially singular, and asymptotic isotropization takes place as t approaches ∞.

case (iv). $\Lambda > 0$ and $\lambda < 0$. In this case, for the initial value of t we find the solution that coincides with (4.34), i.e.,

$$\lim_{t \to 0} \tau \approx [-3\lambda n^2 C_0^n t^2/4]^{1/n}. \quad (4.36)$$

On the other hand, since $\Lambda > 0$, τ should be bounded from above, otherwise the integrand becomes imaginary. Thus, beginning from some $t = t_0$, where t_0 is big enough, we can present the solution in the form

$$\lim_{t \geq t_0} \tau \approx \sin\sqrt{3\Lambda} t. \quad (4.37)$$

Thus, we see that in this case the solution is singular and oscillatory. After the analysis done above, we conclude that the nonlinear term dominates the initial stage of evolution, while the Λ term dominates the asymptotic stage.

(ii) We study the system when $F = F(J)$, which means that in the case considered $\mathcal{D} = 0$. Let us note that, in the unified nonlinear spinor theory of Heisenberg, the massive term remains absent, and according to Heisenberg, the particle mass should be obtained as a result of quantization of spinor prematter [65]. In the nonlinear generalization of classical field equations, the massive term does not possess the significance that it possesses in the linear one, as it by no means defines the total energy (or mass) of the nonlinear field system. Thus, without losing the generality, we can consider massless spinor field putting $m = 0$. Then, from (3.37), one gets

$$P(t) = \frac{D_0}{\tau}, \quad D_0 = \text{const}. \quad (4.38)$$

The system of spinor field equations in this case reads

$$\dot{V}_1 - \mathcal{G}_1 V_3 = 0, \quad (4.39a)$$

$$\dot{V}_2 - \mathcal{G}_1 V_4 = 0, \quad (4.39b)$$

$$\dot{V}_3 + \mathcal{G}_1 V_1 = 0, \quad (4.39c)$$

$$\dot{V}_4 + \mathcal{G}_1 V_2 = 0. \tag{4.39d}$$

Defining $U(\sigma) = V(t)$, where $\sigma = \int \mathcal{G}_1 dt$, we rewrite (4.39) as

$$U_1' - U_3 = 0, \tag{4.40a}$$

$$U_2' - U_4 = 0, \tag{4.40b}$$

$$U_3' + U_1 = 0, \tag{4.40c}$$

$$U_4' + U_2 = 0, \tag{4.40d}$$

where prime (') denotes differentiation with respect to σ. Differentiating the first equation of system (4.40) and taking into account the third one, we get

$$U_1'' + U_1 = 0, \tag{4.41}$$

which leads to the solution

$$U_1 = D_1 e^{i\sigma} + iD_3 e^{-i\sigma},$$

$$U_3 = iD_1 e^{i\sigma} + D_3 e^{-i\sigma}.$$

Analogically for U_2 and U_4, one gets

$$U_2 = D_2 e^{i\sigma} + iD_4 e^{-i\sigma},$$

$$U_4 = iD_2 e^{i\sigma} + D_4 e^{-i\sigma},$$

where D_i are the constants of integration. Finally, we can write

$$\psi_1 = \frac{1}{\sqrt{\tau}}(D_1 e^{i\sigma} + iD_3 e^{-i\sigma}), \tag{4.42a}$$

$$\psi_2 = \frac{1}{\sqrt{\tau}}(D_2 e^{i\sigma} + iD_4 e^{-i\sigma}), \tag{4.42b}$$

$$\psi_3 = \frac{1}{\sqrt{\tau}}(iD_1 e^{i\sigma} + D_3 e^{-i\sigma}), \tag{4.42c}$$

$$\psi_4 = \frac{1}{\sqrt{\tau}}(iD_2 e^{i\sigma} + D_4 e^{-i\sigma}). \tag{4.42d}$$

Putting (4.42) into the expressions (4.38), one comes to

$$D_0 = 2(D_1^2 + D_2^2 - D_3^2 - D_4^2).$$

For the components of the spin current from (3.78), we find

$$j^0 = \frac{2}{\tau}[D_1^2 + D_2^2 + D_3^2 + D_4^2], \tag{4.43a}$$

$$j^1 = \frac{4}{a\tau}[D_2 D_3 + D_1 D_4]\cos[2\int \mathcal{G}_1 dt], \tag{4.43b}$$

$$j^2 = \frac{4}{b\tau}[D_2 D_3 - D_1 D_4]\sin[2\int \mathcal{G}_1 dt], \tag{4.43c}$$

$$j^3 = \frac{4}{c\tau}[D_1 D_3 - D_2 D_4]\cos[2\int \mathcal{G}_1)dt], \tag{4.43d}$$

whereas, for the projection of spin vectors on the X, Y, and Z axes, we find

$$S^{23,0} = \frac{2}{bc\tau}[D_1 D_2 + D_3 D_4], \tag{4.44a}$$

$$S^{31,0} = 0, \tag{4.44b}$$

$$S^{12,0} = \frac{1}{2ab\tau}[D_1^2 - D_2^2 + D_3^2 - D_4^2]. \tag{4.44c}$$

We now choose the nonlinear term as $F = \lambda J^n = \lambda P^{2n}$, with λ being the coupling constant. In this case, for the components of energy-momentum tensor, we get

$$T_0^0 = -\lambda P^n, \quad T_1^1 = T_2^2 = T_3^3 = \lambda(2n-1)P^n. \tag{4.45}$$

In account of (4.45) and (4.38) from (3.52) for τ, we obtain

$$\ddot{\tau} = (3/2)\kappa\lambda(2n-2)D_0^{2n}\tau^{1-2n} - 3\Lambda\tau, \tag{4.46}$$

with the solutions in quadrature

$$\int \frac{d\tau}{\sqrt{y_1^2 - \kappa\lambda D_0^{2n}\tau^{2-2n} - \Lambda\tau^2}} = \sqrt{3}t, \tag{4.47}$$

with y_1^2 being the integration constant such that $y^2 = X_1^2 + X_1X_3 + X_3^2 > 0$.

Let us now analyze the solution obtained here. As one can see, the case $n = 1$ is the linear one. First we consider the case when $\lambda < 0$. Depending on the value of n and Λ, we obtain the following results:

$$\tau(t)|_{t\to 0} \approx \begin{cases} [\sqrt{-3\kappa\lambda}nD_0^n t]^{1/n}, & n > 1, \\ \sqrt{3}y_1 t, & n < 1 \end{cases} \tag{4.48}$$

$$\tau(t)|_{t\to\infty} \approx \begin{cases} [\sqrt{-3\kappa\lambda}nD_0^n t]^{1/n}, & n < 1, \quad \Lambda = 0, \\ \sqrt{3}y_1 t, & n > 1, \quad \Lambda = 0 \end{cases} \tag{4.49}$$

and

$$\tau|_{t\to\infty} \approx e^{\sqrt{-3\Lambda}t}, \quad \Lambda < 0, \tag{4.50}$$

whereas for $\Lambda > 0$, it is bounded from above. Thus we see that for the term F considered with $\lambda < 0$, the solution is initially singular and in the absence of the Λ term, the space-time is asymptotically anisotropic for $n > 1$, while in case of $n < 1$, the isotropization process takes place. Introduction of a negative Λ term gives rise to the asymptotic isotropization process, while the positive Λ term provides oscillatory solutions.

Let us now see what happens to the system when the coupling constant is positive. As one can see from (4.47), for $\lambda > 0$ and $n > 1$, $\tau = 0$ cannot be reached at any moment t, as in this case the integrand turns out to be imaginary. Thus, the solution is always regular. But, as follows from (4.45), the energy density in this case is negative while the pressure components are positive, which is in violation of the energy dominant condition. For $n < 1$, we obtain solutions analogical to those for $\lambda < 0$.

(iii) In this case, we study $F = F(I, J)$. Choosing

$$F = F(K_\pm), \quad K_+ = I + J = I_V = -I_A, \quad K_- = I - J = I_T, \tag{4.51}$$

in the case of massless NLSF, we find

$$\mathcal{D} = 2SF_{K_\pm}, \quad \mathcal{G} = \pm 2PF_{K_\pm}, \quad F_{K_\pm} = dF/dK_\pm.$$

Putting them into (3.37), we find

$$S_0^2 \pm P_0^2 = D_\pm. \tag{4.52}$$

Choosing $F = \lambda K_\pm^n$ for the components of the energy-momentum tensor, we get

$$T_0^0 = -\lambda K_\pm^n, \quad T_1^1 = T_2^2 = T_3^3 = \lambda(2n-1)K_\pm^n. \tag{4.53}$$

In view of (4.53) and (4.52), from (3.52) we obtain

$$\ddot{\tau} = (3/2)\kappa\lambda(2n-2)D_\pm^n \tau^{1-2n} - 3\Lambda\tau, \tag{4.54}$$

with the solution

$$\int \frac{d\tau}{\sqrt{y_1^2 - \kappa\lambda D_\pm^n \tau^{2-2n} - \Lambda\tau^2}} = \sqrt{3}t, \tag{4.55}$$

with $y_1^2 = X_1^2 + X_1X_3 + X_3^2$. From the similarity of the equations (4.47) and (4.55), one comes to the analogical conclusion made for the case when $F = \lambda J^n$, i.e., for a negative coupling constant ($\lambda < 0$), the solution is initially singular and in the absence of the Λ term, the space-time is asymptotically anisotropic for $n > 1$, while in case of $n < 1$, the isotropization process takes place; in case of $\lambda > 0$, we obtain regular solutions with breaking of the dominant energy condition; introduction of a negative Λ term gives rise to the asymptotic isotropization process, while the positive Λ term provides oscillatory solutions. Note that one comes to an analogous conclusion choosing $F = \lambda S^{2n}P^{2n}$.

4.2. Nonlinear Scalar Field in the Absence of a Spinor Field

Let us consider the nonlinear scalar field in absence of a spinor field. As a nonlinear scalar field equation, we choose the Born–Infeld equation, given by the Lagrangian [66]

$$\Psi(\Upsilon) = -\frac{1}{\sigma}(1 - \sqrt{1 + \sigma\Upsilon}), \tag{4.56}$$

with $\Upsilon = \varphi_\alpha\varphi^\alpha$, and σ being the parameter of nonlinearity. From (4.56), we also have

$$\lim_{\sigma \to 0}\Psi(\Upsilon) = \frac{1}{2}\Upsilon\ldots \tag{4.57}$$

Inserting (4.56) into (3.24) for the scalar field, we obtain the equation

$$\dot{\varphi}(t) = \frac{2C_s}{\sqrt{\tau^2 - 4\sigma C_s^2}}, \tag{4.58}$$

that gives

$$\Upsilon = (\dot{\varphi})^2 = \frac{4C_s^2}{\tau^2 - 4\sigma C_s^2}. \tag{4.59}$$

From (4.59), it follows that

$$\Upsilon|_{\tau \to 0} = -\frac{1}{\sigma}, \quad \Upsilon|_{\tau \to +\infty} = 0, \tag{4.60}$$

showing that Υ is kinklike.

For the case considered in this subsection, we have

$$T_{\text{sc}0}^0 = 2\Upsilon\frac{d\Psi}{d\Upsilon} - \Psi = \frac{1}{\sigma}(1 - \sqrt{1 - 4\sigma C_s^2/\tau^2}) \tag{4.61}$$

and

$$T_{\text{sc}1}^1 = T_{\text{sc}2}^2 = T_{\text{sc}3}^3 = -\Psi(\Upsilon) = \frac{1}{\sigma}(1 - 1/\sqrt{1 - 4\sigma C_s^2/\tau^2}). \tag{4.62}$$

For τ in this case, we have

$$\ddot{\tau} = \frac{3\kappa}{2\sigma}(2\tau - \sqrt{\tau^2 - 4\sigma C_s^2} - \tau^2/\sqrt{\tau^2 - 4\sigma C_s^2}) - 3\Lambda\tau, \tag{4.63}$$

with the solution in quadrature

$$\int \frac{d\tau}{\sqrt{(\kappa/\sigma)[\tau^2(1 - \sqrt{1 - 4\sigma C_s^2/\tau^2})] - \Lambda\tau^2 + C}} = \sqrt{3}t. \tag{4.64}$$

Further analysis of (4.64) gives

$$\tau|_{t \to 0} \approx \sqrt{3C}t, \tag{4.65}$$

i.e., the solution is initially singular,

$$\tau|_{t \to \infty} \approx \sqrt{3(2\kappa C_s^2 + C)}t, \quad \Lambda = 0; \tag{4.66}$$

i.e., the asymptotic isotropization process does not take place in the absence of a Λ term,

$$\tau|_{t \to \infty} \approx e^{\sqrt{-3\Lambda}t}, \quad \Lambda < 0; \tag{4.67}$$

i.e., the solution is asymptotically isotropic for $\Lambda < 0$; and finally for $\Lambda > 0$, we find an oscillatory solution, since in that case, τ is nonnegative and bounded from above (otherwise the integrand in (4.64) turns out to be imaginary).

Let us study the energy density distribution of nonlinear scalar field. From (3.40d), we find

$$T^0_{\rm sc0}(t)\big|_{t\to 0} \longrightarrow \infty, \quad T^0_{\rm sc0}(t)\big|_{t\to\infty} \longrightarrow 0, \qquad (4.68)$$

which shows that the energy density of the scalar field is not localized.

4.3. Nonlinear Spinor and Nonlinear Scalar Field with Minimal Coupling

Let us consider the system with nonlinear spinor and scalar field given by the spinor nonlinearity as a function of I, i.e., $F = \lambda S^n$, while the scalar nonlinearity is given by (4.56). The field functions in this case will be the same as those in the two previous subsections, while the function τ in this case is determined by

$$\int \frac{d\tau}{\sqrt{\kappa[mC_0\tau - \lambda C_0^n/\tau^{(n-2)} + (\tau^2/\sigma)[1-\sqrt{1-4\sigma C_s^2/\tau^2}]] - \Lambda\tau^2 + C}} = \sqrt{3}t, \qquad (4.69)$$

with C being the constant of integration. A detailed analysis of (4.69) gives the following results:

$$\tau|_{t\to 0} \approx (-(3/4)\kappa\lambda n^2 C_0^n t^2)^{1/2}, \quad \lambda < 0; \qquad (4.70)$$

i.e., for $\lambda < 0$, the solution is initially singular. For $\lambda > 0$, we come to the conclusion that τ cannot be zero as in this case the integrand will be imaginary. This means that we obtain solutions that are regular in each timepoint, but as in previous subsection, we see that it breaks the dominant energy condition. Thus we see that the initial stage is completely dominated by nonlinear spinor term. For *tto*∞, we find

$$\tau|_{t\to\infty} \approx 3\kappa m C_0 t^2, \quad m \neq 0, \quad \Lambda = 0, \qquad (4.71)$$

i.e., an asymptotically isotropic solution for a massive spinor in the absence of a Λ term,

$$\tau|_{t\to\infty} \approx \sqrt{3(2\kappa C_s^2 + C)}t, \quad m = 0, \quad \Lambda = 0; \quad (4.72)$$

i.e., in case of a massless spinor, the scalar field plays the dominant role at the asymptotic stage in the absence of a Λ term. It should be noted that, as in case of nonlinear scalar field alone, the isotropization process does not take place. Finally, let us see what happens when one introduces a non-zero Λ term. In this case, we find

$$\tau|_{t\to\infty} \approx e^{\sqrt{-3\Lambda}t}, \quad \Lambda < 0, \qquad (4.73)$$

i.e., that the solution is asymptotically isotropic for $\Lambda < 0$; and finally, for $\Lambda > 0$, we find an oscillatory solution, since in that case, τ is nonnegative and bound from above, otherwise the integrand in (4.64) turns out to be imaginary. Setting $F = \lambda P^n$ for the massless spinor field, we come to an analogous conclusions.

4.4. Interacting Spinor and Scalar Field

In this subsection, we give a detailed analysis of the system of interacting spinor and scalar fields. We consider the spinor field setting $F = 0$ that is initially linear and the scalar one that is linear too, i.e., $L_{\rm sc} = (1/2)\Upsilon = \varphi_{,\alpha}\varphi^{,\alpha}$. We choose the interaction term in the form $L_{\rm int} = (1/2)\lambda_1\Upsilon F_1(I, J)$, with λ_1 being the coupling constant. As a result, we obtain the spinor field equations with induced nonlinearity. For the scalar field, we obtain

$$\dot\varphi(t) = \frac{C}{\tau(1+\lambda_1 F_1(I,J))}, \quad C = \text{const}. \qquad (4.74)$$

In view of (4.74), we find

$$\Upsilon = \dot\varphi^2 = \frac{C^2}{\tau^2(1+\lambda_1F_1(I,J))^2}. \qquad (4.75)$$

For the spinor field, we obtain

$$i\bar\gamma^0\left(\frac{\partial}{\partial t} + \frac{\dot\tau}{2\tau}\right)\psi - m\psi + \lambda_1\mathcal{D}_2\psi + \lambda_1\mathcal{G}_2 i\gamma^5\psi = 0, \qquad (4.76)$$

where, in view of (4.75), $\mathcal{D}_2$ and $\mathcal{G}_2$ are as follows:

$$\lambda_1\mathcal{D}_2 = \lambda_1 S\Upsilon\frac{\partial F_1}{\partial I} = -\frac{C^2}{2\tau^2}\frac{\partial F_2}{\partial S},$$

$$\lambda_1\mathcal{G}_2 = \lambda_1 S\Upsilon\frac{\partial F_1}{\partial J} = -\frac{C^2}{2\tau^2}\frac{\partial F_2}{\partial P},$$

with $F_2(I, J) = 1/[1 + \lambda_1F_1(I, J)]$. Thus, we obtain the spinor field equation with induced nonlinearity. Since the solutions to the equation (4.76) coincide with those of the nonlinear spinor field equation with corresponding $F(I, J)$ and $F_1(I, J)$, we simply write the solutions without giving the details. Now, taking into account that the components of the energy-momentum tensor in this case are

$$T_0^0 = mS + \frac{1}{2}\dot\varphi^2 + \lambda_1F_1\dot\varphi^2,$$

$$T_1^1 = T_2^2 = T_3^3 = \lambda_1\mathcal{D}_2S + \lambda_1\mathcal{G}_2P - \frac{1}{2}\dot\varphi^2 - \lambda_1F_1\dot\varphi^2, \qquad (4.77)$$

from (3.52) for τ, we obtain

$$\frac{\ddot\tau}{\tau} = \frac{3\kappa}{2}[mS + \lambda_1\mathcal{D}_2S + \lambda_1\mathcal{G}_2P] - 3\Lambda. \qquad (4.78)$$

For $F_1 = F_1(I)$, we find $S = C_0/\tau$ and

$$\psi_1(t) = (C_1/\sqrt{\tau}\exp[-i\int(m-\lambda_1\mathcal{D}_2)dt], \quad (4.79a)$$

$$\psi_2(t) = (C_2/\sqrt{\tau}\exp[-i\int(m-\lambda_1\mathcal{D}_2)dt], \quad (4.79b)$$

$$\psi_3(t) = (C_3/\sqrt{\tau}\exp[i\int(m-\lambda_1\mathcal{D}_2)dt], \quad (4.79c)$$

$$\psi_4(t) = (C_4/\sqrt{\tau}\exp[i\int(m-\lambda_1\mathcal{D}_2)dt], \quad (4.79d)$$

with C_1, C_2, C_3, C_4 being the integration constants, which are related to C_0 as $C_0 = C_1^2 + C_2^2 - C_3^2 - C_4^2$. In this case, we have $\mathcal{G}_2 = 0$, while $\lambda_1\mathcal{D}_2 = (C^2/2C_0)\partial F_2/\partial\tau$. In account of this, (4.78) can be integrated to write

$$\int\frac{d\tau}{\sqrt{\kappa[mC_0\tau + (C^2/2)F_2] - \Lambda\tau^2 + y_1^2}} = \sqrt{3}t. \quad (4.80)$$

Setting $F_1 = S^n$, i.e., $F_2 = \tau^n/(\tau^n + \lambda_1 C_0^n)$, we estimate

$$\tau(t)|_{t\to 0} \approx \sqrt{3}y_1 t, \quad (4.81)$$

$$\tau(t)|_{t\to\infty} \approx \begin{cases}\sqrt{3(\kappa C^2 + y_1^2)}t, & m = 0, \quad \Lambda = 0,\\ (3/4)\kappa m C_0 t^2, & m \neq 0, \quad \Lambda = 0,\end{cases} \quad (4.82)$$

and

$$\tau|_{t\to\infty} \approx e^{\sqrt{-3\Lambda}t}, \quad \Lambda < 0, \quad (4.83)$$

whereas, for $\Lambda > 0$, it is bounded from above. Thus, we see that the solutions are initially singular; in the absence of a Λ term, only the massive spinor provides asymptotic isotropization; introduction of a Λ term leads to asymptotic isotropization or oscillatory solutions depending on its sign.

Let us study the system for some other choice of interacting term. Note that the Λ term contribution is same for these choices, so we consider the case in the absence of a Λ term.

To investigate the system of spinor and scalar field equations with direct interaction, we consider the interacting term such that

$$F_2(S) = 1 + \lambda S^n = 1 + \lambda\frac{C_0^n}{\tau^n}, \quad (4.84)$$

where λ is the interaction parameter and n is some arbitrary constant. Inserting (4.84) into (3.67), one obtains

$$\int\frac{d\tau}{\sqrt{mC_0\tau + \lambda C^2 C_0^n/2\tau^n + C_2^2}} = \sqrt{3\kappa}t, \quad (4.85)$$

where $C_2^2 = C^2/2 + C_1$.

Let us study different cases of choosing λ and n.

4.4.1. $\lambda > 0$, $n > 0$. In this case, (4.85) leads to the following behavior of $\tau(t)$:

$$\tau|_{t\to 0} \approx \left[\left(\frac{n}{2}+1\right)\sqrt{\frac{3\kappa\lambda C_0^n C^2}{2}}t\right]^{\frac{1}{n/2+1}}, \quad (4.86a)$$

$$\tau|_{t\to 0} \approx \frac{3}{4}\kappa m C_0 t^2; \quad (4.86b)$$

i.e., the solution is initially singular and the asymptotical isotropization of the expansion process of initially anisotropic BI space-time takes place without the influence of the scalar field. Thus, the evolution of the interacting fields system at $\lambda > 0$ and $n > 0$ is qualitatively the same as that of the system with minimal coupling.

4.4.2. $\lambda = -\sigma^2 < 0$, $n > 0$. In this case, from (4.85), we find that the asymptotic expression of τ coincides with (4.86b), whereas $\tau = 0$ cannot be reached, as in this case the denominator of the integrand in (4.85) becomes imaginary at $\tau \longrightarrow 0$. There exists the minimum value $\tau_{\min} = \tau_0 > 0$, which is defined from the equation

$$mC_0\tau_0^{n+1} + C_2^2\tau_0^n - \frac{\sigma^2 C^2 C_0^n}{2} = 0.$$

This means that for $\lambda < 0$ and $n > 0$, there exist regular solutions to the previous system of equations. The absence of the initial singularity in the considered cosmological solution appears to be consistent with the violation for $\lambda < 0$, of the dominant energy condition in the Hawking–Penrose theorem.

4.4.3. $\lambda > 0$, $n = -k^2 < 0$. In this case, the equation (4.85) takes the form

$$\int\frac{d\tau}{\sqrt{mC_0\tau + \lambda C^2\tau^{k^2}/2C_0^{k^2} + C_2^2}} = \sqrt{3\kappa}t. \quad (4.87)$$

Let us study concrete solutions for some values of k^2.

(a) $k^2 = 1$. Then, from (4.87), one gets

$$\tau(t) = \frac{3}{4}MC_0\kappa t^2 - \frac{C_2^2}{MC_0}, \quad M = m + \frac{\lambda C^2}{2C_0^2}. \quad (4.88)$$

The solution is singular at $t_0 = 2C_2/\sqrt{3\kappa}MC_0$ and is asymptotically isotropic.

(b) $k^2 = 2$. In this case, we have

$$\tau(t) = \frac{C_0^2}{\lambda C^2}\left[\Delta\sinh\left(\frac{\sqrt{3\kappa\lambda}C}{\sqrt{2}C_0}t\right) - mC_0\right], \quad (4.89)$$

where $\Delta = \sqrt{2\lambda C^2 C_2^2/C_0^2 - m^2 C_0^2}$. Thus, we see that, in the case considered here, the solution is singular at t_0 with t_0 being the root of the equation

$$\Delta\sinh\left(\frac{\sqrt{3\kappa\lambda}C}{\sqrt{2}C_0}t_0\right) - mC_0 = 0. \quad (4.90)$$

As one sees from (4.89), the asymptotic isotropization of the BI universe takes place.

4.4.4. $\lambda = -\sigma^2 < 0$, $n = -k^2 < 0$. Let us consider concrete solutions for some values of k^2 as in 4.4.3.

(a) $k^2 = \frac{1}{2}$. In this case, one gets

$$\frac{2}{\sqrt{mC_0}}(\sqrt{\tau} + \sqrt{\tau_1}\ln|\sqrt{\tau} - \sqrt{\tau_1}|) = \sqrt{3\kappa}t, \quad (4.91)$$

with $\sqrt{\tau_1} = \sigma^2 C^2/4mC_0^{3/2}$. We can conclude that the solution is initially regular and that it is asymptotically isotropic.

(b) $k^2 = 1$. In this case, we write

$$\int \frac{d\tau}{\sqrt{(m - \sigma^2 C^2/2C_0^2)C_0\tau + C_2^2}} = \sqrt{3\kappa}t. \quad (4.92)$$

If in (4.92) $m - \sigma^2C^2/2C_0^2 > 0$, then the solution coincides with that for 4.4.3, where $M = m - \sigma^2C^2/2C_0^2$. For $m - \sigma^2C^2/2C_0^2 = -T^2 < 0$, from (4.92) one gets

$$\tau(t) = \frac{C_2^2}{T^2C_0} - \frac{3\kappa T^2 C_0}{4}t^2. \quad (4.93)$$

In this case, we have

$$\tau|_{t=0} = \tau_{\max} = \frac{C_2^2}{T^2C_0}, \quad (4.94a)$$

$$\tau|_{t = t_{1,2}} = \tau_{\min} = 0, \quad (4.94b)$$

where $t_{1,2} = \mp 2C_2/\sqrt{3\kappa}T^2C_0$. Thus, the solution obtained describes the cosmological model, which begins to expand at t_1, acquires its maximum at $t = 0$, and then collapses into a point at t_2.

(c) $k^2 = 2$. In this case, for τ, one gets

$$\tau(t) = \frac{C_0^2}{\sigma^2C^2}\left[mC_0 + \Delta\sin\left(\frac{\sigma C\sqrt{3\kappa}t}{\sqrt{2}C_0}\right)\right], \quad (4.95)$$

where $\Delta = C_0^{-1}\sqrt{m^2C_0^4 + 2\sigma^2C^2C_2^2}$.

From (4.95), it follows that the model begins to expand at

$$t_0 = -(\sqrt{2}C_0/\sqrt{3\kappa}\sigma C)\arcsin[mC_0/\Delta],$$

acquires maximum

$$\tau = \tau_{\max} = (C_0/\sigma C)^2[mC_0 + \Delta],$$

at

$$t = t_1 = \pi C_0/\sqrt{6\kappa}\sigma C,$$

and finally at

$$t = t_2 = \pi + (\sqrt{2}C_0/\sqrt{3\kappa}\sigma C)\arcsin[mC_0/\Delta]$$

collapses into a point.

For $F_1 = F_1(J)$ for the massless spinor field, we obtain the solutions

$$\psi_1 = \frac{1}{\sqrt{\tau}}(D_1e^{i\sigma} + iD_3e^{-i\sigma}), \quad (4.96a)$$

$$\psi_2 = \frac{1}{\sqrt{\tau}}(D_2e^{i\sigma} + iD_4e^{-i\sigma}), \quad (4.96b)$$

$$\psi_3 = \frac{1}{\sqrt{\tau}}(iD_1e^{i\sigma} + D_3e^{-i\sigma}), \quad (4.96c)$$

$$\psi_4 = \frac{1}{\sqrt{\tau}}(iD_2e^{i\sigma} + D_4e^{-i\sigma}), \quad (4.96d)$$

with $\sigma = \int \mathcal{G}_2 dt$. Further, choosing $F_1 = \lambda P^n$ for τ, in this case we have

$$\int \frac{d\tau}{\sqrt{\kappa C^2\tau^n/(\tau^n + \lambda_1 D_0^n) - \Lambda\tau^2 + y_1^2}} = \sqrt{3}t, \quad (4.97)$$

where we used $P = D_0/\tau$. From (4.97), we estimate

$$\tau(t)|_{t\to 0} \approx \sqrt{3}y_1t, \quad (4.98)$$

$$\tau(t)|_{t\to\infty} \approx \begin{cases} \sqrt{3(\kappa C^2 + y_1^2)}t, & \Lambda = 0, \\ e^{\sqrt{-3\Lambda}t}, & \Lambda < 0, \end{cases} \quad (4.99)$$

whereas for $\Lambda > 0$, τ is bounded from above. Thus, we see that the solutions are initially singular; in absence of the Λ term, no asymptotic isotropization process takes place; introduction of the Λ term leads to asymptotic isotropization or oscillatory solutions depending on its sign. Contrary to the nonlinear spinor case when nonlinearity provides initially regular solutions depending on the sign of λ, the induced nonlinearity does not give rise to singularity-free solutions.

4.5. BI Universe is Filled with Perfect Fluid Only

Let us now analyze the system filled with perfect fluid. As we saw earlier, the introduction of perfect fluid does not change the field equations, thus leaving the solutions to the NLSF equations externally unchanged. Changes in the solutions performed by perfect fluid are carried out through Einstein equations, namely through τ.

In the absence of material field, in this case, from (3.52) we find

$$\ddot{\tau} = \frac{3\kappa}{2}\frac{(1-\zeta)\varepsilon_0}{\tau^{\zeta}}, \quad (4.100)$$

with the solution

$$\frac{d\tau}{\sqrt{\tau^{(1-\zeta)}+C}} = \sqrt{3\kappa\varepsilon_0}t, \qquad (4.101)$$

where C is an integration constant. From (4.101), one estimates

$$\tau \propto t^2, \quad \text{for} \quad \zeta = 0, \quad \text{(dust)}, \qquad (4.102a)$$

$$\tau \propto t^{3/2}, \quad \text{for} \quad \zeta = 1/3, \quad \text{(radiation)}, \qquad (4.102b)$$

$$\tau \propto t^{6/5}, \quad \text{for} \quad \zeta = 2/3, \quad \text{(hard universe)}, \qquad (4.102c)$$

$$\tau \propto t, \quad \text{for} \quad \zeta = 1, \quad \text{(stiff matter)}. \qquad (4.102d)$$

Let us now consider the system as a whole with the nonlinear term being $F = \lambda S^n$. In this case, we get

$$\int \frac{d\tau}{\sqrt{mC_0\tau - \lambda C_0^n/\tau^{(n-2)} + \varepsilon_0\tau^{(1-\xi)} + g^2}} = \pm\sqrt{3\kappa}t. \qquad (4.103)$$

As one can see in the case of dust ($\xi = 0$), the fluid term can be combined with the massive one, whereas in the case of stiff matter ($\xi = 1$), it mixes with the constant. In the absence of a massive term (e.g., when we consider $F = F(J)$), the asymptotic behavior of τ is determined by

$$\tau|_{t\to \text{infty}} \approx ([\sqrt{\varepsilon_0}(\zeta+1)/2]t)^{2/(\zeta+1)}. \qquad (4.104)$$

As one can see, the space-time is asymptotically isotropic if $\zeta < 1$ and anisotropic if $\zeta = 1$. Thus, in the absence of a Λ term for a massless spinor field, the asymptotic isotropization process of the initially anisotropic space-time depends on the value of ζ, i.e., the matter the space-time is filled with.

5. SYSTEM WITH TIME-DEPENDENT G AND Λ

As has been mentioned earlier, Einstein's theory of gravity contains two parameters, considered as fundamental constants: Newton's gravitational constant G ($\kappa = 8\pi G$) and the cosmological constant Λ [41, 42]. A possible time variation of G has been suggested by Dirac and extensively discussed in the literature [67–71]. The "cosmological constant" Λ as a function of time was studied by many authors. Chen and Wu [72] advocated the possibility that the cosmological constant varies in time as $1/R^2$, with R being the scale factor of the Robertson–Walker model. Further, Abdel–Rahman [73] considered a model with the same kind of variation, while Berman et al. [74–76] stressed that the relation $R \propto t^{-2}$ plays an important role in cosmology. Berman and Gomide [77] also showed that all the phases of the universe, i.e., radiation, inflation, and pressure-free, may be considered as particular cases of the decelaration parameter q = constant type, where

$$q = -R\ddot{R}/\dot{R}^2, \qquad (5.1)$$

where dots stand for the time derivative. This definition was extended by Singh and Agrawal [78] to the Bianchi cosmological models. Perfect fluid cosmological models with time varying constants were also studied in [79, 80]. Models with viscous fluid and time-dependent constants were studied by a number of authors [81–84]. Recently, a self-consistent system of a nonlinear spinor field and a BI gravitational field with time dependent gravitational constant (G) and cosmological constant (Λ) has been studied by the author [40]. In this subsection, we will present the main results obtained in [40]. Note that in this case, the scalar and spinor field equations will remain unchanged. Formally, the spinor and scalar fields as well as the metric functions will be the same as for time independent G and Λ. Changes occur only in the equation for τ, plus we have an extra equation for determining G. Einstein's field equations with variable cosmological and gravitational "constants" Λ and G are given by

$$R_\nu^\mu - \frac{1}{2}\delta_\nu^\mu R = -8\pi G(t)T_\nu^\mu + \Lambda(t)\delta_\nu^\mu. \qquad (5.2)$$

Taking the divergence of (5.2), we obtain

$$8\pi G_{,\mu}T_\nu^\mu + 8\pi G(T_{\nu;\mu}^\mu) - \Lambda, \quad \mu\delta_\nu^\mu = 0, \qquad (5.3)$$

which for the BI metric reduces to

$$8\pi\dot{G}T_0^0 + 8\pi G\Big[\dot{T}_0^0 + T_0^0\Big(\frac{\dot{a}}{a} + \frac{\dot{b}}{b} + \frac{\dot{c}}{c}\Big) - T_1^1\frac{\dot{a}}{a} - T_2^2\frac{\dot{b}}{b} - T_3^3\frac{\dot{c}}{c}\Big] - \dot{\Lambda} = 0. \qquad (5.4)$$

If we suppose the energy conservation law $T_{\nu;\mu}^\mu = 0$ to hold, then (5.4) reduces to

$$\dot{T}_0^0 + T_0^0\Big(\frac{\dot{a}}{a} + \frac{\dot{b}}{b} + \frac{\dot{c}}{c}\Big) - T_1^1\frac{\dot{a}}{a} - T_2^2\frac{\dot{b}}{b} - T_3^3\frac{\dot{c}}{c} = 0, \qquad (5.5a)$$

$$8\pi\dot{G}T_0^0 - \dot{\Lambda} = 0. \qquad (5.5b)$$

Solving (5.5a), we come to (3.65). Let us now define G. Taking into account that $\dot{G} = \dot{\tau}\partial G/\partial\tau$ and $\dot{\Lambda} = \dot{\tau}\partial\Lambda/\partial\tau$, we rewrite (5.5b) as

$$8\pi T_0^0\frac{\partial G}{\partial\tau} = \frac{\partial\Lambda}{\partial\tau}. \qquad (5.6)$$

On the other hand, inserting a, b, c from (3.49) into (3.40), we obtain

$$8\pi T_0^0 G = \frac{\dot{\tau}^2}{3\tau^2} - \frac{\mathcal{X}}{3\tau^2} + \Lambda, \qquad (5.7)$$

where $\mathcal{X} = X_1^2 + X_1X_3 + X_3^2$. Dividing (5.6) by (5.7), we find the following equation for G

$$\frac{\partial G/\partial\tau}{G} = \frac{3\tau^2\partial\Lambda/\partial\tau}{\dot{\tau}^2 - \mathcal{X} + 3\tau^2\Lambda}. \quad (5.8)$$

Now, along with other authors, we suppose Λ to be a given function of τ, namely, $\Lambda = \Lambda_0/\tau^2$. On the other hand, T_1^1 and T_0^0 are also some functions of τ. These allow us to write the solution (3.52) in quadrature, i.e., in the form (3.67). But in this case, (3.52) or (3.67) turn out to be rather complicated, since along with T_1^1 and T_0^0, G also depends on t. So, one has to solve (3.52) and (5.8) simultaneously, which is far too complicated. To this end, we recall the results obtained in previous subsections. As one remembers, the solutions, which are in accord with the Hawking–Penrose theorem, are initially singular with the universe being anisotropic. This can be achieved by setting $\tau = \alpha t$ for $t \longrightarrow 0$, with α being some constant. On the other hand, setting $\tau = \beta t^2$, for $t \longrightarrow \infty$ and β being constant, we get a universe that is in accord with the present day isotropic state. Now, setting $\Lambda = \Lambda_0/\tau^2$ from (5.8), we find [40]

$$G = C/\tau^{6\Lambda_0/(\alpha^2 - \mathcal{X} + 3\Lambda_0)}, \quad \tau = \alpha t, \quad C = \text{const}, \quad (5.9)$$

and

$$G = D\left(\frac{4\beta\tau}{4\beta\tau - \mathcal{X} + 3\Lambda_0}\right)^{6\Lambda_0/(\mathcal{X} - 3\Lambda_0)}, \quad (5.10)$$
$$\tau = \beta t^2, \quad D = \text{const}.$$

Here we would like to emphasize the properties of the Bianchi type-I Universe. As was noticed in [23], this Universe has the agreeable property that near the singularity it behaves like a Kasner Universe, even in the presence of matter, and consequently falls within the general analysis of the singularity. Since in a Kasner Universe $a = a_0 t^{p_1}$, $b = b_0 t^{p_2}$, and $c = c_0 t^{p_3}$, with $p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1$, our assumption to set $\tau \propto t$ at initial time is correct. On the other hand, in a Universe filled with matter for $p = \gamma\varepsilon$, $\gamma < 1$, it has been shown that any initial anisotropy in a BI universe quickly dies away and a BI universe eventually evolves into a FRW universe [24]. Setting $\tau \propto t^2$ is also correct.

If we consider $\Lambda = \Lambda_0/\tau^2$ and G = constant, then the conservation law $T^{\mu}_{\nu;\,\mu} = 0$ doesn't hold separately, as in that case (5.5b) leads to Λ = const, which contradicts our assumption. In this case, from (5.4) we find

$$\dot{\varepsilon} + (1+\zeta)\frac{\dot{\tau}}{\tau} = -\frac{2\Lambda_0\dot{\tau}}{\tau^3}, \quad (5.11)$$

with the solution

$$\varepsilon = \frac{2\Lambda_0}{1-\zeta}\frac{1}{\tau^2}. \quad (5.12)$$

Setting $F = K^n$ with $K = \{I, J, (I \pm J), IJ\}$ from (3.67), we conclude that even in the presence of time dependent Λ in the Einstein's equation, perfect fluid plays no role in the early stage of expansion or isotropization of the BI universe, leaving it to the nonlinear spinor term, which confirms our claim made in [35, 38].

6. INTERACTING SPINOR AND SCALAR FIELD IN A BI UNIVERSE FILLED WITH MAGNETO-FLUID

In this section we consider the case when the BI Universe is filled with magneto-fluid. An LRS BI model containing a magnetic field directed along one axis with a barotropic fluid was investigated by Thorne [85]. Jacobs [86, 87] investigated BI models with magnetic field satisfying a barotropic equation of state. Bali [88] studied the behavior of the magnetic field in a BI universe for perfect fluid distribution. For simplicity, we consider the spinor field setting $F = 0$ that is initially linear and the scalar one that is linear too, i.e., $L_{sc} = (1/2)\Upsilon = \varphi_{,\alpha}\varphi^{,\alpha}$. We choose the interaction term in the form $L_{int} = (1/2)\lambda_1\Upsilon F_1(I, J)$, with λ_1 being the coupling constant. As a result, we obtain the spinor field equations with induced nonlinearity. Actually, the introduction of magneto-fluid brings significant changes in the components of the energy-momentum tensor. Since the spinor and scalar field equations in this case remain unchanged, we confine this study only to solving the Einstein equations and the one for τ. Let us begin with the energy-momentum tensor.

The energy-momentum tensor of the magneto-fluid is chosen to be

$$T^{\nu}_{\mu(m)} = (\varepsilon + p)u_\mu u^\nu - p\delta^\nu_\mu + E^\nu_\mu, \quad (6.1)$$

where $E_{\mu\nu}$ is the electro-magnetic field given by Lichnerowich [89]

$$E^\nu_\mu = \bar{\mu}\left[|h|^2\left(u_\mu u^\nu - \frac{1}{2}\delta^\nu_\mu\right) - h_\mu h^\nu\right]. \quad (6.2)$$

Here u^μ is the flow vector satisfying

$$g_{\mu\nu}u^\mu u^\nu = 1, \quad (6.3)$$

$\bar{\mu}$ is the magnetic permeability, and h_μ is the magnetic flux vector defined by

$$h_\mu = \frac{1}{\bar{\mu}} * F_{\nu\mu}u^\nu, \quad (6.4)$$

where $*F_{\mu\nu}$ is the dual electro-magnetic field tensor defined as

$$*F_{\mu\nu} = \frac{\sqrt{-g}}{2}\varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}. \quad (6.5)$$

Here, $F^{\alpha\beta}$ is the electro-magnetic field tensor and $\varepsilon_{\mu\nu\alpha\beta}$ is the totally anti-symmetric Levi–Civita tensor with $\varepsilon_{0123} = +1$. Here, the comoving coordinates are taken to be $u^0 = 1$, $u^1 = u^2 = u^3 = 0$. I choose the incident magnetic field to be in the direction of the x axis so that the magnetic flux vector has only one nontrivial component, namely $h_1 \neq 0$. In view of the aforementioned assumption from (6.4), one obtains $F_{12} = F_{13} = 0$. I also assume that the conductivity of the fluid is infinite. This leads to $F_{01} = F_{02} = F_{03} = 0$. Thus I have only one nonvanishing component of $F_{\mu\nu}$, which is F_{23}. Then from the first set of Maxwell equations

$$F_{\mu\nu;\beta} + F_{\nu\beta;\mu} + F_{\beta\mu;\nu} = 0, \qquad (6.6)$$

where the semicolon stands for covariant derivative, one finds

$$F_{23} = \mathcal{I}, \quad \mathcal{I} = \text{const}. \qquad (6.7)$$

Then, from (6.4) in account of (6.5), one finds

$$h_1 = \frac{a\mathcal{I}}{\bar{\mu}bc}. \qquad (6.8)$$

Finally, for E_μ^ν, one finds the following non-trivial components

$$E_0^0 = E_1^1 = -E_2^2 = -E_3^3 = \frac{\mathcal{I}^2}{2\bar{\mu}b^2c^2}. \qquad (6.9)$$

ε and p in (6.1) are the energy density and pressure of perfect fluid obeying $p = \zeta\varepsilon$.

Let us now solve the Einstein equations. In doing so, I first write the expressions for the components of the energy-momentum tensor explicitly:

$$T_0^0 = mS + C^2/2\tau^2(1+\lambda_1F_1) + \varepsilon + \frac{\mathcal{I}^2}{2\bar{\mu}b^2c^2}, \qquad (6.10a)$$

$$T_1^1 = \mathcal{D}S + \mathcal{G}P - C^2/2\tau^2(1+\lambda_1F_1) - p + \frac{\mathcal{I}^2}{2\bar{\mu}b^2c^2}, \qquad (6.10b)$$

$$T_2^2 = \mathcal{D}S + \mathcal{G}P - C^2/2\tau^2(1+\lambda_1F_1) - p - \frac{\mathcal{I}^2}{2\bar{\mu}b^2c^2}, \qquad (6.10c)$$

$$T_3^3 = \mathcal{D}S + \mathcal{G}P - C^2/2\tau^2(1+\lambda_1F_1) - p - \frac{\mathcal{I}^2}{2\bar{\mu}b^2c^2}. \qquad (6.10d)$$

Thus, the introduction of magneto-fluid generates nonhomogeneity in the energy-momentum tensor of the material field.

In view of $T_2^2 = T_3^3$ from (3.40b), (3.40c), one finds

$$b = cD\exp\left(X\int\frac{dt}{\tau}\right), \qquad (6.11)$$

with D and X being integration constants.

Following Bali [88], let us assume that the expansion (θ) in the model is proportional to the eigenvalue σ_1^1 of the shear tensor σ_μ^ν. Since for the BI space-time

$$\theta = \frac{\dot{a}}{a} + \frac{\dot{b}}{b} + \frac{\dot{c}}{c}, \qquad (6.12)$$

$$\sigma_1^1 = -\frac{1}{3}\left(4\frac{\dot{a}}{a} + \frac{\dot{b}}{b} + \frac{\dot{c}}{c}\right), \qquad (6.13)$$

the aforementioned condition leads to

$$a = (bc)^N, \qquad (6.14)$$

with N being the proportionality constant.

In account of (3.34) from (6.11) and (6.14), after some manipulation for the metric functions one finds [90]

$$a = \tau^{N/(N+1)}, \qquad (6.15a)$$

$$b = \sqrt{D}\tau^{1/2(N+1)}\exp\left[\frac{X}{2}\int\frac{dt}{\tau}\right], \qquad (6.15b)$$

$$c = \frac{1}{\sqrt{D}}\tau^{1/2(N+1)}\exp\left[-\frac{X}{2}\int\frac{dt}{\tau}\right]. \qquad (6.15c)$$

The equation for τ in this case can be written as

$$\ddot{\tau} = \frac{3}{2}\kappa\left(mC_0 + \mathcal{D}C_0 + \varepsilon_0(1-\zeta)/\tau^\zeta + \frac{2\mathcal{I}^2}{3\bar{\mu}}\tau^{(N-1)/(N+1)}\right) - 3\Lambda. \qquad (6.16)$$

Recalling the definition of $\mathcal{D}$, we write the solution to Eq. (6.16) in quadrature

$$\int\frac{d\tau}{\sqrt{\kappa(mC_0\tau + C^2/2(1+\lambda_1F_1) + \varepsilon_0\tau^{1-\zeta} + ((N+1)\mathcal{I}^2/3\bar{\mu}N)\tau^{2N/(N+1)}) - \Lambda\tau^2 + E}} = \sqrt{3}t, \qquad (6.17)$$

with E being some integration constant. Eqs. (6.16) and (6.17) can be analyzed in the same line as we have done previously. Thus, a self-consistent system of spinor, scalar, and gravitation fields has been studied in the presence of magneto-fluid and cosmological term Λ. With the presence of the F_{23} component of the electromagnetic field tensor, the system can be viewed as one where all four fields, i.e., scalar, electro-magnetic, spinor, and gravitational, are taken into consideration.

7. NUMERICAL ANALYSIS OF THE RESULTS

In this section, we consider the case when both F and F_1 are the functions of $I = S^2$ only, setting $F = \lambda S^p$ and $F_1 = S^q$. As the scalar field Lagrangian, we consider $\Psi = (1/2)\Upsilon$. Note that (i) $\lambda = 0$ and $\lambda_1 = 0$ corresponds to the case with linear spinor field, (ii) $\lambda = 0$ corresponds to interacting linear spinor and scalar field, and (iii) $\lambda_1 = 0$ corresponds to minimal coupling of nonlinear spinor and linear scalar field. Moreover, setting spinor mass $m = 0$, we obtain the system for massless spinor field with the nonlinear term being a function of J or $I \pm J$, e.g. $F = \lambda P^p$.

In this section, we consider the case when both F and F_1 are functions of $I = S^2$ only, setting $F = \lambda S^p$ and $F_1 = S^q$. As the scalar field Lagrangian, we consider $\Psi = (1/2)\Upsilon$. Let us go back to the equations for τ. In general, we have to solve the following system of equations:

$$\ddot{\tau} = \frac{3}{2}\kappa(T_1^1 + T_0^0)\tau - 3\Lambda\tau, \tag{7.1}$$

$$\dot{T}_0^0 = -\frac{\dot{\tau}}{\tau}(T_0^0 - T_1^1). \tag{7.2}$$

Defining the Hubble parameter, this system can be rewritten as

$$\dot{\tau} = 3H\tau, \tag{7.3}$$

$$\dot{H} = -3H^2 + \frac{1}{2}\kappa(T_1^1 + T_0^0)\tau - \Lambda\tau, \tag{7.4}$$

$$\dot{T}_0^0 = -3H(T_0^0 - T_1^1). \tag{7.5}$$

The other way to solve this system is to use the relation (3.88) between the quantities in question established above, i.e., consider the system

$$\dot{H} = -\frac{1}{2}\kappa(T_0^0 - +T_1^1) - C_{00}/\tau^2, \tag{7.6}$$

$$\dot{T}_0^0 = -3H(T_0^0 - T_1^1), \tag{7.7}$$

$$\tau^2 = \frac{C_{00}}{3H^2 + \Lambda - \kappa T_0^0}. \tag{7.8}$$

Since in our case T_0^0 and T_1^1 are functions of τ only and are explicitly analytically established, we proceed as follows:

$$\ddot{\tau} = \mathcal{F}(p), \tag{7.9}$$

where we define

$$\mathcal{F}(p) = \frac{3\kappa}{2}(m + \lambda(p-2)\tau^{1-p} + 4\lambda_1 q\tau^{q-1}/(2\lambda_1 + \tau^q)^2 + \varepsilon_0(1-\zeta)/\tau^\zeta) - 3\Lambda\tau. \tag{7.10}$$

Here, p is the set of problem parameters, namely, $p = \{\kappa, m, \lambda, \lambda_1, p, q, \varepsilon_0, \Lambda\}$. From a mechanical point of view, Eq. (7.9) can be interpreted as an equation of motion of a single particle with unit mass under the force $\mathcal{F}(\tau, p)$. Then the following first integral exists [91]:

$$\dot{\tau} = \pm\sqrt{2[E - U(\tau)]}, \tag{7.11}$$

with the potential

$$U(\tau) = -\frac{3}{2}[\kappa(m\tau - \lambda\tau^{2-p} - 4\lambda_1/(2\lambda_1 + \tau^q) + \varepsilon_0\tau^{1-\zeta}) - \Lambda\tau^2]. \tag{7.12}$$

Here we set $C_s = 1$ and $C_0 = 1$. Giving boundary conditions the Eq. (7.9) can be solved by continuous analog of Newton method [92], whereas given initial (or asymptotic) value, one can numerically solve velocity by the Runge–Kutta method.

Let us formulate the boundary condition. To formulate the boundary conditions, we recall that the BI space-time models an expanding universe, i.e., the size of the universe should be small enough at the initial stage, whereas with time increasing, it becomes bigger and bigger. Since for $\Lambda > 0$ the value of τ is bound from above, the idea of choosing a large value for τ at the present time (which is sufficiently big) is valid only for $\Lambda \le 0$. For a positive Λ, one can choose the periodical boundary conditions. As a result of what has been said above, we see that at the initial stage, perfect fluid plays the principal role and τ at this stage obeys

$$\ddot{\tau} = \frac{3\kappa}{2}\varepsilon_0(1-\zeta)/\tau^\zeta, \tag{7.13}$$

with the solution

$$\tau|_{t\to 0} = \left[\frac{1+\zeta}{2}(\sqrt{3\kappa\varepsilon_0}t + c_1)\right]^{2/(1+\zeta)}, \quad c_1 = \text{const}, \tag{7.14}$$

whereas at large t, we have for τ

$$\ddot{\tau} = \frac{3\kappa}{2}m - 3\Lambda\tau, \tag{7.15}$$

with the solutions

$$\tau|_{t\to\infty} = \begin{cases} (3/4)\kappa m t^2, & \Lambda = 0, \\ \sinh(\sqrt{-3\Lambda}t) + (\kappa m/2\Lambda), & \Lambda < 0. \end{cases} \tag{7.16}$$

Eq. (7.9), together with the boundary conditions formulated above, give the evolutionary picture of a BI universe within the scope of the model itself. For a more realistic picture, the boundary conditions for τ can

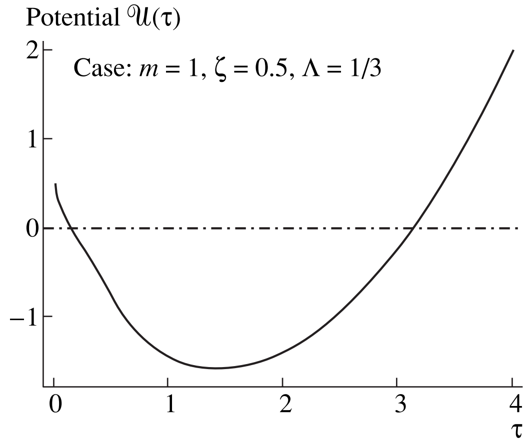

Fig. 1. View of the potential $\mathcal{U}(\tau)$ [Eq. (7.20)] with BI space-time being filled with perfect fluid describing a hard Universe.

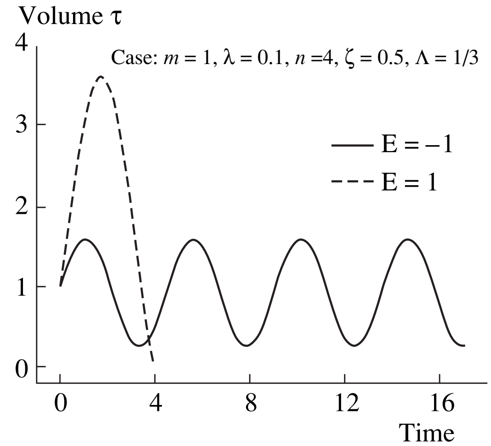

Fig. 2. Evolution of the BI space-time corresponding to the potential given in Fig. 1 for a different choice of E.

be formulated from physical reasoning as well. If we consider that at time zero the size of the universe vanishes, the temperature and density soar to infinity. To avoid this singularity at the beginning moment of the universe, the length of the universe at each spatial direction can be taken as Planck size ($\hbar = 1.616 \times 10^{-33}$ cm). A detailed discussion on this assumption can be found in [93]. To formulate the boundary condition at the other end, we recall that the Hubble constant H is related to τ as

$$3H = \frac{\dot{\tau}}{\tau}. \tag{7.17}$$

Given the present day value of H from (7.17), one finds the value of τ at a time $t = t_N$

$$\tau|_{t = t_N} = \exp[3Ht_N]. \tag{7.18}$$

As was mentioned earlier, the problem (7.9) can be solved numerically by the Runge–Kutta method as well given the initial (or asymptotic) value of τ and its first derivative with time $\dot{\tau}$. In line with the above discussion, at time zero, any small but positive value τ_0 can be taken for τ, whereas $\dot{\tau}$ in this case is the positive root of (7.11) for the given τ_0. For a backward direction Runge–Kutta method, the asymptotic value of τ can be evaluated from (7.18) with $\dot{\tau}$ being the negative root of (7.11) for the corresponding τ_N. In case of $\Lambda > 0$, the positivity of the radical in velocity imposes additional restrictions on the choice of initial (asymptotic) value of τ. For example, with other parameters fixed, for a definite integration constant $\mathcal{E}$ we have a finite range of τ_0.

In what follows, we analyze Eqs. (7.9) and (7.10) for a different choice of $F(I)$ as well as for different problem parameters p.

1. $F = S^n$

Let us first choose F to be a power law of S (or I), setting $F = S^n$. In this case, setting $C_0 = 1$ and $C = 1$, we rewrite $\mathcal{F}$ as

$$\mathcal{F} = \frac{3\kappa}{2}\left(m + \frac{\lambda n \tau^{n-1}}{2(\lambda + \tau^n)^2} + \varepsilon_0 \frac{(1-\zeta)}{\tau^{\zeta}}\right) - 3\Lambda\tau, \tag{7.19}$$

with the potential

$$\mathcal{U} = -\frac{3}{2}\left\{\kappa\left[m\tau - \frac{\lambda}{2(\lambda + \tau^n)} + \varepsilon_0 \tau^{1-\zeta}\right] - \Lambda\tau^2\right\}. \tag{7.20}$$

Note that the nonnegativity of the radical in Eq. (7.11) in view of Eq. (7.20) imposes a restriction on τ from above in the case of $\Lambda > 0$. This means that in the case of $\Lambda > 0$ the value of τ runs between 0 and some τ_{max}, where τ_{max} is the maximum value of τ for the given p. This equation has been studied for different values of parameters p. Here we demonstrate the evolution of τ for different choices of τ_0 for fixed "energy" E and vice versa.

As the first example, we consider a massive spinor field with $m = 1$. Other parameters are chosen in the following way: coupling constant $\lambda = 0.1$, power of nonlinearity $n = 4$, and cosmological constant $\Lambda = 1/3$. We also choose $\zeta = 0.5$, describing a hard Universe.

In Fig. 1, we plot corresponding potential $\mathcal{U}(\tau)$ multiplied by the factor 2/3. As is seen from Figs. 1 and 2, choosing the integration constant E we may obtain two different types of solutions. For $E > 0.5$, solutions are nonperiodic, whereas for $E_{min} < E \leq 0.5$, the evolution of the Universe is oscillatory.

As a second example, we consider the massless spinor field. Other parameters of the problem are left unaltered, with the exception of ζ. Here we choose $\zeta = 1$ describing stiff matter. It should be noted that this par-

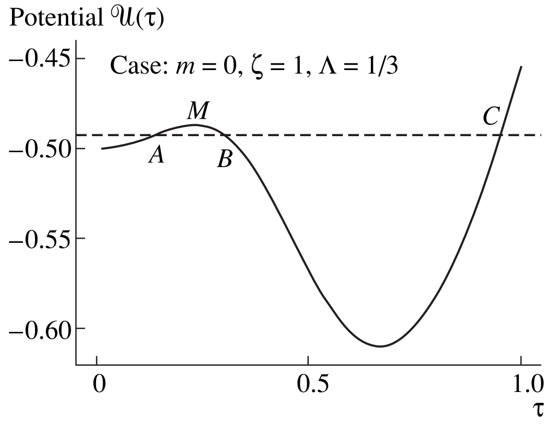

Fig. 3. View of the potential $\mathcal{U}(\tau)$ [Eq. (7.20)] with BI space-time being filled with stiff matter.

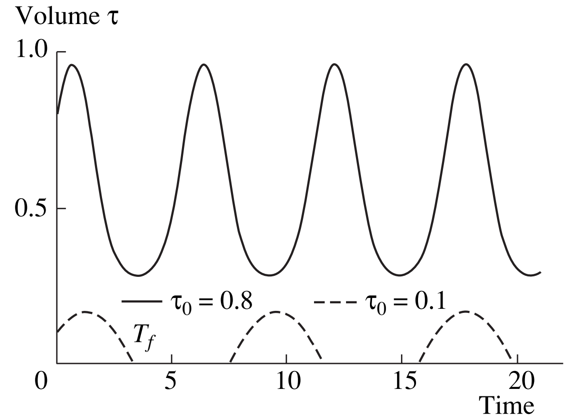

Fig. 4. Evolution of the BI space-time corresponding to the potential given in Fig. 3 in the case of a massless spinor field for different choices of τ_0 with $E \in (-0.5, M)$.

ticular choice of ζ gives rise to a local maximum. This results in two types of solutions for a single choice of E.

As can be seen from Fig. 3, if E is taken to be above the level M, there exist only nonperiodic solutions, whereas for $E_{\min} < E < \mathcal{U}(\tau = 0) = -0.5$, the solutions are always oscillatory. For $E \in (-0.5, M)$, there exit two types of solutions depending on the choice of τ_0. In Fig. 4, we plot the evolution of τ for $E \in (-0.5, M)$. As is seen, for $\tau_0 \in (0, A)$ (here $\tau_0 = 0.1$), we have mathematical solutions that are oscillatory and τ in this case becomes negative in some interval of time. Since by definition τ is non-negative, we plot only the part of the solution where $\tau \geq 0$ (cf. Fig. 4, dashed curve). Note that only that part of τ defined in the interval of time $t \in (0, T_f)$ is physically relevant. For $\tau_0 \in (B, C)$, we again have the oscillatory mode of the evolution of τ. These two regions are separated by the no-solution zone (A, B).

Let us also consider the case with $\Lambda < 0$. For a negative Λ, as well as in the absence of the Λ term, the evolution of τ is always exponential, as is seen in Fig. 5. In this case, the initial anisotropy of the BI space-time quickly dies away and the Universe becomes isotropic.

Let us analyze the dominant energy condition in the Hawking–Penrose theorem [52, 63]. For a BI Universe, the dominant energy condition can be written in the form [53]

$$T_0^0 \geq T_1^1 a^2 + T_2^2 b^2 + T_3^3 c^2, \tag{7.21a}$$

$$T_0^0 \geq T_1^1 a^2, \tag{7.21b}$$

$$T_0^0 \geq T_2^2 b^2, \tag{7.21c}$$

$$T_0^0 \geq T_3^3 c^2. \tag{7.21d}$$

Let us note that in [53] we considered a self-consistent system of nonlinear spinor and BI gravitational fields in the presence of a perfect fluid and a Λ term. It was shown that in this case the regular solutions can be obtained by virtue of the spinor field nonlinearity and/or a positive Λ term. It was shown also that the absence of initial singularity in the considered cosmological solution is consistent with the violation of the dominant energy condition in the Hawking–Penrose theorem. Note that regular solutions obtained for a linear spinor field by means of a positive Λ term do not violate this condition.

Let us now analyze the dominant energy condition for the system at hand. To analyze this condition for the system of the interacting spinor and scalar fields, we rewrite the components of the energy momentum tensor. For energy density in this case we have

$$T_0^0 = \frac{mC_0}{\tau} + \frac{C^2\tau^{n-2}}{2(\tau^n + \lambda C_0^n)} + \frac{\varepsilon_0}{\tau^{1+\zeta}}. \tag{7.22}$$

As one sees from Eq. (7.22) for any positive value of τ, energy density is always positive and definite. As

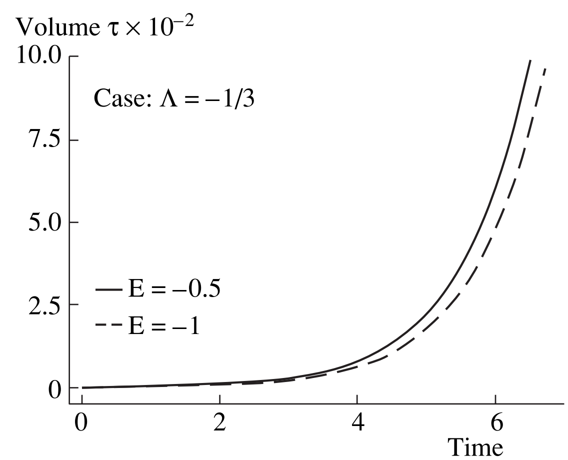

Fig. 5. Evolution of the BI Universe for a negative Λ. As can be seen, the evolution of the Universe in this case takes exponential character and the initial anisotropy of the BI space-time quickly dies away.

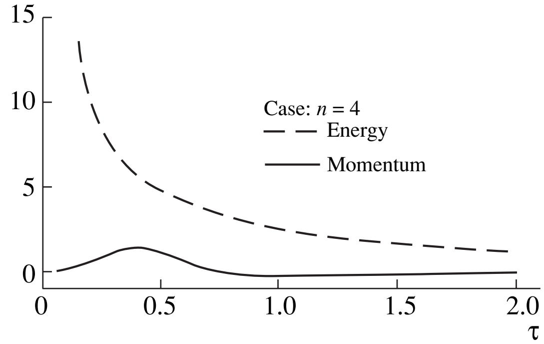

Fig. 6. Comparing T_0^0 and T_1^1 for a positive n, one sees that for a small value of n it is possible to construct a regular solution without violating the dominant energy condition.

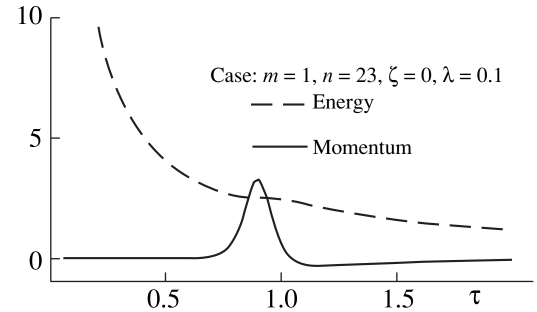

Fig. 7. For a large n, there exists some value of τ where the pressure component prevails over energy. In this case, the dominant energy condition breaks down.

$\tau \longrightarrow 0$, $T_0^0 \longrightarrow \infty$, whereas T_0^0 decreases as τ increases. For the pressure components in this case we have

$$T_1^1 = T_2^2 = T_3^3$$
$$= \frac{C^2\tau^{n-2}}{2(\tau^n + \lambda C_0^n)^2}[\lambda C_0^n(n-1) - \tau^n] - \frac{\zeta\varepsilon_0}{\tau^{1+\zeta}}. \tag{7.23}$$

The second term in Eq. (7.23) is always positive; this means that T_1^1 has a greater value when the BI Universe is filled with dust, i.e., when $\zeta = 0$. To investigate the dominant energy condition we study the pressure term (since $T_1^1 = T_2^2 = T_3^3$, hereafter we mention it as T_1^1) at length. For simplicity, we set $C = 1$ and $C_0 = 1$. It is clear from Eq. (7.23) that if

$$\tau^n > \lambda(n-1), \tag{7.24}$$

we have $T_1^1 < 0$. In this case, the dominant energy condition remains unbroken. From Eq. (7.24), we see for $\lambda = 0$ that the foregoing inequality holds for any $\tau > 0$. This means that, like the linear spinor field [53], the system with minimally coupled scalar and spinor fields possesses regular solutions without broken dominant energy condition. For an interacting system, this condition holds for any negative n with a positive λ and vice versa. Let us now see what happens when both n and λ are positive (negative). Note that the coupling constant λ may take any value. The magnitude of λ defines the strength of interaction.

Let us go back to Eq. (7.24). As can be seen, for any reasonable value of λ the inequality (7.24) holds at large τ. On the other hand, as $\tau \longrightarrow 0$, the corresponding energy density T_0^0 tends to infinity. So the conditions (4.23) hold for small τ as well. Finally, let us analyze the situation in the neighborhood of $\tau = 1$. The energy density T_0^0 at this point is reasonably small, whereas, as is shown in Fig. 7, violation of the dominant energy condition, i.e., the situation when T_1^1 dominates over T_0^0, may occur only for a relatively large value of n. Thus we conclude that in case of interacting spinor and scalar fields, it is possible to construct regular solutions [94] without violating the dominant energy condition of the Hawking–Penrose theorem (cf. Fig. 6).

2. $F = \sin S$

Let us now consider the case with F being a trigonometric function of S, namely, $F = \sin S$. In this case, for $\mathcal{F}$ we have

$$\mathcal{F} = \frac{3\kappa}{2}\left(m + \frac{\lambda\cos S}{2\tau^2(\lambda + \sin S)^2} + \varepsilon_0\frac{(1-\zeta)}{\tau^\zeta}\right) - 3\Lambda\tau, \tag{7.25}$$
$$S = \frac{1}{\tau},$$

with the potential

$$\mathcal{U} = -\frac{3}{2}\left\{\kappa\left[m\tau + \frac{1}{2(1+\lambda\sin S)} + \varepsilon_0\tau^{1-\zeta}\right] - \Lambda\tau^2\right\}. \tag{7.26}$$

It should be noted that unlike the case with F being a power law of $S = 1/\tau$, where the nonlinearity appears in the region with a large value of τ, in the case under consideration, a number of interesting properties emerge in the region where $0 < \tau < 1$, namely, in the vicinity of the singular point $\tau = 0$. A graphical view of the potential $\mathcal{U}(\tau)$ (Eq. (7.26)) is given in Figs. 8 and 9. Here we choose the problem parameters as follows: $\kappa = 2/3$, spinor mass $m = 1$, coupling constant $\lambda = 0.01$, cosmological constant $\Lambda = 2/3$, $\varepsilon_0 = 1$, and $\zeta = 2/3$. Since $S = 1/\tau$ and $\mathcal{U}(\tau) \propto 1/\sin(S)$, a large number of small oscillations occur as $\tau \longrightarrow 0$ [cf. Fig. 9].

It is clear from Figs. 8 and 9 that depending on the choice of integration constant E we have two types of

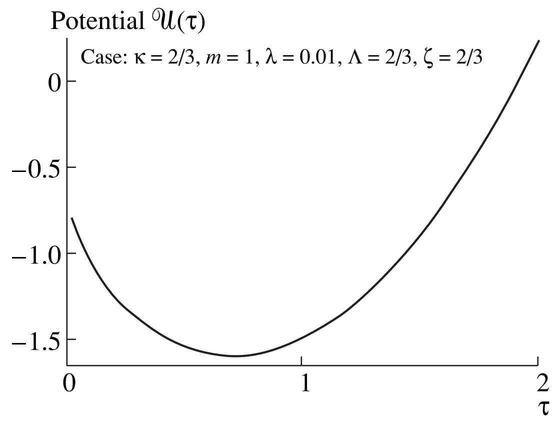

Fig. 8. The potential $\mathcal{U}(\tau)$ [Eq. (7.26)] with BI space-time being filled with perfect fluid describing a hard Universe.

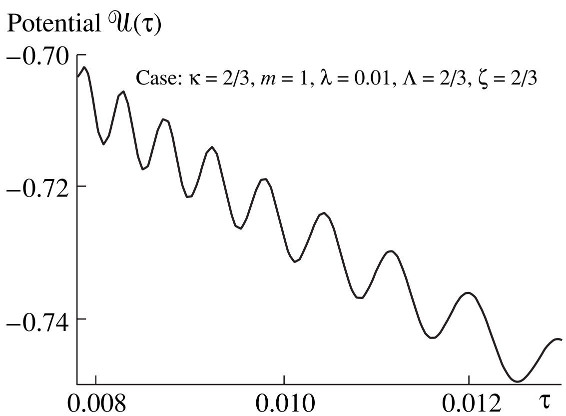

Fig. 9. Fragment of the potential (7.26) in the vicinity of the point $\tau = 0$ that occurs due to the nonlinear term F.

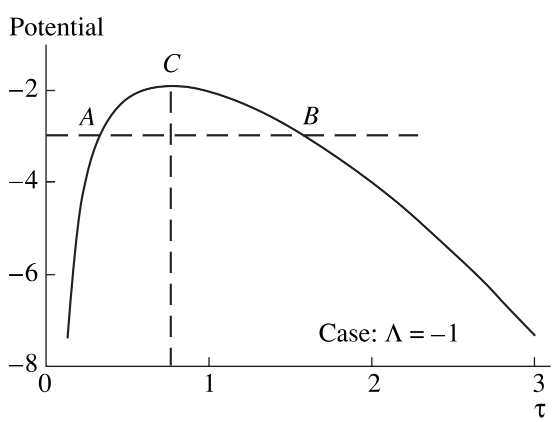

Fig. 10. View of the potential $\mathcal{U}(\tau)$ [Eq. (7.26)] with a negative Λ.

solutions demonstrated in Fig. 2. Moreover, for some values of E there exist more than one periodic solution.

Let us now study the system for a negative Λ. Contrary to the case with $F = S^n$, where all the solutions for a negative Λ grow exponentially, in this case, an interesting situation occurs for some special choice of parameters.

As one sees from Fig. 10, depending on the integration constant and the initial value of τ, the mode of evolution can be both finite and exponential. For the integration constant being at the level AB in Fig. 10 (here it is –3), with $\tau_0 \in (0, \tau_A)$, the evolution of τ is finite and similar to the one illustrated in Fig. 2 corresponding to $E = 1$, whereas, for $\tau_0 > \tau_B$ we have an exponentially expanding τ. Thus we conclude that for the interacting term being a trigonometric function of its arguments, the system even with a negative Λ admits a nonexponential mode of evolution.

To investigate the dominant energy condition let us write the components of the energy momentum tensor. For simplicity we set $C_0 = 1$ and in terms of S for the energy density we write

$$T_0^0 = mS + \frac{S^2}{2(1+\lambda\sin S)} + \varepsilon_0 S^{1+\zeta}. \qquad (7.27)$$

Since τ is a positive quantity, S is positive as well. As one sees from Eq. (7.27), for any positive value of S and $\lambda < 1$, energy density is always positive, definite, and proportional to S^2. Since $S = 1/\tau$, this means that T_0^0 reaches a maximum as $\tau \longrightarrow 0$ and tends to zero as $\tau \longrightarrow \infty$.

For the pressure components, we have

$$T_1^1 = T_2^2 = T_3^3$$
$$= \frac{\lambda S^3 \cos S}{2(1+\lambda\sin S)^2} - \frac{S^2}{2(1+\lambda\sin S)} - \varepsilon_0\zeta S^{1+\zeta}. \qquad (7.28)$$

As can be seen, for a $\lambda < 1$, the pressure T_1^1 may be either positive or negative depending on the sign of $\cos S$. Moreover, its maximum value is proportional to S^3. Thus, in the case of $F = \sin S$, for any ζ defined as in Eq. (3.33) and any nontrivial λ, there exist intervals (S_i, S_{i+1}) such that for $S \in (S_i, S_{i+1})$, the inequality $T_0^0 < T_1^1$ takes place as is shown in Fig. 11. Therefore, we conclude that the regular solutions obtained in this case result in the broken dominant energy condition.

8. CONCLUSIONS AND PROSPECTS

We consider a system of interacting spinor and scalar fields in a Bianchi type-I (BI) cosmological model. The nonlinearity in the spinor field Lagrangian is given by an arbitrary function of the invariants generated from the bilinear spinor forms $S = \bar{\psi}\psi$ and $P = i\bar{\psi}\gamma^5\psi$; the scalar Lagrangian is chosen as an arbitrary function of the scalar invariant $\Upsilon = \varphi_{,\alpha}\varphi^{,\alpha}$, which becomes linear at weak field limit. Self-consistent solutions to the nonlinear spinor, scalar, and BI gravitational field equations have been obtained. The problems of initial singu-

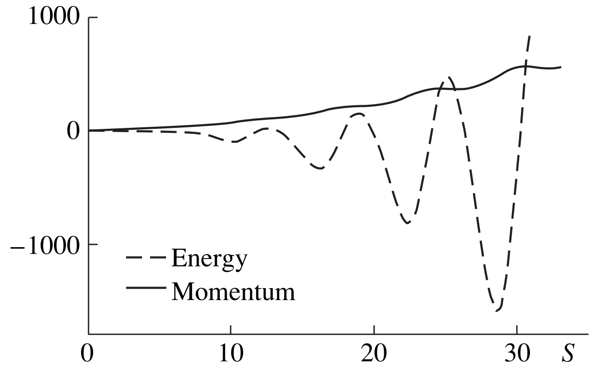

Fig. 11. Relative behavior of T_0^0 and T_1^1 with $F = \sin(S)$. This picture clearly shows the violation of the dominant energy condition that takes place in the case considered.

larity and asymptotical isotropization process have been thoroughly studied. It has been shown that, for some special type of nonlinearity, the model provides a regular solution. If the nonlinear term occurs due to spinor field self-action, this singularity-free solution results in violating the dominant energy condition in the Hawking–Penrose theorem, whereas, in the case of interacting spinor and scalar fields, it is possible to attain regular solutions without violating the dominant energy condition. It has been established that the spinor field nonlinearity plays a crucial role in the initial stage, while the spinor mass or scalar field nonlinearity in case of massless spinor is crucial at the asymptotic stage if the Λ term remains absent. It has also been shown that the introduction of Λ term in the Lagrangian generates oscillations of the BI model even in case of linear spinor field, whereas oscillatory solutions in case of nonlinear spinor field without the Λ term are subject to the choice of nonlinearity. Moreover, for the linear spinor field, the Λ term provides oscillatory solutions; these are regular everywhere without violating the dominant energy condition. Note that in the present paper we confined our study to the perfect fluid and magneto-fluid. Models with viscous and Van der Waals fluid, as well as dark energy, have recently been studied by us in a number of papers [95–99], and some other works are under preparation. We plan to review these in the near future. The nonlinear spinor field has also been studied within the framework of Bianchi VI [100] and plane-symmetric space-time [66, 101]. It was shown that the introduction of a nonlinear spinor field can accelerate the growth of the Universe. In connection with the recent acceleration of the Universe, this fact impels us to consider the spinor field as a possible candidate for the so-called source of dark energy. Some works with this assumption can be found in literature, e.g., Ribas et al. [102].

REFERENCES

1. A. F. Ranada, "Classical Nonlinear Dirac Field Models of Extended Particles," in *Quantum Theory, Groups, Fields and Particles*, Ed. by A. O. Barut (Reidel, Dordrecht, 1983), pp. 271–291.
2. D. D. Ivanenko, "Introduction to the Elementray Particle Theory," Usp. Fiz. Nauk. **32** (2), 149–184 (1947).
3. D. D. Ivanenko, "Introduction to the Elementray Particle Theory," Usp. Fiz. Nauk. **32** (3), 261–315 (1947).
4. V. Rodichev, "Twisted Space and Nonlinear Field Equations," Zh. Eksp. Teor. Fiz. **40**, 1469–1472 (1961) [Sov. Phys. JETP **13**, 1029 (1961)].
5. H. Weyl, "A Remark on the Coupling of Gravitation and Electron," Phys. Rev. **77**, 699–701 (1950).
6. R. Utiyama, "Invariant Theoretical Interpretation of Interaction," Phys. Rev. **101**, 1597–1607 (1956).
7. T. W. B. Kibble, "Lorentz Invariance and the Gravitational Filed," J. Math. Phys. **2**, 212–221 (1961).
8. D. W. Sciama, *Festschrift for Infeld* (Pergamon, Oxford, 1960), pp. 415–439.
9. A. F. Ranada and M. Soler, "Elementary Spinorial Excitations in a Model Universe," J. Math. Phys. **13**, 671–675 (1972).
10. F. W. Hehl, P. Heyde, and G. D. Kerlick, "General Relativity with Spin and Torsion: Foundations and Prospects," Rev. Mod. Phys. **48**, 393–416 (1976).
11. W. Heisenberg, "Doubts and Hopes in Quantum-Electrodynamics," Physica **19**, 897–908 (1953).
12. W. Heisenberg, "Quantum Theory of Fields and Elementary Particles," Rev. Mod. Phys. **29** (3), 269–278 (1957).
13. D. J. Gross and A. Neveu, "Dynamical Symmetry Breaking in Asymptotically Free Field Theories," Phys. Rev. D **10**, 3235–3253 (1974).
14. R. Finkelstein, R. LeLevier, and M. Ruderman, "Nonlinear Spinor Fields," Phys. Rev. **83**, 326–332 (1951).
15. C. Armendáriz-Picón and P. B. Greene, "Spinors, Inflation, and Non-Singular Cyclic Cosmologies," Gen. Relativ. Gravit. **35**, 1637–1658 (2003).
16. L. Parker, "Quantized Fields and Particle Creation in Expanding Universes. I," Phys. Rev. **183**, 1057–1068 (1969).
17. L. Parker, "Quantized Fields and Particle Creation in Expanding Universes. II," Phys. Rev. D **3**, 346–356 (1971).
18. C. W. Misner, "The Isotropy of the Universe," Astrophys. J. **151**, 431–457 (1968).
19. Ya. B. Zel'dovich, "Creation of Particles in Cosmology," Pis'ma Zh. Eksp. Teor. Fiz. **12**, 443 (1970) [JETP Lett. **12**, 307 (1970)].
20. V. N. Lukash and A. A. Starobinsky, "Isotropization of Cosmological Expansion Due to Particle Production," Zh. Eksp. Teor. Fiz. **66**, 1515 (1974) [Sov. Phys. JETP **39**, 742 (1974)].
21. V. N. Lukash, I. D. Novikov, A. A. Starobinsky, and Ya. B. Zel'dovich, "Quantum Effects and Evolution of Cosmological Models," Nuovo Cimento **35**, 293–307 (1976).
22. B. L. Hu and L. Parker, "Anisotropy Damping Through Quantum Effects in the Early Universe," Phys. Rev. D **17**, 933–945 (1978).
23. V. A. Belinskii, I. M. Khalatnikov, and E. M. Lifshitz, "Oscillatory Approach to a Singular Point in the Relativistic Cosmology," Adv. Phys. **19**, 525–573 (1970).

24. K. C. Jacobs, "Spatially Homogeneous and Euclidean Cosmological Models with Shear," Astrophys. J. **153**, 661–678 (1968).
25. L. P. Chimento and M. S. Mollerach, "Dirac Equation in Bianchi I Metrics," Phys. Lett. A **121** (1), 7–10 (1987).
26. M. A. Castagnino, C. D. El Hasi, F. D. Mazzitelli, and J. P. Paz, "On the Dirac Equation in Anisotropic Backgrounds," Phys. Lett. A **128** (1), 25–28 (1988).
27. M. Henneaux, "Bianchi Type-I Cosmologies and Spinor Fields," Phys. Rev. D **21**, 857–863 (1980).
28. M. Henneaux, "Univers De Bianchi Et Champs Spinoriels," Ann. Inst. Henri Poincaré **34**, 329–349 (1981).
29. B. L. Hu, "Gravitational Waves in a Bianchi Type-I Universe," Phys. Rev. D **18**, 969–982 (1978).
30. P. G. Miedema and W. A. van Leeuwen, "Cosmological Perturbations in Bianchi Type-I Universes," Phys. Rev. D **47**, 3151–3164 (1993).
31. H. T. Cho and A. D. Speliotopoulos, "Gravitational Waves in Bianchi Type-I Universes: The Classical Theory," Phys. Rev. D **5**, 5445–5458 (1995).
32. G. N. Shikin, "Nonlinear Spinor Fields in External Cosmological Field and the Problem of Elimination of Initial Singularity," Preprint No. 19, IPBRAE (Academy of Science of USSR, 1991), pp. 1–21.
33. Yu. P. Rybakov, B. Saha, and G. N. Shikin, "Nonlinear Spinor Field in External Bianchi-I Type Gravitational Field and the Problem of Eliminating Initial Singularity," PFU Rep., Phys. **2** (2), 61–78 (1994).
34. Yu. P. Rybakov, B. Saha, and G. N. Shikin, "Exact Self-Consistent Solutions to Nonlinear Spinor Field Equations in Bianchi Type-I Space-Time," Commun. Theor. Phys. **3**, 199–210 (1994).
35. B. Saha and G. N. Shikin, "Nonlinear Spinor Field in Bianchi Type-I Universe Filled with Perfect Fluid: Exact Self-Consistent Solutions," J. Math. Phys. **38**, 5305–5318 (1997).
36. R. Alvarado, Yu. P. Rybakov, B. Saha, and G. N. Shikin, "Exact Self-Consistent Solutions to the Interacting Spinor and Scalar Field Equations in Bianchi Type-I Space-Time," Commun. Theor. Phys. **4**, 247–262 (1995).
37. R. Alvarado, Yu. P. Rybakov, B. Saha, and G. N. Shikin, "Interacting Spinor and Scalar Fields in Bianchi Type-I Space-Time: Exact Self-Consistent Solutions," Izv. Vyssh. Uchebn. Zaved., Fiz. **38**, 53–58 (1995).
38. B. Saha and G. N. Shikin, "Interacting Spinor and Scalar Fields in Bianchi Type I Universe Filled with Perfect Fluid: Exact Self-Consistent Solutions," Gen. Relativ. Gravit. **29**, 199–1112 (1997); gr-gc/9609056.
39. B. Saha and G. N. Shikin, "On the Role of Λ Term in the Evolution of Bianchi-I Cosmological Model with Nonlinear Spinor Field," PFU Rep., Phys., No. 8, 17–20 (2000).
40. B Saha, "Dirac Spinor in Bianchi-I Universe with time dependent Gravitational and Cosmological Constants," Mod. Phys. Lett. A **16**, 1287–1296 (2001).
41. A. Einstein, "Kosmologische Betrachtungen Zur Allgemeinen Relativita Tstheorie," Sitzungsber. Preuss. Acad. Wiss. **1**, 142–152 (1917).
42. A. Einstein, "Spielen Die Gravitationsfelder Im Aufbau Der Materiellen Elementarteilchen Eine Wesentliche Rolle?," Sitzungsber. Preuss. Acad. Wiss. **1**, 349–356 (1919).
43. M. Tsamparlis and P. S. Apostolopoulos, "Symmetries of Bianchi I Space-Times," J. Math. Phys. **41**, 7573–7588 (2000).
44. G. F. R. Ellis, "Dynamics of Pressure-Free Matter in General Relativity," J. Math. Phys. **8** (5), 1171–1194 (1967).
45. J. M. Stewart and G. F. R. Ellis, "Solutions of Einstein's Equations for a Fluid Which Exhibit Local Rotational Symmetry," J. Math. Phys. **9** (7), 1072–1082 (1968).
46. V. B. Berestetskii, E. M. Lifshitz, and L. P. Pitaevskii, *Quantum Electrodynamics* (Nauka, Moscow, 1989; Pergamon, Oxford, 1982).
47. V. A. Zhelnorovich, *Spinor Theory and Its Application in Physics and Mechanics* (Nauka, Moscow, 1982) [in Russian].
48. D. Brill and J. Wheeler, "Interaction of Neutrinos and Gravitational Fields," Rev. Mod. Phys. **29**, 465–479 (1957).
49. N. N. Bogoliubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields* (Nauka, Moscow, 1976; Wiley, New York, 1980).
50. P. A. M. Dirac, *General Theory of Relativity* (Wiley, New York, 1975; Atomizdat, Moscow, 1978).
51. G. A. Milekhin, "Nonlinear Scalar Fields and Multiple Particle Creation," Izv. Akad. Nauk SSSR, Ser. Fiz. **26**, 635–641 (1962).
52. Ya. B. Zel'dovich and I. D. Novikov, *Structure and Evolution of the Universe* (Nauka, Moscow, 1975) [in Russian].
53. B. Saha, "Spinor Field in Bianchi Type-I Universe: Regular Solutions," Phys. Rev. D **64**, 123501 (2001); gr-qc/0107013.
54. E. Kamke, *Differentialgleichungen Losungsmethoden und Losungen* (Akademische Verlagsgesellschaft, Leipzig, 1957).
55. V. N. Mitskevich, A. P. Efremov, and A. I. Nesterov, *Dynamics of Fields in General Relativity* (Energoatomizdat, Moscow, 1985) [in Russian].
56. K. A. Bronnikov and G. N. Shikin, "Cylindrically Symmetric Solitons with Nonlinear Self-Gravitating Scalar Fields," Gravit. Cosmol. **7**, 231–240 (2001).
57. S. Fay, "Sufficient Conditions for Curvature Invariants to Avoid Divergences in Hyperextended Scalar-Tensor Theory for Bianchi Models," Class. Quantum. Grav. **17**, 2663–2673 (2000).
58. A. L. Zel'manov and V. G. Agakov, *Elements of the General Theory of Relativity* (Nauka, Moscow, 1989) [in Russian].
59. N. V. Mitskievich, *Physical Fields in General Relativity* (Nauka, Moscow, 1969) [in Russian].
60. D. D. Rabounski and L. B. Borisova, "Particles Here and Beyond the Mirror," gr-qc/0304018.
61. A. Guth, "Inflationary Universe: A Possible Solution to the Horizon and Flatness Problems," Phys. Rev. D **23**, 347–356 (1981).
62. T. Padmanabhan, "Cosmological Constant—the Weight of the Vacuum," Phys. Rep. **38** (5–6), 235–320 (2003).
63. S. W. Hawking and R. Penrose, "The Singularities of Gravitational Collapse and Cosmology," Proc. R. Soc. Math. Phys. Sci. London **314**, 529–548 (1970).

64. D. D. Ivanenko, *An Attempt to construct the Unified Nonlinear Spinor Theory of Matter: Nonlinear Quantum Theory of Fields* (Inostrannaya Literatura, Moscow, 1959) [in Russian].

65. W. Heisenberg, *Introduction to the Unifled Field Theory of Elementary Particles* (Wiley, New York, 1966; Mir, Moscow, 1968).

66. B. Saha and G. N. Shikin, "Plane-Symmetric Solitons of Spinor and Scalar Fields," Czech. J. Phys. **54**, 597–620 (2004).

67. H. Dicke, "Dirac's Cosmology and Mach's Principle," Nature (London) **192**, 440–441 (1961).

68. S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (Wiley, New York, 1972; Mir, Moscow, 1975).

69. Y. S. Wu and Z. Wang, "Time Variation of Newton's Gravitational Constant in Superstring Theories," Phys. Rev. Lett. **57**, 1978–1981 (1986).

70. K. Maeda, "On Time Variation of Fundamental Constants in Superstring Theories," Mod. Phys. Lett. A **3** (3), 243–249 (1988).

71. T. Damour, G. W. Gibbons, and J. H. Taylor, "Limits on the Variability of G Using Binary-Pulsar Data," Phys. Rev. Lett. **61** (10), 1151–1154 (1988).

72. W. Chen and Y. S. Wu, "Implications of a Cosmological Constant Varying as R^{-2}," Phys. Rev. D **41**, 695–698 (1990).

73. A. M. M. Abdel-Rahman, "A Critical Density Cosmological Model with Varying Gravitational and Cosmological "Constants"," Gen. Relativ. Gravit. **22**, 655–663 (1990).

74. M. S. Berman, "Static Universe in a Modified Brans-Dicke Cosmology," Int. J. Theor. Phys. **29**, 567–570 (1990).

75. M. S. Berman, "Kantowski-Sachs Cosmological Models with Constant Deceleration Parameter," Nuovo Cimento B **105**, 239–242 (1990).

76. M. S. Berman, M. M. Som, and F. M. Gomide, "Brans-Dicke Static Universe," Gen. Relativ. Gravit. **21**, 287–292 (1989).

77. M. S. Berman and F. M. Gomide, "Cosmological Models with Constant Deceleration Parameters," Gen. Relativ. Gravit. **20**, 191–198 (1988).

78. T. Singh and A. K. Agrawal, "Homogeneous Anisotropic Cosmological Models with Variable Gravitational and Cosmological Constants," Int. J. Theor. Phys. **32**, 1041–1059 (1993).

79. J. A. Belinchon, "Perfect Fluid LRS Bianchi I with Time Varying Constants," gr-qc/0411005 (2004).

80. J. A. Belinchon and I. Chakrabarty, "Perfect Fluid Cosmological Models with Time-Varying Constants," Int. J. Mod. Phys. D **12**, 1113–1129 (2003).

81. J. A. Belinchon and I. Chakrabarty, "Full Causal Bulk Viscous Cosmologies with Time-Varying Constants," Int. J. Mod. Phys. D **12**, 861–883 (2003).

82. A. Pradhan and P. Pandey, "Plane-Symmetric Inhomogeneous Magnetized Viscous Fluid Universe with a Variable," Czech. J. Phys. **55**, 749–764 (2005).

83. A. Pradhan, P. Pandey, G. P. Singh, and R. V. Deshpandey, "Causal Bulk Viscous LRS Bianchi I Models With Variable Gravitational and Cosmological "Constant"," gr-qc/03100023 (2003).

84. A. Pradhan, S. K. Srivastav, and R. S. Singh, "Tilted Bianchi Type V Bulk Viscous Cosmological Models with Varying Λ-Term," gr-qc/0408043 (2004).

85. K. S. Thorne, "Primordial Element Formation, Primordial Magnetic Fields, and the Isotropy of the Universe," Astrophys. J. **148**, 51–68 (1967).

86. K. C. Jacobs, "Spatially Homogeneous and Euclidean Cosmological Models with Shear," Astrophys. J. **153**, 661–678 (1968).

87. K. C. Jacobs, "Cosmologies of Bianchi Type I with a Uniform Magnetic Field," Astrophys. J. **155**, 379–391 (1969).

88. R. Bali, "Magnetized Cosmological Model," Int. J. Theor. Phys **25**, 755–761 (1986).

89. A. Lichnerowicz, *Relativistic Hydrodynamics and Magnetohydrodynamics: Lectures on the Existence of Solutions* (Benjamin, New York, 1967).

90. B. Saha, "Interacting Scalar and Spinor Fields in Bianchi Type I Universe Filled with Magneto-Fluid," J. Astrophys. Space Sci. **299**, 149–158 (2005); gr-qc/0309062.

91. L. D. Landau and E. M. Lifshitz, *Course of Theoretical Physics*, Vol. 1: *Mechanics* (Nauka, Moscow, 1982; Pergamon, New York, 1988).

92. E. P. Zhidkov and I. V. Puzynin, "Solution to the Boundary Value Problems of Second Order Nonlinear Differential Equations with Stabilization Methods," Dokl. Akad. Nauk USSR **174**, 271–273 (1967).

93. B. Greene, *The Elegant Universe* (Vintage, London, 2000).

94. B. Saha and T. Boyadjiev, "Bianchi Type-I Cosmology with Scalar and Spinor Fields," Phys. Rev. D **69**, 124010 (2004).

95. B. Saha, "Anisotropic Cosmological Models with Perfect Fluid and Dark Energy," Chin. J. Phys. **43**, 1035–1043 (2005); gr-qc/0412078.

96. B. Saha, "Anisotropic Cosmological Models with a Perfect fluid and a Λ Term," Astrophys. Space Sci. (in press); gr-qc/0411080.

97. B. Saha and V. Rikhvitsky, "Bianchi Type I Universe with Viscous Fluid: A Qualitative Analysis," gr-qc/0410056.

98. B. Saha, "Bianchi Type Universe with Viscous Fluid," Mod. Phys. Lett. A **20**, 2127–2143 (2005); gr-qc/0409104.

99. B. Saha, "Nonlinear Spinor Field in Bianchi Type-I Universe Filled with Viscous Fluid: Some Special Solutions," Rom. Rep. Phys. **57**, 7–24 (2005).

100. B. Saha, "Nonlinear Spinor Field in Cosmology," Phys. Rev. D **69**, 124006 (2004); gr-qc/0308088.

101. B. Saha and G. N. Shikin, "Static Plane-Symmetric Nonlinear Spinor and Scalar Fields in GR," Int. J. Theor. Phys. 44, 1459 (2005).

102. M. O. Ribas, F. P. Devecchi, and G. M. Kremer, "Fermions as Sources of Accelerated Regimes in Cosmology," gr-qc/0511099.