Spinor field in a Bianchi type-I universe: Regular solutions

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Self-consistent solutions to the nonlinear spinor field equations in general relativity are studied for the case of Bianchi type-I (BI) space-time. It is shown that, for some special type of nonlinearity the model provides a regular solution, but this singularity-free solution is attained at the cost of breaking the dominant energy condition in the Hawking-Penrose theorem. It is also shown that the introduction of a Λ term in the Lagrangian generates oscillations of the BI model, which is not the case in the absence of a Λ term. Moreover, for the linear spinor field, the Λ term provides oscillatory solutions, which are regular everywhere, without violating the dominant energy condition.

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I. INTRODUCTION

Nonlinear phenomena have been one of the most popular topics during recent years. Nevertheless, it must be admitted that nonlinear classical fields have not received general consideration. This is probably due to the mathematical difficulties which arise because of the nonrenormalizability of the Fermi and other nonlinear couplings [1]. Nonlinear self-couplings of the spinor fields may arise as a consequence of the geometrical structure of the space-time and, more precisely, because of the existence of torsion. As early as 1938, Ivanenko [2–4] showed that a relativistic theory imposes in some cases a fourth-order self-coupling. In 1950, Weyl [5] proved that, if the affine and the metric properties of the space-time are taken as independent, the spinor field obeys either a linear equation in space with torsion or a nonlinear one in a Riemannian space. As the self-action is of spin-spin type, it allows the assignment of a dynamical role to the spin and offers a clue about the origin of the nonlinearities. This question was further clarified in some important papers by Utiyama, Kibble, and Sciama [6–8]. In the simplest scheme, the self-action is of pseudovector type, but it can be shown that one can also get a scalar coupling [9]. An excellent review of the problem may be found in [10]. Nonlinear quantum Dirac fields were used by Heisenberg [11,12] in his ambitious unified theory of elementary particles. They are presently the object of renewed interest since the widely known paper by Gross and Neveu [13].

The quantum field theory in curved space-time has been a matter of great interest in recent years because of its applications to cosmology and astrophysics. The evidence of the existence of strong gravitational fields in our Universe led to the study of the quantum effects of material fields in external classical gravitational field. Since the appearance of Parker's paper on scalar fields [14] and spin-$\frac{1}{2}$ fields [15], several authors have studied this subject. The present cosmology is based largely on Friedmann's solutions of the Einstein equations, which describe the completely uniform and isotropic universe (''closed'' and ''open'' models, i.e., bounded or unbounded universe). The main feature of these solutions is their nonstationarity. The idea of an expanding Universe, following from this property, is confirmed by the astronomical observations and it is now safe to assume that the isotropic model provides, in its general features, an adequate description of the present state of the Universe. Although the Universe seems homogenous and isotropic at present, it does not necessarily mean that it is also suitable for description of the early stages of the development of the Universe and there are no observational data guaranteeing the isotropy in the era prior to the recombination. In fact, there are theoretical arguments that support the existence of an anisotropic phase that approaches an isotropic one [16]. Interest in studying Klein-Gordon and Dirac equations in anisotropic models has increased since Hu and Parker [17] have shown that the creation of scalar particles in anisotropic backgrounds can dissipate the anisotropy as the Universe expands.

A Bianchi type-I (BI) universe, being the straightforward generalization of the flat Robertson-Walker (RW) universe, is one of the simplest models of an anisotropic universe that describes a homogenous and spatially flat universe. Unlike the RW universe, which has the same scale factor for each of the three spatial directions, a BI universe has a different scale factor in each direction, thereby introducing an anisotropy to the system. It moreover has the agreeable property that near the singularity it behaves like a Kasner universe, even in the presence of matter, and consequently falls within the general analysis of the singularity given by Belinskii *et al.* [18]. Also in a universe filled with matter for $p=\zeta\varepsilon,\ \zeta<1$, it has been shown that any initial anisotropy in a BI universe quickly dies away and a BI universe eventually evolves into a Friedmann-RW (FRW) universe [19]. Since the present-day universe is surprisingly isotropic, this feature of the BI universe makes it a prime candidate for studying the possible effects of an anisotropy in the early universe on present-day observations. In light of the importance mentioned above, several authors have studied a BI universe from different aspects.

In [20], Chimento and Mollerach studied the Dirac equations in a BI universe and obtained their classical solutions. They also claimed that for each value of the momentum only two independent solutions exist and they showed that it is not possible to obtain the solutions from those of a FRW universe only by perturbation. One of the solutions obtained

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would describe a particle with a given helicity, while the other one would represent antiparticles with the opposite helicity. This fact posed a very interesting problem. Spin-$\frac{1}{2}$ particles cannot live in a BI, at least if they keep their well-known properties of flat space-time. This problem was handled by Castagnino *et al.* [21], where they showed that if the Dirac equation is separable, the number of independent solutions is four, contrary to the claim made in [20]. A spinor field in a BI universe was also studied by Belinskii and Khalatnikov [22]. In this paper they solved Einstein-Dirac equations when both the cosmological constant and the mass of the spinor field vanish (neutrinos). They also noticed that for BI models filled with neutrinos, the principal directions of expansion vary with time. Using Hamiltonian techniques, Henneaux studied class-A Bianchi universes generated by a spinor source [23,24]. In [23], he derived the general solution to the massive Dirac equation in Bianchi type-I space-time with a cosmological constant [23], which was further extended for the Bianchi type-II model [24].

In a number of papers [25–27], several authors studied the behavior of gravitational waves (GWs) in a BI universe. In [26] the evolution equations for small perturbations in the metric, energy density, and material velocity were derived for an anisotropic viscous BI universe. It has been shown that the results were independent of the equation of state of the cosmic fluid and its viscosity. They also showed that the GWs need not necessarily be transversal in an anisotropically expanding BI universe and the longitudinal components of the gravitational waves have no physical significance. In [27], Cho and Speliotopoulos studied the propagation of classical gravitational waves in a BI universe. They found that GWs in a BI universe are not equivalent to two minimally coupled massless scalar fields as in a FRW universe. Because of its tensorial nature, the GW is much more sensitive to the anisotropy in space-time than the scalar field is and it gains an effective mass term. Moreover, they found a coupling between the two polarization states of the GW, which is not present in a FRW universe.

A nonlinear spinor field (NLSF) in an external FRW cosmological gravitational field was first studied by Shikin in 1991 [28]. The main purpose of introducing a nonlinear term in the spinor field Lagrangian is to study the possibility of the elimination of initial singularity. Following [28], we analyzed the nonlinear spinor field equations in an external BI universe [29]. In that paper, we consider the nonlinear term in the spinor field Lagrangian as an arbitrary function of all possible invariants generated from spinor bilinear forms. There we also studied the possibility of the elimination of initial singularity, especially for the Kasner universe. For a few years we studied the behavior of a self-consistent NLSF in a BI universe [30,31] both in the presence of perfect fluid and without it, which was followed by Refs. [32–34], where we studied the self-consistent system of interacting spinor and scalar fields. Recently, we studied [35,36] the role of the cosmological constant (Λ) in the Lagrangian, which, together with Newton's gravitational constant (G), is considered to be the fundamental constant in Einstein's theory of gravity [37].

II. REVIEW OF BI COSMOLOGY

A diagonal Bianchi type-I space-time (hereafter BI) is a spatially homogeneous space-time, which admits an Abelian group G_3, acting on spacelike hypersurfaces, generated by the spacelike Killing vectors $\boldsymbol{\xi}_1=\partial_1$, $\boldsymbol{\xi}_2=\partial_2$, and $\boldsymbol{\xi}_3=\partial_3$. In synchronous coordinates, the metric is [38,39]

$$ds^2=dt^2-\sum_{i=1}^{3} a_i^2(t)dx_i^2\,. \tag{2.1}$$

If the three scale factors are equal (i.e., $a_1=a_2=a_3$), Eq. (2.1) describes an isotropic and spatially flat Friedmann-Robertson-Walker (FRW) universe. The BI universe has a different scale factor in each direction, thereby introducing an anisotropy to the system. Thus, a Bianchi type-I universe, being the straightforward generalization of the flat FRW universe, is one of the simplest models of an anisotropic universe that describes a homogeneous and spatially flat universe. When two of the metric functions are equal (e.g., $a_2=a_3$), the BI space-time is reduced to the important class of plane symmetric space-time (a special class of the locally rotational symmetric space-times [40,41]), which admits a G_4 group of isometries acting multiply transitively on the spacelike hypersurfaces of homogeneity generated by the vectors $\boldsymbol{\xi}_1$, $\boldsymbol{\xi}_2$, $\boldsymbol{\xi}_3$, and $\boldsymbol{\xi}_4=x^2\partial_3-x^3\partial_2$. The BI has the agreeable property that near the singularity it behaves like a Kasner universe, given by

$$a_1(t)=a_1^0t^{p_1}, \quad a_2(t)=a_2^0t^{p_2}, \quad a_3(t)=a_3^0t^{p_3}, \tag{2.2}$$

with p_j being the parameters of the BI space-time which measure the relative anisotropy between any two asymmetry axes and satisfy the constraints

$$p_1+p_2+p_3=1, \tag{2.3a}$$

$$p_1^2+p_2^2+p_3^2=1. \tag{2.3b}$$

Thus out of three parameters, only one is arbitrary. One particular choice of parametrization is

$$p_1=\frac{-p}{p^2+p+1}, \tag{2.4a}$$

$$p_2=\frac{p(p+1)}{p^2+p+1}, \tag{2.4b}$$

$$p_3=\frac{p+1}{p^2+p+1}. \tag{2.4c}$$

The condition $0\leqslant p\leqslant 1$ on p then yields the condition $-\frac{1}{3}\leqslant p_1\leqslant 0$, $0\leqslant p_2\leqslant\frac{2}{3}$, $\frac{2}{3}\leqslant p_3\leqslant 1$. Another particular parametrization can be given using an angle on the unit circle, since Eqs. (2.3) describe the intersection of a sphere with a plane in the parameter space (p_1,p_2,p_3):

$$p_1=\frac{1}{3}(1+\cos\vartheta+\sqrt{3}\sin\vartheta), \tag{2.5a}$$

$$p_2=\frac{1}{3}(1+\cos\vartheta-\sqrt{3}\sin\vartheta), \tag{2.5b}$$

$$p_3=\frac{1}{3}(1-2\cos\vartheta). \tag{2.5c}$$

Although ϑ ranges over the unit circle, the labeling of each p_j is quite arbitrary. Thus the unit circle can be divided into six equal parts, each of which span 60°, and the choice of p_j is unique within each section separately. For $\vartheta=0$, $p_1=p_2=\frac{2}{3}$ and $p_3=-\frac{1}{3}$ while for $\vartheta=\pi/3$, $p_1=1$ and $p_2=p_3=0$.

Let us now go back to the BI metric. The nontrivial Christoffel symbols for Eq. (2.1) are

$$\Gamma_{ii}^{0}=a_i\dot{a}_i, \quad \Gamma_{0i}^{i}=\Gamma_{i0}^{i}=\frac{\dot{a}_i}{a_i}, \tag{2.6}$$

while the components of the nontrivial Ricci tensor read

$$R_{00}=-\sum_{i=1}^{3}\frac{\ddot{a}_i}{a_i}, \quad R_{ii}=\left[\frac{\ddot{a}_i}{a_i}+\frac{\dot{a}_i}{a_i}\left(\frac{\dot{a}_j}{a_j}+\frac{\dot{a}_k}{a_k}\right)\right]a_i^2,$$

$$i,j,k=1,2,3, \quad i\neq j\neq k. \tag{2.7}$$

The Ricci scalar for the BI universe has the form

$$R=-2\left(\frac{\ddot{a}_1}{a_1}+\frac{\ddot{a}_2}{a_2}+\frac{\ddot{a}_3}{a_3}+\frac{\dot{a}_1}{a_1}\frac{\dot{a}_2}{a_2}+\frac{\dot{a}_2}{a_2}\frac{\dot{a}_3}{a_3}+\frac{\dot{a}_3}{a_3}\frac{\dot{a}_1}{a_1}\right). \tag{2.8}$$

Sometimes it proves convenient to introduce a new time parameter η by

$$\eta=\int^{t}a^{-1}(\bar{t})d\bar{t}, \tag{2.9}$$

where we define

$$[a(t)]^2=C(t)\equiv(a_1a_2a_3)^{2/3}=(C_1C_2C_3)^{1/3}, \tag{2.10}$$

with $C_i\equiv a_i^2$. Note that in the isotropic limit, i.e., $a_1=a_2=a_3$, η reduces to conformal time. Further, defining

$$d_i=\frac{C_i'}{C_i}, \quad D\equiv\frac{1}{3}\sum_{i=1}^{3}d_i=\frac{C'}{C}, \quad Q\equiv\frac{1}{72}\sum_{i<j}(d_i-d_j)^2, \tag{2.11}$$

where the prime denotes differentiation with respect to η, we get the following nonzero Christoffel symbols for the metric (2.1):

$$\Gamma_{\eta\eta}^{\eta}=\frac{1}{2}D, \quad \Gamma_{ii}^{\eta}=\frac{1}{2}\frac{d_iC_i}{C}, \quad \Gamma_{i\eta}^{i}=\Gamma_{\eta i}^{i}=\frac{1}{2}d_i. \tag{2.12}$$

The nonzero components of the Ricci tensor now read

$$R_{\eta\eta}=\frac{3}{2}D'+6Q, \quad R_{ii}=-\frac{C_i}{2C}(d_i'+d_iD) \tag{2.13}$$

and the Ricci scalar

$$R=C^{-1}\left(3D'+\frac{3}{2}D^2+6Q\right). \tag{2.14}$$

Note that in the sections to follow, we work with the usual time t.

III. FUNDAMENTAL EQUATIONS AND GENERAL SOLUTIONS

The action of the nonlinear spinor and gravitational fields can be written as

$$\mathcal{S}(g;\psi,\bar{\psi})=\int L\sqrt{-g}\,d\Omega \tag{3.1}$$

with

$$L=L_g+L_{\rm sp}+L_{\rm m}. \tag{3.2}$$

Here L_g corresponds to the gravitational field

$$L_g=\frac{R+2\Lambda}{2\kappa}, \tag{3.3}$$

where R is the scalar curvature, $\kappa=8\pi G$, with G being Einstein's gravitational constant and Λ is the cosmological constant. The spinor field Lagrangian $L_{\rm sp}$ is given by

$$L_{\rm sp}=\frac{i}{2}[\bar{\psi}\gamma^{\mu}\nabla_{\mu}\psi-\nabla_{\mu}\bar{\psi}\gamma^{\mu}\psi]-m\bar{\psi}\psi+L_N, \tag{3.4}$$

where the nonlinear term L_N describes the self-interaction of a spinor field and can be presented as some arbitrary functions of invariants generated from the real bilinear forms of a spinor field. Since ψ and $\psi^{\star}$ (complex conjugate of ψ) have four component each, one can construct $4\times4=16$ independent bilinear combinations. They are

$$S=\bar{\psi}\psi \quad (\text{scalar}), \tag{3.5a}$$

$$P=i\bar{\psi}\gamma^5\psi \quad (\text{pseudoscalar}), \tag{3.5b}$$

$$v^{\mu}=(\bar{\psi}\gamma^{\mu}\psi) \quad (\text{vector}), \tag{3.5c}$$

$$A^{\mu}=(\bar{\psi}\gamma^5\gamma^{\mu}\psi) \quad (\text{pseudovector}), \tag{3.5d}$$

$$T^{\mu\nu}=(\bar{\psi}\sigma^{\mu\nu}\psi) \quad (\text{antisymmetric tensor}), \tag{3.5e}$$

where $\sigma^{\mu\nu}=(i/2)[\gamma^{\mu}\gamma^{\nu}-\gamma^{\nu}\gamma^{\mu}]$. Invariants, corresponding to the bilinear forms, are

$$I=S^2, \tag{3.6a}$$

$$J=P^2, \tag{3.6b}$$

$$I_v=v_{\mu}v^{\mu}=(\bar{\psi}\gamma^{\mu}\psi)g_{\mu\nu}(\bar{\psi}\gamma^{\nu}\psi), \tag{3.6c}$$

$$I_A=A_{\mu}A^{\mu}=(\bar{\psi}\gamma^5\gamma^{\mu}\psi)g_{\mu\nu}(\bar{\psi}\gamma^5\gamma^{\nu}\psi), \tag{3.6d}$$

$$I_T = T_{\mu\nu}T^{\mu\nu} = (\bar{\psi}\sigma^{\mu\nu}\psi)g_{\mu\alpha}g_{\nu\beta}(\bar{\psi}\sigma^{\alpha\beta}\psi). \tag{3.6e}$$

According to the Pauli-Fierz theorem [42], among the five invariants only I and J are independent as all others can be expressed by them: $I_v = -I_A = I+J$ and $I_T = I-J$. Therefore, we choose the nonlinear term F to be the function of I and J only, i.e., $L_N = F(I,J)$, thus claiming that it describes the nonlinearity in its most general form. L_m is the Lagrangian of a perfect fluid.

Variation of Eq. (3.1) with respect to a spinor field, ψ ($\bar{\psi}$) gives the nonlinear spinor field equations

$$i\gamma^\mu \nabla_\mu \psi - m\psi + \mathcal{D}\psi + \mathcal{G}i\gamma^5\psi = 0, \tag{3.7a}$$

$$i\nabla_\mu \bar{\psi}\gamma^\mu + m\bar{\psi} - \mathcal{D}\bar{\psi} - \mathcal{G}i\bar{\psi}\gamma^5 = 0, \tag{3.7b}$$

where we denote

$$\mathcal{D} = 2S\frac{\partial F}{\partial I}, \quad \mathcal{G} = 2P\frac{\partial F}{\partial J}.$$

Varying Eq. (3.1) with respect to the metric tensor $g_{\mu\nu}$, one finds Einstein's field equation

$$R^\mu_\nu - \frac{1}{2}\delta^\mu_\nu R = -\kappa T^\mu_\nu + \Lambda\delta^\mu_\nu, \tag{3.8}$$

where R^μ_ν is the Ricci tensor, $R = g^{\mu\nu}R_{\mu\nu}$ is the Ricci scalar, and T^μ_ν is the energy-momentum tensor of the material field given by

$$T^\nu_\mu = T^\nu_{\mathrm{sp}\,\mu} + T^\nu_{m\,\mu}. \tag{3.9}$$

Here $T^\nu_{\mathrm{sp}\,\mu}$ is the energy-momentum tensor of the spinor field

$$T^\rho_{\mathrm{sp}\,\mu} = \frac{i}{4}g^{\rho\nu}(\bar{\psi}\gamma_\mu\nabla_\nu\psi + \bar{\psi}\gamma_\nu\nabla_\mu\psi - \nabla_\mu\bar{\psi}\gamma_\nu\psi - \nabla_\nu\bar{\psi}\gamma_\mu\psi) - \delta^\rho_\mu L_{sp}, \tag{3.10}$$

where L_{sp} with respect to Eqs. (3.7) takes the form

$$L_{sp} = -(\mathcal{D}S + \mathcal{G}P) + F(I,J). \tag{3.11}$$

$T^\nu_{\mathrm{m}\mu}$ is the energy-momentum tensor of a perfect fluid. For a universe filled with perfect fluid, in the concomitant system of reference ($u^0 = 1, u^i = 0, i = 1,2,3$), we have

$$T^\nu_{\mathrm{m}\mu} = (p+\varepsilon)u_\mu u^\nu - \delta^\nu_\mu p = (\varepsilon, -p, -p, -p), \tag{3.12}$$

where energy ε is related to the pressure p by the equation of state $p = \zeta\varepsilon$. The general solution has been derived by Jacobs [19]. Here ζ varies between the interval $0 \leqslant \zeta \leqslant 1$, whereas $\zeta = 0$ describes the dust universe, $\zeta = \frac{1}{3}$ presents the radiation universe, $\frac{1}{3} < \zeta < 1$ ascribes the hard universe, and $\zeta = 1$ corresponds to the stiff matter.

In Eqs. (3.7) and (3.9), ∇_μ denotes the covariant differentiation; its explicit form depends on the quantity it acts on. This covariant differentiation has the standard properties

$$\nabla_\mu(AB) = (\nabla_\mu A)B + A(\nabla_\mu B), \tag{3.13a}$$

$$\nabla_\mu(A^*) = (\nabla_\mu A)^*, \tag{3.13b}$$

$$\nabla_\mu\gamma_\nu = 0, \tag{3.13c}$$

where the asterisk denotes the Hermitian adjoint (the transpose of the complex conjugate). The explicit form of the covariant derivative of a spinor is [43,44]

$$\nabla_\mu\psi = \frac{\partial\psi}{\partial x^\mu} - \Gamma_\mu\psi, \tag{3.14a}$$

$$\nabla_\mu\bar{\psi} = \frac{\partial\bar{\psi}}{\partial x^\mu} + \bar{\psi}\Gamma_\mu, \tag{3.14b}$$

where $\Gamma_\mu(x)$ are spinor affine connection matrices. γ matrices in the above equations obey the algebra

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} \tag{3.15}$$

and are connected with the flat space-time Dirac matrices $\bar{\gamma}$ in the following way:

$$g_{\mu\nu}(x) = e^a_\mu(x)e^b_\nu(x)\eta_{ab}, \quad \gamma_\mu(x) = e^a_\mu(x)\bar{\gamma}_a, \tag{3.16}$$

where $\eta_{ab} = \mathrm{diag}(1,-1,-1,-1)$ and e^a_μ is a set of tetrad 4-vectors. The spinor affine connection matrices $\Gamma_\mu(x)$ are uniquely determined up to an additive multiple of the unit matrix by the equation

$$\nabla_\mu\gamma_\nu = \frac{\partial\gamma_\nu}{\partial x^\mu} - \Gamma^\rho_{\nu\mu}\gamma_\rho - \Gamma_\mu\gamma_\nu + \gamma_\nu\Gamma_\mu = 0, \tag{3.17}$$

with the solution

$$\Gamma_\mu(x) = \frac{1}{4}g_{\rho\sigma}(x)(\partial_\mu e^b_\delta e^\rho_b - \Gamma^\rho_{\mu\delta})\gamma^\sigma\gamma^\delta. \tag{3.18}$$

Let us now write the γ's and Γ_μ's explicitly for the BI metric (2.1) that we rewrite in the form [45]

$$ds^2 = dt^2 - a^2(t)dx^2 - b^2(t)dy^2 - c^2(t)dz^2. \tag{3.19}$$

For the metric (3.19) from Eq. (3.16) one finds

$$\gamma_0 = \bar{\gamma}_0, \quad \gamma_1 = a(t)\bar{\gamma}_1, \quad \gamma_2 = b(t)\bar{\gamma}_2, \quad \gamma_3 = c(t)\bar{\gamma}_3,$$
$$\gamma^0 = \bar{\gamma}^0, \quad \gamma^1 = \bar{\gamma}^1/a(t), \quad \gamma^2 = \bar{\gamma}^2/b(t), \quad \gamma^3 = \bar{\gamma}^3/c(t). \tag{3.20}$$

For the affine spinor connections from Eq. (3.18) we find

$$\Gamma_0 = 0, \quad \Gamma_1 = \frac{1}{2}\dot{a}(t)\bar{\gamma}^1\bar{\gamma}^0, \quad \Gamma_2 = \frac{1}{2}\dot{b}(t)\bar{\gamma}^2\bar{\gamma}^0,$$
$$\Gamma_3 = \frac{1}{2}\dot{c}(t)\bar{\gamma}^3\bar{\gamma}^0. \tag{3.21}$$

We will choose flat space-time matrices $\overline{\gamma}$ in the form, given in [46],

$$\overline{\gamma^0}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix},\quad \overline{\gamma^1}=\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&-1&0&0\\-1&0&0&0\end{pmatrix},$$

$$\overline{\gamma^2}=\begin{pmatrix}0&0&0&-i\\0&0&i&0\\0&i&0&0\\-i&0&0&0\end{pmatrix},\quad \overline{\gamma^3}=\begin{pmatrix}0&0&1&0\\0&0&0&-1\\-1&0&0&0\\0&1&0&0\end{pmatrix}.$$

Defining γ^5 as follows:

$$\gamma^5=-\frac{i}{4}E_{\mu\nu\sigma\rho}\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho,\quad E_{\mu\nu\sigma\rho}=\sqrt{-g}\,\varepsilon_{\mu\nu\sigma\rho},$$

$$\varepsilon_{0123}=1,$$

$$\gamma^5=-i\sqrt{-g}\,\gamma^0\gamma^1\gamma^2\gamma^3=-i\overline{\gamma^0}\,\overline{\gamma^1}\,\overline{\gamma^2}\,\overline{\gamma^3}=\overline{\gamma^5},$$

we obtain

$$\overline{\gamma^5}=\begin{pmatrix}0&0&-1&0\\0&0&0&-1\\-1&0&0&0\\0&-1&0&0\end{pmatrix}.$$

For the space-time (3.19), the Einstein equations (3.8) now read

$$\frac{\ddot b}{b}+\frac{\ddot c}{c}+\frac{\dot b}{b}\frac{\dot c}{c}=\kappa T_1^1-\Lambda, \tag{3.22a}$$

$$\frac{\ddot c}{c}+\frac{\ddot a}{a}+\frac{\dot c}{c}\frac{\dot a}{a}=\kappa T_2^2-\Lambda, \tag{3.22b}$$

$$\frac{\ddot a}{a}+\frac{\ddot b}{b}+\frac{\dot a}{a}\frac{\dot b}{b}=\kappa T_3^3-\Lambda, \tag{3.22c}$$

$$\frac{\dot a}{a}\frac{\dot b}{b}+\frac{\dot b}{b}\frac{\dot c}{c}+\frac{\dot c}{c}\frac{\dot a}{a}=\kappa T_0^0-\Lambda, \tag{3.22d}$$

where an overdot denotes differentiation with respect to t.

We will study the space-independent solutions to the spinor field equations (3.7) so that $\psi=\psi(t)$. Setting

$$\tau=abc=\sqrt{-g} \tag{3.23}$$

we rewrite the spinor field equation (3.7a) as

$$i\overline{\gamma^0}\left(\frac{\partial}{\partial t}+\frac{\dot\tau}{2\tau}\right)\psi-m\psi+\mathcal{D}\psi+\mathcal{G}i\gamma^5\psi=0. \tag{3.24}$$

Setting $V_j(t)=\sqrt{\tau}\psi_j(t),\ j=1,2,3,4$, from Eq. (3.24) one deduces the following system of equations:

$$\dot V_1+i(m-\mathcal{D})V_1-\mathcal{G}V_3=0, \tag{3.25a}$$

$$\dot V_2+i(m-\mathcal{D})V_2-\mathcal{G}V_4=0, \tag{3.25b}$$

$$\dot V_3-i(m-\mathcal{D})V_3+\mathcal{G}V_1=0, \tag{3.25c}$$

$$\dot V_4-i(m-\mathcal{D})V_4+\mathcal{G}V_2=0. \tag{3.25d}$$

Using the solutions obtained one can write the components of a spinor current:

$$j^\mu=\overline{\psi}\gamma^\mu\psi. \tag{3.26}$$

Taking into account that $\overline{\psi}=\psi^\dagger\overline{\gamma^0}$, where $\psi^\dagger=(\psi_1^*,\psi_2^*,\psi_3^*,\psi_4^*)$ and $\psi_j=V_j/\sqrt{\tau}$, $j=1,2,3,4$ for the components of a spin current, we write

$$j^0=\frac{1}{\tau}[V_1^*V_1+V_2^*V_2+V_3^*V_3+V_4^*V_4], \tag{3.27a}$$

$$j^1=\frac{1}{a\tau}[V_1^*V_4+V_2^*V_3+V_3^*V_2+V_4^*V_1], \tag{3.27b}$$

$$j^2=\frac{-i}{b\tau}[V_1^*V_4-V_2^*V_3+V_3^*V_2-V_4^*V_1], \tag{3.27c}$$

$$j^3=\frac{1}{c\tau}[V_1^*V_3-V_2^*V_4+V_3^*V_1-V_4^*V_2]. \tag{3.27d}$$

The component j^0 defines the charge density of a spinor field that has the following chronometric-invariant form:

$$\varrho=(j_0j^0)^{1/2}. \tag{3.28}$$

The total charge of a spinor field is defined as

$$Q=\int\varrho\sqrt{-{}^3g}\,dx\,dy\,dz. \tag{3.29}$$

Let us consider the spin tensor [46]

$$S^{\mu\nu,\epsilon}=\frac{1}{4}\overline{\psi}\{\gamma^\epsilon\sigma^{\mu\nu}+\sigma^{\mu\nu}\gamma^\epsilon\}\psi. \tag{3.30}$$

We write the components $S^{ik,0}$ $(i,k=1,2,3)$, defining the spatial density of a spin vector explicitly. From Eq. (3.30), we have

$$S^{ij,0}=\frac{1}{4}\overline{\psi}\{\gamma^0\sigma^{ij}+\sigma^{ij}\gamma^0\}\psi=\frac{1}{2}\overline{\psi}\gamma^0\sigma^{ij}\psi, \tag{3.31}$$

which defines the projection of a spin vector on the k axis. Here i,j,k takes the value 1,2,3 and $i\neq j\neq k$. Thus, for the projection of spin vectors on the X,Y, and Z axis we find

$$S^{23,0}=\frac{1}{2bc\tau}[V_1^*V_2+V_2^*V_1+V_3^*V_4+V_4^*V_3], \tag{3.32a}$$

$$S^{31,0}=\frac{-i}{2ca\tau}[V_1^*V_2-V_2^*V_1+V_3^*V_4-V_4^*V_3], \tag{3.32b}$$

$$S^{12,0}=\frac{1}{2ab\tau}[V_1^*V_1-V_2^*V_2+V_3^*V_3-V_4^*V_4]. \tag{3.32c}$$

The chronometric invariant spin tensor takes the form

$$S_{\rm ch}^{ij,0}=(S_{ij,0}S^{ij,0})^{1/2}, \tag{3.33}$$

and the projection of the spin vector on the k axis is defined by

$$S_k=\int_{-\infty}^{\infty}S_{\rm ch}^{ij,0}\sqrt{-{}^3g}\,dx\,dy\,dz. \tag{3.34}$$

From Eqs. (3.7) we also write the equations for the invariants $S=\bar\psi\psi$, $P=i\bar\psi\gamma^5\psi$, and $A=\bar\psi\bar\gamma^5\bar\gamma^0\psi$,

$$\dot S_0-2\mathcal{G}A_0=0, \tag{3.35a}$$

$$\dot P_0-2(m-\mathcal{D})A_0=0, \tag{3.35b}$$

$$\dot A_0+2(m-\mathcal{D})P_0+2\mathcal{G}S_0=0, \tag{3.35c}$$

where $S_0=\tau S$, $P_0=\tau P$, and $A_0=\tau A$, leading to the following relation:

$$S^2+P^2+A^2=C^2/\tau^2,\quad C^2=\text{const}. \tag{3.36}$$

Let us now solve the Einstein equations. To do it we first write the expressions for the components of the energy-momentum tensor explicitly. Using the property of flat space-time Dirac matrices and the explicit form of the covariant derivative ∇_μ, one can easily find

$$T_0^0=mS-F(I,J)+\varepsilon,$$
$$T_1^1=T_2^2=T_3^3=\mathcal{D}S+\mathcal{G}P-F(I,J)-p. \tag{3.37}$$

Summation of the Einstein equations (3.22a), (3.22b), (3.22c), and (3.22d) multiplied by 3 gives

$$\frac{\ddot\tau}{\tau}=\frac{3}{2}\kappa(T_1^1+T_0^0)-3\Lambda. \tag{3.38}$$

For the right-hand side of Eq. (3.38) to be a function of τ only, the solution to this equation is well known [47]. As we see in the next section, the right-hand side of Eq. (3.38) is indeed a function of τ. Given the explicit form of L_N from Eq. (3.38) one finds the concrete solution for τ in quadrature.

Let us express a,b,c through τ. For this we notice that subtraction of Einstein equations (3.22b) and (3.22a) leads to the equation

$$\frac{\ddot a}{a}-\frac{\ddot b}{b}+\frac{\dot a\dot c}{ac}-\frac{\dot b\dot c}{bc}=\frac{d}{dt}\left(\frac{\dot a}{a}-\frac{\dot b}{b}\right)+\left(\frac{\dot a}{a}-\frac{\dot b}{b}\right)\left(\frac{\dot a}{a}+\frac{\dot b}{b}+\frac{\dot c}{c}\right)=0 \tag{3.39}$$

with the solution

$$\frac{a}{b}=D_1\exp\left(X_1\int\frac{dt}{\tau}\right),\quad D_1=\text{const},\quad X_1=\text{const}. \tag{3.40}$$

Analogically, one finds

$$\frac{a}{c}=D_2\exp\left(X_2\int\frac{dt}{\tau}\right),\quad \frac{b}{c}=D_3\exp\left(X_3\int\frac{dt}{\tau}\right), \tag{3.41}$$

where D_2,D_3,X_2,X_3 are integration constants. In view of Eq. (3.23) we find the following functional dependence between the constants D_1,D_2,D_3,X_1,X_2,X_3:

$$D_2=D_1D_3,\qquad X_2=X_1+X_3.$$

Finally, from Eqs. (3.40) and (3.41) we write $a(t)$, $b(t)$, and $c(t)$ in the explicit form

$$a(t)=(D_1^2D_3)^{1/3}\tau^{1/3}\exp\left[\frac{2X_1+X_3}{3}\int\frac{dt}{\tau(t)}\right], \tag{3.42a}$$

$$b(t)=(D_1^{-1}D_3)^{1/3}\tau^{1/3}\exp\left[-\frac{X_1-X_3}{3}\int\frac{dt}{\tau(t)}\right], \tag{3.42b}$$

$$c(t)=(D_1D_3^2)^{-1/3}\tau^{1/3}\exp\left[-\frac{X_1+2X_3}{3}\int\frac{dt}{\tau(t)}\right]. \tag{3.42c}$$

Thus the system of Einstein's equations is completely integrated.

Defining the Hubble constant in analogy with a FRW universe from Eqs. (3.42) we obtain

$$H_j=\frac{\dot a_j}{a_j}=\frac{\dot\tau+Y_j}{3\tau},\quad j=1,2,3, \tag{3.43}$$

or a generalized one,

$$H=(H_1+H_2+H_3)/3=\dot\tau/3\tau. \tag{3.44}$$

Here $a_1=a$, $a_2=b$, $a_3=c$, $Y_1=2X_1+X_3$, $Y_2=-X_1+X_3$, and $Y_3=-X_1-2X_3$. The deceleration parameter given by

$$q=-\frac{\ddot RR}{\dot R^2} \tag{3.45}$$

for a FRW universe with R being the scale factor can also be generalized for the BI space-time to obtain

$$q_i = -\frac{\ddot{a}_i a_i}{\dot{a}_i^2} = -\left[\left(\frac{\ddot{a}_i}{a_i}\right) \Big/ \left(\frac{\dot{a}_i}{a_i}\right)^2\right] = -\left[1 + \left(\frac{\dot{a}_i}{a_i}\right)^{\cdot} \Big/ \left(\frac{\dot{a}_i}{a_i}\right)^2\right]. \tag{3.46}$$

Inserting Eqs. (3.42) into Eq. (3.46), one obtains

$$q_i = -\frac{\ddot{\tau} - 2\dot{\tau}^2 - Y_i\dot{\tau} + Y_i^2}{\dot{\tau}^2 + 2Y_i\dot{\tau} + Y_i^2}, \qquad i = 1,2,3. \tag{3.47}$$

Let us now go back to the Einstein equation (3.8). Taking the divergence of the Einstein equation, we obtain

$$T^{\nu}_{\mu;\nu} = T^{\nu}_{\mu,\nu} + \Gamma^{\nu}_{\rho\nu} T^{\rho}_{\mu} - \Gamma^{\rho}_{\mu\nu} T^{\nu}_{\rho} = 0, \tag{3.48}$$

which in our case reads

$$\dot{T}_0^0 + \frac{\dot{\tau}}{\tau}(T_0^0 - T_1^1) = 0. \tag{3.49}$$

Putting T_0^0 and T_1^1 into Eq. (3.49), we obtain

$$\dot{\varepsilon} + (\varepsilon + p)\frac{\dot{\tau}}{\tau} + (m - \mathcal{D})\dot{S}_0 - \mathcal{G}\dot{P}_0 = 0, \tag{3.50}$$

where $S_0 = \tau S$ and $P_0 = \tau P$. From Eqs. (3.35a) and (3.35b), we have $(m-\mathcal{D})\dot{S}_0 - \mathcal{G}\dot{P}_0 = 0$. Further, taking into account the equation of state, i.e., $p = \zeta\varepsilon$, we find

$$\frac{d\varepsilon}{(1+\zeta)\varepsilon} + \frac{d\tau}{\tau} = 0, \tag{3.51}$$

with the solutions

$$\varepsilon = \frac{\varepsilon_0}{\tau^{1+\zeta}}, \quad p = \frac{\zeta\varepsilon_0}{\tau^{1+\zeta}}, \tag{3.52}$$

where ε_0 is the integration constant. Note that the relation (3.52) holds for any combination of the material field Lagrangian, e.g., spinor or scalar or interacting spinor and scalar fields. Thus we see that the right-hand side of Eq. (3.38) is a function of τ only. Then Eq. (3.38), multiplied by $2\dot{\tau}$, can be written as

$$2\dot{\tau}\ddot{\tau} = \{3[\kappa(T_1^1 + T_0^0) - 2\Lambda]\tau\}\dot{\tau} = \Psi(\tau)\dot{\tau}. \tag{3.53}$$

We write the solution to Eq. (3.53) in quadrature,

$$\int \frac{d\tau}{\sqrt{\int \Psi(\tau) d\tau}} = t. \tag{3.54}$$

Given the explicit form of $F(I,J)$, from Eq. (3.54) one finds the concrete function $\tau(t)$. Once the value of τ is obtained, one can get expressions for components $\psi_j(t)$, $j = 1,2,3,4$. Thus the initial systems of Einstein and Dirac equations have been completely integrated.

Further we will investigate the existence of singularity (singular point) of the gravitational case, which can be done by investigating the invariant characteristics of the space-time. In general relativity these invariants are composed from the curvature tensor and the metric one. Contrary to the electrodynamics, where there are two invariants only ($J_1 = F_{\mu\nu}F^{\mu\nu}$ and $J_2 = \star F_{\mu\nu}F^{\mu\nu}$), in 4D Riemann space-time there are 14 independent invariants. They are [48]

$$I_1 = R, \tag{3.55a}$$

$$I_2 = R_{\mu\nu}R^{\mu\nu}, \tag{3.55b}$$

$$I_3 = R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}, \tag{3.55c}$$

$$I_4 = \star R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}, \tag{3.55d}$$

$$I_5 = R^{\alpha}_{\beta}R^{\beta}_{\mu}R^{\mu}_{\alpha}, \tag{3.55e}$$

$$I_6 = R^{\alpha\beta}R^{\mu\nu}R_{\alpha\mu\beta\nu}, \tag{3.55f}$$

$$I_7 = R^{\alpha\beta}R^{\mu\nu}\star R_{\alpha\mu\beta\nu}, \tag{3.55g}$$

$$I_8 = R^{\alpha\beta\mu\nu}R_{\alpha\beta\sigma\rho}R^{\sigma\rho}{}_{\mu\nu}, \tag{3.55h}$$

$$I_9 = \star R^{\alpha\beta\mu\nu}R_{\alpha\beta\sigma\rho}R^{\sigma\rho}{}_{\mu\nu}, \tag{3.55i}$$

$$I_{10} = R^{\beta}_{\alpha}R^{\alpha\mu}R_{\mu\nu}R^{\nu}_{\beta}, \tag{3.55j}$$

$$I_{11} = R^{\mu}_{\nu}R_{\rho\mu}{}^{\sigma\alpha}R_{\sigma\alpha}{}^{\beta[\nu}R^{\rho]}_{\beta}, \tag{3.55k}$$

$$I_{12} = R^{\mu}_{\nu}\star R^{\sigma\alpha}{}_{\rho\mu}R_{\sigma\alpha}{}^{\beta[\nu}R^{\rho]}_{\beta}, \tag{3.55l}$$

$$I_{13} = R^{\mu\nu}{}_{\alpha\beta}(A^{\alpha\beta}{}_{\mu\nu} + R^{\alpha}_{\rho}R^{\rho}_{\sigma}R^{\sigma}_{\eta}R^{\eta}_{\mu}\delta^{\beta}_{\nu}), \tag{3.55m}$$

$$I_{14} = \star R_{\alpha\beta}{}^{\mu\nu}A^{\alpha\beta}{}_{\mu\nu}, \tag{3.55n}$$

where $A^{\alpha\beta}{}_{\mu\nu} = 4R^{\alpha}_{\rho}R^{\rho}_{\sigma}R^{\sigma}_{\mu}R^{\beta}_{\nu} + 3R^{\alpha}_{\rho}R^{\rho}_{\mu}R^{\beta}_{\sigma}R^{\sigma}_{\nu}$ and $\star R_{\alpha\beta\mu\nu} = \frac{1}{2}E_{\alpha\beta\sigma\rho}R^{\sigma\rho}{}_{\mu\nu} = \frac{1}{2}E_{\sigma\rho\mu\nu}R_{\alpha\beta}{}^{\sigma\rho}$, $\star R_{\alpha\beta}{}^{\mu\nu} = \frac{1}{2}\underline{E_{\alpha\beta\sigma\rho}}R^{\sigma\rho\mu\nu}$ with $E_{\alpha\beta\mu\nu} = \sqrt{-g}\varepsilon_{\alpha\beta\mu\nu}$ and $E^{\alpha\beta\mu\nu} = (-1/\sqrt{-g})\varepsilon^{\alpha\beta\mu\nu}$. Here $\varepsilon_{\alpha\beta\mu\nu}$ is the totally antisymmetric Levi-Civita tensor with $\varepsilon_{0123} = 1$. Instead of analyzing all 14 invariants mentioned above, one can confine this study only to 3, namely the scalar curvature $I_1 = R$, $I_2 = R^{R}_{\mu\nu}\mu\nu$, and the Kretschmann scalar $I_3 = R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$. At any regular space-time point, these three invariants I_1, I_2, I_3 should be finite. Let us rewrite these invariants in detail.

For the Bianchi I metric one finds the scalar curvature

$$I_1 = R = -2\frac{\ddot{\tau} - \dot{a}\dot{b}c - \dot{b}\dot{c}a - \dot{c}\dot{a}b}{\tau}. \tag{3.56}$$

Since the Ricci tensor for the Bianchi I metric is diagonal, the invariant $I_2 = R_{\mu\nu}R^{\mu\nu} \equiv R^{\nu}_{\mu}R^{\mu}_{\nu}$ is a sum of squares of diagonal components of Ricci tensor, i.e.,

$$I_2 = [(R_0^0)^2 + (R_1^1)^2 + (R_2^2)^2 + (R_3^3)^2], \tag{3.57}$$

with

$$R_0^0 = -\frac{\ddot{a}bc + a\ddot{b}c + ab\ddot{c}}{\tau}, \tag{3.58a}$$

$$R^1_1 = -\frac{\ddot{a}bc + \dot{a}\dot{b}c + \dot{a}b\dot{c}}{\tau}, \tag{3.58b}$$

$$R^2_2 = -\frac{a\ddot{b}c + \dot{a}\dot{b}c + a\dot{b}\dot{c}}{\tau}, \tag{3.58c}$$

$$R^3_3 = -\frac{ab\ddot{c} + a\dot{b}\dot{c} + \dot{a}b\dot{c}}{\tau}. \tag{3.58d}$$

Analogically, for the Kretschmann scalar in this case we have $I_3 = R^{\mu\nu}{}_{\alpha\beta}R^{\alpha\beta}{}_{\mu\nu}$, a sum of squared components of all nontrivial $R^{\mu\nu}{}_{\mu\nu}$:

$$\begin{aligned} I_3 &= 4[(R^{01}{}_{01}{}^2 + (R^{01}{}_{01})^2 + + (R^{02}{}_{02})^2 + (R^{03}{}_{03})^2 \\ &\quad + (R^{12}{}_{12})^2 + (R^{23}{}_{23})^2 + (R^{31}{}_{31})^2 \\ &= \frac{4}{\tau^2}[(\ddot{a}bc)^2 + (a\ddot{b}c)^2 + (ab\ddot{c})^2 + (\dot{a}\dot{b}c)^2 \\ &\quad + (\dot{a}b\dot{c})^2 + (a\dot{b}\dot{c})^2], \quad \tau = abc. \end{aligned} \tag{3.59}$$

From Eqs. (3.42) we have

$$a_i = A_i \tau^{1/3} \exp\left((Y_i/3) \int \tau^{-1}\, dt \right), \tag{3.60a}$$

$$\dot{a}_i = \frac{Y_i + 1}{3} \frac{a_i}{\tau} \quad (i = 1,2,3), \tag{3.60b}$$

$$\ddot{a}_i = \frac{(Y_i + 1)(Y_i - 2)}{9} \frac{a_i}{\tau^2}, \tag{3.60c}$$

i.e., the metric functions a, b, c and their derivatives are in functional dependence with τ. As we see from Eqs. (3.60), at any space-time point, where $\tau = 0$ the invariants I_1, I_2, I_3 become infinity, hence the space-time becomes singular at this point.

IV. ANALYSIS OF THE RESULTS

In this section we shall analyze the general results obtained in the preceding section. In the following subsections, we will study the system with linear and nonlinear spinor fields, respectively.

A. Linear spinor field in a BI universe

In this subsection, we study the linear spinor field in a BI universe. The reason for getting the solution to the self-consistent system of equations for the linear spinor and gravitational fields is the necessity of comparing this solution with that for the system of equations for the nonlinear spinor and gravitational fields, which permits clarifications of the role of nonlinear spinor terms in the evolution of the cosmological model in question.

In this case we get explicit expressions for the components of spinor field functions and metric functions:

$$\psi_1(t) = (C_1/\sqrt{\tau})\exp[-imt], \tag{4.1a}$$

$$\psi_2(t) = (C_2/\sqrt{\tau})\exp[-imt], \tag{4.1b}$$

$$\psi_3(t) = (C_3/\sqrt{\tau})\exp[imt], \tag{4.1c}$$

$$\psi_4(t) = (C_4/\sqrt{\tau})\exp[imt], \tag{4.1d}$$

with C_1, C_2, C_3, C_4 being the integration constants. On the other hand, from Eqs. (3.35) we find

$$S = \frac{C_0}{\tau}, \tag{4.2}$$

where C_0 is an integration constant and related to the previous ones as $C_0 = C_1^2 + C_2^2 - C_3^2 - C_4^2$. For the components of the spin current from Eqs. (3.27) we find

$$j^0 = \frac{1}{\tau}[C_1^2 + C_2^2 + C_3^2 + C_4^2], \tag{4.3a}$$

$$j^1 = \frac{2}{a\tau}[C_1C_4 + C_2C_3]\cos(2mt), \tag{4.3b}$$

$$j^2 = \frac{2}{b\tau}[C_1C_4 - C_2C_3]\sin(2mt), \tag{4.3c}$$

$$j^3 = \frac{2}{c\tau}[C_1C_3 - C_2C_4]\cos(2mt), \tag{4.3d}$$

whereas, for the projection of spin vectors on the X, Y, and Z axis, we find

$$S^{23,0} = \frac{1}{bc\tau}[C_1C_2 + C_3C_4], \tag{4.4a}$$

$$S^{31,0} = 0, \tag{4.4b}$$

$$S^{12,0} = \frac{1}{2ab\tau}[C_1^2 - C_2^2 + C_3^2 - C_4^2]. \tag{4.4c}$$

From Eq. (3.29) we find the charge of the system in a volume $\mathcal{V}$,

$$Q = [C_1^2 + C_2^2 + C_3^2 + C_4^2]\mathcal{V}. \tag{4.5}$$

Thus we see that the total charge of the system in a finite volume is always finite.

Let us now determine the function τ. In the absence of perfect fluid for the linear spinor field, we have

$$T_0^0 = mS, \quad T_1^1 = T_2^2 = T_3^3 = 0. \tag{4.6}$$

Taking Eq. (4.6) into account, for τ we write

$$\ddot{\tau} = M - 3\Lambda\tau \tag{4.7}$$

with the solutions

$$\tau = \begin{cases} (1/3\Lambda)[M - q_1 \sinh(\sqrt{-3\Lambda}t)], & \Lambda < 0, \\ (1/2)Mt^2 + y_1 t + y_0, & \Lambda = 0, \\ (1/3\Lambda)[M - q_2 \sin(\sqrt{3\Lambda}t)], & \Lambda > 0, \end{cases} \tag{4.8}$$

where $M = \frac{3}{2}\kappa m C_0$ and y_1, y_0, q_1, q_2 are the constants. Let us now analyze the solutions obtained.

First we study the case when $\Lambda = 0$. It can be shown that [31]

$$y_1^2 - 2My_0 = (X_1^2 + X_1X_3 + X_3^2)/3 > 0. \tag{4.9}$$

This means that the quadratic polynomial $(1/2)Mt^2 + y_1 t + y_0 = 0$ possesses real roots, i.e., $\tau(t)$ in the case of $\Lambda = 0$ becomes zero at $t = t_{1,2} = -y_1/M \pm \sqrt{(y_1/M)^2 - 2y_0/M}$ and the solution obtained is the singular one. At $t \to \infty$ in this case we have

$$\tau(t) \approx \frac{3}{4}\kappa m C_0 t^2, \quad a(t) \approx b(t) \approx c(t) \approx t^{2/3},$$

which leads to the conclusion about the asymptotical isotropization of the expansion process for the initially anisotropic BI space. Thus the solution to the self-consistent system of equations for the linear spinor and gravitational fields is the singular one at the time $t = t_{1,2}$. In the initial state of evolution of the field system the expansion process of space is anisotropic, but at $t \to \infty$ the isotropization of the expansion process takes place. As one can see, the components of spin current and the projections of spin vector are singular at space-time points $t_{1,2}$ where τ vanishes. A qualitative picture of this case has been given in Fig. 1.

For $\Lambda < 0$, we see that the solution is singular at $t = t_0 = (1/\sqrt{-3\Lambda})\text{arcsinh}(M/q_1)$ and the isotropization of the expansion process takes place as $t \to \infty$. Note that the izotropization process in this case is rather rapid (cf. Fig. 2).

For $\Lambda > 0$, we have the oscillatory solutions (cf. Fig. 3). Taking into account that τ is a non-negative quantity, it can be shown that the model has singular solutions at $t = (4k + 1)\pi/2\sqrt{3\Lambda}$, $k = 0,1,2,3, \ldots$ with $M = q_2$. For $M > q_2$, we have τ, which is always positive definite, i.e., the solutions obtained are regular at each space-time point.

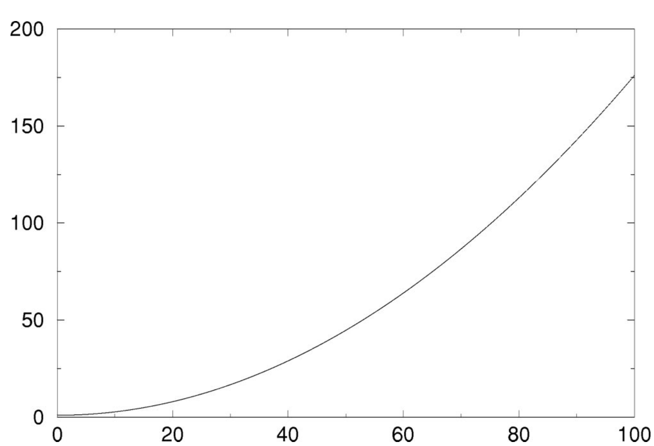

FIG. 1. Perspective view of τ for a linear spinor field in the absence of a Λ term.

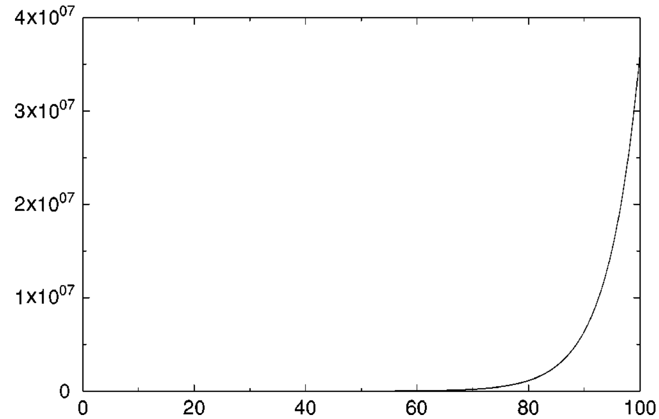

FIG. 2. Perspective view of τ for a linear spinor field with $\Lambda < 0$.

B. Nonlinear spinor field

Let us now go back to the nonlinear case. We consider the following forms of the nonlinear term: (i) $L_N = F(I)$; (ii) $L_N = F(J)$; (iii) $L_N = F(K_\pm)$ with $K_\pm = I \pm J$.

(i) Let us consider the case when $L_N = F(I)$. From Eqs. (3.35) we find in this case

$$S = \frac{C_0}{\tau}, \quad C_0 = \text{const.} \tag{4.10}$$

Note that in this case we denote the constants in the same way as we did for the linear case, but the constants in these cases are not necessarily identical. Spinor field equations in this case read

$$\dot{V}_1 + i(m - \mathcal{D})V_1 = 0, \tag{4.11a}$$

$$\dot{V}_2 + i(m - \mathcal{D})V_2 = 0, \tag{4.11b}$$

$$\dot{V}_3 - i(m - \mathcal{D})V_3 = 0, \tag{4.11c}$$

$$\dot{V}_4 - i(m - \mathcal{D})V_4 = 0. \tag{4.11d}$$

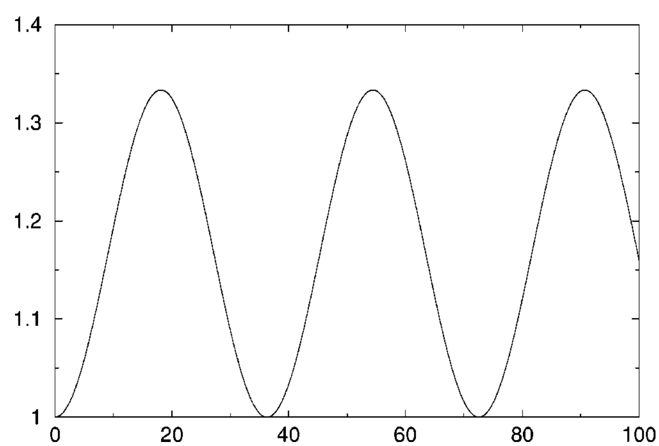

FIG. 3. Perspective view of τ for a linear spinor field with $\Lambda > 0$.

As in the considered case when $L_N=F$ depends only on S, from Eq. (4.10) it follows that $F(I)$ and $\mathcal{D}$ are functions of τ only. Taking this fact into account, we get explicit expressions for the components of spinor field functions,

$$\psi_1(t)=(C_1/\sqrt{\tau})\exp\left(-i\int(m-\mathcal{D})dt\right), \tag{4.12a}$$

$$\psi_2(t)=(C_2/\sqrt{\tau})\exp\left(-i\int(m-\mathcal{D})dt\right), \tag{4.12b}$$

$$\psi_3(t)=(C_3/\sqrt{\tau})\exp\left(i\int(m-\mathcal{D})dt\right), \tag{4.12c}$$

$$\psi_4(t)=(C_4/\sqrt{\tau})\exp\left(i\int(m-\mathcal{D})dt\right), \tag{4.12d}$$

with C_1,C_2,C_3,C_4 being the integration constants and related to C_0 as $C_0=C_1^2+C_2^2-C_3^2-C_4^2$. For the components of the spin current from Eqs. (3.27) we find

$$j^0=\frac{1}{\tau}[C_1^2+C_2^2+C_3^2+C_4^2], \tag{4.13a}$$

$$j^1=\frac{2}{a\tau}[C_1C_4+C_2C_3]\cos\left(2\int(m-\mathcal{D})dt\right), \tag{4.13b}$$

$$j^2=\frac{2}{b\tau}[C_1C_4-C_2C_3]\sin\left(2\int(m-\mathcal{D})dt\right), \tag{4.13c}$$

$$j^3=\frac{2}{c\tau}[C_1C_3-C_2C_4]\cos\left(2\int(m-\mathcal{D})dt\right), \tag{4.13d}$$

whereas, for the projection of spin vectors on the X, Y, and Z axis, we find

$$S^{23,0}=\frac{1}{bc\tau}[C_1C_2+C_3C_4], \tag{4.14a}$$

$$S^{31,0}=0, \tag{4.14b}$$

$$S^{12,0}=\frac{1}{2ab\tau}[C_1^2-C_2^2+C_3^2-C_4^2]. \tag{4.14c}$$

We now study the equation for τ in detail choosing the nonlinear spinor term as $F(I)=\lambda I^{(n/2)}=\lambda S^n$ with λ being the coupling constant and $n>1$. In this case for τ one gets

$$\ddot{\tau}=(3/2)\kappa[mC_0+\lambda(n-2)C_0^n/\tau^{n-1}]-3\Lambda\tau. \tag{4.15}$$

The first integral of the foregoing equation takes the form

$$\dot{\tau}^2=3\kappa[mC_0\tau-\lambda C_0^n/\tau^{n-2}+g^2]-3\Lambda\tau^2. \tag{4.16}$$

Here g^2 is the integration constant that is positively defined and connected with the constants X_i as $g^2=(X_1^2+X_1X_3+X_3^2)/9\kappa C_0$ [31]. The sign C_0 is determined by the positivity of the energy-density T_0^0 of a linear spinor field, i.e.,

$$T_0^0=mC_0/\tau>0. \tag{4.17}$$

It is obvious from Eq. (4.17) that $C_0>0$. Now one can write the solution to Eq. (4.16) in quadratures:

$$\int\frac{\tau^{(n-2)/2}d\tau}{\sqrt{\kappa[mC_0\tau^{n-1}+g^2\tau^{n-2}-\lambda C_0^n]-\Lambda\tau^n}}=\sqrt{3}\,t. \tag{4.18}$$

The constant of integration in Eq. (4.18) has been taken to be zero, as it only gives the shift of the initial time. Let us study the properties of the solution obtained for a different choice of n, λ, and Λ. First we study the case with $\Lambda=0$.

For $n>2$ from Eq. (4.18) one gets

$$\tau(t)|_{t\to\infty}\approx(3/4)\kappa mC_0t^2. \tag{4.19}$$

It leads to the conclusion about isotropization of the expansion process of the BI space-time. It should be remarked that the isotropization takes place if and only if the spinor field equation contains the massive term [cf. the parameter m in Eq. (4.18)]. This is not the case for a massless spinor field, since from Eq. (4.18) we get

$$\tau(t)|_{t\to\infty}\approx\sqrt{3\kappa C_0g^2}\,t. \tag{4.20}$$

Substituting Eq. (4.20) into Eqs. (3.42), one comes to the conclusion that the functions $a(t)$, $b(t)$, and $c(t)$ are different.

Let us consider the properties of solutions to Eq. (4.15) when $t\to0$. For $\lambda<0$ from Eq. (4.18) we get

$$\tau(t)=[(3/4)n^2\kappa|\lambda|C_0^n]^{1/n}t^{2/n}\to0, \tag{4.21}$$

i.e., solutions are singular. For $\lambda>0$, from Eq. (4.18) it follows that $\tau=0$ cannot be reached for any value of t as in this case when the denominator of the integrand in Eq. (4.18) becomes imaginary. It means that for $\lambda>0$ there exist regular solutions to the previous system of equations [30]. The absence of the initial singularity in the considered cosmological solution appears to be consistent with the violation for $\lambda>0$ of the dominant energy condition in the Hawking-Penrose theorem [49], which reads as follows.

Theorem. A space-time $\mathcal{M}$ cannot be causally, geodesically complete if the GTR equations hold and if the following conditions are satisfied.

(i) The space-time $\mathcal{M}$ does not contain closed timelike lines.

(ii) The conditions (dominant energy condition)

$$T_{00}+T_{11}+T_{22}+T_{33}\geqslant0, \tag{4.22a}$$

$$T_{00}+T_{11}\geqslant0, \tag{4.22b}$$

$$T_{00}+T_{22}\geqslant0, \tag{4.22c}$$

$$T_{00}+T_{33}\geqslant0, \tag{4.22d}$$

on the equations of state are fulfilled, where T_{00} is the energy density and T_{11}, T_{22}, and T_{33} are three principal values of pressure tensor.

(iii) On each timelike or null geodesic, there is at least one point for which

$$K_{[a}R_{b]cd[e}K_{f]}K^cK^d \neq 0, \tag{4.23}$$

where K_a is the tangent to the curve at the given point and where the brackets on the subscripts imply antisymmetrization.

(iv) The space-time $\mathcal{M}$ contains either (a) a point P such that all diverging rays from this point begin to converge if one traces them back to the past, or (b) a compact spacelike hypersurface.

Proof. To prove that in the case considered the dominant energy condition is violated, we rewrite Eq. (4.22) in the following form:

$$T_0^0 \geqslant T_1^1 a^2 + T_2^2 b^2 + T_3^3 c^2, \tag{4.24a}$$

$$T_0^0 \geqslant T_1^1 a^2, \tag{4.24b}$$

$$T_0^0 \geqslant T_2^2 b^2, \tag{4.24c}$$

$$T_0^0 \geqslant T_3^3 c^2. \tag{4.24d}$$

Let us go back to the energy density of a spinor field. From

$$T_0^0 = \frac{mC_0}{\tau} - \frac{\lambda C_0^n}{\tau^n} \tag{4.25}$$

it follows that at

$$\tau^{n-1} < \frac{\lambda C_0^{n-1}}{m} \tag{4.26}$$

the energy density of the spinor field becomes negative. On the other hand, we have

$$T_1^1 = T_2^2 = T_3^3 = \frac{\lambda(n-1)C_0^n}{\tau^n} > 0 \tag{4.27}$$

for any non-negative value of τ. Thus, we see all four conditions in (4.24) are violated, i.e., the absence of initial singularity in the considered cosmological solution appears to be consistent with the violation of the dominant energy condition in the Hawking-Penrose theorem.

Let us consider the Heisenberg-Ivanenko equation [50] setting $n=2$ in Eq. (4.15). In this case the equation for $\tau(t)$ does not contain the nonlinear term and its solution coincides with that of the linear one. The spinor field functions in this case are written as follows:

$$V_1 = \frac{C_1}{\sqrt{\tau}} e^{-imt} Z^{4i\lambda C_0/B}, \tag{4.28a}$$

$$V_2 = \frac{C_2}{\sqrt{\tau}} e^{-imt} Z^{4i\lambda C_0/B}, \tag{4.28b}$$

$$V_3 = \frac{C_3}{\sqrt{\tau}} e^{imt} Z^{-4i\lambda C_0/B}, \tag{4.28c}$$

$$V_4 = \frac{C_4}{\sqrt{\tau}} e^{imt} Z^{-4i\lambda C_0/B}, \tag{4.28d}$$

where $Z=(t-t_1)/(t-t_2)$, $B=M(t_1-t_2)$, and $t_{1,2} = -y_1/M \pm \sqrt{(y_1/M)^2 - 2y_0/M}$ are the roots of the quadratic equation $Mt^2+2y_1t+2y_0=0$. As in the linear case, the obtained solution is singular at time $t=t_{1,2}$ and asymptotically isotropic as $t\to\infty$.

We now study the properties of solutions to Eq. (4.15) for $1<n<2$. In this case it is convenient to present the solution (4.18) in the form

$$\int \frac{d\tau}{\sqrt{m\tau - \lambda\tau^{2-n}C_0^{n-1} + g^2}} = \sqrt{3\kappa C_0}\, t. \tag{4.29}$$

As $t\to\infty$, from Eq. (4.29) we get the equality (4.19), leading to the isotropization of the expansion process. If $m=0$ and $\lambda>0$, $\tau(t)$ lies on the interval

$$0 \leqslant \tau(t) \leqslant (g^2/\lambda C_0^{n-1})^{1/(2-n)}.$$

If $m=0$ and $\lambda<0$, the relation (4.29) at $t\to\infty$ leads to the equality

$$\tau(t) \approx [(3/4)n^2\kappa|\lambda|C_0^n]^{1/n} t^{2/n}. \tag{4.30}$$

Substituting Eq. (4.30) into Eqs. (3.42) and taking into account that at $t\to\infty$

$$\int \frac{dt}{\tau} \approx \frac{n(3\kappa|\lambda|n^2C_0^n)^{1/n}}{(n-2)2^{2/n}} t^{-2/n+1} \to 0$$

due to $-2/n+1<0$, we obtain

$$a(t) \sim b(t) \sim c(t) \sim [\tau(t)]^{1/3} \sim t^{2/3n} \to \infty. \tag{4.31}$$

This means that the solution obtained tends to the isotropic one. In this case the isotropization is provided not by the massive parameter, but by the degree n in the term $L_N = \lambda S^n$. Equation (4.29) implies

$$\tau(t)|_{t\to 0} \approx \sqrt{3\kappa C_0 g^2}\, t \to 0, \tag{4.32}$$

which means the solution obtained is initially singular. Thus for $1<n<2$ there exist only singular solutions at initial time. At $t\to\infty$ the isotropization of the expansion process of the BI space takes place both for $m\neq 0$ and for $m=0$.

Finally, let us study the properties of the solution to Eq. (4.15) for $0<n<1$. In this case we use the solution in the form (4.29). Since now $2-n>1$, then with the increasing of $\tau(t)$ in the denominator of the integrand in Eq. (4.29) the second term $\lambda\tau^{2-n}C_0^{n-1}$ increases faster than the first one.

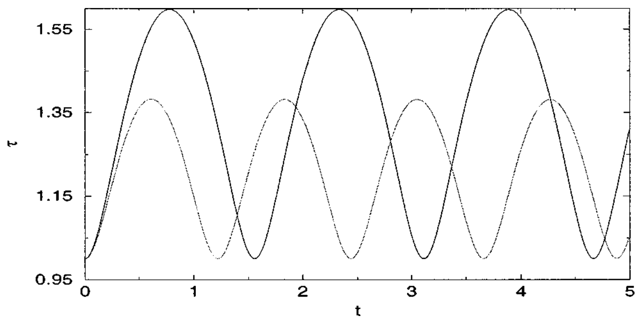

FIG. 4. Perspective view of τ showing the initially nonsingular and oscillating behavior of the solutions. The continuous and dash lines correspond to the massive and massless spinor field, respectively.

Therefore, the solution describing the space expansion can be possible only for $\lambda<0$. In this case at $t\to\infty$, for $m=0$ as well as for $m\neq0$, one can get the asymptotic representation (4.30) of the solution. This solution, as for the choice $1<n<2$, provides an asymptotically isotropic expansion of the BI space-time. For $t\to0$ in this case we shall get only the singular solution of the form (4.32).

For a nonzero Λ term we study the following situations depending on the sign of Λ and λ.

Case (i). $\Lambda=-\epsilon^2<0$, $\lambda>0$. In this case for $n>2$ and $t\to\infty$ we find

$$\tau(t)\approx e^{\sqrt{3}\epsilon t}. \tag{4.33}$$

Thus we see that the asymptotic behavior of τ does not depend on n and is defined by the Λ term. From Eqs. (3.42) it is obvious that the asymptotic isotropization takes place.

From Eq. (4.18) it also follows that τ cannot be zero at any moment, since the integrand turns out to be imaginary as τ approaches zero. Thus the solution obtained is a nonsingular one thanks to the nonlinear term in the Dirac equation and asymptotically isotropic. As it has been noted earlier, the absence of initial singularity in the considered cosmological model results in the violation of the dominant energy condition.

Case (ii). $\Lambda>0$ and $\lambda>0$. For $n>2$, Eq. (4.18) admits only nonsingular oscillating solutions, since $\tau>0$ and is bound from above. Consider the case with $n=4$ and for simplicity set $m=0$. Then from Eq. (4.18) one gets

$$\tau(t)=\frac{1}{\sqrt{2\Lambda}}[\kappa C_0\tau_0+\sqrt{\kappa^2C_0^2\tau_0^2+4\Lambda\lambda C_0^4}\sin 2\sqrt{3\Lambda}t]^{1/2}. \tag{4.34}$$

For a massive spinor field with $\Lambda>0$ and $\lambda>0$ and $n=10$, a perspective view of τ is shown in Fig. 4. The period for the massive field is greater than that for the massless one. As it occurs, the order of nonlinearity (n) has a direct effect on the period (the more in n the less is the period).

Case (iii). $\Lambda<0$ and $\lambda<0$. The solution is singular at initial moment, that is,

$$\lim_{t\to0}\tau\approx[\sqrt{-3\lambda n^2C_0^n/4t}]^{2/n} \tag{4.35}$$

and at $t\to\infty$ asymptotic isotropization takes place since

$$\lim_{t\to\infty}\tau\approx e^{\sqrt{3\Lambda t}}. \tag{4.36}$$

Case (iv). $\Lambda>0$ and $\lambda<0$. The solution is initially singular as

$$\lim_{t\to0}\tau\approx[\sqrt{-3\lambda n^2C_0^n/4t}]^{2/n} \tag{4.37}$$

and is bound from the above, i.e., oscillating, since

$$\lim_{t\to\infty}\tau\approx\sin\sqrt{3\Lambda}t. \tag{4.38}$$

(ii) We study the system when $L_N=F(J)$, which means in the case considered $\mathcal{D}=0$. Let us note that, in the unified nonlinear spinor theory of Heisenberg, the massive term remains absent, and according to Heisenberg, the particle mass should be obtained as a result of quantization of spinor prematter [51]. In the nonlinear generalization of classical field equations, the massive term does not possess the significance that it possesses in the linear one, as it by no means defines total energy (or mass) of the nonlinear field system. Thus without losing the generality we can consider the massless spinor field putting $m=0$. Then from Eqs. (3.35) one gets

$$P(t)=\frac{D_0}{\tau},\quad D_0=\text{const}. \tag{4.39}$$

The system of spinor field equations in this case reads

$$\dot V_1-\mathcal{G}V_3=0, \tag{4.40a}$$

$$\dot V_2-\mathcal{G}V_4=0, \tag{4.40b}$$

$$\dot V_3+\mathcal{G}V_1=0, \tag{4.40c}$$

$$\dot V_4+\mathcal{G}V_2=0. \tag{4.40d}$$

Defining $U(\sigma)=V(t)$, where $\sigma=\int\mathcal{G}\,dt$, we rewrite Eqs. (4.40) as

$$U_1'-U_3=0, \tag{4.41a}$$

$$U_2'-U_4=0, \tag{4.41b}$$

$$U_3'+U_1=0, \tag{4.41c}$$

$$U_4'+U_2=0, \tag{4.41d}$$

where the primes denote differentiation with respect to σ. Differentiating the first equation of system (4.41) and taking into account the third one, we get

$$U_1''+U_1=0, \tag{4.42}$$

which leads to the solution

$$U_1=D_1e^{i\sigma}+iD_3e^{-i\sigma},$$

$$U_3=iD_1e^{i\sigma}+D_3e^{-i\sigma}.$$

Analogically for U_2 and U_4 one gets

$$U_2=D_2e^{i\sigma}+iD_4e^{-i\sigma},$$

$$U_4=iD_2e^{i\sigma}+D_4e^{-i\sigma},$$

where D_i are the constants of integration. Finally, we can write

$$\psi_1=\frac{1}{\sqrt{\tau}}(D_1e^{i\sigma}+iD_3e^{-i\sigma}), \tag{4.43a}$$

$$\psi_2=\frac{1}{\sqrt{\tau}}(D_2e^{i\sigma}+iD_4e^{-i\sigma}), \tag{4.43b}$$

$$\psi_3=\frac{1}{\sqrt{\tau}}(iD_1e^{i\sigma}+D_3e^{-i\sigma}), \tag{4.43c}$$

$$\psi_4=\frac{1}{\sqrt{\tau}}(iD_2e^{i\sigma}+D_4e^{-i\sigma}). \tag{4.43d}$$

Putting Eqs. (4.43) into the expressions (4.39), one comes to

$$D_0=2\,(D_1^2+D_2^2-D_3^2-D_4^2).$$

For the components of the spin current from Eqs. (3.27) we find

$$j^0=\frac{2}{\tau}[D_1^2+D_2^2+D_3^2+D_4^2], \tag{4.44a}$$

$$j^1=\frac{4}{a\tau}[D_2D_3+D_1D_4]\cos\left(2\int\mathcal{G}\,dt\right), \tag{4.44b}$$

$$j^2=\frac{4}{b\tau}[D_2D_3-D_1D_4]\sin\left(2\int\mathcal{G}\,dt\right), \tag{4.44c}$$

$$j^3=\frac{4}{c\tau}[D_1D_3-D_2D_4]\cos\left(2\int\mathcal{G}\right)dt), \tag{4.44d}$$

whereas, for the projection of spin vectors on the *X*, *Y*, and *Z* axis, we find

$$S^{23,0}=\frac{2}{bc\tau}[D_1D_2+D_3D_4], \tag{4.45a}$$

$$S^{31,0}=0, \tag{4.45b}$$

$$S^{12,0}=\frac{1}{2ab\tau}[D_1^2-D_2^2+D_3^2-D_4^2]. \tag{4.45c}$$

Let us now estimate τ using the equation

$$\ddot{\tau}/\tau=3\kappa\lambda(n-1)P^{2n}, \tag{4.46}$$

where we chose $L_N=\lambda P^{2n}$. Putting the value of P into Eq. (4.46) and integrating, one gets

$$\dot{\tau}^2=-3\kappa\lambda D_0^{2n}\tau^{2-2n}+y^2, \tag{4.47}$$

where y^2 is the integration constant having the form $y^2=(X_1^2+X_1X_3+X_3^2)/3>0$. The solution to Eq. (4.47) in quadrature reads

$$\int\frac{d\tau}{\sqrt{-3\kappa\lambda D_0^{2n}\tau^{2-2n}+y^2}}=t. \tag{4.48}$$

Let us now analyze the solution obtained here. As one can see, the case $n=1$ is the linear one. In the case of $\lambda<0$ for $n>1$, i.e., $2-2n<0$, we get

$$\tau(t)|_{t\to 0}\approx[(\sqrt{3\kappa|\lambda|}D_0^n n)t]^{1/n}$$

and

$$\tau|_{t\to\infty}\approx\sqrt{3\kappa y^2}\,t.$$

This means that for the term L_N considered with $\lambda<0$ and $n>1$, the solution is initially singular and the space-time is anisotropic at $t\to\infty$. Let us now study it for $n<1$. In this case we obtain

$$\tau|_{t\to 0}\approx\sqrt{3\kappa y^2}\,t$$

and

$$\tau|_{t\to\infty}\approx[(\sqrt{3\kappa|\lambda|}D_0^n n)t]^{1/n}.$$

The solution is initially singular as in the previous case, but as far as $1/n>1$, it provides an asymptotically isotropic expansion of BI space-time. The analysis for $\Lambda\neq 0$ completely coincides with those for $F=\lambda S^n$ with $m=0$.

(iii) In this case we study $L_N=F(I,J)$. Choosing

$$L_N=F(K_\pm),\quad K_+=I+J=I_v=-I_A,\quad K_-=I-J=I_T, \tag{4.49}$$

in the case of a massless NLSF we find

$$\mathcal{D}=2SF_{K_\pm},\quad \mathcal{G}=\pm 2PF_{K_\pm},\quad F_{K_\pm}=dF/dK_\pm\,.$$

Putting them into Eqs. (3.35), we find

$$S_0^2\pm P_0^2=D_\pm\,. \tag{4.50}$$

Choosing $F=\lambda K_\pm^n$ from Eq. (3.38) we get

$$\ddot{\tau}=3\kappa\lambda(n-1)\,D_\pm^n\,\tau^{1-2n}, \tag{4.51}$$

with the solution

$$\int \frac{\tau^{n-1}\,d\tau}{\sqrt{g^2\tau^{2n-2}-3\kappa\lambda D_{\pm}^n}}=t, \tag{4.52}$$

where $g^2=(X_1^2+X_1X_3+X_3^2)/3$. Let us study the case with $\lambda<0$. For $n<1$ from Eq. (3.33) one gets

$$\tau(t)|_{t\to 0}\approx gt\to 0, \tag{4.53}$$

i.e., the solutions are initially singular, and

$$\tau(t)|_{t\to\infty}\approx[\sqrt{(3\kappa|\lambda|D_{\pm}^n)}t]^{1/n}, \tag{4.54}$$

which means that the anisotropy disappears as the Universe expands. In the case of $n>1$ we get

$$\tau(t)|_{t\to 0}\approx t^{1/n}\to 0$$

and

$$\tau(t)|_{t\to\infty}\approx gt,$$

i.e., the solutions are initially singular and the metric functions $a(t)$, $b(t)$, and $c(t)$ are different at $t\to\infty$, i.e., the isotropization process remains absent. For $\lambda>0$ we get that the solutions are initially regular, but it violates the dominant energy condition in the Hawking-Penrose theorem [49]. Note that one comes to the analogical conclusion choosing $L_N=\lambda S^{2n}P^{2n}$.

C. Analysis of the results obtained when the BI universe is filled with perfect fluid

Let us now analyze the system filled with perfect fluid. In the absence of other matter, i.e., spinor field, in this case from Eq. (3.38) we find

$$\ddot{\tau}=\frac{3\kappa}{2}\frac{(1-\zeta)\varepsilon_0}{\tau^{\zeta}}, \tag{4.55}$$

with the first integral

$$\dot{\tau}=\sqrt{3\kappa\varepsilon_0\tau^{(1-\zeta)}+C}, \tag{4.56}$$

where C is an integration constant. From Eq. (4.56) one estimates

$$\tau\propto t^2 \quad \text{for} \quad \zeta=0 \quad (\text{dust}), \tag{4.57a}$$

$$\tau\propto t^{3/2} \quad \text{for} \quad \zeta=\tfrac{1}{3} \quad (\text{radiation}), \tag{4.57b}$$

$$\tau\propto t^{6/5} \quad \text{for} \quad \zeta=\tfrac{2}{3} \quad (\text{hard universe}), \tag{4.57c}$$

$$\tau\propto t \quad \text{for} \quad \zeta=1 \quad (\text{stiff matter}). \tag{4.57d}$$

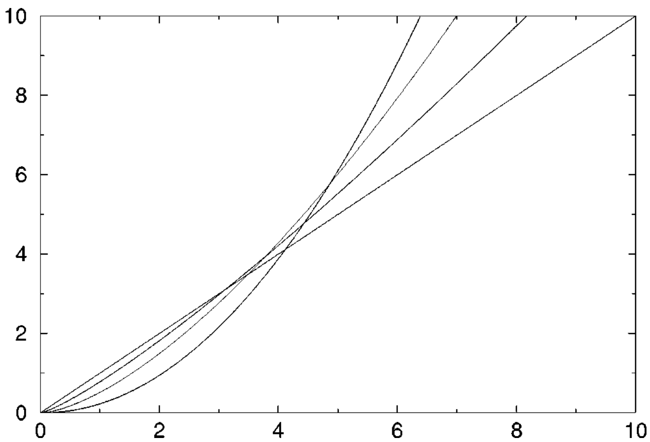

FIG. 5. Perspective view of τ when the BI universe is filled with perfect fluid only. The lines from left to right at the upper corner correspond to dust ($\zeta=0$), radiation ($\zeta=\frac{1}{3}$), hard universe ($\zeta=\frac{2}{3}$), and stiff matter ($\zeta=1$), respectively.

A perspective view of these solutions is given in Fig. 5.

Let us now consider the system as a whole with the nonlinear term being $L_{\rm N}=\lambda S^n$. In this case we get

$$\int \frac{d\tau}{\sqrt{mC_0\tau-\lambda C_0^n/\tau^{(n-2)}+\varepsilon_0\tau^{(1-\xi)}+g^2}}=\pm\sqrt{3\kappa}t. \tag{4.58}$$

As one can see in the case of dust ($\xi=0$), the fluid term can be combined with the massive one, whereas in the case of stiff matter ($\xi=1$), it mixes up with the constant. Analyzing Eq. (4.58) one concludes that in the presence of a spinor field, perfect fluid plays a secondary role in the evolution of a BI universe.

V. CONCLUSION

Within the framework of the simplest nonlinear model of a spinor field it has been shown that the Λ term plays a very important role in Bianchi-I cosmology. In particular, it invokes oscillations in the model, which is not the case when the Λ term remains absent. It should be noted that regularity of the solutions obtained by virtue of the Λ term, especially for the linear spinor field, does not violate the dominant energy condition, while this is not the case when regular solutions are attained by means of a nonlinear term. The growing interest in studying the role of the Λ term by present-day physicists of various disciplines indicates its fundamental value. For details on the time-dependent Λ term, one may consult [36] and references therein.

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