Nonlinear spinor field in cosmology

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Within the scope of the Bianchi type VI models, the self-consistent system of nonlinear spinor field and gravitational fields is considered. Exact self-consistent solutions to the spinor and gravitational field equations are obtained for a special choice of spatial inhomogeneity and nonlinear spinor term. The role of inhomogeneity in the evolution of spinor and gravitational field is studied. Some solutions allow an oscillating behavior of the Universe's volume. It should be emphasized that for a suitable choice of spinor field nonlinearity some of these solutions are nonsingular at all space-time points.

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I. INTRODUCTION

The problem of an initial singularity still remains at the center of modern day cosmology. Though the Big Bang theory is deeply rooted among the scientists dealing with the cosmology of the early Universe, it is natural to reconsider models of a universe free from initial singularities. In doing so, we previously considered a self-consistent system of nonlinear spinor and gravitational fields in the presence of a perfect fluid [1–5]. The nonlinear term in the corresponding Lagrangian mainly described the self-interaction of a spinor field [1–4], whereas in Ref. [5] we studied the case when the spinor field nonlinearity is induced by an interacting scalar field. As a gravitational field we chose a Bianchi type-I (BI) anisotropic cosmological model. We studied also the role of the cosmological constant Λ in the formation of oscillatory modes of evolution of the BI universe.

Why study a nonlinear spinor field? It is well known that the nonlinear generalization of classical field theory remains one possible way to overcome the difficulties of a theory that considers elementary particles as mathematical points. In this approach elementary particles are modeled by regular (solitonlike) solutions of the corresponding nonlinear equations. The gravitational field equation is nonlinear by nature and the field itself is universal and unscreenable. These properties lead to a definite physical interest in the gravitational field that goes with these matter fields. We prefer a spinor field to scalar or electromagnetic fields, as the spinor field is the most sensitive to the gravitational field.

Why study an anisotropic universe? Though spatially homogeneous and isotropic, Friedmann-Robertson-Walker (FRW) models are widely considered as a good approximation of the present and early stages of the Universe. However, the large scale matter distribution in the observable Universe, largely manifested in the form of discrete structures, does not exhibit a high degree of homogeneity. Recent space investigations detect anisotropy in the cosmic microwave background. The Cosmic Background Explorer's differential radiometer has detected and measured cosmic microwave background anisotropies at different angular scales.

These anisotropies are supposed to contain in their fold the entire history of cosmic evolution dating back to the recombination era and are being considered as indicative of the geometry and the content of the Universe. More information about cosmic microwave background anisotropy is expected to be uncovered by the investigations of the microwave anisotropy probe. There is widespread consensus among cosmologists that cosmic microwave background anisotropies at small angular scales are the key to the formation of discrete structures. The theoretical arguments [6] and recent experimental data that support the existence of an anisotropic phase that approaches an isotropic phase leads one to consider universe models with an anisotropic background. Its simplicity, the Kasner-universe-like behavior near the singularity [7] and evolution into a FRW universe when filled with matter obeying the equation of state $p=\zeta\varepsilon$, $\zeta<1$ [8] make the Bianchi type I (BI) model a prime candidate for studying the possible effects of an anisotropy in the early Universe on present-day observations. But there are a few other models, which describe an anisotropic space-time and generate particular interest among physicists [9–16]. In Ref. [10] methods of dynamical systems analysis were used to show that the presence of a magnetic field orthogonal to the two commuting Killing vector fields in any spatially homogeneous Bianchi type VI_0 vacuum solution to Einstein's equation changes the evolution towards the singularity from collapse to bounce. The authors in Ref. [11] studied the problem of isotropization of scalar field Bianchi models with an exponential potential(s). Other papers mentioned above are devoted to tilted perfect fluid solutions, chaotic singularities, and conditional symmetries.

In this paper we study the self-consistent system of the nonlinear spinor field and an anisotropic inhomogeneous gravitational field in order to clarify the role of the spinor field nonlinearity and the space-time inhomogeneity in the formation of a singularity-free universe. As an anisotropic space-time we choose a Bianchi type-VI (BVI) model, since a suitable choice of its parameters yields a few other Bianchi models including BI and FRW universes. It can be noted that unlike the BI universe, the BVI space-time is inhomogeneous. Inclusion of inhomogeneity in the gravitational field significantly complicates the search for an exact solution to the system. In Sec. II, we write the equations for nonlinear spinor fields and the system of Einstein equations. In this section we also give their solutions in some general form; more precisely, we write the solutions in terms of a time-

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dependent function that can be defined only when the concrete form of the spinor field nonlinearity is given. In Sec. III, we give exact solutions to the equations for some special choices of spinor field nonlinearity and space-time inhomogeneity. Beside this, we also present some numerical solutions in graphical form.

II. FUNDAMENTAL EQUATIONS AND GENERAL SOLUTIONS

We shall investigate a self-consistent system of nonlinear spinor and Einstein gravitational fields. These two fields are to be codetermined by the following action:

$$\mathcal{S}(\mathrm{g};\psi,\bar{\psi}) = \int \mathcal{L}\sqrt{-\mathrm{g}}\,d\Omega \tag{2.1}$$

with

$$\mathcal{L} = \mathcal{L}_{\rm grav} + \mathcal{L}_{\rm spinor}. \tag{2.2}$$

The gravitational part of the Lagrangian (2.2) is given by a BVI space-time, while the spinor part is a usual spinor field Lagrangian with an arbitrary nonlinear term.

A. Spinor field Lagrangian

For a spinor field ψ, symmetry between ψ and $\bar{\psi}$ appears to demand that one should choose a symmetrized Lagrangian [17]. Accordingly we choose the spinor field Lagrangian with a nonlinear term in (2.3) as follows:

$$\mathcal{L}_{\rm spinor} = \frac{i}{2}[\bar{\psi}\gamma^{\mu}\nabla_{\mu}\psi - \nabla_{\mu}\bar{\psi}\gamma^{\mu}\psi] - M\bar{\psi}\psi + F. \tag{2.3}$$

Here M is the spinor mass.The nonlinear term F describes the self-interaction of the spinor field and can be presented as some arbitrary functions of invariants generated from the real bilinear forms of a spinor field having the form

$$S = \bar{\psi}\psi \quad (\text{scalar}), \tag{2.4a}$$

$$P = i\bar{\psi}\gamma^5\psi \quad (\text{pseudoscalar}), \tag{2.4b}$$

$$v^{\mu} = (\bar{\psi}\gamma^{\mu}\psi) \quad (\text{vector}), \tag{2.4c}$$

$$A^{\mu} = (\bar{\psi}\gamma^5\gamma^{\mu}\psi) \quad (\text{pseudovector}), \tag{2.4d}$$

$$Q^{\mu\nu} = (\bar{\psi}\sigma^{\mu\nu}\psi) \quad (\text{antisymmetric tensor}), \tag{2.4e}$$

where $\sigma^{\mu\nu} = (i/2)[\gamma^{\mu}\gamma^{\nu} - \gamma^{\nu}\gamma^{\mu}]$. Invariants, corresponding to the bilinear forms, are

$$I = S^2, \tag{2.5a}$$

$$J = P^2, \tag{2.5b}$$

$$I_v = v_{\mu}v^{\mu} = (\bar{\psi}\gamma^{\mu}\psi)g_{\mu\nu}(\bar{\psi}\gamma^{\nu}\psi), \tag{2.5c}$$

$$I_A = A_{\mu}A^{\mu} = (\bar{\psi}\gamma^5\gamma^{\mu}\psi)g_{\mu\nu}(\bar{\psi}\gamma^5\gamma^{\nu}\psi), \tag{2.5d}$$

$$I_Q = Q_{\mu\nu}Q^{\mu\nu} = (\bar{\psi}\sigma^{\mu\nu}\psi)g_{\mu\alpha}g_{\nu\beta}(\bar{\psi}\sigma^{\alpha\beta}\psi). \tag{2.5e}$$

According to the Pauli-Fierz theorem [18] among the five invariants only I and J are independent as all others can be expressed by them: $I_V = -I_A = I + J$ and $I_Q = I - J$. Therefore, we choose the nonlinear term F as a function of the invariants I and J, i.e., $F = F(I,J)$, thus claiming that it describes the nonlinearity in the most general form.

B. The gravitational field

The gravitational part of the Lagrangian in (2.2) has the form:

$$\mathcal{L}_{\rm grav} = \frac{R}{2\kappa}, \tag{2.6}$$

Here R is the scalar curvature and κ is Einstein's gravitational constant. The gravitational field in our case is given by a BVI metric:

$$ds^2 = dt^2 - a^2e^{-2mz}dx^2 - b^2e^{2nz}dy^2 - c^2dz^2, \tag{2.7}$$

with a,b,c being functions of time only. Here m,n are some arbitrary constants and the velocity of light is taken to be unity. It should be emphasized that the BVI metric models a universe that is anisotropic and inhomogeneous. A suitable choice of m,n as well as the metric functions a,b,c in the BVI metric given by (2.7) generates Bianchi-type universes discussed in the following:

(1) For $m = n$ the BVI metric transforms into a Bianchi-type V (BV) universe, i.e., $m = n$, BVI⇒BV ∈ open FRW with the line element(s)

$$ds^2 = dt^2 - a^2e^{-2mz}dx^2 - b^2e^{2mz}dy^2 - c^2dz^2. \tag{2.8}$$

(2) For $n = 0$ the BVI metric transforms into a Bianchi-type III (BIII) universe, i.e., $n = 0$, BVI⇒BIII with the line element(s)

$$ds^2 = dt^2 - a^2e^{-2mz}dx^2 - b^2dy^2 - c^2dz^2. \tag{2.9}$$

(3) For $m = n = 0$ the BVI metric transforms into a Bianchi-type I (BI) universe, i.e., $m = n = 0$, BVI⇒BI with the line element(s)

$$ds^2 = dt^2 - a^2dx^2 - b^2dy^2 - c^2dz^2. \tag{2.10}$$

(4) For $m = n = 0$ and an equal scale factor in all three directions the BVI metric transforms into a FRW universe, i.e., $m = n = 0$ and $a = b = c$, BVI⇒FRW with the line element(s)

$$ds^2 = dt^2 - a^2(dx^2 + dy^2 + dz^2). \tag{2.11}$$

Let us write the nontrivial components of Ricci and Riemann tensors as well as Christoffel symbols of the BVI metric. The nontrivial Christoffel symbols of the BVI metric read

$$\Gamma_{01}^{1}=\dot{a}/a, \quad \Gamma_{02}^{2}=\dot{b}/b, \quad \Gamma_{03}^{3}=\dot{c}/c,$$

$$\Gamma_{11}^{0}=a\dot{a}e^{-2mz}, \quad \Gamma_{22}^{0}=b\dot{b}e^{2nz}, \quad \Gamma_{33}^{0}=c\dot{c},$$

$$\Gamma_{31}^{1}=-m, \quad \Gamma_{32}^{2}=n, \quad \Gamma_{11}^{3}=\frac{ma^2}{c^2}e^{-2mz},$$

$$\Gamma_{22}^{3}=-\frac{nb^2}{c^2}e^{2mz}.$$

The nontrivial components of Riemann tensor are

$$R^{01}{}_{01}=-\frac{\ddot{a}}{a}=-\frac{\ddot{a}bc}{\tau}, \quad R^{02}{}_{02}=-\frac{\ddot{b}}{b}=-\frac{a\ddot{b}c}{\tau},$$

$$R^{03}{}_{03}=-\frac{\ddot{c}}{c}=-\frac{ab\ddot{c}}{\tau},$$

$$R^{12}{}_{12}=-\frac{mn}{c^2}-\frac{\dot{a}}{a}\frac{\dot{b}}{b}=-\frac{mn}{c^2}-\frac{\dot{a}\dot{b}c}{\tau},$$

$$R^{13}{}_{13}=\frac{m^2}{c^2}-\frac{\dot{c}}{c}\frac{\dot{a}}{a}=\frac{m^2}{c^2}-\frac{b\dot{c}\dot{a}}{\tau},$$

$$R^{23}{}_{23}=\frac{n^2}{c^2}-\frac{\dot{b}}{b}\frac{\dot{c}}{c}=\frac{n^2}{c^2}-\frac{\dot{b}\dot{c}a}{\tau}.$$

The nontrivial components of the Ricci tensor are

$$R_3^0=\left(m\frac{\dot{a}}{a}-n\frac{\dot{b}}{b}-(m-n)\frac{\dot{c}}{c}\right)$$
$$=\frac{1}{\tau}[m\dot{a}bc-n\dot{b}ca-(m-n)\dot{c}ab],$$

$$R_0^0=-\left(\frac{\ddot{a}}{a^2}+\frac{\ddot{b}}{b^2}+\frac{\ddot{c}}{c^2}\right)$$
$$=-\frac{1}{\tau}[\ddot{a}bc+\ddot{b}ca+\ddot{c}ab],$$

$$R_1^1=-\left(\frac{\ddot{a}}{a}+\frac{\dot{a}}{a}\frac{\dot{b}}{b}+\frac{\dot{a}}{a}\frac{\dot{c}}{c}-\frac{m^2-mn}{c^2}\right)$$
$$=-\frac{1}{\tau}[\ddot{a}bc+\dot{a}\dot{b}c+\dot{c}\dot{a}b]+\frac{m^2-mn}{c^2},$$

$$R_2^2=-\left(\frac{\ddot{b}}{b}+\frac{\dot{a}}{a}\frac{\dot{b}}{b}+\frac{\dot{b}}{b}\frac{\dot{c}}{c}-\frac{n^2-mn}{c^2}\right)$$
$$=-\frac{1}{\tau}[a\ddot{b}c+\dot{a}\dot{b}c+\dot{b}\dot{c}a]+\frac{n^2-mn}{c^2},$$

$$R_3^3=-\left(\frac{\ddot{c}}{c}+\frac{\dot{a}}{a}\frac{\dot{c}}{c}+\frac{\dot{b}}{b}\frac{\dot{c}}{c}-\frac{m^2+n^2}{c^2}\right)$$
$$=-\frac{1}{\tau}[\ddot{c}ab+\dot{c}\dot{a}b+\dot{b}\dot{c}a]+\frac{m^2+n^2}{c^2},$$

where we define

$$\tau=abc. \tag{2.12}$$

To investigate the existence of a singularity (singular point), one has to study the invariant characteristics of space-time. As we know, in general relativity, these invariants are composed of the curvature tensor and the metric one. Although in 4D Riemann space there are 14 independent invariants [4,19], it is sufficient to study only three of them, namely the scalar curvature $I_1=R$, $I_2=R_{\mu\nu}R^{\mu\nu}$ and the Kretschmann scalar $I_3=R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$ [20,21].

From the Riemann and Ricci tensors written above one finds

$$I_1=R=-\frac{2}{\tau}\left[\ddot{\tau}-\dot{a}\dot{b}c-a\dot{b}\dot{c}-\dot{a}b\dot{c}-\frac{ab}{c}(m^2-mn+n^2)\right], \tag{2.13a}$$

$$I_2=(R_0^0)^2+(R_1^1)^2+(R_2^2)^2+(R_3^3)^2+R_3^0R_0^3, \tag{2.13b}$$

$$I_3=4[(R_{01}^{01})^2+(R_{02}^{02})^2+(R_{03}^{03})^2+(R_{12}^{12})^2+(R_{31}^{31})^2+(R_{23}^{23})^2]. \tag{2.13c}$$

From (2.13) it follows that $I_1\propto 1/\tau$, $I_2\propto 1/\tau^2$, and $I_3\propto 1/\tau^2$. Note that the remaining 11 invariants are composed of two or more Ricci and/or Riemann tensors and hence are inversely proportional to $(\tau)^p$, where p is the number of tensors in the corresponding invariant. Thus we see that at any space-time point where $\tau=0$, the invariants I_1,I_2,I_3 become infinity; hence the space-time becomes singular at this point.

C. Field equations

The field equations for the spinor and gravitational fields can be obtained from the variational principle. Variation of the Lagrangian (2.3) with respect to the field functions $\psi(\bar{\psi})$ gives the nonlinear spinor field equations:

$$i\gamma^\mu\nabla_\mu\psi-M\psi+\mathcal{D}\psi+i\mathcal{G}\gamma^5\psi=0, \tag{2.14a}$$

$$i\nabla_\mu\bar{\psi}\gamma^\mu+M\bar{\psi}-\mathcal{D}\bar{\psi}-i\mathcal{G}\bar{\psi}\gamma^5=0, \tag{2.14b}$$

where $\mathcal{D}=2SF_I$ and $\mathcal{G}=2PF_J$.

Varying (2.2) with respect to metric function $(g_{\mu\nu})$, we find Einstein's field equation

$$R^{\mu}_{\nu} - \frac{1}{2}\delta^{\mu}_{\nu}R = \kappa T^{\mu}_{\nu}, \tag{2.15}$$

where R^{μ}_{ν} is the Ricci tensor, R is the Ricci scalar, and T^{μ}_{ν} is the energy-momentum tensor of the spinor field. In our case, where space-time is given by a BVI metric (2.7), the equations for the metric functions a,b,c read

$$\frac{\ddot b}{b} + \frac{\ddot c}{c} + \frac{\dot b}{b}\frac{\dot c}{c} - \frac{n^2}{c^2} = \kappa T^1_1, \tag{2.16a}$$

$$\frac{\ddot c}{c} + \frac{\ddot a}{a} + \frac{\dot c}{c}\frac{\dot a}{a} - \frac{m^2}{c^2} = \kappa T^2_2, \tag{2.16b}$$

$$\frac{\ddot a}{a} + \frac{\ddot b}{b} + \frac{\dot a}{a}\frac{\dot b}{b} + \frac{mn}{c^2} = \kappa T^3_3, \tag{2.16c}$$

$$\frac{\dot a}{a}\frac{\dot b}{b} + \frac{\dot b}{b}\frac{\dot c}{c} + \frac{\dot c}{c}\frac{\dot a}{a} - \frac{m^2 - mn + n^2}{c^2} = \kappa T^0_0, \tag{2.16d}$$

$$m\frac{\dot a}{a} - n\frac{\dot b}{b} - (m-n)\frac{\dot c}{c} = \kappa T^0_3. \tag{2.16e}$$

Here over dots denote differentiation with respect to time (t). The energy-momentum tensor of the material field T^{ν}_{μ} has the form:

$$T^{\rho}_{\mu} = \frac{i}{4}g^{\rho\nu}(\bar\psi\gamma_\mu\nabla_\nu\psi + \bar\psi\gamma_\nu\nabla_\mu\psi - \nabla_\mu\bar\psi\gamma_\nu\psi - \nabla_\nu\bar\psi\gamma_\mu\psi) - \delta^{\rho}_{\mu}L_{\rm spinor}. \tag{2.17}$$

Here $L_{\rm spinor}$ is the spinor field Lagrangian, which on account of the spinor field equations (2.14) takes the form:

$$L_{\rm spinor} = -\mathcal{D}S - \mathcal{G}P + F. \tag{2.18}$$

In the expressions above ∇_μ denotes the covariant derivative on spinors, having the form [22]

$$\nabla_\mu\psi = \partial_\mu\psi - \Gamma_\mu\psi, \tag{2.19}$$

where Γ_μ is the spin connection. The spin affine connection matrices $\Gamma_\mu(x)$ are uniquely determined up to an additive multiple of the unit matrix by the following equation [23],

$$\partial_\mu\gamma_\nu - \Gamma^{\alpha}_{\nu\mu}\gamma_\alpha - \Gamma_\mu\gamma_\nu + \gamma_\nu\Gamma_\mu = 0, \tag{2.20}$$

with the solution

$$\Gamma_\mu(x) = \frac{1}{4}g_{\rho\sigma}(x)(\partial_\mu e^b_\delta e^\rho_b - \Gamma^\rho_{\mu\delta})\gamma^\sigma\gamma^\delta. \tag{2.21}$$

Here e^a_μ is a set of tetrad four-vectors defined as

$$g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}, \quad \eta_{ab} = \mathrm{diag}(1, -1, -1, -1). \tag{2.22}$$

For the metric element (2.7) this gives

$$\Gamma_0 = 0,$$

$$\Gamma_1 = \frac{1}{2}\left[\dot a\bar\gamma^1\bar\gamma^0 - m\frac{a}{c}\bar\gamma^1\bar\gamma^3\right]e^{-mz}$$

$$\Gamma_2 = \frac{1}{2}\left[\dot b\bar\gamma^2\bar\gamma^0 + n\frac{b}{c}\bar\gamma^2\bar\gamma^3\right]e^{nz},$$

$$\Gamma_3 = \frac{1}{2}\dot c\bar\gamma^3\bar\gamma^0$$

It is easy to show that

$$\gamma^\mu\Gamma_\mu = -\frac{1}{2}\frac{\dot\tau}{\tau}\bar\gamma^0 + \frac{m-n}{2c}\bar\gamma^3.$$

The Dirac matrices $\gamma^\mu(x)$ of curved space-time are connected with those of Minkowski space-time as follows:

$$\gamma^0 = \bar\gamma^0, \quad \gamma^1 = \bar\gamma^1 e^{mz}/a, \quad \gamma^2 = \bar\gamma^2/be^{nz}, \quad \gamma^3 = \bar\gamma^3/c,$$

with

$$\bar\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \bar\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix},$$

$$\gamma^5 = \bar\gamma^5 = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix},$$

where σ_i are the Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Note that the $\bar\gamma$ and the σ matrices obey the following relations:

$$\bar\gamma^i\bar\gamma^j + \bar\gamma^j\bar\gamma^i = 2\eta^{ij}, \quad i,j = 0,1,2,3$$

$$\bar\gamma^i\bar\gamma^5 + \bar\gamma^5\bar\gamma^i = 0, \quad (\bar\gamma^5)^2 = I, \quad i = 0,1,2,3$$

$$\sigma^j\sigma^k = \delta_{jk} + i\varepsilon_{jkl}\sigma^l, \quad j,k,l = 1,2,3$$

where $\eta_{ij} = \{1,-1,-1,-1\}$ is the diagonal matrix, δ_{jk} is the Kronecker symbol, and ε_{jkl} is the totally antisymmetric tensor with $\varepsilon_{123} = +1$.

Let us consider the spinors to be functions of t and z only, such that

$$\psi(t,z) = v(t)e^{ikz}, \quad \bar\psi(t,z) = \bar v(t)e^{-ikz}. \tag{2.23}$$

Inserting (2.23) into (2.14) for the nonlinear spinor field, we find

$$\bar{\gamma}^0\left(\dot{v}+\frac{\dot{\tau}}{2\tau}v\right)-\left(\frac{m-n}{2c}-i\frac{k}{c}\right)\bar{\gamma}^3 v+i\Phi v+\mathcal{G}\bar{\gamma}^5 v=0, \tag{2.24a}$$

$$\left(\dot{\bar{v}}+\frac{\dot{\tau}}{2\tau}\bar{v}\right)\bar{\gamma}^0-\left(\frac{m-n}{2c}+i\frac{k}{c}\right)\bar{v}\bar{\gamma}^3-i\Phi\bar{v}-\mathcal{G}\bar{v}\bar{\gamma}^5=0. \tag{2.24b}$$

Here we define $\Phi=M-\mathcal{D}$. Let us introduce a new function

$$u_j(t)=\sqrt{\tau}v_j(t).$$

Then for the components of the nonlinear spinor field from (2.24), one obtains

$$\dot{u}_1+i\Phi u_1-\left[\frac{m-n}{2c}-i\frac{k}{c}+\mathcal{G}\right]u_3=0, \tag{2.25a}$$

$$\dot{u}_2+i\Phi u_2+\left[\frac{m-n}{2c}-i\frac{k}{c}-\mathcal{G}\right]u_4=0, \tag{2.25b}$$

$$\dot{u}_3-i\Phi u_3-\left[\frac{m-n}{2c}-i\frac{k}{c}-\mathcal{G}\right]u_1=0, \tag{2.25c}$$

$$\dot{u}_4-i\Phi u_4+\left[\frac{m-n}{2c}-i\frac{k}{c}+\mathcal{G}\right]u_2=0. \tag{2.25d}$$

Using the spinor field equations (2.14) and (2.24), it can be shown that the bilinear spinor forms, defined by (2.4), i.e.,

$$S=\bar{\psi}\psi=\bar{v}v,\quad P=i\bar{\psi}\bar{\gamma}^5\psi=i\bar{v}\bar{\gamma}^5 v,$$

$$A^0=\bar{\psi}\bar{\gamma}^5\bar{\gamma}^0\psi=\bar{v}\bar{\gamma}^5\bar{\gamma}^0 v,$$

$$A^3=\bar{\psi}\bar{\gamma}^5\bar{\gamma}^3\psi=\bar{v}\bar{\gamma}^5\bar{\gamma}^3 v,\quad V^0=\bar{\psi}\bar{\gamma}^0\psi=\bar{v}\bar{\gamma}^0 v,$$

$$V^3=\bar{\psi}\bar{\gamma}^3\psi=\bar{v}\bar{\gamma}^3 v,\quad Q^{30}=i\bar{\psi}\bar{\gamma}^3\bar{\gamma}^0\psi=i\bar{v}\bar{\gamma}^3\bar{\gamma}^0 v,$$

$$Q^{21}=\bar{\psi}\bar{\gamma}^0\bar{\gamma}^3\bar{\gamma}^5\psi=i\bar{\psi}\bar{\gamma}^2\bar{\gamma}^1\psi=i\bar{v}\bar{\gamma}^2\bar{\gamma}^1 v,$$

obey the following system of equations:

$$\dot{S}_0-2\frac{k}{c}Q_0^{30}-2\mathcal{G}A_0^0=0, \tag{2.26a}$$

$$\dot{P}_0-2\frac{k}{c}Q_0^{21}-2\Phi A_0^0=0, \tag{2.26b}$$

$$\dot{A}_0^0-\frac{m-n}{c}A_0^3+2\Phi P_0+2\mathcal{G}S_0=0, \tag{2.26c}$$

$$\dot{A}_0^3-\frac{m-n}{c}A_0^0=0, \tag{2.26d}$$

$$\dot{V}_0^0-\frac{m-n}{c}V_0^3=0, \tag{2.26e}$$

$$\dot{V}_0^3-\frac{m-n}{c}V_0^0+2\Phi Q_0^{30}-2\mathcal{G}Q_0^{21}=0, \tag{2.26f}$$

$$\dot{Q}_0^{30}+2\frac{k}{c}S_0-2\Phi V_0^3=0, \tag{2.26g}$$

$$\dot{Q}_0^{21}+2\frac{k}{c}P_0+2\mathcal{G}V_0^3=0, \tag{2.26h}$$

where we use the notation $F_0=\tau F$. Combining these equations and taking the first integral one gets

$$(S_0)^2+(P_0)^2+(A_0^0)^2-(A_0^3)^2-(V_0^0)^2+(V_0^3)^2+(Q_0^{30})^2+(Q_0^{21})^2=C=\text{const.} \tag{2.27}$$

Before dealing with the Einstein equations (2.16) let us go back to (2.25). From the first and the third equations of the system (2.25) one finds

$$\dot{u}_{13}=(\mathcal{G}-Q)u_{13}^2-2i\Phi u_{13}+(\mathcal{G}+Q), \tag{2.28}$$

where we denote $u_{13}=u_1/u_3$ and $Q=[m-n-2ik]/2c$. Equation (2.28) is of the Riccati type [24] with variable coefficients. A transformation [25]

$$v_{13}=\exp\left(-\int(\mathcal{G}-Q)u_{13}dt\right), \tag{2.29}$$

leads from the general Riccati equation (2.28) to a second order linear one, namely,

$$(\mathcal{G}-Q)\ddot{v}_{13}+[2i\Phi(\mathcal{G}-Q)-\dot{\mathcal{G}}+\dot{Q}]\dot{v}_{13}+(\mathcal{G}-Q)^2(\mathcal{G}+Q)v_{13}=0. \tag{2.30}$$

Sometimes it is easier to solve a linear second order differential equation than a first order nonlinear equation. Here we give a general solution to (2.28). For this purpose we rewrite (2.28) in the form

$$\dot{w}_{13}=(\mathcal{G}-Q)w_{13}^2e^{-2i\int\Phi(t)dt}+(\mathcal{G}+Q)e^{2i\int\Phi(t)dt}, \tag{2.31}$$

where we set $u_{13}=w_{13}\exp[-2i\int\Phi(t)dt]$. This is an inhomogeneous nonlinear differential equation for w_{13}. The solution for the homogeneous part of (2.31), i.e.,

$$\dot{w}_{13}=(\mathcal{G}-Q)w_{13}^2\exp\left(-2i\int\Phi(t)dt\right) \tag{2.32}$$

reads

$$w_{13}=-\left[\int(\mathcal{G}-Q)\exp\left(-2i\int\Phi(t)dt\right)dt+C\right]^{-1}, \tag{2.33}$$

where C is an arbitrary constant. Then the general solution to the inhomogeneous equation (2.31) can be presented as

$$w_{13}=-\left[\int(\mathcal{G}-Q)\exp-\left(2i\int\Phi(t)dt\right)dt+C(t)\right]^{-1}, \tag{2.34}$$

with the time dependent parameter $C(t)$ to be determined from

$$\dot{C}=\left[\int(\mathcal{G}-Q)\exp\left(-2i\int\Phi(t)dt\right)dt+C(t)\right]^{2}\times(\mathcal{G}+Q)e^{2i\int\Phi(t)dt}. \tag{2.35}$$

Thus given a concrete nonlinear term in the Lagrangian and the solutions of the Einstein equations, one finds the relation between u_1 and u_3 (u_2 and u_4 as well), hence the components of the spinor field.

Now we study the Einstein equations (2.16). In doing so, we write the components of the energy-momentum tensor, which in our case read

$$T_0^{\,0}=MS-F+\frac{k}{c}V^3, \tag{2.36a}$$

$$T_1^1=T_2^2=\mathcal{D}S+\mathcal{G}P-F, \tag{2.36b}$$

$$T_3^3=\mathcal{D}S+\mathcal{G}P-F-\frac{k}{c}V^3, \tag{2.36c}$$

$$T_3^{\,0}=-kV^0. \tag{2.36d}$$

Let us demand the energy-momentum tensor to be conserved, i.e.,

$$T^{\mu}_{\nu;\mu}=T^{\mu}_{\nu,\mu}+\Gamma^{\mu}_{\beta\mu}T^{\beta}_{\nu}-\Gamma^{\beta}_{\nu\mu}T^{\mu}_{\beta}=0. \tag{2.37}$$

Taking into account that T_{μ}^{ν} is a function of t only, from (2.37) we find

$$\Phi\dot{S}_0-\mathcal{G}\dot{P}_0+\frac{k}{c}\dot{V}_0^3-\frac{k}{c}\frac{m-n}{c}V_0^0=0, \tag{2.38a}$$

$$\dot{V}_0^0-\frac{m-n}{c}V_0^3=0. \tag{2.38b}$$

As one can easily verify, Eqs. (2.38) are consistent with those of (2.26).

Let us go back to Eqs. (2.16). In view of (2.36), from (2.16e) one obtains the following relation between the metric functions a,b,c:

$$\left(\frac{a}{c}\right)^m=\left(\frac{b}{c}\right)^n\mathcal{N}\exp\left(-\kappa k\int V^0dt\right),\quad \mathcal{N}=\text{const.} \tag{2.39}$$

Subtracting (2.16a) from (2.16b) we find

$$\frac{d}{dt}\left[\tau\frac{d}{dt}\left\{\ln\left(\frac{a}{b}\right)\right\}\right]=\frac{m^2-n^2}{c^2}\tau \tag{2.40}$$

Analogously, subtraction of (2.16a) from (2.16c) and (2.16b) from (2.16c) gives

$$\frac{d}{dt}\left[\tau\frac{d}{dt}\left\{\ln\left(\frac{a}{c}\right)\right\}\right]=-\frac{mn+n^2}{c^2}\tau-\frac{\kappa k}{c}V^3\tau \tag{2.41}$$

and

$$\frac{d}{dt}\left[\tau\frac{d}{dt}\left\{\ln\left(\frac{b}{c}\right)\right\}\right]=-\frac{mn+m^2}{c^2}\tau-\frac{\kappa k}{c}V^3\tau, \tag{2.42}$$

respectively. It can be shown that, in view of (2.26) and (2.39), Eqs. (2.40), (2.41), and (2.42) are interchangeable.

Taking into account that $\tau=abc$, from (2.39) we can write a and b in terms of c, such that

$$a=\left[\tau^n c^{m-2n}\mathcal{N}\exp\left(-\kappa k\int V^0dt\right)\right]^{1/(m+n)}, \tag{2.43}$$

and

$$b=\left\{\tau^m c^{n-2m}\Big/\left[\mathcal{N}\exp\left(-\kappa k\int V^0dt\right)\right]\right\}^{1/(m+n)}. \tag{2.44}$$

In view of (2.43), (2.44), and (2.26) from (2.40) one finds

$$\frac{\ddot{\tau}}{\tau}=3\frac{\dot{\tau}}{\tau}\frac{\dot{c}}{c}+3\left(\frac{\ddot{c}}{c}-\frac{\dot{c}^2}{c^2}\right)-2\frac{\kappa k}{c}V^3-\frac{(m+n)^2}{c^2}. \tag{2.45}$$

In getting (2.45) we employ only four out of five Einstein equations, leaving (2.16d) unused. On the other hand, adding (2.16a), (2.16b), (2.16c), and (2.16d), multiplied by 3 we get the equation for τ, which in view of (2.36) takes the form

$$\frac{\ddot{\tau}}{\tau}=2\frac{m^2-mn+n^2}{c^2}+\frac{\kappa}{2}\left[3(MS+\mathcal{D}S+\mathcal{G}P-2F)+2\frac{k}{c}V^3\right]. \tag{2.46}$$

Thus we are left with two equations, namely (2.45) and (2.46), for two unknowns c and τ. These two equations can be combined to get

$$\frac{\ddot{c}}{c}-\frac{\dot{c}^2}{c^2}+\frac{\dot{\tau}}{\tau}\frac{\dot{c}}{c}=\frac{\kappa k}{c}V^3+\frac{m^2+n^2}{c^2}+\frac{\kappa}{2}[MS+\mathcal{D}S+\mathcal{G}P-2F]. \tag{2.47}$$

Thus we have come to Eq. (2.47) where all the equations at hand, both spinor and gravitational, are employed. Assuming c as a function of τ (or vice versa) and given a concrete form of the spinor field nonlinearity one finds the solution of (2.47). This is exactly what we do in the next section.

III. ANALYSIS OF THE RESULTS

In the preceding section we derived the fundamental equations for nonlinear spinor fields and metric functions. Comparing the equation with those in a BI universe (see e.g., Ref. [4]) we conclude that introduction of inhomogeneity both in gravitational (through m and n) and spinor (through k) fields significantly complicates the whole picture. In what follows, we will write the solutions explicitly.

Let us first consider Eq. (2.47). As one sees, there are two unknown functions in this equation, namely, c and τ, with τ defined as $\tau = abc$. As a first step, we demand an additional assumption relating c and τ, namely, $c=\tau$ or $c=\sqrt{\tau}$. Note that such an assumption imposes some restrictions on the metric functions, though leaving the space-time anisotropic. In what follows, we study Eq. (2.47) under the assumptions made above for different types of nonlinear spinor terms.

A. Case I

Let us assume that

$$c=\tau. \tag{3.1}$$

Under this assumption in view of $\tau=abc$, we should have $a=1/b$. Indeed, from (2.43) and (2.44) we find

$$a=\left[\tau^{m-n}\mathcal{N}\exp\left(-\kappa k\int V^0 dt\right)\right]^{1/(m+n)}, \tag{3.2}$$

and

$$b=\left\{\tau^{n-m}\Big/\left[\mathcal{N}\exp\left(-\kappa k\int V^0 dt\right)\right]\right\}^{1/(m+n)}. \tag{3.3}$$

With regard to (3.1), from (2.47) we obtain

$$\frac{\ddot{\tau}}{\tau}=\frac{\kappa k}{\tau}V^3+\frac{m^2+n^2}{\tau^2}+\frac{\kappa}{2}[MS+\mathcal{D}S+\mathcal{G}P-2F]. \tag{3.4}$$

Let us now study (3.4) for some special choice of spinor field nonlinearity.

1. Linear spinor field

To begin with we consider the linear case setting $F(I,J)=0$. It immediately leads to $\mathcal{D}=0$ and $\mathcal{G}=0$. Equation (3.4) now takes the form

$$\frac{\ddot{\tau}}{\tau}=\frac{\kappa k}{\tau}V^3+\frac{m^2+n^2}{\tau^2}+\frac{\kappa}{2}MS. \tag{3.5}$$

As one sees, to solve (3.5), we have to find V^3 and S first. From (2.26) for the linear spinor field we have

$$\dot{S}_0-2\frac{k}{c}Q_0^{30}=0, \tag{3.6a}$$

$$\dot{P}_0-2\frac{k}{c}Q_0^{21}-2MA_0^0=0, \tag{3.6b}$$

$$\dot{A}_0^0-\frac{m-n}{c}A_0^3+2MP_0=0, \tag{3.6c}$$

$$\dot{A}_0^3-\frac{m-n}{c}A_0^0=0, \tag{3.6d}$$

$$\dot{V}_0^0-\frac{m-n}{c}V_0^3=0, \tag{3.6e}$$

$$\dot{V}_0^3-\frac{m-n}{c}V_0^0+2MQ_0^{30}=0, \tag{3.6f}$$

$$\dot{Q}_0^{30}+2\frac{k}{c}S_0-2MV_0^3=0, \tag{3.6g}$$

$$\dot{Q}_0^{21}+2\frac{k}{c}P_0=0, \tag{3.6h}$$

with the first integrals

$$(S_0)^2+(V_0^3)^2+(Q_0^{30})^2-(V_0^0)^2=0, \tag{3.7a}$$

$$(P_0)^2+(A_0^0)^2+(Q_0^{21})^2-(A_0^3)^2=0. \tag{3.7b}$$

Thus we see that even in the case of a linear spinor field with $k\neq 0$ we cannot write V^3 or S explicitly. In order to express S or P, hence the massive term or spinor field nonlinearity, in terms of τ, we now consider the spinor field to be space independent setting $k=0$.

From (2.26) in this case one obtains,

$$S=\frac{C_0}{\tau}, \tag{3.8}$$

with C_0 being the integration constant. Equation (3.4) in this case takes the form

$$\ddot{\tau}=\frac{m^2+n^2}{\tau}+\frac{\kappa}{2}MC_0. \tag{3.9}$$

The solution of (3.9) can be written in quadrature as

$$\int\frac{d\tau}{\sqrt{2(m^2+n^2)\ln\tau+\kappa MC_0\tau+E}}=t,\quad E=\text{const}. \tag{3.10}$$

For the solution to be meaningful, the integrand in (3.10) should be positive. This means that for τ to have an initial value close to zero, one has to set small values for m and n, while the constant E should be large enough.

The components of the spinor field can be obtained from (2.33). In the case considered, $\mathcal{G}=0$, $Q=(m-n)/2\tau$ and $\Phi=M$.

2. Nonlinear spinor field with $k=0$

Let us now consider the nonlinear spatially independent spinor field. We first choose the nonlinear term as a function of $I=S^2$ only, followed by a massless spinor field with the nonlinear term being a function of $J=P^2$.

If the nonlinear spinor term is given as $F=F(I)=\lambda S^{\eta}$, where λ is the (self-) coupling constant, then in view of $S=C_0/\tau$ for τ, we find

$$\ddot{\tau}=\frac{m^2+n^2}{\tau}+\frac{\kappa}{2}MC_0+\frac{\kappa\lambda(\eta-2)C_0^{\eta}}{2\tau^{\eta-1}}, \tag{3.11}$$

with the solution in quadrature

$$\int\frac{d\tau}{\sqrt{2(m^2+n^2)\ln\tau+\kappa MC_0\tau-\kappa\lambda C_0^{\eta}\tau^{2-\eta}+E}}=t. \tag{3.12}$$

As one sees, the inclusion of the nonlinear term sets an additional restriction on the smallness of the initial value of τ. The components of the spinor field, as in linear case, can be obtained from (2.33). In the case considered, $\mathcal{G}=0$ and $Q=(m-n)/2c$.

For the massless spinor field, if the nonlinear term is chosen as $F=F(J)=\lambda P^{\eta}$, from (2.26) for P we find

$$P=D_0/\tau. \tag{3.13}$$

The equation for τ then takes the form

$$\ddot{\tau}=\frac{m^2+n^2}{\tau}+\frac{\kappa\lambda(\eta-2)D_0^{\eta}}{2\tau^{\eta-1}} \tag{3.14}$$

with the solution in quadrature

$$\int\frac{d\tau}{\sqrt{2(m^2+n^2)\ln\tau-\kappa\lambda D_0^{\eta}\tau^{2-\eta}+E}}=t. \tag{3.15}$$

The components of the spinor field can be obtained from (2.33). In the case considered, $Q=(m-n)/2c$ and $\Phi=0$.

B. Case II

Let us now consider the case (setting)

$$c=\sqrt{\tau}. \tag{3.16}$$

This leads to the following expressions for a and b:

$$a=\left[\tau^{m/2}\mathcal{N}\exp\left(-\kappa k\int V^0dt\right)\right]^{1/(m+n)} \tag{3.17}$$

and

$$b=\left\{\tau^{n/2}\Big/\left[\mathcal{N}\exp\left(-\kappa k\int V^0dt\right)\right]\right\}^{1/(m+n)}. \tag{3.18}$$

Under this assumption from (2.47) we get

$$\ddot{\tau}=2\kappa kV^3\sqrt{\tau}+2(m^2+n^2)+\kappa[MS+\mathcal{D}S+\mathcal{G}P-2F]\tau. \tag{3.19}$$

This equation can be solved exactly as in the previous case if we set $k=0$ and choose the spinor field nonlinearity as $F=F(I)$ or in case of a massless spinor field $F=F(J)$ or $F=F(I\pm J)$.

For the reason that will be given afterwards, we consider the nonlinear spinor field in a BV universe setting $m=n$ in the corresponding equations. To begin with we write the equations for bilinear spinor forms. Setting $m=n$ in (2.26) one finds

$$\dot{S}_0-2\mathcal{G}A_0^0=0, \tag{3.20a}$$

$$\dot{P}_0-2\Phi A_0^0=0, \tag{3.20b}$$

$$\dot{A}_0^0+2\Phi P_0+2\mathcal{G}S_0=0, \tag{3.20c}$$

$$\dot{A}_0^3=0, \tag{3.20d}$$

$$\dot{V}_0^0=0, \tag{3.20e}$$

$$\dot{V}_0^3+2\Phi Q_0^{30}-2\mathcal{G}Q_0^{21}=0, \tag{3.20f}$$

$$\dot{Q}_0^{30}-2\Phi V_0^3=0, \tag{3.20g}$$

$$\dot{Q}_0^{21}+2\mathcal{G}V_0^3=0, \tag{3.20h}$$

with the following relations between spinor bilinear forms,

$$(S_0)^2+(P_0)^2+(A_0^0)^2=B_1, \tag{3.21a}$$

$$A_0^3=B_2, \tag{3.21b}$$

$$V_0^0=B_3, \tag{3.21c}$$

$$(V_0^3)^2+(Q_0^{30})^2+(Q_0^{21})^2=B_4, \tag{3.21d}$$

where B_i are the constants of integration.

Let us now go back to Einstein's equations. Equation (2.16e) in this case takes the form

$$\frac{\dot{a}}{a}-\frac{\dot{b}}{b}=-\frac{\kappa k}{m}V^0. \tag{3.22}$$

Unlike the BVI universe, where the corresponding equation, i.e., (2.16e), connects all the three metric functions a,b,c, Eq. (3.22) relates only a and b between them:

$$a=\mathcal{N}\exp\left[-(\kappa k/m)\int V^0dt\right]b. \tag{3.23}$$

Recalling $\tau=abc$ in view of (3.23) and (3.16) we can now express a and b in terms of τ:

$$a=\mathcal{N}^{1/2}\tau^{1/4}\exp\left[-(\kappa k/2m)\int V^0dt\right], \tag{3.24}$$

$$b=\mathcal{N}^{-1/2}\tau^{1/4}\exp\left[(\kappa k/2m)\int V^0 dt\right]. \tag{3.25}$$

In view of (3.24), (3.25) and the fact that $\dot{V}_0^0=0$, from

$$\frac{d}{dt}\left[\tau\frac{d}{dt}\left\{\ln\left(\frac{b}{c}\right)\right\}\right]=-\frac{2m^2}{c^2}\tau \tag{3.26}$$

one obtains

$$\frac{\ddot{\tau}}{\tau}=3\frac{\dot{\tau}}{\tau}\frac{\dot{c}}{c}+3\left(\frac{\ddot{c}}{c}-\frac{\dot{c}^2}{c^2}\right)-4\frac{m^2}{c^2}. \tag{3.27}$$

On the other hand, (2.46) in this case has the form

$$\frac{\ddot{\tau}}{\tau}=2\frac{m^2}{c^2}+\frac{\kappa}{2}[3(MS+\mathcal{D}S+\mathcal{G}P-2F)]+\frac{\kappa k}{c}V^3. \tag{3.28}$$

Combining (3.27) and (3.28) we obtain

$$\frac{\ddot{c}}{c}-\frac{\dot{c}^2}{c^2}+\frac{\dot{\tau}}{\tau}\frac{\dot{c}}{c}=\frac{\kappa k}{3c}V^3+2\frac{m^2}{c^2}+\frac{\kappa}{2}[MS+\mathcal{D}S+\mathcal{G}P-2F]. \tag{3.29}$$

Thus we see that a straightforward insertion of $m=n$ into (2.47) does not lead to (3.29), since Eq. (2.16e) for different Bianchi type space-times gives different relations between the metric functions. Here we simply note that for a BIII metric, where $n=0$, Eq. (2.16e) relates a and c, whereas for a BI universe, as well as for a FRW universe, there is no such equation. Note that, though in a BV space-time where $m=n$, many equations in question become significantly simpler, it is not enough to write the solutions explicitly, since V^3, S, and P are still not explicitly defined. As in the previous case, we again consider only a time-dependent spinor field setting $k=0$. It will give us enough ground to solve both spinor and gravitational field equations explicitly.

Before studying Eq. (3.29) in detail, we go back to the nonlinear spinor field equations. With $m=n$ and $k=0$ for the spinor field we immediately find

$$\dot{u}_1+i\Phi u_1-\mathcal{G}u_3=0, \tag{3.30a}$$

$$\dot{u}_2+i\Phi u_2-\mathcal{G}u_4=0, \tag{3.30b}$$

$$\dot{u}_3-i\Phi u_3+\mathcal{G}u_1=0, \tag{3.30c}$$

$$\dot{u}_4-i\Phi u_4+\mathcal{G}u_2=0. \tag{3.30d}$$

As in BVI space-time we consider the nonlinear term to be $F=F(I)$, or for a massless spinor field $F=F(J)$ or $F=F(I\pm J)$. The spinor field equation (3.30) completely coincides with those for a BI metric. So in what follows we simply write the corresponding results without any details. A detailed analysis of these results can be found in Ref. [4].

Thus, for the nonlinear term in the Lagrangian given as $F=F(I)$, the components of the spinor field take the form [4]

$$\psi_1(t)=(C_1/\sqrt{\tau})\exp\left[-i\int(M-\mathcal{D})dt\right], \tag{3.31a}$$

$$\psi_2(t)=(C_2/\sqrt{\tau})\exp\left[-i\int(M-\mathcal{D})dt\right], \tag{3.31b}$$

$$\psi_3(t)=(C_3/\sqrt{\tau})\exp\left[i\int(M-\mathcal{D})dt\right], \tag{3.31c}$$

$$\psi_4(t)=(C_4/\sqrt{\tau})\exp\left[i\int(M-\mathcal{D})dt\right]. \tag{3.31d}$$

Here C_1, C_2, C_3, C_4 are the integration constants, such that

$$C_1^2+C_2^2-C_3^2-C_4^2=C_0,$$

with $C_0=S\tau$.

In case, the nonlinear term is given by $F=F(J)$, and the components of the spinor field have the form

$$\psi_1=\frac{1}{\sqrt{\tau}}(D_1e^{i\sigma}+iD_3e^{-i\sigma}), \tag{3.32a}$$

$$\psi_2=\frac{1}{\sqrt{\tau}}(D_2e^{i\sigma}+iD_4e^{-i\sigma}), \tag{3.32b}$$

$$\psi_3=\frac{1}{\sqrt{\tau}}(iD_1e^{i\sigma}+D_3e^{-i\sigma}), \tag{3.32c}$$

$$\psi_4=\frac{1}{\sqrt{\tau}}(iD_2e^{i\sigma}+D_4e^{-i\sigma}). \tag{3.32d}$$

Here $\sigma=\int\mathcal{G}dt$, and the integration constants D_i obey

$$2(D_1^2+D_2^2-D_3^2-D_4^2)=D_0,$$

with D_0 to be determined from $P=D_0/\tau$. Thus we see that in the cases considered here the spinor bilinear forms are inversely proportional to τ, i.e., $S=C_0/\tau$ and $P=D_0/\tau$.

Let us now go back to (3.29). As one sees, for $k=0$, the assumption $\tau=c$ makes no sense, since in this case the metric functions a and b turn out to be constant. So as was mentioned earlier, we consider the case with $c=\sqrt{\tau}$. Under this assumption from (3.29) we get

$$\ddot{\tau}=4m^2+\kappa[MS+\mathcal{D}S+\mathcal{G}P-2F]\tau. \tag{3.33}$$

Consider the case with $F=\lambda S^\eta$. Taking into account that $S=C_0/\tau$ and $\mathcal{G}=0$, from (3.33) one derives

$$\ddot{\tau} = 4m^2 + \kappa[MC_0 + \lambda C_0^{\eta}(\eta-2)\tau^{1-\eta}], \qquad (3.34)$$

with the solution in quadrature

$$\int \frac{d\tau}{\sqrt{(8m^2+2\kappa MC_0)\tau - 2\lambda\kappa C_0^{\eta}\tau^{2-\eta} + E}} = t. \qquad (3.35)$$

Note that, for a linear spinor field and for the massless spinor field with $F=\lambda P^{\eta}$ one has to put $\lambda=0$ and $M=0$, respectively, into (3.35). It should be noted that for a positive constant E, in case of a linear spinor field τ may have even a trivial initial value.

C. Numerical solutions

Let us now demonstrate some numerical solutions to the Eqs. (3.11) and (3.34). For simplicity we consider the case with $F=F(I)$, since by setting the self-coupling constant $\lambda=0$ one comes to linear case, while setting the spinor mass $M=0$ we have the case with $F=F(J)$.

1. Case I

Let us first consider the case with $F=\lambda S^{\eta}$. In this case the equation in question has the following form:

$$\ddot{\tau} = \mathcal{F}(p), \qquad (3.36)$$

where we define

$$\mathcal{F}(p) = q_1 \frac{m^2+n^2}{\tau} + q_2 m^2 + \kappa[q_3 M + q_4 \lambda(\eta-2)\tau^{(1-\eta)}]. \qquad (3.37)$$

Here p is the set of problem parameters, namely, $p=\{\kappa,m,n,M,\lambda,\eta\}$. Equation (3.36) admits the following first integral,

$$\dot{\tau} = \sqrt{2[E-U(\tau)]}, \qquad (3.38)$$

with the potential

$$U(\tau) = -[q_1(m^2+n^2)\ln(\tau) + q_2 m^2 \tau + \kappa(q_3 M\tau - q_4\lambda\tau^{(2-\eta)})]. \qquad (3.39)$$

Note that setting $q_1=1$, $q_2=0$, $q_3=0.5$, $q_4=0.5$ and $q_1=0$, $q_2=4$, $q_3=1$, $q_4=1$, we get Eqs. (3.11) and (3.34) corresponding to $c=\tau$ in a BVI universe and $c=\sqrt{\tau}$ in a BV universe, respectively.

Here we illustrate some numerical results obtained for cases considered above. The parameters of the equations are taken as follows: for spatial inhomogeneity parameters we set $m=2$ and $n=1$, whereas Einstein's gravitational constant κ is taken to be unity. For the nonlinear spinor field we choose $\lambda=0.1$. As one sees from (3.39) and (3.38), for a negative η the value of τ should be bound from above; on the other hand, since the metric is positively defined τ should be non-negative as well. In Figs. 1 and 2 we show some numerical results for some negative value of η.

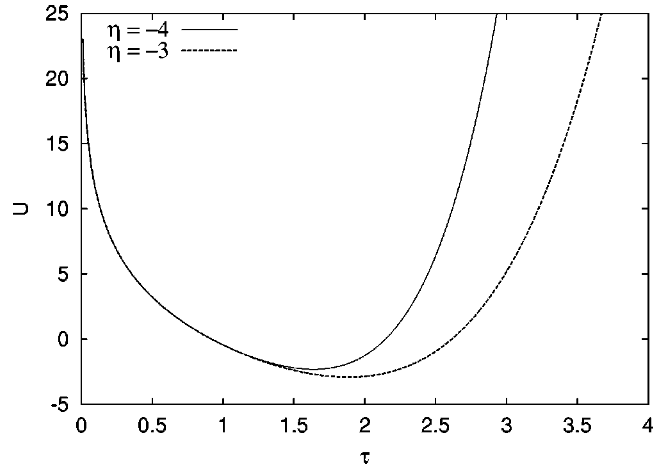

FIG. 1. View of the potential $U(\tau)$ (3.39) for different values of η, namely $\eta=-3$ and $\eta=-4$ in a BVI model with $m=2$, $n=1$, and $\lambda=0.1$.

As one sees from Fig. 2, a negative η gives rise to an oscillatory mode of evolution. Depending on the value of E we have two type of solutions: periodic (corresponding to $E=0$ and $E=10$) and bounded in a finite interval (corresponding to $E=25$).

Let us now see whether the dominant energy condition holds here. The dominant energy condition in a BVI universe has the form

$$T_0^0 \geqslant T_1^1 a^2 e^{-2mz} + T_2^2 b^2 e^{2nz} + T_3^3 c^2, \qquad (3.40a)$$

$$T_0^0 \geqslant T_1^1 a^2 e^{-2mz}, \qquad (3.40b)$$

$$T_0^0 \geqslant T_2^2 b^2 e^{2nz}, \qquad (3.40c)$$

$$T_0^0 \geqslant T_3^3 c^2. \qquad (3.40d)$$

For $c=\tau$ and $k=0$ with regard to (3.2) and (3.3), Eqs. (3.40) in a BVI space-time can be written as

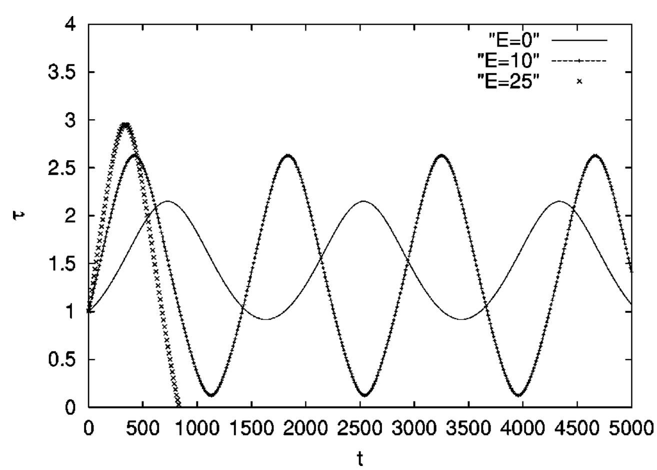

FIG. 2. Evolution of τ as a solution of (3.38) with the potential given in Fig. 1 for different values of E. Here $\eta=-4$.

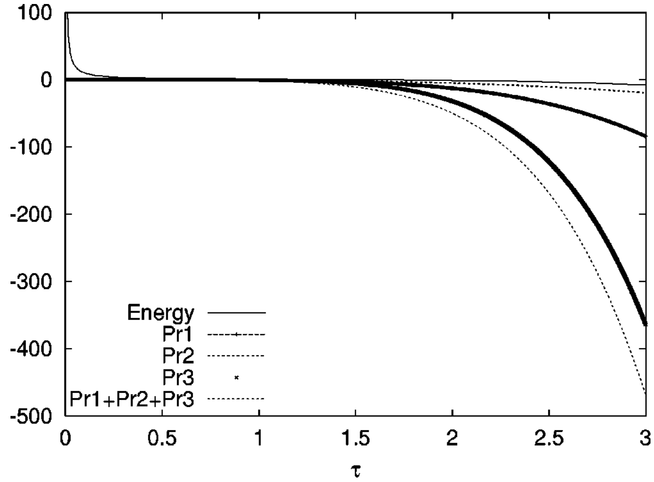

FIG. 3. View of energy T_{00} and pressure components T_{ii}, $i=1,2,3$ corresponding to (3.44) for $\eta=-4$. Here Pr1, Pr2, and Pr3 denote the pressure components along x, y, and z axes, respectively.

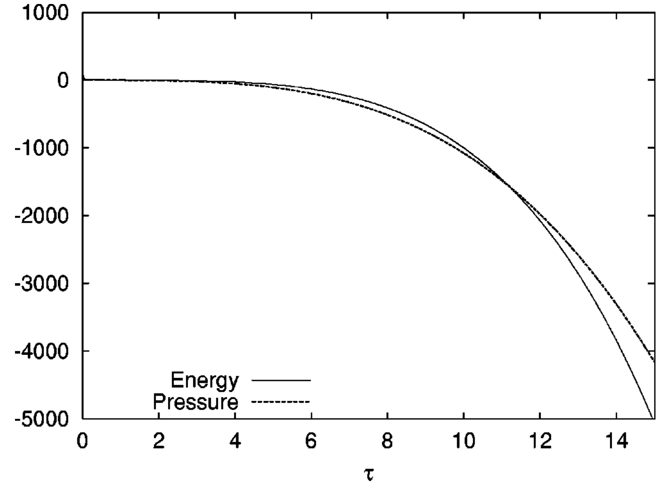

FIG. 4. View of energy T_{00} and pressure component T_{22} for $\eta=-4$. It clearly shows the dominance of the pressure component beginning at some value of τ.

$$T_0^0 \geqslant T_1^1 \tau^{2(m-n)/(m+n)} e^{-2mz} + T_2^2 \tau^{-2(m-n)/(m+n)} e^{2nz} + T_3^3 \tau^2, \tag{3.41a}$$

$$T_0^0 \geqslant T_1^1 \tau^{2(m-n)/(m+n)} e^{-2mz}, \tag{3.41b}$$

$$T_0^0 \geqslant T_2^2 \tau^{-2(m-n)/(m+n)} e^{2nz}, \tag{3.41c}$$

$$T_0^0 \geqslant T_3^3 \tau^2, \tag{3.41d}$$

whereas, for $c=\sqrt{\tau}$ and $k=0$ on account of (3.24) and (3.25), Eqs. (3.40) in a BVI space-time take the form

$$T_0^0 \geqslant T_1^1 \sqrt{\tau}\mathcal{N} e^{-2mz} + T_2^2 (\sqrt{\tau}/\mathcal{N}) e^{2nz} + T_3^3 \tau, \tag{3.42a}$$

$$T_0^0 \geqslant T_1^1 \sqrt{\tau}\mathcal{N} e^{-2mz}, \tag{3.42b}$$

$$T_0^0 \geqslant T_2^2 (\sqrt{\tau}/\mathcal{N}) e^{2nz}, \tag{3.42c}$$

$$T_0^0 \geqslant T_3^3 \tau. \tag{3.42d}$$

The components of the energy-momentum tensor for $k=0$ are

$$T_0^0 = MS - F, \quad T_1^1 = T_2^2 = T_3^3 = \mathcal{D}S - F. \tag{3.43}$$

For $F=\lambda S^\eta$, (3.43) in account of $S=C_0/\tau$ reads

$$T_0^0 = \frac{M}{\tau} - \frac{\lambda}{\tau^\eta}, \quad T_1^1 = T_2^2 = T_3^3 = \frac{\lambda(\eta-1)}{\tau^\eta}, \tag{3.44}$$

For simplicity here we set $C_0=1$. As one sees from (3.44), for a negative η (say $\eta=-\eta_1$), the energy density of the system T_0^0 becomes negative for $\tau>(M/\lambda)^{1/(1+\eta_1)}$ and decreases as τ^{η_1}. On the other hand, all the pressure components are negative, but the components along the x and y axes decreases as $\tau^{[\eta_1+2(m-n)/(m+n)}$ and $\tau^{[\eta_1-2(m-n)/(m+n)}$ (see Fig. 3). It means sooner or later one of the pressure components becomes dominant. A graphical view of it is given in Fig. 4. It should be noted that the energy density and the pressure components of the system are independent of the integration constant E, whereas the value of τ in case of a negative η strongly depends on it. As shown in Fig. 2, for some values of E, τ runs between 0 and 3, while the largest pressure component in the case in question becomes dominant only for $\tau>10$. Figure 3 shows the dominance of energy in the region $\tau\in(0,3)$. This means that for a suitable choice of integration constant E it is still possible to construct regular solutions without breaking the dominant energy condition.

Let us now illustrate the behavior of τ for the case $c=\sqrt{\tau}$ in a BV space-time (see Figs. 5 and 6). As one sees from Fig. 6, a negative η gives rise to the oscillatory mode of evolution. Depending on the value of E we have two types of solutions. Unlike in the previous case here we have a solution where the process may repeat after some interval of time(s) (as it is seen for $E=25$ in Figs. 2 and 6). As in the previous case the dominant energy conditions holds for a negative η.

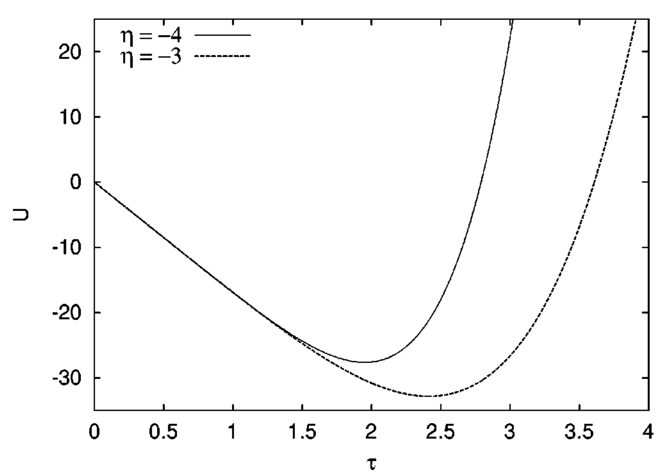

FIG. 5. Potential $U(\tau)$ (3.39) for different values of η, namely $\eta=-3$ and $\eta=-4$ for a BV model with $m=n=2$ and $\lambda=0.1$.

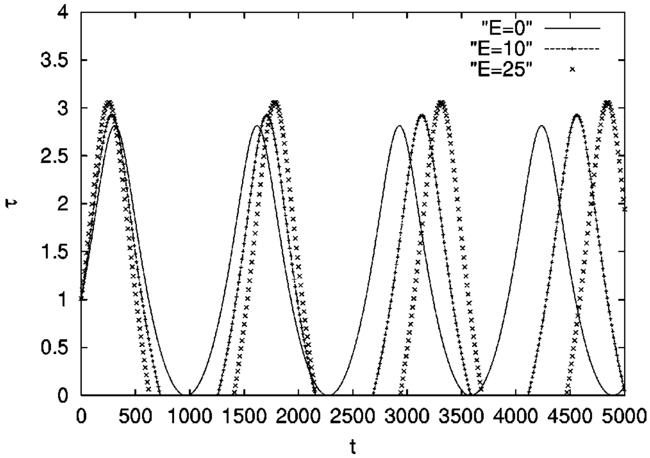

FIG. 6. Evolution of τ as a solution of (3.38) for different values of E with $\eta=-4$. The corresponding potential in this case is the one given in Fig. 5.

Finally we compare the models for a positive η (see Figs. 7 and 8). As was expected, in this case the universe expands monotonically. In case of a BV model the expansion process is rather rapid.

2. Case II

We now consider the case when $F=\sin(S)$. In this case we have

$$\mathcal{F}=q_1\frac{m^2+n^2}{\tau}+q_2m^2+\kappa\{q_3M+q_4\lambda[\cos(1/\tau)$$
$$-2\tau\ \sin(1/\tau)]\} \tag{3.45}$$

with the potential

$$U=-\{q_1(m^2+n^2)\ln(\tau)+q_2m^2\tau$$
$$+\kappa[q_3M\tau-q_4\lambda\tau^2\ \sin(1/\tau)]\}. \tag{3.46}$$

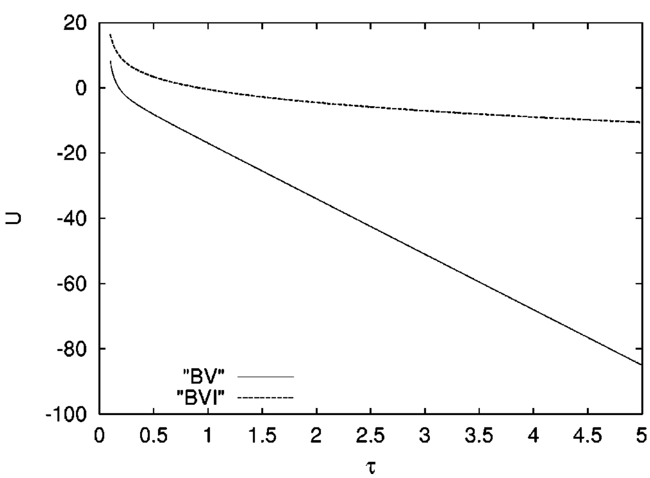

FIG. 7. Potential of the system for a positive η, namely $\eta=4$ for both BV ($m=2$, $n=1$) and BVI ($m=n=2$) models for $\lambda=0.1$. Unlike the case with negative η, the potential in this case is not bounded from the right, allowing τ expand infinitely.

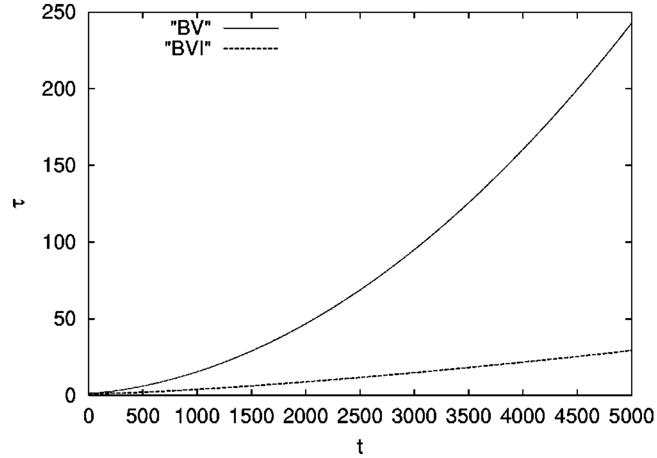

FIG. 8. Evolution of τ for a positive η in both BV and BVI space-times. As one sees, the evolution in BV is rather rapid.

For simplicity we only consider the case in a BVI metric with $c=\tau$. Note that for the case with $c=\sqrt{\tau}$ we come to (the) similar results.

3. Case III

We now consider the case when $F=\exp(S)$. In this case we have

$$\mathcal{F}=q_1\frac{m^2+n^2}{\tau}+q_2m^2+\kappa\{q_3M+q_4\lambda[1-2\tau]\exp(1/\tau)]\} \tag{3.47}$$

with the potential

$$U=-\{q_1(m^2+n^2)\ln(\tau)+q_2m^2\tau$$
$$+\kappa[q_3M\tau-q_4\lambda\tau^2\ \exp(1/\tau)]\}. \tag{3.48}$$

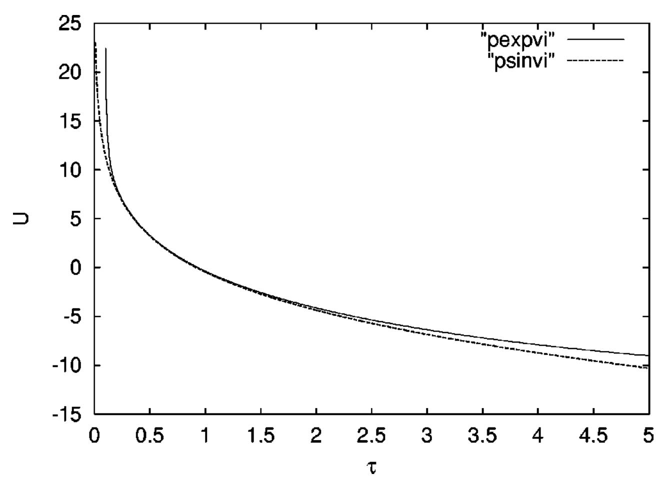

FIG. 9. View of the potential $U(\tau)$ given by (3.46) ("psinvi") and (3.48) ("pexpvi"), respectively, in a BVI model with $m=2$, $n=1$, and $\lambda=0.1$.

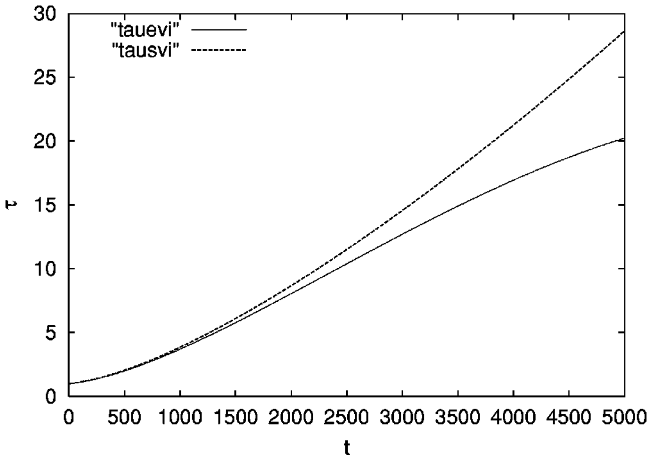

FIG. 10. Evolution of τ for the potentials given in Fig. 9. Here "tausvi" corresponds to $F=\lambda\,\sin(S)$ and "tauevi" corresponds to $F=\lambda\,\exp(S)$.

As in the previous case we consider only the case with $c=\tau$ in a BVI metric.

In Fig. 9 the corresponding potentials with the nonlinear term being a sinusoidal or exponential function of S are shown for the case when we assume that $c=\tau$. It is clear from these figures that with the nonlinear terms considered here the process of evolution is similar to that of a power law nonlinearity with a positive η. (See Figs. 8 and 10).

IV. CONCLUSION

A self-consistent system of nonlinear spinor and gravitational fields, modeled by a BVI space-time, is studied. Exact solutions of the spinor and gravitational field equations are obtained for a special choice of the spinor field nonlinearity and the space-time inhomogeneity. It is shown that if the nonlinear spinor term is chosen to be a function of the invariants $I=S^2$ or $J=P^2$, with a negative power, the model provides an oscillatory mode of expansion. For a suitable value of the integration constant E these solutions are singularity-free at any space-time point. We showed also that though a suitable choice of m and n in a BVI metric yields other Bianchi models, namely, BV, BIII and BI, the solutions of Einstein equations in these universes cannot be obtained by simply setting m and n in the corresponding solutions obtained in a BVI universe, since in different models the metric functions are connected to each other differently. Indeed, it follows from Eq. (2.16e),

$$m\frac{\dot{a}}{a}-n\frac{\dot{b}}{b}-(m-n)\frac{\dot{c}}{c}=\kappa T_{3}^{0}. \tag{4.1}$$

that for a BVI model the metric functions a,b,c are connected with each other by (2.39), whereas, for a BV universe, (4.1) gives a relation between a and b by (3.23) and for a BIII space-time it connects a and c. For BI or FRW models Eq. (4.1) does not exist.

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