Material equations for electromagnetism with toroidal polarizations

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With regard to the toroid contributions, a modified system of equations of electrodynamics moving continuous media has been obtained. Alternative formalisms to introduce the toroid moment contributions in the equations of electromagnetism has been worked out. The two four-potential formalism has been developed. Lorentz transformation laws for the toroid polarizations has been given. Covariant form of equations of electrodynamics of continuous media with toroid polarizations has been written.

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I. INTRODUCTION

The history of electromagnetism is the history of the struggle of different rival concepts from the very early days of its existence. Though, after the historical observation by Hertz, all main investigations in electromagnetism were based on Maxwell equations, nevertheless this theory still suffers from some shortcomings inherent to its predecessors. Several attempts were made to remove the internal inconsistencies of the theory. To be brief, we refer to very few of them. One of the attempts to modify the theory of electromagnetism was connected with the introduction of magnetic charge in Maxwell equation by Dirac [1,2], while keeping the usual definition of **E** and **B** in terms of the gauge potentials. A very interesting work in this direction was done by Miller [3]. In this paper he showed the mutual substitution of the sources as follows:

$$\rho^e \to \rho^e \mathbf{v} = \mathbf{j}^e = c\,\mathrm{curl}\,\mathbf{M} \to -\mathrm{div}\,\mathbf{M} = \rho^m \to \rho^m \mathbf{v} = j^m = c\,\mathrm{curl}\,\mathbf{P} \to -\mathrm{div}\,\mathbf{P} = \rho^e \to .$$

Recently Singleton [4,5] gave an alternative formulation of classical electromagnetism with magnetic and electric charges by introducing two four-vector potentials $A^\mu = (\phi_e, \mathbf{A})$ and $C^\mu = (\phi_m, \mathbf{C})$ and defining **E** and **B** fields as

$$\mathbf{E} = -\nabla \phi_e - \frac{\partial \mathbf{A}}{\partial t} - \mathrm{curl}\,\mathbf{C}, \tag{1.1a}$$

$$\mathbf{B} = -\nabla \phi_m - \frac{\partial \mathbf{C}}{\partial t} + \mathrm{curl}\,\mathbf{A}. \tag{1.1b}$$

Inserting these newly defined vector potentials into the generalized Maxwell equations [6]

$$\mathrm{curl}\,\mathbf{B} = \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J}_e, \tag{1.2a}$$

$$\mathrm{div}\,\mathbf{E} = \rho_e, \tag{1.2b}$$

$$-\mathrm{curl}\,\mathbf{E} = \frac{\partial \mathbf{B}}{\partial t} + \mathbf{J}_m, \tag{1.2c}$$

$$\mathrm{div}\,\mathbf{B} = \rho_m, \tag{1.2d}$$

where ρ_m and $\mathbf{J}_m$ are the magnetic charge and current, respectively, and imposing the Lorentz gauge condition

$$\frac{\partial \phi_e}{\partial t} + \mathrm{div}\,\mathbf{A} = 0, \quad \frac{\partial \phi_m}{\partial t} + \mathrm{div}\,\mathbf{C} = 0, \tag{1.3}$$

he arrived at the wave form of Maxwell's equations with electric and magnetic charges. Note that a similar theory (two potential formalism) was developed by us few years ago (we will come back to it in Sec. III). Though this treatment avoids the use of singular, nonlocal, variables in electrodynamics with magnetic charge and makes the Maxwell system more symmetric, since both the charges in this approach are gauge charges, the main defect of this theory in our view is that the existence of magnetic charge still lacks experimental support, hence can be considered as a mathematically convenient one only. Here we would like to note that a similar work was done by Han and Biedenharn [7] in 1971. In that paper the authors gave a manifestly dyality-invariant formulation of electromagnetic theory, that is, they formulated a theory which is invariant under a dual transformation on fields and sources *simultaneously*. Within the

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framework of that formalism they also introduced two four-vector potentials and a mixed gauge transformation between them.

Recently Chubykalo and co-workers made an effort to modify the electromagnetic theory by invoking both the transverse and longitudinal (explicitly time-independent) fields simultaneously, thus giving an equal footing to both the Maxwell-Hertz and Maxwell-Lorentz equations [8–10]. To remove all ambiguities related to the applications of Maxwell's displacement current they substituted all partial derivatives in Maxwell-Lorenz equations by the *total* ones, such that

$$\frac{d}{dt}=\frac{\partial}{\partial t}-(\mathbf{V}\cdot \mathrm{grad}), \tag{1.4}$$

where $\mathbf{V}=d\mathbf{r}_q/dt$ and $\mathbf{r}_q(t)$ are the velocity and the coordinate of the charge, respectively, at the instant t. Further they separated all field quantities into *two independent classes* with explicit $\{\}^*$ and implicit $\{\}_0$ time dependence, respectively. Thus, the component $\mathbf{E}_0$ of the total electric field $\mathbf{E}$ in every point is understood to depend *only* on the position of source at a given instant. In other words, $\mathbf{E_0}$ is rigidly linked with the location of the charge. From this point of view, the partial time derivative in Eq. (1.4) must be related *only* with the explicit time-dependent component $\mathbf{E}^*$ whereas the convection derivative *only* with $\mathbf{E}_0$:

$$\frac{d\mathbf{E}}{dt}=\frac{\partial \mathbf{E}^*}{\partial t}-(\mathbf{V}\cdot \mathrm{grad})\mathbf{E}_0, \quad \mathbf{E}=\mathbf{E}^*(\mathbf{r},t)+\mathbf{E}_0(\mathbf{R}(t)), \tag{1.5}$$

where $\mathbf{r}$ is a fixed distance from the origin of the reference system at rest to the point of observation and $\mathbf{R}(t)=\mathbf{r}-\mathbf{r}_q(t)$. Here we suggest the modification of the equations of electromagnetism in connection with the existence of the third family of multipole moments, namely the *toroid* one. This theory was developed by us during the recent years. Recently we introduced toroid moments in Maxwell equations exploiting Lagrangian formalism [11]. In Sec. II of this paper we give a brief description of this formalism. Moreover, here we develop an alternative method to introduce toroid moments in the equation of electromagnetism. In Sec. III we develop the two-potential formalism suggested by us earlier.

II. INTRODUCTION OF TOROID MOMENTS IN THE EQUATIONS OF ELECTROMAGNETISM

In the early 1950s, while solving the problem of the multipole radiation of a spatially bounded source, Franz and Wallace [12,13] found a contribution to the electric part of radiation at the expense of magnetization. Further Zel'dovich [14] pointed out the noncorrespondence between the existence of two known multipole sets, Coulomb and magnetic, and the number of form factors for a spin-$\frac{1}{2}$ charged particles. Following the parity nonconservation law in weak interactions Zel'dovich suggested a third form factor in the parametrization of the Dirac spinor particle current. As a classical counterpart of this form factor he introduced the *anapole* in connection with the global electromagnetic properties of a toroid coil that are impossible to describe within the charge or magnetic dipole moments in spite of explicit axial symmetry of the toroid coil. In 1963 Shirokov and Cheshkov [15] constructed the parametrization for relativistic matrix elements of currents of charged and spinning particles, which contain the third set of form factors. Finally, in 1974 Dubovik and Cheskov [16] determined the toroid moment in the framework of classical electrodynamics. Note that *anapole* and *toroid dipole* are not the different names of one and the same thing. They are indeed quite different in nature. For example, the anapole cannot radiate at all while the toroid coil and its pointlike model, toroid dipole, can. The matter is that the anapole is some composition of electric dipole and actual toroid dipole giving destructive interference of their radiation. Thus it comes out that the toroid moment corresponds to the pointlike toroidal solenoid, whereas the anapole contains, in addition to the toroid moment, a linear element of direct current centered in it [16,17]. Toroid polarization is made evident in different condensed matter by a large number of investigations. For magnetic media, we note the recent measurement of the toroid moment in $Ga_{2-x}Fe_xO_3$ [18] and Cr_2O_3 [19].

Moreover, a principally new type of magnetism known as *aromagnetism* was observed in a class of organic substances, suspended either in water or in other liquids [20]. Later, it was shown that this phenomena of aromagnetism cannot be explained in a standard way, e.g., by ferromagnetism, since the organic molecules do not possess magnetic moments of either orbital or spin origin. It was also shown that the origin of aromagnetism is the interaction of a vortex electric field induced by an alternative magnetic one with the axial toroid moments of the fragment C_6 in aromatic elements [21]. In a recent work Dubovik and Kuznetsov [22] calculated the toroid moment of Majorana neutrino. It was also pointed out that the magnitude of the toroid dipole moment of a Dirac neutrino (ν_d) is just half of that of a Majorana one (ν_m) and both of them possesses nontrivial toroid moments even if $m_\nu=0$ [23]. The study of toroid moments in high-energy physics indicates its importance in modern physics. Beside the works mentioned above we would like to refer to the paper by Rubin [24], about applications of toroidal moments in relativistic anyons theory and the need for considering generalized toroidal four moments as effective four-vector potentials. The latest theoretical and experimental development forces the introduction of toroid moments in the framework of conventional classical electrodynamics that in its part inevitably leads to the modification of the equations of electromagnetism and the equations of motion of particles in an external electromagnetic field. In the rest of this section we give two alternative schemes of introduction of toroid polarizations in the electromagnetic equations. To begin with we write the Maxwell equations for electromagnetic fields in vacuum, in the presence of extraneous electric charge ρ and electric current, i.e., charge-in-motion, of density $\mathbf{j}$:

$$\mathrm{curl}\,\mathbf{B}-\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}=\frac{4\pi}{c}\mathbf{j}, \tag{2.1a}$$

$$\mathrm{div}\,\mathbf{E}=4\pi\rho, \tag{2.1b}$$

$$\mathrm{curl}\,\mathbf{E}+\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}=\mathbf{0}, \tag{2.1c}$$

$$\operatorname{div}\mathbf{B}=0, \tag{2.1d}$$

where $\mathbf{E}$ and $\mathbf{B}$ are the flux densities of electric field and magnetic induction, respectively. Note that the electric charges and electric currents, being distributed in vacuum, construct the electromagnetic structure of matter [25] and are related to the elementary charge e in the following way:

$$\rho(\mathbf{r})=\sum_n e_n\delta(\mathbf{r}-\mathbf{q}_n), \tag{2.2a}$$

$$\mathbf{j}(\mathbf{r})=\sum_n e_n\dot{\mathbf{q}}_n\delta(\mathbf{r}-\mathbf{q}_n). \tag{2.2b}$$

So, to describe the system as a whole Eqs. (2.1) and (2.2) should be supplemented by the equation of motion of microparticles, i.e.,

$$m_n\ddot{\mathbf{q}}_n=e_n\mathbf{E}+\frac{e_n}{c}\dot{\mathbf{q}}_n\times\mathbf{B}. \tag{2.3}$$

In this process there occurs a large number of problems connected with the different areas of this vast field such as (i) to write the classical and quantum analogies of the equations of motion of a pointlike particle possessing a toroid dipole (with the usual properties); (ii) to solve the boundary-value problems for the model with all the polarizations, i.e., electric, magnetic and toroid ones; (iii) to formulate the electrodynamics of continuous infinite media for the latter case. Since the electromagnetic field in media generates bound charges and bound currents, the source parts in Eqs. (2.1a) and (2.1b) should be supplemented as follows [26]:

$$\rho_{\text{total}}=\rho+\rho_{\text{bound}}=\rho-\operatorname{div}\mathbf{P}, \tag{2.4a}$$

$$\mathbf{j}_{\text{total}}=\mathbf{J}=\mathbf{j}+\frac{\partial\mathbf{P}}{\partial t}+c\operatorname{curl}\mathbf{M}. \tag{2.4b}$$

Then we may write the Maxwell equations for electromagnetic fields in media as

$$\operatorname{curl}\mathbf{H}-\frac{1}{c}\frac{\partial\mathbf{D}}{\partial t}=\frac{4\pi}{c}\mathbf{j}, \tag{2.5a}$$

$$\operatorname{div}\mathbf{D}=4\pi\rho, \tag{2.5b}$$

$$\operatorname{curl}\mathbf{E}+\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}=\mathbf{0}, \tag{2.5c}$$

$$\operatorname{div}\mathbf{B}=0, \tag{2.5d}$$

where we denote $\mathbf{D}=\mathbf{E}+4\pi\mathbf{P}$ and $\mathbf{H}=\mathbf{B}-4\pi\mathbf{M}$ are the flux densities of electric displacement vector and magnetic field, respectively.

Note that $\partial P/\partial t$, known as the bound-charge current density is independent of the details of the model and is a conduction current. The difference between an ‘‘ordinary’’ conduction current density and the current density $\partial P/\partial t$ is that the first involves *free* charge ρ (foreign charge over which we have some control, i.e., charge that can be added to or remove from an object) in motion, when the second *bound* charge (the integral parts of atoms or molecules of the dielectric) in motion. Another obvious practical distinction is that it is impossible to get a *steady* bound charge current that goes forever unchanged. The *bound* currents $c\operatorname{curl}\mathbf{M}$ are associated with molecular or atomic magnetic moments, including the intrinsic magnetic moment of particles with spin, whereas *free* currents $\mathbf{j}$ are ‘‘ordinary’’ conduction currents flowing on macroscopic paths and can be started or stopped with a switch and measured with an ammeter. Let us begin with the case when both electric polarization and magnetization remain absence, i.e., $\mathbf{D}=\mathbf{E}$ and $\mathbf{H}=\mathbf{B}$. In this case Eqs. (2.5) can be obtained from a Lagrangian describing the interacting system of electromagnetic field and nonrelativistic test charged particle (see, e.g., [27])

$$L=\frac{1}{2}\sum_j m_j\mathbf{q}_j^{\cdot 2}-V(q)+\frac{1}{8\pi}\int[\mathbf{E}^2-\mathbf{B}^2]d\mathbf{r}$$
$$+\frac{1}{c}\int\mathbf{J}(\mathbf{r})\cdot\mathbf{A}(\mathbf{r})d\mathbf{r}-\int\rho\varphi d\mathbf{r} \tag{2.6}$$

with $\mathbf{E}=-c^{-1}\dot{\mathbf{A}}-\operatorname{grad}\varphi$, $\mathbf{B}=\operatorname{curl}\mathbf{A}$, and $V(q)=\frac{1}{2}\Sigma(e_je_k/|q_j-q_k|)$. Here the total current of the system $\mathbf{J}$ coincides with the current $\mathbf{j}=\Sigma_j e_j\dot{q}_j$, generated by the free charges (charged particles) in motion since there is no polarization current.

It can be easily verified that variation of the Lagrangian (2.6) with respect to the particle coordinates gives the second law of Newton with the Lorentz force, i.e., Eq. (2.3)

$$m\ddot{\mathbf{q}}=e\mathbf{E}(\mathbf{q},t)+\frac{e}{c}\dot{\mathbf{q}}\times\mathbf{B}(\mathbf{q},t),$$

whereas the variations with respect to field variables φ and $\mathbf{A}$ give

$$\operatorname{curl}\mathbf{B}-\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}=-\frac{4\pi}{c}\mathbf{j}, \tag{2.7a}$$

$$\operatorname{div}\mathbf{E}=4\pi\rho. \tag{2.7b}$$

The Hamiltonian, corresponding to the Lagrangian (2.6) reads

$$H[\mathbf{\Pi},\mathbf{A};p,q]=\mathbf{p}\cdot\dot{\mathbf{q}}+\int\mathbf{\Pi}\cdot\dot{\mathbf{A}}d\mathbf{r}-L=\frac{1}{2m}\left[\mathbf{p}-\frac{e}{c}\mathbf{A}(\mathbf{q},t)\right]^2$$
$$+V(q)+\frac{1}{8\pi}\int[(4\pi c\mathbf{\Pi})^2+(\operatorname{curl}\mathbf{A})^2]d\mathbf{r}, \tag{2.8}$$

where the corresponding conjugate momenta are $\mathbf{p}=m\dot{\mathbf{q}}+(e/c)\mathbf{A}(\mathbf{q},t)$, $\mathbf{\Pi}(\mathbf{r})=(4\pi c^2)^{-1}\dot{\mathbf{A}}$. Note that the Hamiltonian contains two more terms $\int\rho\varphi d\mathbf{r}+(1/4\pi)\int\mathbf{E}\cdot\operatorname{grad}\varphi d\mathbf{r}$. However, the expression $\mathbf{E}\cdot\operatorname{grad}\varphi$ is equal to $\operatorname{div}(\varphi\mathbf{E})-\varphi\operatorname{div}\mathbf{E}$ and by remembering that $\operatorname{div}\mathbf{E}=4\pi\rho$, we see that these two terms are just equal to $(1/4\pi)\operatorname{div}(\varphi\mathbf{E})$. Since the integral of a divergence over all space equals the net outflow integral at infinity, which is zero, we simply exclude these terms from the Hamiltonian to obtain Eq. (2.8). It is well known that in classical dynamics the addition of a total time derivative to a Lagrangian leads to a new

Lagrangian with the equations of motion unaltered. Lagrangians obtained in this manner are said to be equivalent. In general, the Hamiltonians following from the equivalent Lagrangians are different. Even the relationship between the conjugate and the kinetic momenta may be changed [28]. Let us now construct an equivalent Lagrangian to that of Eq. (2.6) in the following way:

$$L_1 = L - \frac{1}{c}\frac{d}{dt}\int \mathbf{P}(\mathbf{r})\cdot\mathbf{A}(\mathbf{r})d\mathbf{r} - \int \operatorname{div}(\mathbf{M}\times\mathbf{A})d\mathbf{r}. \tag{2.9}$$

The new Lagrangian is a function of the variables $\mathbf{q}$, $\dot{\mathbf{q}}$, and a functional of the field variables $\mathbf{A}$, $\dot{\mathbf{A}}$, and the equations of motion follow from the variational principle. We have

$$\frac{\partial L_1}{\partial \mathbf{A}_i} = \frac{1}{c}(\mathbf{J} - \dot{\mathbf{P}} - c\,\operatorname{curl}\mathbf{M})_i = \mathbf{j}_i,$$

$$\sum_j \frac{\partial}{\partial x_j}\frac{\partial L_1}{\partial(\partial \mathbf{A}_i/\partial x_j)} = \frac{1}{4\pi}(\operatorname{curl}\mathbf{B} - 4\pi\,\operatorname{curl}\mathbf{M})_i = \frac{1}{4\pi}(\operatorname{curl}\mathbf{H})_i,$$

$$\frac{\partial}{\partial t}\frac{\partial L_1}{\partial(\partial \mathbf{A}_i/\partial t)} = -\frac{1}{4\pi c}\frac{\partial}{\partial t}(\mathbf{E} + 4\pi\mathbf{P})_i = -\frac{1}{4\pi c}\frac{\partial \mathbf{D}_i}{\partial t}.$$

Applying the Euler-Lagrange equations of motion

$$\frac{\partial}{\partial t}\frac{\partial L_1}{\partial \dot{\mathbf{A}}_i(\mathbf{r})} + \sum_j \frac{\partial}{\partial x_j}\frac{\partial L_1}{\partial(\partial \mathbf{A}_i/\partial x_j)} - \frac{\partial L_1}{\partial \mathbf{A}_i} = \mathbf{0},$$

we obtain Eq. (2.5a), i.e.,

$$\operatorname{curl}\mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} + \frac{4\pi}{c}\mathbf{j}.$$

The corresponding Hamiltonian now can be written as

$$H_1[\mathbf{\Pi},\mathbf{A};p,q] = \frac{1}{2m}\left[\mathbf{p} - \frac{e}{c}\mathbf{A}(\mathbf{q},t)\right]^2 + \frac{1}{8\pi}\int [\mathbf{D}^2 + \mathbf{B}^2]d\mathbf{r} + V(q) - \int \mathbf{D}\cdot\mathbf{P}d\mathbf{r} + 2\pi\int \mathbf{P}^2 d\mathbf{r} - \int \mathbf{M}\cdot\mathbf{B}d\mathbf{r}. \tag{2.10}$$

The system (2.5) contains four unknown vector quantities $\mathbf{D}$, $\mathbf{E}$, $\mathbf{B}$, and $\mathbf{H}$, therefore, is not a closed one. And one has to add constitutive relations, those connecting $\mathbf{D}$ and $\mathbf{J}$ with $\mathbf{E}$ (as well as $\mathbf{B}$ with $\mathbf{H}$), to Eq. (2.5) to close it. For most of the materials, to a great order of accuracy, one can consider these connections to be linear, i.e.,

$$\mathbf{D} = \varepsilon\mathbf{E}, \quad \mathbf{B} = \mu\mathbf{H}, \tag{2.11}$$

with ε and μ being the electric permittivity and magnetic permeability, respectively. Further we will use the equivalent description of Eq. (2.11) by means of generalized susceptibility $\alpha^{\mathbf{PD}}$ and $\alpha^{\mathbf{MB}}$:

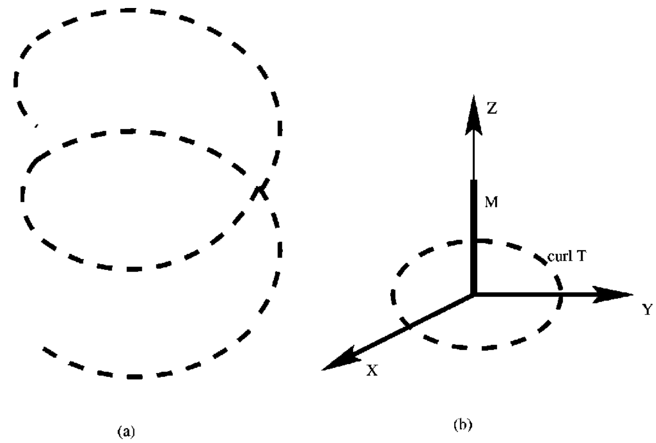

FIG. 1. A large number of small magnets have been pasted together around a cylinder to construct a spiral-like configuration

$$\mathbf{P} = \alpha^{\mathbf{PD}}\mathbf{D}, \quad \mathbf{M} = \alpha^{\mathbf{MB}}\mathbf{B}. \tag{2.12}$$

Relations between ε, μ, $\alpha^{\mathbf{PD}}$, and $\alpha^{\mathbf{MB}}$ can be easily established. The constitutive relations (2.11) can be deduced from the Hamiltonian (2.10) and the terms $\int(\mathbf{P}\cdot\mathbf{D})d\mathbf{r}$ and $\int(\mathbf{M}\cdot\mathbf{B})d\mathbf{r}$ in Eq. (2.10), following the theory of generalized susceptibility illustrated in [29], can be interpreted as the indicator of external forces (in this case $\mathbf{D}$ and $\mathbf{B}$) those lead to corresponding polarizations (here $\mathbf{P}$ and $\mathbf{M}$).

Proceedings from the linearity of the relations between the polarizations and generalized external forces one can write

$$\begin{pmatrix}\mathbf{P}\\ \mathbf{M}\end{pmatrix} = \begin{pmatrix}\mathbf{P}_0\\ \mathbf{M}_0\end{pmatrix} + \begin{pmatrix}\alpha^{\mathbf{PD}} & \alpha^{\mathbf{PB}}\\ \alpha^{\mathbf{MD}} & \alpha^{\mathbf{MB}}\end{pmatrix}\begin{pmatrix}\mathbf{D}\\ \mathbf{B}\end{pmatrix}. \tag{2.13}$$

Here $\mathbf{P}_0$ and $\mathbf{M}_0$ take into account the possible polarization of the media in the absence of external fields (e.g., in ferromagnetics, ferroelectrics), whereas the ''mixed'' susceptibilities $\alpha^{\mathbf{PB}}$ and $\alpha^{\mathbf{MD}}$ say that some material may be electrically polarized by magnetic field and vice versa. In the literature these effects are known as magnetoelectric. Since the analysis of susceptibilities and a detailed clarification of their physical meanings are beyond the scope of this paper, we merely note that, in accord with general theory [29], susceptibilities are connected among themselves by means of symmetry relations. Further, we use an analogical discussion to close the equations of electrodynamics with toroid polarizations.

Before introducing the toroid polarizations in the system let us consider an example visually demonstrating such a possibility.

Let a large number of small magnets be pasted spiral-wise around a cylinder [Fig. 1(a)]. For simplicity we assume the axis of the cylinder to coincide with the Z axis. As one sees, the configuration has a circlelike (close) configuration on the XY-plane [Fig. 1(b)]. Therefore, the Z projection of the configuration can be given by a vector $\mathbf{M} = (0,0,M)$, whereas the projection on the XY plane describes a vector with zero divergence (see, e.g., [30]) and can be given as the Z component of the curl of some vector, e.g., $(\operatorname{curl}\mathbf{T}^m)_z$. In general this type of configuration can be written as the sum of two vectors

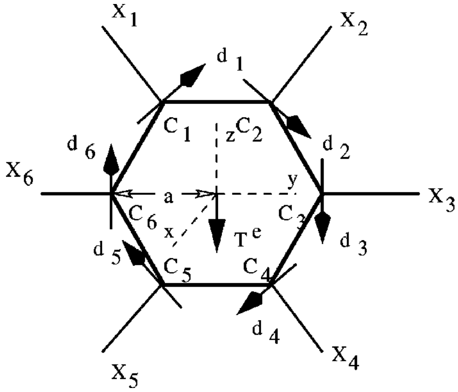

FIG. 2. Fragment of an aromatic molecule where C_i is a carbon atom, X_i is a hydrogen atom or its substitute. The vector d_i denotes the electric dipole moment. It can be shown that the molecules in this state possess axial toroid moment $\mathbf{T}^e$. If the benzene molecule is considered to be laid on the XY plane with the origin placed at the center as shown in the figure, the axial toroid moment has only the nontrivial Z component directed downward. A simple calculation gives $\mathbf{T}^e=\{0,0,-(a/2)\Sigma_i d_i\}$, with a being the distance between the center and the carbon atom C_i.

$$\mathbf{M}+\mathrm{curl}\,\mathbf{T}^m.$$

We would like to note that this type of configurations (with null divergence) often occurs in nature. As a particular example, where electromagnetic properties of molecules are described by the axial toroid moment, we point out the phenomenon of ‘‘aromagnetism’’ [20]. There exist substances with closed chains of atoms like the benzene ring (Fig. 2). In the experiment, the microcrystals of the aromatic series substances (antrazen, fenantren, etc.), suspended either in water or in other liquids, were reoriented by an alternating magnetic field so that the modulation of polarized light, propagating through the given media, was observed. This reorientation can be easily explained if the aromatic molecules are considered to possess axial toroid moment $\mathbf{T}^{(e)}$ as an electromagnetic order parameter. The latter can be clarified as follows. All molecular wave functions Ψ of the benzene ring, being the main fragment of aromatic molecules, are first classified through the irreducible representations of the point-group symmetry D_{6h} and then the asymmetric representations (E_{2g} and E_{1u}) are selected among them. When such molecules are packed into a molecular crystal, the parallel orientation of their axial toroid moments are preferred and the crystal as a whole can possess a macroscopic axial toroid moment [21]. As another example we consider a cubic perovskite, where the crystallographic plane (1,1,1) contains the triangular sublattice of oxygen atoms (Fig. 3).

Magnetic moments of these atoms can be in two states of orientation symmetry that are separated by the temperature phase transition (Fig. 4). As one sees, the magnetic moments of oxygen atoms are oriented in such a way that they generate in each triangle a toroid moment directed ‘‘up’’ and ‘‘down’’ by turns thus building the antitoroic. It is noteworthy to mention that the antitoroic can be transformed into a toroic (with all the toroid moments directed ‘‘up’’ or ‘‘down’’) by applying external force, e.g., current.

The other important case of molecular aromagnetism is the vortex distribution of dipole moments of oxygen or nitrogen atoms. In Fig. 5 the stereomer of the molecule of phloroglycine is represented and dipole moments of the oxygen atoms are shown. The stability of this stereomer was demonstrated by numerical calculation by the standard molecular-dynamics method. It follows from the picture that vortex distribution of the electrical dipole moments on the oxygen atoms exists and that this stereomer of phloroglycine is an aromagnetic one. As other examples we may refer to IPB molecules, irons (cf. Fig. 10.28 in [26]). Thus we argue that $\mathbf{P}$ and $\mathbf{M}$ in Eq. (2.9) should be substituted by

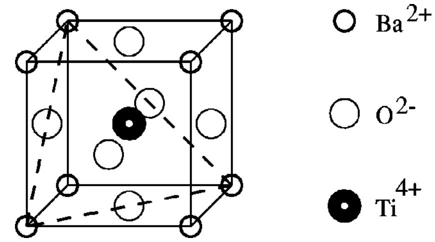

FIG. 3. Structure of a perovskite-type compound barium titanate $BaTiO_3$.

$$\mathbf{P}\Rightarrow\mathbf{P}+\mathrm{curl}\,\mathbf{T}^e,\quad \mathbf{M}\Rightarrow\mathbf{M}+\mathrm{curl}\,\mathbf{T}^m, \tag{2.14}$$

where $\mathbf{T}^e$ and $\mathbf{T}^m$ are the toroid polarizations of electric and magnetic types, respectively. Here we would like to note that the addition of a divergence-free magnetization distribution to the original magnetization was suggested by Beardsley to reconstruct the magnetization in a thin film [31]. Indeed, it can be viewed as a partial application of the Helmholtz theorem [32] which states that any vector field can be uniquely decomposed into the sum of curl-free and divergence-free parts.

Noticing that the total current in the account of toroid polarizations has the form [33] (also see [7])

$$\mathbf{j}_{\mathrm{total}}=\mathbf{J}=\mathbf{j}+\frac{\partial\mathbf{P}}{\partial t}+\frac{\partial\,\mathrm{curl}\,\mathbf{T}^e}{\partial t}+c\,\mathrm{curl}\,\mathbf{M}+c\,\mathrm{curl}\,\mathrm{curl}\,\mathbf{T}^m, \tag{2.15}$$

we obtain the following system of equations:

$$\mathrm{curl}\,\boldsymbol{\mathcal{H}}=\frac{1}{c}\frac{\partial\boldsymbol{\mathcal{D}}}{\partial t}+\frac{4\pi}{c}\mathbf{j}, \tag{2.16a}$$

$$\mathrm{div}\,\boldsymbol{\mathcal{D}}=4\pi\rho, \tag{2.16b}$$

$$\mathrm{curl}\,\boldsymbol{\mathcal{E}}=-\frac{1}{c}\frac{\partial\boldsymbol{\mathcal{B}}}{\partial t}, \tag{2.16c}$$

$$\mathrm{div}\,\boldsymbol{\mathcal{B}}=0, \tag{2.16d}$$

where we define

$$\begin{gathered}\boldsymbol{\mathcal{E}}=\mathbf{E}+4\pi\,\mathrm{curl}\,\mathbf{T}^e,\quad \boldsymbol{\mathcal{H}}=\mathbf{H}-4\pi\,\mathrm{curl}\,\mathbf{T}^m,\\ \boldsymbol{\mathcal{D}}=\mathbf{D}+4\pi\,\mathrm{curl}\,\mathbf{T}^e,\quad \boldsymbol{\mathcal{B}}=\mathbf{B}-4\pi\,\mathrm{curl}\,\mathbf{T}^m, \qquad (2.17)\\ \boldsymbol{\mathcal{D}}=\boldsymbol{\mathcal{E}}+4\pi\,\mathbf{P},\quad \boldsymbol{\mathcal{H}}=\boldsymbol{\mathcal{B}}-4\pi\,\mathbf{M},\\ \mathbf{D}=\mathbf{E}+4\pi\,\mathbf{P},\quad \mathbf{H}=\mathbf{B}-4\pi\,\mathbf{M}.\end{gathered}$$

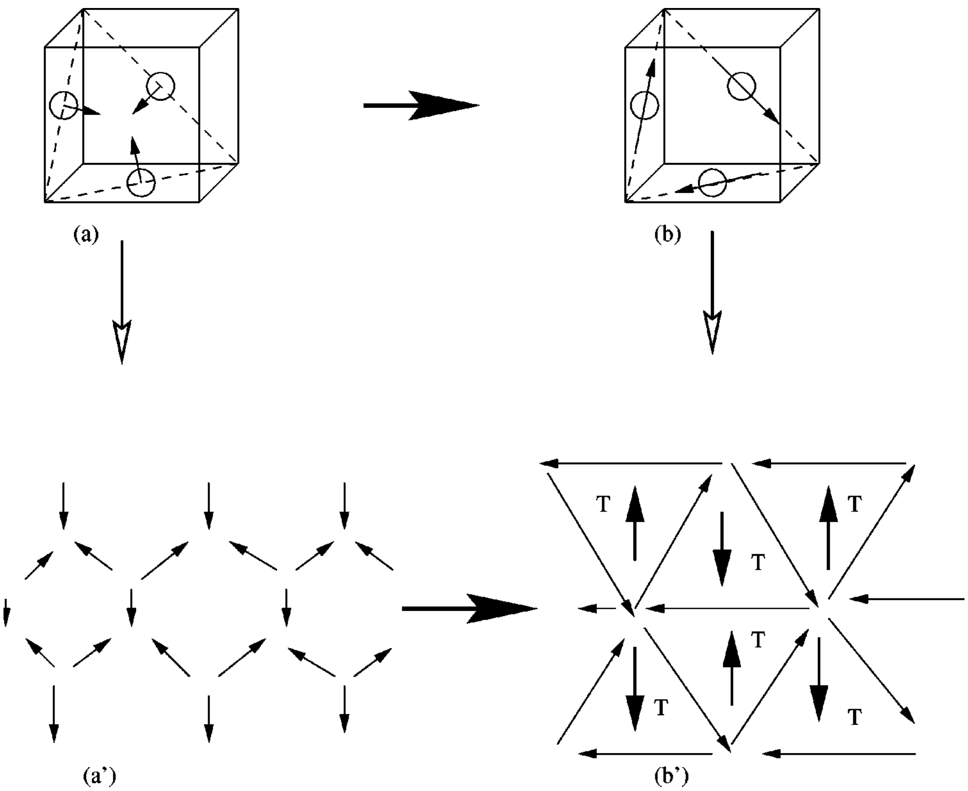

FIG. 4. Possible orientations of the (1,1,1) crystallographic plane and (orientational) dielectric phase transition [(a)→(b), or (a′)→(b′)]. (b′) shows the antitoroic ordering.

Indeed, due to the introduction of toroid polarizations, having an independent origin in terms of atomic and molecular current and charge distributions, the quantities **B** and **D** as well as **E** and **H** lost their initial meaning. The existence of the vorticities $\mathbf{T}^e$ and $\mathbf{T}^m$, generally speaking, can be imputed to the one and the same physical volume. The equation of motion Eq. (2.3) now reads

$$m\ddot{\mathbf{q}}=e\boldsymbol{\mathcal{E}}+\frac{e}{c}\dot{\mathbf{q}}\times\boldsymbol{\mathcal{B}}=e\mathbf{E}+\frac{e}{c}\dot{\mathbf{q}}\times\mathbf{B}+4\pi e\times\left(\mathrm{curl}\,\mathbf{T}^e-\frac{1}{c}\dot{\mathbf{q}}\times\mathrm{curl}\,\mathbf{T}^m\right). \tag{2.18}$$

Equation (2.16a) can be derived as earlier by constructing a Lagrangian equivalent to L_1 such that

$$L_2=L_1-\frac{1}{c}\frac{d}{dt}\int \mathrm{curl}\,\mathbf{T}^e(\mathbf{r})\cdot\mathbf{A}(\mathbf{r})d\mathbf{r}-\int \mathrm{div}(\mathrm{curl}\,\mathbf{T}^m\times\mathbf{A})d\mathbf{r}. \tag{2.19}$$

The corresponding Hamiltonian reads

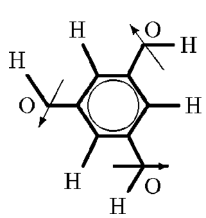

FIG. 5. Distribution of dipole moments of oxygen atoms of the molecule phloroglycine. It can be estimated from this figure that the molecule has an axial toroid moment.

$$H_2[\mathbf{\Pi},\mathbf{A};p,q]=\frac{1}{2m}\left[\mathbf{p}-\frac{e}{c}\mathbf{A}(\mathbf{q},t)\right]^2+\frac{1}{8\pi}\int[\boldsymbol{\mathcal{D}}^2+\boldsymbol{\mathcal{B}}^2]d\mathbf{r}+V(q)-\int\boldsymbol{\mathcal{D}}\cdot[\mathbf{P}+\mathrm{curl}\,\mathbf{T}^e]d\mathbf{r}+2\pi\times\int[\mathbf{P}+\mathrm{curl}\,\mathbf{T}^e]^2d\mathbf{r}-\int[\mathbf{M}+\mathrm{curl}\,\mathbf{T}^m]\cdot\boldsymbol{\mathcal{B}}d\mathbf{r}. \tag{2.20}$$

The system (2.16) containing four unknown vectors $\boldsymbol{\mathcal{E}},\boldsymbol{\mathcal{D}},\boldsymbol{\mathcal{B}},\boldsymbol{\mathcal{H}}$ is not closed and should be supplemented with corresponding constitutive relations. As in the case of standard electrodynamics of continuous media, we confine ourselves to linear approximation. In correspondence with the theory of generalized susceptibility [29] we rewrite the terms in the Hamiltonian (2.20), responsible for the interaction as follows:

$$H_i=-\int(\boldsymbol{\mathcal{D}}\cdot\mathbf{P})d\mathbf{r}-\int(\boldsymbol{\mathcal{B}}\cdot\mathbf{M})d\mathbf{r}-\int(\boldsymbol{\mathcal{D}}\cdot\mathrm{curl}\,\mathbf{T}^e)d\mathbf{r}-\int(\boldsymbol{\mathcal{B}}\cdot\mathrm{curl}\,\mathbf{T}^m)d\mathbf{r}. \tag{2.21}$$

Transforming the last two terms of Eq. (2.21) as follows:

$$\int(\boldsymbol{\mathcal{D}}\cdot\mathrm{curl}\,\mathbf{T}^e)d\mathbf{r}=\int\mathrm{div}[\mathbf{T}^e\times\boldsymbol{\mathcal{D}}]d\mathbf{r}+\int(\mathbf{T}^e\cdot\mathrm{curl}\,\boldsymbol{\mathcal{D}})d\mathbf{r},$$

$$\int(\boldsymbol{\mathcal{B}}\cdot\mathrm{curl}\,\mathbf{T}^m)d\mathbf{r}=\int\mathrm{div}[\mathbf{T}^m\times\boldsymbol{\mathcal{B}}]d\mathbf{r}+\int(\mathbf{T}^m\cdot\mathrm{curl}\,\boldsymbol{\mathcal{B}})d\mathbf{r}, \tag{2.22}$$

and performing surface integral instead of the volume one (Gauss' theorem), we can exclude the div terms in Eq. (2.22). Here we use the fact that beyond the surface, used for the surface integral, polarizations vanish. From Eq. (2.20) in account of Eq. (2.21) we conclude that the spatial derivatives of $\mathcal{D}$ and $\mathcal{B}$, namely, curl $\mathcal{D}$ and curl $\mathcal{B}$ play the role of generalized forces. Based on this, the simplest relation between polarizations and generalized forces can be written as

$$\mathbf{P}=\alpha^{\mathbf{P}\mathcal{D}}\mathcal{D},\quad \mathbf{T}^e=\alpha^{\mathbf{T}^e\mathcal{D}^c}\mathrm{curl}\,\mathcal{D},$$

$$\mathbf{M}=\alpha^{\mathbf{M}\mathcal{B}}\mathcal{B},\quad \mathbf{T}^m=\alpha^{\mathbf{T}^m\mathcal{B}^c}\mathrm{curl}\,\mathcal{B},$$

where beside the standard generalized susceptibilities $\alpha^{\mathbf{P}\mathcal{D}}$ and $\alpha^{\mathbf{M}\mathcal{B}}$ we introduced toroidal susceptibilities $\alpha^{\mathbf{T}^e\mathcal{D}^c}\alpha^{\mathbf{T}^m\mathcal{B}^c}$. In general, relation between polarizations $\mathbf{P},\mathbf{M},\mathbf{T}^e,\mathbf{T}^m$ and corresponding fields $\mathcal{D},\mathcal{B}$, curl $\mathbf{T}^e$, curl $\mathbf{T}^m$ can be presented in the form

$$\begin{pmatrix}\mathbf{P}\\ \mathbf{M}\\ \mathbf{T}^e\\ \mathbf{T}^m\end{pmatrix}=\begin{pmatrix}\mathbf{P}_0\\ \mathbf{M}_0\\ \mathbf{T}_0^e\\ \mathbf{T}_0^m\end{pmatrix}+\begin{pmatrix}\alpha^{\mathbf{P}\mathcal{D}} & \alpha^{\mathbf{P}\mathcal{B}} & \alpha^{\mathbf{P}\mathcal{D}^c} & \alpha^{\mathbf{P}\mathcal{B}^c}\\ \alpha^{\mathbf{M}\mathcal{D}} & \alpha^{\mathbf{M}\mathcal{B}} & \alpha^{\mathbf{M}\mathcal{D}^c} & \alpha^{\mathbf{M}\mathcal{B}^c}\\ \alpha^{\mathbf{T}^e\mathcal{D}} & \alpha^{\mathbf{T}^e\mathcal{B}} & \alpha^{\mathbf{T}^e\mathcal{D}^c} & \alpha^{\mathbf{T}^e\mathcal{B}^c}\\ \alpha^{\mathbf{T}^m\mathcal{D}} & \alpha^{\mathbf{T}^m\mathcal{B}} & \alpha^{\mathbf{T}^m\mathcal{D}^c} & \alpha^{\mathbf{T}^m\mathcal{B}^c}\end{pmatrix}\times\begin{pmatrix}\mathcal{D}\\ \mathcal{B}\\ \mathrm{curl}\,\mathcal{D}\\ \mathrm{curl}\,\mathcal{B}\end{pmatrix}. \tag{2.23}$$

Here we take into account that the media can possess electric, magnetic, and toroid polarizations $\mathbf{P}_0,\mathbf{M}_0,\mathbf{T}_0^e,\mathbf{T}_0^m$ even in the absence of field as it took place for aromagnetic, ferromagnetic, and other media considered above. Beside these, Eq. (2.23) contains "mixed" susceptibilities responsible for different effects of polarizations of media with (without) definite symmetry. Here we again note that the detailed analysis of these effects are beyond our scope. We simply confine ourselves with some general remarks. (1) Generalized susceptibilities satisfy the symmetry relations following from the general properties of these quantities [29]. If we introduce matrix notation

$$|p\rangle=\begin{pmatrix}\mathbf{P}\\ \mathbf{M}\\ \mathbf{T}^e\\ \mathbf{T}^m\end{pmatrix},\quad |\hat{\mathcal{D}}\rangle=\begin{pmatrix}\mathcal{D}\\ \mathcal{B}\\ \mathrm{curl}\,\mathcal{D}\\ \mathrm{curl}\,\mathcal{B}\end{pmatrix},$$

$$\hat{\alpha}=\begin{pmatrix}\alpha^{\mathbf{P}\mathcal{D}} & \alpha^{\mathbf{P}\mathcal{B}} & \alpha^{\mathbf{P}\mathcal{D}^c} & \alpha^{\mathbf{P}\mathcal{B}^c}\\ \alpha^{\mathbf{M}\mathcal{D}} & \alpha^{\mathbf{M}\mathcal{B}} & \alpha^{\mathbf{M}\mathcal{D}^c} & \alpha^{\mathbf{M}\mathcal{B}^c}\\ \alpha^{\mathbf{T}^e\mathcal{D}} & \alpha^{\mathbf{T}^e\mathcal{B}} & \alpha^{\mathbf{T}^e\mathcal{D}^c} & \alpha^{\mathbf{T}^e\mathcal{B}^c}\\ \alpha^{\mathbf{T}^m\mathcal{D}} & \alpha^{\mathbf{T}^m\mathcal{B}} & \alpha^{\mathbf{T}^m\mathcal{D}^c} & \alpha^{\mathbf{T}^m\mathcal{B}^c}\end{pmatrix},$$

and rewrite Eq. (2.23) in the form

$$|p\rangle=|p_0\rangle+\hat{\alpha}|\hat{\mathcal{D}}\rangle, \tag{2.24}$$

then the symmetry relation of the susceptibility can be formulated as the hermiticity of the matrix $\hat{\alpha}$:

$$\hat{\alpha}=\hat{\alpha}^{\dagger}.$$

These properties of susceptibility applied to the magnetic part only, i.e.,

$$\begin{pmatrix}\mathbf{M}\\ \mathbf{T}^m\end{pmatrix}=\begin{pmatrix}\alpha^{\mathbf{M}\mathcal{B}} & \alpha^{\mathbf{M}\mathcal{B}^c}\\ \alpha^{\mathbf{T}^m\mathcal{B}} & \alpha^{\mathbf{T}^m\mathcal{B}^c}\end{pmatrix}\begin{pmatrix}\mathcal{B}\\ \mathrm{curl}\mathcal{B}\end{pmatrix} \tag{2.25}$$

were analyzed in detail in [34] in connection with the construction of the theory of toroid response of nuclear spin subsystem on external action and analysis of the condition for nuclear toroid resonance. (2) As it was noticed earlier the polarizations $\mathbf{P},\mathbf{M},\mathbf{T}^e,\mathbf{T}^m$ and also the fields $\mathcal{D}$, $\mathcal{B}$, curl $\mathcal{D}$, and curl $\mathcal{B}$ possess different temporal and spatial parities. So the susceptibilities may be zero only in that case when the medium possesses corresponding symmetry. For example, the susceptibility $\alpha^{\mathbf{P}\mathcal{D}^c}$, responsible for electric polarization $\mathbf{P}$ in the vortex electrical field curl $\mathcal{D}$ may differ from zero only in the medium that is not invariant under space inversion since it connects the vector $\mathbf{P}$ with the pseudovector curl $\mathcal{D}$, and therefore, is the pseudotensor of rank 2 (or pseudoscalar for isotropic medium). Media, constructed from chiral molecules, possess this property. As is known [35] optical activeness is observed in these media, it means, the susceptibility considered, describes this effect. A more detailed analysis of the equations of media, possessing toroid polarization, we give elsewhere.

III. TWO POTENTIAL FORMALISM

It is generally inferred that the divergence equations of the Maxwell system are "redundant" since they are the consequences of curl equations under the condition of continuity [36]. Recently Krivsky *et al.* [37] claimed that to describe the free electromagnetic field it is sufficient to consider the curl subsystem of Maxwell equations since the equalities $\mathrm{div}\,\mathbf{E}=0$ and $\mathrm{div}\,\mathbf{B}=0$ are fulfilled identically. Contrary to this statement, it has been proved that the divergence equations are not redundant and that neglecting these equations is at the origin of spurious solutions in computational electromagnetics [38,39]. Here we construct a generalized formulation of Maxwell equations including both curl and divergence subsystems. Note that in the classical electrodynamics of charged particles the fields $\mathbf{E}$ and $\mathbf{B}$ (or $F^{\mu\nu}$) completely determine the properties of the system and the knowledge of A^{μ} is redundant there, because it is determined only up to gauge transformations which do not effect $F^{\mu\nu}$. This is not the case in quantum theory. After the discovery of the Aharonov-Bohm effect the significance of A^{μ} has been radically changed. Here we to cite the papers by Dvoeglazov [40] where he gives a detailed analysis about the increasing role of four-vector potential A^{μ} both in quantum and classical theory. In view of the growing interest in A^{μ} in electromagnetic theory in this section we develop two-potential formalism (a similar formalism was developed by us earlier with the curl subsystem taken into account only). Note that in the ordinary one-potential formalism $(\mathbf{A},\varphi)$ the second set of Maxwell equations are fulfilled identically. So that all four

Maxwell equations bring their contribution individually, in our view, one has to rewrite the Maxwell equation in terms of two vector and two scalar potentials. To this end we introduce the double potential [41,42,11] to the system (2.16) rewriting it as follows:

$$\mathrm{curl}\,\boldsymbol{\mathcal{B}}=\frac{1}{c}\frac{\partial\boldsymbol{\mathcal{E}}}{\partial t}+\frac{4\pi}{c}\left(\mathbf{j}_{\mathrm{free}}+\frac{\partial\mathbf{P}}{\partial t}+c\,\mathrm{curl}\,\mathbf{M}\right),\quad(3.1a)$$

$$\mathrm{div}\,\boldsymbol{\mathcal{E}}=4\pi(\rho-\mathrm{div}\,\mathbf{P}),\quad(3.1b)$$

$$\mathrm{curl}\,\boldsymbol{\mathcal{E}}=-\frac{1}{c}\frac{\partial\boldsymbol{\mathcal{B}}}{\partial t},\quad(3.1c)$$

$$\mathrm{div}\,\boldsymbol{\mathcal{B}}=0.\quad(3.1d)$$

Before developing the two-potential formalism we first rewrite the system (2.1) in terms of vector and scalar potentials $\mathbf{A},\phi$ such that $\mathbf{B}=\mathrm{curl}\,\mathbf{A}$, $\mathbf{E}=-\boldsymbol{\nabla}\phi-(1/c)(\partial\mathbf{A}/\partial t)$. Following any text book we can write system (2.1) as

$$\Box\mathbf{A}=-\frac{4\pi}{c}\mathbf{j}_{\mathrm{tot}}=-\frac{4\pi}{c}\left[\mathbf{j}_{\mathrm{free}}+\frac{\partial\mathbf{P}}{\partial t}+c\,\mathrm{curl}\,\mathbf{M}\right],\quad(3.2a)$$

$$\Box\phi=-4\pi[\rho-\mathrm{div}\,\mathbf{P}],\quad(3.2b)$$

under Lorentz gauge, i.e., $\mathrm{div}\,\mathbf{A}+(1/c)(\partial\phi/\partial t)=0$. Here $\Box=\nabla^2-(1/c^2)(\partial^2/\partial t^2)$. Note that to obtain Eq. (3.2) it is sufficient to consider Eqs. (2.1a) and (2.1b) only since the two others are fulfilled identically. Let us now develop two-potential formalism. The two-potential formalism was introduced in [41] and further developed in [42,11]. In both papers we introduce only two vector potentials $\boldsymbol{\alpha}^m,\boldsymbol{\alpha}^e$ and use only the curl subsystem of the Maxwell equations with the additional condition $\mathrm{div}\boldsymbol{\alpha}^{m,e}=0$. Thus, in our view our previous version of two-potential formalism lacks completeness. In the present paper together with the vector potentials $\boldsymbol{\alpha}^m,\boldsymbol{\alpha}^e$ we introduce two scalar potentials φ^m and φ^e such that

$$\boldsymbol{\mathcal{B}}=\mathrm{curl}\,\boldsymbol{\alpha}^m+\frac{1}{c}\frac{\partial\boldsymbol{\alpha}^e}{\partial t}+\boldsymbol{\nabla}\varphi^m,\quad(3.3a)$$

$$\boldsymbol{\mathcal{E}}=\mathrm{curl}\,\boldsymbol{\alpha}^e-\frac{1}{c}\frac{\partial\boldsymbol{\alpha}^m}{\partial t}-\boldsymbol{\nabla}\varphi^e.\quad(3.3b)$$

Inserting Eq. (3.3) into the system of equations (3.1) we find

$$\Box\boldsymbol{\alpha}^m=-\frac{4\pi}{c}\left(\mathbf{j}_{\mathrm{free}}+\frac{\partial\mathbf{P}}{\partial t}+c\,\mathrm{curl}\,\mathbf{M}\right),\quad(3.4a)$$

$$\Box\varphi^m=0,\quad(3.4b)$$

$$\Box\boldsymbol{\alpha}^e=0,\quad(3.4c)$$

$$\Box\varphi^e=-4\pi(\rho-\mathrm{div}\,\mathbf{P}),\quad(3.4d)$$

under $\mathrm{div}\,\boldsymbol{\alpha}^{m,e}+(1/c)(\partial\varphi^{e,m}/\partial t)=0$. Note that introduction of free magnetic current $\mathbf{j}^m_{\mathrm{free}}$ and magnetic charge ρ^m in Eqs. (3.1c) and (3.1d), respectively, leads to the equations obtained by Singleton [5]. The general solutions to the systems (3.4) can be written as (see, for example, [11,30]):

$$F(\mathbf{r},t)=-\frac{1}{4\pi}\int_{\text{all space}}\frac{f(\mathbf{r}',t')d\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}\Big|_{t'=t-|\mathbf{r}-\mathbf{r}'|/c},\quad(3.5)$$

where $F(\mathbf{r},t)$ stands for the unknowns $\boldsymbol{\alpha}^e,\varphi^e,\boldsymbol{\alpha}^m,\varphi^m$, whereas f is the right-hand side of Eqs. (3.4). Here we would also like to note that in our previous works [41,42,11] we introduced two potentials in the following way:

$$\boldsymbol{\mathcal{B}}=\mathrm{curl}\,\boldsymbol{\alpha}^m+\frac{1}{c}\frac{\partial\boldsymbol{\alpha}^e}{\partial t},\quad(3.6a)$$

$$\boldsymbol{\mathcal{D}}=\mathrm{curl}\,\boldsymbol{\alpha}^e-\frac{1}{c}\frac{\partial\boldsymbol{\alpha}^m}{\partial t}.\quad(3.6b)$$

As was mentioned earlier the scalar parts $\varphi^{m,e}$ has not been taken into account. Since we are dealing with moving media and want the equations of electromagnetism with toroid moments to be Lorentz covariant, we should consider the pair $(\boldsymbol{\mathcal{B}},\boldsymbol{\mathcal{E}})$ as in one-potential case rather than the pair $(\boldsymbol{\mathcal{B}},\boldsymbol{\mathcal{D}})$.

IV. LORENTZ TRANSFORMATIONS

Let us now consider the case when the coordinate variables and fields are subject to Lorentz transformation. For history's sake, we note that one of the earliest attempts to give an alternative description of electrodynamics of moving bodies was undertaken by Cohn as early as 1902 [43]. Here we will work within the framework of Lorentz transformation and write the transformation laws for those fields which leave the system (3.4) Lorentz covariant. We connect the fields in the stationary frame (unprimed) with those in a moving one (primed) in the following way:

$$\mathbf{P}=\gamma(\mathbf{P}'+\boldsymbol{\beta}\times\mathbf{M}')-\frac{\gamma-1}{\beta^2}(\mathbf{P}'\cdot\boldsymbol{\beta})\boldsymbol{\beta},\quad(4.1a)$$

$$\mathbf{M}=\gamma(\mathbf{M}'-\boldsymbol{\beta}\times\mathbf{P}')-\frac{\gamma-1}{\beta^2}(\mathbf{M}'\cdot\boldsymbol{\beta})\boldsymbol{\beta},\quad(4.1b)$$

$$\boldsymbol{\alpha}^m=\boldsymbol{\alpha}^{m\prime}+\gamma\boldsymbol{\beta}\varphi^{e\prime}+\frac{\gamma-1}{\beta^2}(\boldsymbol{\alpha}^{m\prime}\cdot\boldsymbol{\beta})\boldsymbol{\beta},\quad(4.1c)$$

$$\varphi^e=\gamma(\varphi^{e\prime}+\boldsymbol{\beta}\cdot\boldsymbol{\alpha}^{m\prime}),\quad(4.1d)$$

$$\boldsymbol{\alpha}^e=\boldsymbol{\alpha}^{e\prime}+\gamma\boldsymbol{\beta}\varphi^{m\prime}+\frac{\gamma-1}{\beta^2}(\boldsymbol{\alpha}^{e\prime}\cdot\boldsymbol{\beta})\boldsymbol{\beta},\quad(4.1e)$$

$$\varphi^m=\gamma(\varphi^{m\prime}+\boldsymbol{\beta}\cdot\boldsymbol{\alpha}^{e\prime}),\quad(4.1f)$$

$$\rho=\gamma\left(\rho'+\frac{1}{c}(\boldsymbol{\beta}\cdot\mathbf{J}')\right),\quad(4.1g)$$

$$\mathbf{J}=\mathbf{J}'+\gamma\boldsymbol{\beta}\rho'+\frac{\gamma-1}{\beta^2}(\mathbf{J}'\cdot\boldsymbol{\beta})\boldsymbol{\beta},\quad(4.1h)$$

one can easily show that the system (3.4) is Lorentz covariant. Inserting Eqs. (4.1c)–(4.1f) into Eq. (3.3) one finds that the vectors $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{B}}$ transform in the following way:

$$\boldsymbol{\mathcal{E}}=\gamma(\boldsymbol{\mathcal{E}}'+\boldsymbol{\beta}\times\boldsymbol{\mathcal{B}}')-\frac{\gamma-1}{\beta^2}(\boldsymbol{\mathcal{E}}'\cdot\boldsymbol{\beta})\boldsymbol{\beta},$$

$$\boldsymbol{\mathcal{B}}=\gamma(\boldsymbol{\mathcal{B}}'-\boldsymbol{\beta}\times\boldsymbol{\mathcal{E}}')-\frac{\gamma-1}{\beta^2}(\boldsymbol{\mathcal{B}}'\cdot\boldsymbol{\beta})\boldsymbol{\beta}. \tag{4.2}$$

Note that the pairs $(\boldsymbol{\mathcal{D}},\boldsymbol{\mathcal{H}})$, $(\mathbf{E},\mathbf{B})$, and $(\mathbf{D},\mathbf{H})$ obey the same transformation law as Eq. (4.2). Finally, we write the transformation law for toroid dipole polarizations. In doing so, we underline a very original relation between the vector potential and toroid moment [44]

$$\text{curl curl}\,\mathbf{A}=4\pi c\,\text{curl curl}\,\mathbf{T}\delta(\mathbf{r}). \tag{4.3}$$

On the other hand, we determine

$$\boldsymbol{\mathcal{B}}=\mathbf{B}+\text{curl}\,\mathbf{T}^m=\text{curl}\,\mathbf{A}+\text{curl}\,\mathbf{T}^m.$$

All these facts suggest to us to handle $\mathbf{A}$ and $\mathbf{T}$ in the same manner. In other words, $\mathbf{T}^{e,m}$ should form the space part of some four vectors. To this end, we introduce two four vectors

$$\Pi^e=(\phi^e,\mathbf{T}^e), \quad \Pi^m=(\phi^m,\mathbf{T}^m) \tag{4.4}$$

with $\phi^{e,m}$ being the scalar parts of the four-vectors $\Pi^{e,m}$. Now we can write down the transformation law for toroid polarization, which is exactly the same for vector potentials, precisely,

$$\mathbf{T}^m=\mathbf{T}^{m'}+\gamma\boldsymbol{\beta}\phi^{m'}+\frac{\gamma-1}{\beta^2}(\mathbf{T}^{m'}\cdot\boldsymbol{\beta})\boldsymbol{\beta}, \tag{4.5a}$$

$$\phi^m=\gamma(\phi^{m'}+\boldsymbol{\beta}\cdot\mathbf{T}^{m'}), \tag{4.5b}$$

$$\mathbf{T}^e=\mathbf{T}^{e'}+\gamma\boldsymbol{\beta}\phi^{e'}+\frac{\gamma-1}{\beta^2}(\mathbf{T}^{e'}\cdot\boldsymbol{\beta})\boldsymbol{\beta}, \tag{4.5c}$$

$$\phi^e=\gamma(\phi^{e'}+\boldsymbol{\beta}\cdot\mathbf{T}^{e'}) \tag{4.5d}$$

with the additional condition

$$\text{curl}\,\mathbf{T}^{e,m}=\pm\left(\frac{1}{c}\frac{\partial\mathbf{T}^{m,e}}{\partial t}+\boldsymbol{\nabla}\phi^{m,e}\right) \tag{4.6}$$

to be fulfilled. Relation (4.6) demands some comments. Both $\mathbf{T}^e$ and $\mathbf{T}^m$ represent the closed isolated lines of electric and magnetic fields. So they have to obey the usual differential relations similar to the free Maxwell equations [45,46]). However, we remark that the signs here are opposite to the corresponding ones in Maxwell equations because the direction of the electric dipole is accepted to be chosen opposite to its inner electric field [47]. It should be emphasized that the relation (4.6) is a local one and it is not necessary to demand the condition (4.6) to be held to introduce toroid polarizations to the electromagnetic equations. Naturally, the question arises ''what do the $\phi^{e,m}$ stand for?'' To answer this question let us consider the following problem. Let four magnetic dipoles with magnitude M and length $2a$ form a quadrangle with the origin at its center [Fig. 6(a)]. For simplicity we choose the Z axis to be perpendicular to the quadrangle. As we showed earlier, this type of configuration possesses toroid dipole moment,

$$T_i^m=\frac{1}{2}\int[\mathbf{r}\times\mathbf{M}]_i d\mathbf{r}. \tag{4.7}$$

As one sees, in the case considered, the only nontrivial component of the toroid dipole moment is $T_z^m=4Ma^2$. Now let the quadrangle move along the Y axis with constant velocity v. In the moving system of coordinates, using Eq. (4.1) we find $\mathbf{M}'=\{\gamma\mathbf{M}_x,\mathbf{M}_y,0\}$ and $\mathbf{P}'=\{0,0,\gamma\beta\mathbf{M}_x\}$ [Fig. 6(b)], where $\beta=v/c$ and $\gamma=1/\sqrt{1-\beta^2}$. Now we get $\mathbf{T}_z'^m=2M(\gamma+1)a'^2$, whereas the electric dipole moments give quadrupole moments that can be calculated using

$$D_{ij}^{(2)}=\int\left\{\mathbf{r}_i\mathbf{P}_j+\mathbf{r}_j\mathbf{P}_i-\frac{2}{3}(\mathbf{r}\cdot\mathbf{P})\delta_{ij}\right\}d\mathbf{r},$$

which gives, e.g., $D_{yz}^{(2)}=-2\gamma Ma'l$. Here a' and l are the lengths of magnetic and electric dipole moments in the moving system of coordinates, respectively. In the case where the quadrangle moves downward along the Z axis, we find $\mathbf{M}'=\{\mathbf{M}_x,\mathbf{M}_y,0\}$ and $\mathbf{P}'=\{-\gamma\beta\mathbf{M}_y,\gamma\beta\mathbf{M}_x,0\}$ [Fig. 6(d)]. In this case we obtain $\mathbf{T}_z'^m=4Ma'^2$. The electric dipoles in this case give toroid polarization of the electric type

$$T_i^e=\frac{1}{2}\int[\mathbf{r}\times\mathbf{P}]_i d\mathbf{r}, \tag{4.8}$$

which reads $\mathbf{T}_z'^e=-4\gamma Ma'^2$.

Thus we see that $\phi^{e,m}$ is not uniquely defined and may be considered as the scalar mean-square radii, quadrupole moment, toroid moment, even dipole moment depending on the orientation of motion of the torus itself [48]. In general, the transformation of the toroid polarizations can be written as follows:

$$T_i'^m=\gamma T_i^m+\kappa\beta_j D_{ij}^{(2)}+\mu\beta_i D^{(0)}+\nu[\boldsymbol{\beta}\times\mathbf{T}^e]_i. \tag{4.9}$$

Here κ,μ,ν are the constants depend on the orientation of motion. Analogous formula can be written for $\mathbf{T}^e$.

V. COVARIANT FORM OF THE EQUATIONS OF ELECTROMAGNETISM

Let us write the Maxwell equations in covariant form. In doing so, we introduce the following antisymmetric tensors

$$\mathcal{F}^{\mu\nu}=\begin{pmatrix}0 & -\boldsymbol{\mathcal{E}}_1 & -\boldsymbol{\mathcal{E}}_2 & -\boldsymbol{\mathcal{E}}_3\\ \boldsymbol{\mathcal{E}}_1 & 0 & -\boldsymbol{\mathcal{B}}_3 & \boldsymbol{\mathcal{B}}_2\\ \boldsymbol{\mathcal{E}}_2 & \boldsymbol{\mathcal{B}}_3 & 0 & -\boldsymbol{\mathcal{B}}_1\\ \boldsymbol{\mathcal{E}}_3 & -\boldsymbol{\mathcal{B}}_2 & \boldsymbol{\mathcal{B}}_1 & 0\end{pmatrix}, \tag{5.1}$$

$$F^{\mu\nu}=\begin{pmatrix}0 & -\mathbf{E}_1 & -\mathbf{E}_2 & -\mathbf{E}_3\\ \mathbf{E}_1 & 0 & -\mathbf{B}_3 & \mathbf{B}_2\\ \mathbf{E}_2 & \mathbf{B}_3 & 0 & -\mathbf{B}_1\\ \mathbf{E}_3 & -\mathbf{B}_2 & \mathbf{B}_1 & 0\end{pmatrix},$$

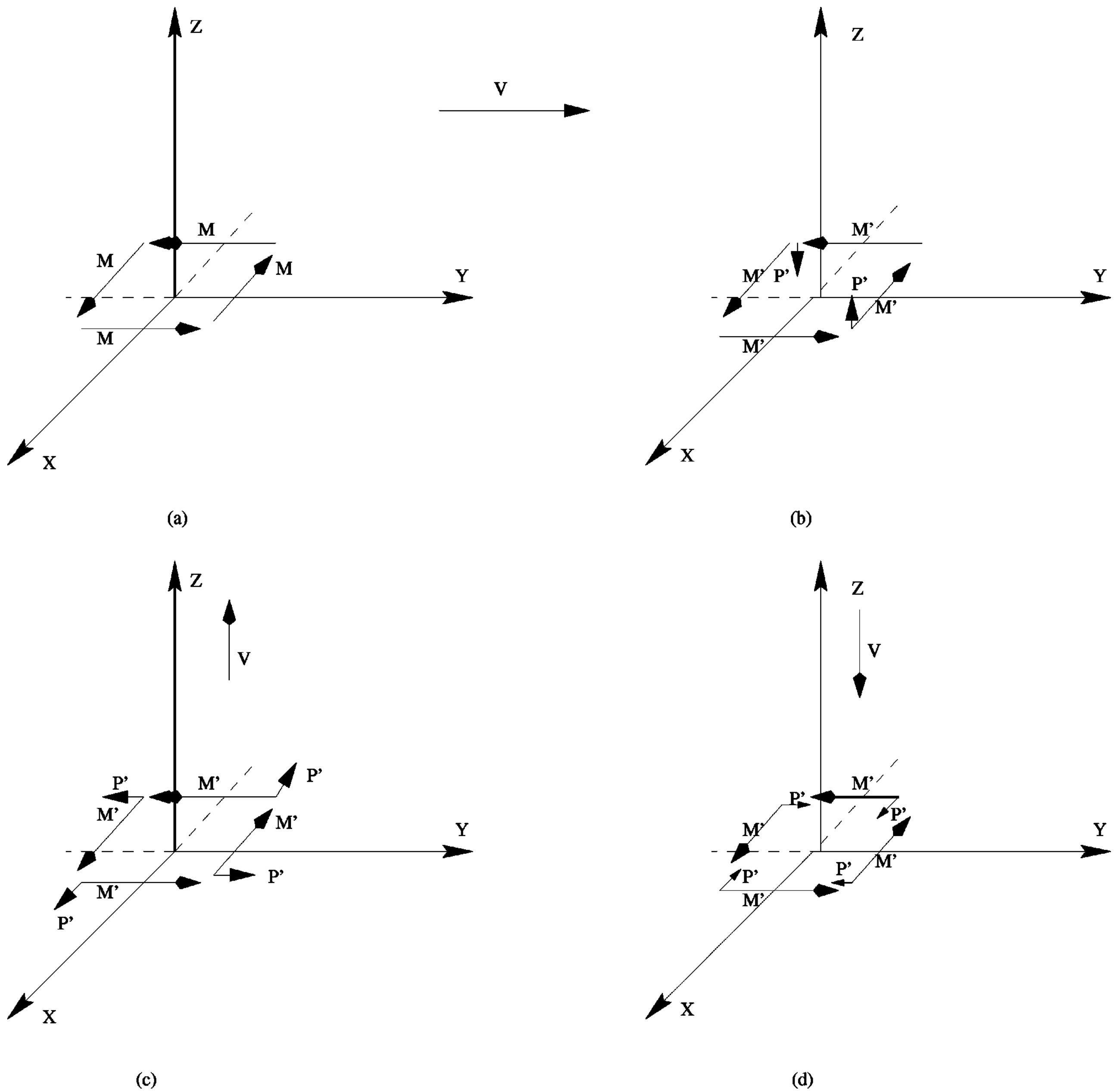

FIG. 6. A quadrangle of magnetic dipoles is in motion along different directions.

$$\mathcal{G}^{\mu\nu}=\begin{pmatrix} 0 & -\boldsymbol{\mathcal{D}}_1 & -\boldsymbol{\mathcal{D}}_2 & -\boldsymbol{\mathcal{D}}_3 \\ \boldsymbol{\mathcal{D}}_1 & 0 & -\boldsymbol{\mathcal{H}}_3 & \boldsymbol{\mathcal{H}}_2 \\ \boldsymbol{\mathcal{D}}_2 & \boldsymbol{\mathcal{H}}_3 & 0 & -\boldsymbol{\mathcal{H}}_1 \\ \boldsymbol{\mathcal{D}}_3 & -\boldsymbol{\mathcal{H}}_2 & \boldsymbol{\mathcal{H}}_1 & 0 \end{pmatrix},$$

$$G^{\mu\nu}=\begin{pmatrix} 0 & -\mathbf{D}_1 & -\mathbf{D}_2 & -\mathbf{D}_3 \\ \mathbf{D}_1 & 0 & -\mathbf{H}_3 & \mathbf{H}_2 \\ \mathbf{D}_2 & \mathbf{H}_3 & 0 & -\mathbf{H}_1 \\ \mathbf{D}_3 & -\mathbf{H}_2 & \mathbf{H}_1 & 0 \end{pmatrix},$$

$$S^{\mu\nu}=\begin{pmatrix} 0 & \mathbf{P}_1 & \mathbf{P}_2 & \mathbf{P}_3 \\ -\mathbf{P}_1 & 0 & -\mathbf{M}_3 & \mathbf{M}_2 \\ -\mathbf{P}_2 & \mathbf{M}_3 & 0 & -\mathbf{M}_1 \\ -\mathbf{P}_3 & -\mathbf{M}_2 & \mathbf{M}_1 & 0 \end{pmatrix},$$

$$\mathcal{T}_{\mu\nu}=(\partial_\mu \Pi^m_\nu - \partial_\nu \Pi^m_\mu),$$

together with the additional condition (4.6). Setting

$$\mathcal{F}^{\mu\nu}=F^{\mu\nu}-4\pi\mathcal{T}^{\mu\nu}, \quad \mathcal{G}^{\mu\nu}=G^{\mu\nu}-4\pi\mathcal{T}^{\mu\nu},$$

$$\mathcal{G}^{\mu\nu}=\mathcal{F}^{\mu\nu}-4\pi S^{\mu\nu}, \quad G^{\mu\nu}=F^{\mu\nu}-4\pi S^{\mu\nu},$$

we write Eqs. (2.16) in the following covariant form:

$$\partial_\mu \mathcal{G}^{\mu\nu}=\frac{4\pi}{c}\mathbf{j}^\nu, \tag{5.2a}$$

$$\partial_\rho \mathcal{F}_{\mu\nu}+\partial_\mu \mathcal{F}_{\nu\rho}+\partial_\nu \mathcal{F}_{\rho\mu}=0. \tag{5.2b}$$

VI. CONCLUSION

The modified equations of electrodynamics have been obtained to account for toroid moment contributions. The two-potential formalism has been further developed for the equations obtained. It has been shown that the modified system is Lorentz covariant. We would also like to point out that the Hamiltonian formalism that has been developed here can be improved further using the recipe given in [49]. The rapid use of toroid polarizations in various fields of physics explains the increasing interest in it. On the other hand, its ambiguous transformation under motion makes the thing more puzzling, leaving a lot of questions behind it. We plan to come back to these questions and answer some of them in our forthcoming works.

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