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ORIGINAL ARTICLE

Nonlinear spinor field in Bianchi type-I Universe filled with viscous fluid: numerical solutions

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Abstract We consider a system of nonlinear spinor and a Bianchi type I gravitational fields in presence of viscous fluid. The nonlinear term in the spinor field Lagrangian is chosen to be λF, with λ being a self-coupling constant and F being a function of the invariants I an J constructed from bilinear spinor forms S and P. Self-consistent solutions to the spinor and BI gravitational field equations are obtained in terms of τ, where τ is the volume scale of BI universe. System of equations for τ and ε, where ε is the energy of the viscous fluid, is deduced. This system is solved numerically for some special cases.

Keywords Spinor field · Bianchi type I (BI) model · Cosmological constant

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1 Introduction

The investigation of relativistic cosmological models usually has the energy momentum tensor of matter generated by a perfect fluid. To consider more realistic models, one must take into account the viscosity mechanisms, which have already attracted attention of many researchers. Misner (1967, 1968) suggested that strong dissipative due to the neutrino viscosity may considerably reduce the anisotropy of the black-body radiation. Viscosity mechanism in cosmology can explain the anomalously high entropy per baryon in the present universe (Weinberg 1972a, 1972b). Bulk viscosity associated with the grand-unified-theory phase transition (Langacker 1981) may lead to an inflationary scenario (Waga et al. 1986; Pacher et al. 1987; Guth 1981).

A uniform cosmological model filled with fluid which possesses pressure and second (bulk) viscosity was developed by Murphy (1973). The solutions that he found exhibit an interesting feature that the big bang type singularity appears in the infinite past. Exact solutions of the isotropic homogeneous cosmology for open, closed and flat universe have been found by Santos et al. (1985), with the bulk viscosity being a power function of energy density.

The nature of cosmological solutions for homogeneous Bianchi type I (BI) model was investigated by Belinski and Khalatnikov (1975) by taking into account a dissipative process due to viscosity. They showed that viscosity cannot remove the cosmological singularity but results in a qualitatively new behavior of the solutions near singularity. They found the remarkable property that during the time of the *big bang* matter is created by the gravitational field. BI solutions in case of stiff matter with a shear viscosity being the power function of energy density were obtained by Banerjee (1985), whereas BI models with bulk viscosity (η) that is a power function of energy density ε and when the universe is filled with stiff matter were studied by Huang (1990). The effect of bulk viscosity, with a time varying bulk viscous coefficient, on the evolution of isotropic FRW models was investigated in the context of open thermodynamics system was studied by Desikan (1997). This study was further developed by Krori and Mukherjee (2000) for anisotropic Bianchi models. Cosmological solutions with nonlinear bulk viscosity were obtained in Chimento et al. (1997). Models with both shear and bulk viscosity were investigated in van Elst et al. (1995), Gavrilov et al. (1997).

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Though Murphy (1973) claimed that the introduction of bulk viscosity can avoid the initial singularity at finite past, results obtained in Barrow (1988) show that, it is, in general, not valid, since for some cases big bang singularity occurs in finite past. To eliminate the initial singularities a self-consistent system of nonlinear spinor and BI gravitational field was considered by us in a series of papers (Saha and Shikin 1997a, 1997b; Saha 2001a, 2001b). For some cases we were able to find field (both matter and gravitational) configurations those were always regular. In the papers mentioned above we considered the system of interacting nonlinear spinor and/or scalar fields in a BI universe filled with perfect fluid. We also study the above system in presence of cosmological constant Λ (both constant and time varying Saha 2001b). A nonlinear spinor field, suggested by the symmetric coupling between nucleons, muons, and leptons, has been investigated by Finkelstein et al. (1951) in the classical approximation. Although the existence of spin-1/2 fermion is both theoretically and experimentally undisputed, these are described by *quantum* spinor fields. Possible justifications for the existence of classical spinors has been addressed in Armendáriz-Picón and Greene (2003). In view of what has been mentioned above, it would be interesting to study the influence of viscous fluid to the system of material (say spinor and/or scalar) and BI gravitational fields in presence of a Λ-term as well. In a recent paper we studied the Bianchi type-I universe filled with viscous fluid in presence of a Λ term (Saha 2005a). This study was further developed in Saha and Rikhvitsky (2006) where we present qualitative analysis of the corresponding system of equations. Finally in Saha (2005b) we introduced spinor field into the system and solved the system for some special choice of viscosity. The purpose of this paper is to further developed those results for more general cases and give some numerical results. It should be noted the in the process there occurs a very rich system of equations for volume scale, Hubble constant and energy density. The qualitative analysis of this system is under active study and we plan to present those results soon.

2 Derivation of basic equations

In this section we derive the fundamental equations for the interacting spinor, scalar and gravitational fields from the action and write their solutions in term of the volume scale τ defined bellow (2.16). We also derive the equation for τ which plays the central role here.

We consider a system of nonlinear spinor, scalar and BI gravitational field in presence of perfect fluid given by the action

$$\mathscr{S}(g;\psi,\bar\psi) = \int \mathscr{L}\sqrt{-g}d\Omega \tag{2.1}$$

with

$$\mathscr{L} = \mathscr{L}_{\rm g} + \mathscr{L}_{\rm sp} + \mathscr{L}_{\rm m}. \tag{2.2}$$

The gravitational part of the Lagrangian (2.2) is given by a Bianchi type I (BI hereafter) space-time, whereas $\mathscr{L}_{\rm sp}$ describes the spinor field Lagrangian and $\mathscr{L}_{\rm m}$ stands for the Lagrangian density of viscous fluid.

2.1 Material field Lagrangian

For a spinor field ψ, symmetry between ψ and $\bar\psi$ appears to demand that one should choose the symmetrized Lagrangian (Kibble 1961). Keep it in mind we choose the spinor field Lagrangian as

$$\mathscr{L}_{\rm sp} = \frac{i}{2}[\bar\psi\gamma^\mu\nabla_\mu\psi - \nabla_\mu\bar\psi\gamma^\mu\psi] - m\bar\psi\psi + \lambda F. \tag{2.3}$$

Here m is the spinor mass, λ is the self-coupling constant and $F = F(I,J)$ with $I = S^2 = (\bar\psi\psi)^2$ and $J = P^2 = (i\bar\psi\gamma^5\psi)^2$. According to the Pauli–Fierz theorem (Berestetski et al. 1989) among the five invariants only I and J are independent as all other can be expressed by them: $I_V = -I_A = I + J$ and $I_Q = I - J$. Therefore, the choice $F = F(I,J)$, describes the nonlinearity in the most general of its form (Saha 2001a). Note that setting $\lambda = 0$ in (2.3) we come to the case with linear spinor field.

2.2 The gravitational field

As a gravitational field we consider the Bianchi type I (BI) cosmological model. It is the simplest model of anisotropic universe that describes a homogeneous and spatially flat space-time. It was first shown in Jacobs (1968) that if filled with perfect fluid with the equation of state $p = \zeta\varepsilon$, $\zeta < 1$, a BI universe eventually evolves into a FRW one. Recently Chimento (2003) investigated a cosmological model with perfect fluid and obtained general solution to the Einstein's field equation. In particular he showed that owing to the spatial isotropy of the stress-energy tensor, the initial anisotropy of the BI model dissipated as the Universe expands. The isotropy of present-day universe makes BI model a prime candidate for studying the possible effects of an anisotropy in the early universe on modern-day data observations. In view of what has been mentioned above we choose the gravitational part of the Lagrangian (2.2) in the form

$$\mathscr{L}_{\rm g} = \frac{R}{2\kappa}, \tag{2.4}$$

where R is the scalar curvature, $\kappa = 8\pi G$ being the Einstein's gravitational constant. The gravitational field in our case is given by a Bianchi type I (BI) metric

$$ds^2 = dt^2 - a^2dx^2 - b^2dy^2 - c^2dz^2, \tag{2.5}$$

with a, b, c being the functions of time t only. Here the speed of light is taken to be unity.

2.3 Field equations

Let us now write the field equations corresponding to the action (2.1).

Variation of (2.1) with respect to spinor field ψ ($\bar\psi$) gives spinor field equations

$$i\gamma^\mu\nabla_\mu\psi - m\psi + \mathscr{D}\psi + \mathscr{G}i\gamma^5\psi = 0, \tag{2.6a}$$

$$i\nabla_\mu\bar\psi\gamma^\mu + m\bar\psi - \mathscr{D}\bar\psi - \mathscr{G}i\bar\psi\gamma^5 = 0, \tag{2.6b}$$

where we denote

$$\mathscr{D} = 2\lambda S\frac{\partial F}{\partial I}, \qquad \mathscr{G} = 2\lambda P\frac{\partial F}{\partial J}.$$

Variation of (2.1) with respect to metric tensor $g_{\mu\nu}$ gives the Einstein's equations which in account of the Λ-term for the BI space-time (2.5) can be rewritten as

$$\frac{\ddot b}{b} + \frac{\ddot c}{c} + \frac{\dot b}{b}\frac{\dot c}{c} = \kappa T_1^1 + \Lambda, \tag{2.7a}$$

$$\frac{\ddot c}{c} + \frac{\ddot a}{a} + \frac{\dot c}{c}\frac{\dot a}{a} = \kappa T_2^2 + \Lambda, \tag{2.7b}$$

$$\frac{\ddot a}{a} + \frac{\ddot b}{b} + \frac{\dot a}{a}\frac{\dot b}{b} = \kappa T_3^3 + \Lambda, \tag{2.7c}$$

$$\frac{\dot a}{a}\frac{\dot b}{b} + \frac{\dot b}{b}\frac{\dot c}{c} + \frac{\dot c}{c}\frac{\dot a}{a} = \kappa T_0^0 + \Lambda, \tag{2.7d}$$

where over dot means differentiation with respect to t and T_ν^μ is the energy-momentum tensor of the material field given by

$$T_\mu^\nu = T_{\mathrm{sp}\mu}^\nu + T_{\mathrm{m}\mu}^\nu. \tag{2.8}$$

Here $T_{\mathrm{sp}\mu}^\nu$ is the energy-momentum tensor of the spinor field which with regard to (2.6) has the form

$$T_{\mathrm{sp}\mu}^\rho = \frac{i}{4}g^{\rho\nu}(\bar\psi\gamma_\mu\nabla_\nu\psi + \bar\psi\gamma_\nu\nabla_\mu\psi - \nabla_\mu\bar\psi\gamma_\nu\psi - \nabla_\nu\bar\psi\gamma_\mu\psi) + \delta_\mu^\rho(\mathscr{D}S + \mathscr{G}P - \lambda F). \tag{2.9}$$

$T_{\mathrm{m}\mu}^\nu$ is the energy-momentum tensor of a viscous fluid having the form

$$T_{\mathrm{m}\mu}^\nu = (\varepsilon + p')u_\mu u^\nu - p'\delta_\mu^\nu + \eta g^{\nu\beta}[u_{\mu;\beta} + u_{\beta;\mu} - u_\mu u^\alpha u_{\beta;\alpha} - u_\beta u^\alpha u_{\mu;\alpha}], \tag{2.10}$$

where

$$p' = p - \left(\xi - \frac{2}{3}\eta\right)u^\mu_{;\mu}. \tag{2.11}$$

Here ε is the energy density, p—pressure, η and ξ are the coefficients of shear and bulk viscosity, respectively. In a comoving system of reference such that $u^\mu = (1,0,0,0)$ we have

$$T_{\mathrm{m}0}^0 = \varepsilon, \tag{2.12a}$$

$$T_{\mathrm{m}1}^1 = -p' + 2\eta\frac{\dot a}{a}, \tag{2.12b}$$

$$T_{\mathrm{m}2}^2 = -p' + 2\eta\frac{\dot b}{b}, \tag{2.12c}$$

$$T_{\mathrm{m}3}^3 = -p' + 2\eta\frac{\dot c}{c}. \tag{2.12d}$$

In (2.6) and (2.9) ∇_μ is the covariant derivatives acting on a spinor field as (Zhelnorovich 1982; Brill and Wheeler 1957)

$$\nabla_\mu\psi = \frac{\partial\psi}{\partial x^\mu} - \Gamma_\mu\psi, \qquad \nabla_\mu\bar\psi = \frac{\partial\bar\psi}{\partial x^\mu} + \bar\psi\Gamma_\mu, \tag{2.13}$$

where Γ_μ are the Fock–Ivanenko spinor connection coefficients defined by

$$\Gamma_\mu = \frac{1}{4}\gamma^\sigma(\Gamma_{\mu\sigma}^\nu\gamma_\nu - \partial_\mu\gamma_\sigma). \tag{2.14}$$

For the metric (2.5) one has the following components of the spinor connection coefficients

$$\begin{aligned}&\Gamma_0 = 0, \qquad \Gamma_1 = \frac{1}{2}\dot a(t)\bar\gamma^1\bar\gamma^0,\\ &\Gamma_2 = \frac{1}{2}\dot b(t)\bar\gamma^2\bar\gamma^0, \qquad \Gamma_3 = \frac{1}{2}\dot c(t)\bar\gamma^3\bar\gamma^0.\end{aligned} \tag{2.15}$$

The Dirac matrices $\gamma^\mu(x)$ of curved space-time are connected with those of Minkowski one as follows:

$$\gamma^0 = \bar\gamma^0, \qquad \gamma^1 = \bar\gamma^1/a, \qquad \gamma^2 = \bar\gamma^2/b, \qquad \gamma^3 = \bar\gamma^3/c$$

with

$$\bar\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I\end{pmatrix}, \qquad \bar\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0\end{pmatrix},$$

$$\gamma^5 = \bar\gamma^5 = \begin{pmatrix} 0 & -I \\ -I & 0\end{pmatrix},$$

where σ_i are the Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix},$$

$$\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}.$$

Note that the $\bar\gamma$ and the σ matrices obey the following properties:

$$\bar\gamma^i\bar\gamma^j + \bar\gamma^j\bar\gamma^i = 2\eta^{ij}, \quad i,j = 0,1,2,3,$$

$$\bar{\gamma}^i\bar{\gamma}^5+\bar{\gamma}^5\bar{\gamma}^i=0,\quad (\bar{\gamma}^5)^2=I,\ i=0,1,2,3,$$
$$\sigma^j\sigma^k=\delta_{jk}+i\varepsilon_{jkl}\sigma^l,\quad j,k,l=1,2,3,$$

where $\eta_{ij}=\{1,-1,-1,-1\}$ is the diagonal matrix, δ_{jk} is the Kronekar symbol and ε_{jkl} is the totally antisymmetric matrix with $\varepsilon_{123}=+1$.

We study the space-independent solutions to the spinor field equations (2.6) so that $\psi=\psi(t)$. Here we define

$$\tau=abc=\sqrt{-g} \tag{2.16}$$

The spinor field equation (2.6a) in account of (2.13) and (2.15) takes the form

$$i\bar{\gamma}^0\left(\frac{\partial}{\partial t}+\frac{\dot{\tau}}{2\tau}\right)\psi-m\psi+\mathscr{D}\psi+\mathscr{G}i\gamma^5\psi=0. \tag{2.17}$$

Setting $V_j(t)=\sqrt{\tau}\psi_j(t)$, $j=1,2,3,4$, from (2.17) one deduces the following system of equations:

$$\dot{V}_1+i(m-\mathscr{D})V_1-\mathscr{G}V_3=0, \tag{2.18a}$$
$$\dot{V}_2+i(m-\mathscr{D})V_2-\mathscr{G}V_4=0, \tag{2.18b}$$
$$\dot{V}_3-i(m-\mathscr{D})V_3+\mathscr{G}V_1=0, \tag{2.18c}$$
$$\dot{V}_4-i(m-\mathscr{D})V_4+\mathscr{G}V_2=0. \tag{2.18d}$$

From (2.6a) we also write the equations for the invariants S, P and $A=\bar{\psi}\bar{\gamma}^5\bar{\gamma}^0\psi$

$$\dot{S}_0-2\mathscr{G}A_0=0, \tag{2.19a}$$
$$\dot{P}_0-2(m-\mathscr{D})A_0=0, \tag{2.19b}$$
$$\dot{A}_0+2(m-\mathscr{D})P_0+2\mathscr{G}S_0=0, \tag{2.19c}$$

where $S_0=\tau S$, $P_0=\tau P$, and $A_0=\tau A$. Equation (2.19) leads to the following relation

$$S^2+P^2+A^2=C^2/\tau^2,\quad C^2=\text{const}. \tag{2.20}$$

Giving the concrete form of F from (2.18) one writes the components of the spinor functions in explicitly and using the solutions obtained one can write the components of spinor current:

$$j^\mu=\bar{\psi}\gamma^\mu\psi. \tag{2.21}$$

The component j^0

$$j^0=\frac{1}{\tau}[V_1^*V_1+V_2^*V_2+V_3^*V_3+V_4^*V_4], \tag{2.22}$$

defines the charge density of spinor field that has the following chronometric-invariant form

$$\rho=(j_0\cdot j^0)^{1/2}. \tag{2.23}$$

The total charge of spinor field is defined as

$$Q=\int_{-\infty}^{\infty}\rho\sqrt{-^3g}dxdydz=\rho\tau\mathscr{V}, \tag{2.24}$$

where $\mathscr{V}$ is the volume. From the spin tensor

$$S^{\mu\nu,\epsilon}=\frac{1}{4}\bar{\psi}\{\gamma^\epsilon\sigma^{\mu\nu}+\sigma^{\mu\nu}\gamma^\epsilon\}\psi. \tag{2.25}$$

One finds chronometric invariant spin tensor

$$S_{\text{ch}}^{ij,0}=(S_{ij,0}S^{ij,0})^{1/2}, \tag{2.26}$$

and the projection of the spin vector on k axis

$$S_k=\int_{-\infty}^{\infty}S_{\text{ch}}^{ij,0}\sqrt{-^3g}dxdydz=S_{\text{ch}}^{ij,0}\tau V. \tag{2.27}$$

Let us now solve the Einstein equations. To do it, we first write the expressions for the components of the energy-momentum tensor explicitly:

$$T_0^0=mS-\lambda F+\varepsilon\equiv\tilde{T}_0^0, \tag{2.28a}$$
$$T_1^1=\mathscr{D}S+\mathscr{G}P-\lambda F-p'+2\eta\frac{\dot{a}}{a}\equiv\tilde{T}_1^1+2\eta\frac{\dot{a}}{a}, \tag{2.28b}$$
$$T_2^2=\mathscr{D}S+\mathscr{G}P-\lambda F-p'+2\eta\frac{\dot{b}}{b}\equiv\tilde{T}_1^1+2\eta\frac{\dot{b}}{b}, \tag{2.28c}$$
$$T_3^3=\mathscr{D}S+\mathscr{G}P-\lambda F-p'+2\eta\frac{\dot{c}}{c}\equiv\tilde{T}_1^1+2\eta\frac{\dot{c}}{c}. \tag{2.28d}$$

In account of (2.28) subtracting (2.7a) from (2.7b), one finds the following relation between a and b:

$$\frac{a}{b}=D_1\exp\left(X_1\int\frac{e^{-2\kappa\int\eta dt}dt}{\tau}\right). \tag{2.29}$$

Analogically, one finds

$$\frac{b}{c}=D_2\exp\left(X_2\int\frac{e^{-2\kappa\int\eta dt}dt}{\tau}\right),$$
$$\frac{c}{a}=D_3\exp\left(X_3\int\frac{e^{-2\kappa\int\eta dt}dt}{\tau}\right). \tag{2.30}$$

Here D_1, D_2, D_3, X_1, X_2, X_3 are integration constants, obeying

$$D_1D_2D_3=1,\qquad X_1+X_2+X_3=0. \tag{2.31}$$

In view of (2.31) from (2.29) and (2.30) we write the metric functions explicitly (Saha 2001a)

$$a(t)=(D_1/D_3)^{1/3}\tau^{1/3}\times\exp\left[\frac{X_1-X_3}{3}\int\frac{e^{-2\kappa\int\eta dt}}{\tau(t)}dt\right], \tag{2.32a}$$

$$b(t) = (D_1^2 D_3)^{-1/3}\tau^{1/3}$$
$$\times \exp\left[-\frac{2X_1+X_3}{3}\int \frac{e^{-2\kappa\int \eta dt}}{\tau(t)}dt\right], \tag{2.32b}$$
$$c(t) = (D_1 D_3^2)^{1/3}\tau^{1/3}$$
$$\times \exp\left[\frac{X_1+2X_3}{3}\int \frac{e^{-2\kappa\int \eta dt}}{\tau(t)}dt\right]. \tag{2.32c}$$

As one sees from (2.32a), (2.32b) and (2.32c), for $\tau = t^n$ with $n > 1$ the exponent tends to unity at large t, and the anisotropic model becomes isotropic one.

Further we will investigate the existence of singularity (singular point) of the gravitational case, which can be done by investigating the invariant characteristics of the space-time. In general relativity these invariants are composed from the curvature tensor and the metric one. In a 4D Riemann space-time there are 14 independent invariants. Instead of analyzing all 14 invariants, one can confine this study only in 3, namely the scalar curvature $I_1 = R$, $I_2 = R^R_{\mu\nu}\mu\nu$, and the Kretschmann scalar $I_3 = R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$. At any regular space-time point, these three invariants I_1, I_2, I_3 should be finite. One can easily verify that

$$I_1 \propto \frac{1}{\tau^2}, \qquad I_2 \propto \frac{1}{\tau^4}, \qquad I_3 \propto \frac{1}{\tau^4}.$$

Thus we see that at any space-time point, where $\tau = 0$ the invariants I_1, I_2, I_3, as well as the scalar and spinor fields become infinity, hence the space-time becomes singular at this point.

In what follows, we write the equation for τ and study it in details.

Summation of Einstein equations (2.7a), (2.7b), (2.7c) and (2.7d) multiplied by 3 gives

$$\ddot\tau = \frac{3}{2}\kappa(\tilde T_0^0 + \tilde T_1^1)\tau + 3\kappa\eta\dot\tau + 3\Lambda\tau, \tag{2.33}$$

which can be rearranged as

$$\ddot\tau - \frac{3}{2}\kappa\xi\dot\tau$$
$$= \frac{3}{2}\kappa(mS + \mathscr{D}S + \mathscr{G}P - 2\lambda F + \varepsilon - p)\tau + 3\Lambda\tau. \tag{2.34}$$

For the right-hand-side of (2.34) to be a function of τ only, the solution to this equation is well-known (Kamke 1957).

On the other hand from Bianchi identity $G^\nu_{\mu;\nu} = 0$ one finds

$$T^\nu_{\mu;\nu} = T^\nu_{\mu,\nu} + \Gamma^\nu_{\rho\nu}T^\rho_\mu - \Gamma^\rho_{\mu\nu}T^\nu_\rho = 0, \tag{2.35}$$

which in our case has the form

$$\frac{1}{\tau}(\tau T_0^0)^{\cdot} - \frac{\dot a}{a}T_1^1 - \frac{\dot b}{b}T_2^2 - \frac{\dot c}{c}T_3^3 = 0. \tag{2.36}$$

This equation can be rewritten as

$$\dot{\tilde T}{}_0^0 = \frac{\dot\tau}{\tau}(\tilde T_1^1 - \tilde T_0^0) + 2\eta\left(\frac{\dot a^2}{a^2} + \frac{\dot b^2}{b^2} + \frac{\dot c^2}{c^2}\right). \tag{2.37}$$

Recall that (2.19) gives

$$(m - \mathscr{D})\dot S_0 - \mathscr{G}\dot P_0 = 0.$$

In view of that after a little manipulation from (2.37) we obtain

$$\dot\varepsilon + \frac{\dot\tau}{\tau}\omega - \left(\xi + \frac{4}{3}\eta\right)\frac{\dot\tau^2}{\tau^2} + 4\eta(\kappa T_0^0 + \Lambda) = 0, \tag{2.38}$$

where

$$\omega = \varepsilon + p \tag{2.39}$$

is the thermal function. For further purpose we would like to note that in absence of shear viscosity from (2.33) and (2.37) one finds

$$\kappa\tilde T_0^0 = 3H^2 - \Lambda + C_{00}, \quad C_{00} = \text{const}, \tag{2.40}$$

where in analogy with Hubble constant introduce the quantity H, such that

$$\frac{\dot\tau}{\tau} = \frac{\dot a}{a} + \frac{\dot b}{b} + \frac{\dot c}{c} = 3H. \tag{2.41}$$

Then (2.34) and (2.38) in account of (2.28) can be rewritten as

$$\dot H = \frac{\kappa}{2}(3\xi H - \omega) - (3H^2 - \kappa\varepsilon - \Lambda)$$
$$+ \frac{\kappa}{2}(mS + \mathscr{D}S + \mathscr{G}P - 2\lambda F), \tag{2.42a}$$
$$\dot\varepsilon = 3H(3\xi H - \omega) + 4\eta(3H^2 - \kappa\varepsilon - \Lambda)$$
$$- 4\eta\kappa(mS - \lambda F). \tag{2.42b}$$

Thus, the metric functions are found explicitly in terms of τ and viscosity. To write τ and components of spinor field as well and scalar one we have to specify F in $\mathscr{L}_{\rm sp}$. In the next section we explicitly solve (2.18) and (2.42) for some concrete value of F.

Equations (2.42) can be written in terms of dynamical scalar as well. For this purpose let us introduce the dynamical scalars such as the expansion and the shear scalar as usual

$$\theta = u^\mu_{;\mu}, \qquad \sigma^2 = \frac{1}{2}\sigma_{\mu\nu}\sigma^{\mu\nu}, \tag{2.43}$$

where

$$\sigma_{\mu\nu} = \frac{1}{2}(u_{\mu;\alpha}P^\alpha_\nu + u_{\nu;\alpha}P^\alpha_\mu) - \frac{1}{3}\theta P_{\mu\nu}. \tag{2.44}$$

Here P is the projection operator obeying

$$P^2 = P. \tag{2.45}$$

For the space-time with signature $(+,-,-,-)$ it has the form

$$P_{\mu\nu} = g_{\mu\nu} - u_\mu u_\nu, \qquad P^\mu_\nu = \delta^\mu_\nu - u^\mu u_\nu. \tag{2.46}$$

For the BI metric the dynamical scalar has the form

$$\theta = \frac{\dot a}{a} + \frac{\dot b}{b} + \frac{\dot c}{c} = \frac{\dot\tau}{\tau}, \tag{2.47}$$

and

$$2\sigma^2 = \frac{\dot a^2}{a^2} + \frac{\dot b^2}{b^2} + \frac{\dot c^2}{c^2} - \frac{1}{3}\theta^2. \tag{2.48}$$

In account of (2.32) one can also rewrite share scalar as

$$2\sigma^2 = \frac{6(X_1^2 + X_1X_3 + X_3^2)}{9\tau^2} e^{-4\kappa\int\eta dt}. \tag{2.49}$$

From (2.7d) now yields

$$\frac{1}{3}\theta^2 - \sigma^2 = \kappa[mS - \lambda F + \varepsilon] + \Lambda. \tag{2.50}$$

Equations (2.42) now can be written in terms of θ and σ as follows

$$\dot\theta = \frac{3\kappa}{2}(\xi\theta - \omega) - \frac{3\kappa}{2}(mS - \mathscr{D}S - \mathscr{G}P) - 3\sigma^2, \tag{2.51a}$$

$$\dot\varepsilon = \theta(\xi\theta - \omega) + 4\eta\sigma^2. \tag{2.51b}$$

Note that (2.51) without spinor and scalar field contributions coincide with the ones given in Banerjee et al. (1985).

3 Some special solutions

In this section we first solve the spinor field equations for some special choice of F, which will be given in terms of τ. Thereafter, we will study the system (2.42) in details and give explicit solution for some special cases.

3.1 Solutions to the spinor field equations

As one sees, introduction of viscous fluid has no direct effect on the system of spinor field equations (2.18). Viscous fluid has an implicit influence on the system through τ. A detailed analysis of the system in question can be found in Saha (2001a). Here we just write the final results.

3.1.1 Case with $F = F(I)$

Here we consider the case when the nonlinear spinor field is given by $F = F(I)$. As in the case with minimal coupling from (2.19a) one finds

$$S = \frac{C_0}{\tau}, \quad C_0 = \text{const.} \tag{3.1}$$

For components of spinor field we find (Saha 2001a)

$$\begin{aligned} \psi_1(t) &= \frac{C_1}{\sqrt{\tau}} e^{-i\beta}, \qquad \psi_2(t) = \frac{C_2}{\sqrt{\tau}} e^{-i\beta}, \\ \psi_3(t) &= \frac{C_3}{\sqrt{\tau}} e^{i\beta}, \qquad \psi_4(t) = \frac{C_4}{\sqrt{\tau}} e^{i\beta}, \end{aligned} \tag{3.2}$$

with C_i being the integration constants and are related to C_0 as $C_0 = C_1^2 + C_2^2 - C_3^2 - C_4^2$. Here $\beta = \int(m - \mathscr{D})dt$.

For the components of the spin current from (2.21) we find

$$\begin{aligned} j^0 &= \frac{1}{\tau}[C_1^2 + C_2^2 + C_3^2 + C_4^2], \\ j^1 &= \frac{2}{a\tau}[C_1C_4 + C_2C_3]\cos(2\beta), \\ j^2 &= \frac{2}{b\tau}[C_1C_4 - C_2C_3]\sin(2\beta), \\ j^3 &= \frac{2}{c\tau}[C_1C_3 - C_2C_4]\cos(2\beta), \end{aligned}$$

whereas, for the projection of spin vectors on the X, Y and Z axis we find

$$\begin{aligned} S^{23,0} &= \frac{C_1C_2 + C_3C_4}{bc\tau}, \qquad S^{31,0} = 0, \\ S^{12,0} &= \frac{C_1^2 - C_2^2 + C_3^2 - C_4^2}{2ab\tau}. \end{aligned}$$

The total charge of the system in a volume $\mathcal{V}$ in this case is

$$Q = [C_1^2 + C_2^2 + C_3^2 + C_4^2]\mathcal{V}. \tag{3.3}$$

Thus, for $\tau \neq 0$ the components of spin current and the projection of spin vectors are singularity-free and the total charge of the system in a finite volume is always finite. Note that, setting $\lambda = 0$, i.e., $\beta = mt$ in the foregoing expressions one get the results for the linear spinor field.

3.1.2 Case with $F = F(J)$

Here we consider the case with $F = F(J)$. In this case we assume the spinor field to be massless. Note that, in the unified nonlinear spinor theory of Heisenberg, the massive term remains absent, and according to Heisenberg, the particle mass should be obtained as a result of quantization of spinor prematter (Heisenberg 1966). In the nonlinear generalization of classical field equations, the massive term does not possess the significance that it possesses in the linear one, as it by no means defines total energy (or mass) of the nonlinear field system. Thus without losing the generality we can consider massless spinor field putting $m=0$. Then from (2.19b) one gets

$$P = D_0/\tau, \quad D_0 = \text{const.} \tag{3.4}$$

In this case the spinor field components take the form

$$\begin{aligned}\psi_1 &= \frac{1}{\sqrt{\tau}}(D_1 e^{i\sigma} + iD_3 e^{-i\sigma}),\\ \psi_2 &= \frac{1}{\sqrt{\tau}}(D_2 e^{i\sigma} + iD_4 e^{-i\sigma}),\\ \psi_3 &= \frac{1}{\sqrt{\tau}}(iD_1 e^{i\sigma} + D_3 e^{-i\sigma}),\\ \psi_4 &= \frac{1}{\sqrt{\tau}}(iD_2 e^{i\sigma} + D_4 e^{-i\sigma}).\end{aligned} \tag{3.5}$$

The integration constants D_i are connected to D_0 by $D_0 = 2(D_1^2 + D_2^2 - D_3^2 - D_4^2)$. Here we set $\sigma = \int \mathcal{G} dt$.

For the components of the spin current from (2.21) we find

$$\begin{aligned}j^0 &= \frac{2}{\tau}[D_1^2 + D_2^2 + D_3^2 + D_4^2],\\ j^1 &= \frac{4}{a\tau}[D_2D_3 + D_1D_4]\cos(2\sigma),\\ j^2 &= \frac{4}{b\tau}[D_2D_3 - D_1D_4]\sin(2\sigma),\\ j^3 &= \frac{4}{c\tau}[D_1D_3 - D_2D_4]\cos(2\sigma),\end{aligned}$$

whereas, for the projection of spin vectors on the X, Y and Z axis we find

$$S^{23,0} = \frac{2(D_1D_2 + D_3D_4)}{bc\tau}, \qquad S^{31,0} = 0,$$
$$S^{12,0} = \frac{D_1^2 - D_2^2 + D_3^2 - D_4^2}{2ab\tau}.$$

We see that for any nontrivial τ as in previous case the components of spin current and the projection of spin vectors are singularity-free and the total charge of the system in a finite volume is always finite.

3.2 Determination of τ

In this section we simultaneously solve the system of equations for τ and ε. Since setting $m=0$ in the equations for $F = F(I)$ one comes to the case when $F = F(J)$, we consider the case with F being the function of I only. Let F be the power function of S, i.e., $F = S^n$. As it was established earlier, in this case $S = C_0/\tau$, or setting $C_0 = 1$ simply $S = 1/\tau$. Evaluating $\mathcal{D}$ in terms of τ we then come to the following system of equations

$$\ddot{\tau} = \frac{3\kappa}{2}\xi\dot{\tau} + \frac{3\kappa}{2}\left(\frac{m}{\tau} + \frac{\lambda(n-2)}{\tau^n} + \varepsilon - p\right)\tau + 3\Lambda\tau, \tag{3.6a}$$

$$\dot{\varepsilon} = -\frac{\dot{\tau}}{\tau}\omega + \left(\xi + \frac{4}{3}\eta\right)\frac{\dot{\tau}^2}{\tau^2} - 4\eta\left[\kappa\left(\frac{m}{\tau} - \frac{\lambda}{\tau^n}\right) + \Lambda\right], \tag{3.6b}$$

or in terms of H

$$\dot{\tau} = 3H\tau, \tag{3.7a}$$

$$\dot{H} = \frac{1}{2}(3\xi H - \omega) - (3H^2 - \kappa\varepsilon - \Lambda) + \frac{\kappa}{2}\left(\frac{m}{\tau} + \frac{\lambda(n-2)}{\tau^n}\right), \tag{3.7b}$$

$$\dot{\varepsilon} = 3H(3\xi H - \omega) + 4\eta(3H^2 - \kappa\varepsilon - \Lambda) - 4\eta\kappa\left[\frac{m}{\tau} - \frac{\lambda}{\tau^n}\right]. \tag{3.7c}$$

Here η and ξ are the bulk and shear viscosity, respectively and they are both positively definite, i.e.,

$$\eta > 0, \qquad \xi > 0. \tag{3.8}$$

They may be either constant or function of time or energy. We consider the case when

$$\eta = A\varepsilon^\alpha, \qquad \xi = B\varepsilon^\beta, \tag{3.9}$$

with A and B being some positive quantities. For p we set as in perfect fluid,

$$p = \zeta\varepsilon, \quad \zeta \in (0, 1]. \tag{3.10}$$

Note that in this case $\zeta \neq 0$, since for dust pressure, hence temperature is zero, that results in vanishing viscosity.

The system (3.7) without spinor field have been extensively studied in literature either partially (Murphy 1973; Huang 1990; Banerjee et al. 1985) or as a whole (Belinski and Khalatnikov 1975). Here we try to solve the system (3.6) for some particular choice of parameters.

3.2.1 *Case with bulk viscosity*

Let us first consider the case with bulk viscosity alone setting coefficient of shear viscosity $\eta = 0$. We also demand the coefficient of bulk viscosity be inverse proportional to expansion, i.e.,

$$\xi\theta = 3\xi H = C_2, \quad C_2 = \text{const.} \tag{3.11}$$

Inserting $\eta = 0$, (3.11) and (3.10) into (3.7c) one finds

$$\varepsilon = \frac{1}{1+\zeta}\left[C_2 - \frac{C_3}{\tau^{1+\zeta}}\right], \quad C_3 = \text{const.} \tag{3.12}$$

Then from (3.6a) we get the following equation for determining τ:

$$\ddot{\tau} = \frac{3\kappa}{2}m + 3\left[\frac{C_2}{2}\kappa + \Lambda\right]\tau + \frac{3\kappa(1-\zeta)}{2(1+\zeta)}\frac{C_2\tau^{1+\zeta} - C_3}{\tau^\zeta} + \frac{3\kappa}{2}\frac{\lambda(n-2)}{\tau^{n-1}} \equiv \mathscr{F}(q,\tau), \tag{3.13}$$

where q is the set of problem parameters. As one sees, the right hand side of (3.13) is a function of τ, hence can be solved in quadrature (Kamke 1957). We solve (3.13) numerically. It can be noted that (3.13) can be viewed as one describing the motion of a single particle. Sometimes it is useful to plot the potential of the corresponding equation which in this case is

$$\mathscr{U}(q,\tau) = -2\int \mathscr{F}(q,\tau)d\tau. \tag{3.14}$$

The problem parameters are chosen as follows: $\kappa = 1$, $m = 1$, $\lambda = 0.5$, $\zeta = 1/3$, $n = 4$, $C_2 = 2$ and $C_3 = 1$. Here we consider the cases with different Λ, namely with $\Lambda = -2, 0, 1$, respectively. The initial value of τ is taken to be a small one, whereas, the first derivative of τ, i.e., $\dot{\tau}$ at that point of time is calculated from (2.40). In Fig. 1 we have illustrated the potential corresponding to (3.13). As one sees, independent to the sign of Λ we have the expanding mode of evolution, though a positive Λ accelerates the process, while the negative one decelerates. Corresponding behavior of τ is given in Fig. 2.

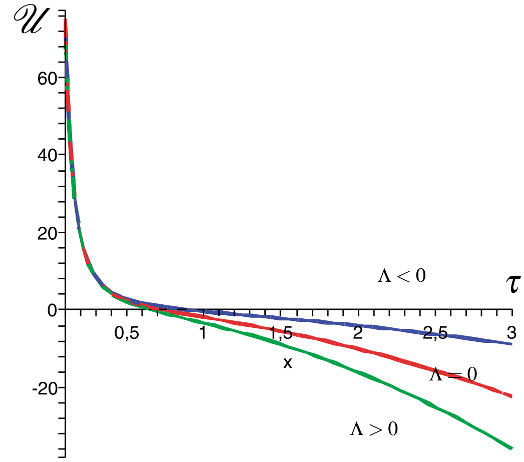

Fig. 1 View of the potential corresponding to the different sign of the Λ term

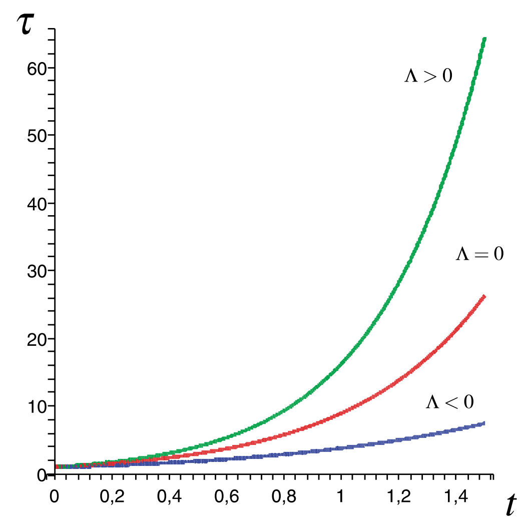

Fig. 2 Evolution of τ depending on the signs of the Λ term

3.2.2 *Case with bulk and shear viscosities*

Let us consider a more general case. Following (Saha 2005a) we choose the shear viscosity being proportional to the expansion, namely,

$$\eta = -\frac{3}{2\kappa}H = -\frac{1}{2\kappa}\theta. \tag{3.15}$$

In absence of spinor field this assumption leads to

$$3H^2 = \kappa\varepsilon + C_4, \quad C_4 = \text{const.} \tag{3.16}$$

It can be shown that the relation (3.16) in our case can be achieved only for massless spinor field with the nonlinear term being

$$F = F_0 S^{2(\kappa-1)/\kappa}.$$

Equation for τ in this case has the form

$$\tau\ddot{\tau} - 0.5(1-\zeta)\dot{\tau}^2 - 1.5\kappa\xi\tau\dot{\tau} - 3[\Lambda - 0.5(1-\zeta)C_4 - \lambda F_0\tau^{2(1-\kappa)/\kappa}]\tau^2 = 0. \tag{3.17}$$

In case of $\xi = \text{const.}$ and $\lambda = 0$ there exists several special solutions available in handbooks on differential equations. But for nonzero λ we can investigate this equation only numerically. We consider the case when the bulk viscosity is given by a constant. Taking this into account for problem parameters we set $\zeta = 1/3$, $\xi = 1$, $F_0 = 1$, $\lambda = 0.5$ and $C_4 = 1$. We study the role of Λ term. In doing this we consider the cases with positive, negative and trivial Λ. Since the nonlinear term in this case depends on κ, we also consider the cases with different κ, namely with $\kappa > 1$ and $\kappa < 1$. In Figs. 3 and 4 the evolution of τ is illustrated for $\kappa < 1$ and $\kappa > 1$, respectively. In case of $\kappa < 1$ we have non-periodic mode of evolution for all Λ, while for $\kappa > 1$ a negative Λ gives a non-periodic mode of expansion. A non-negative Λ in this case gives an ever expanding mode of evolution.

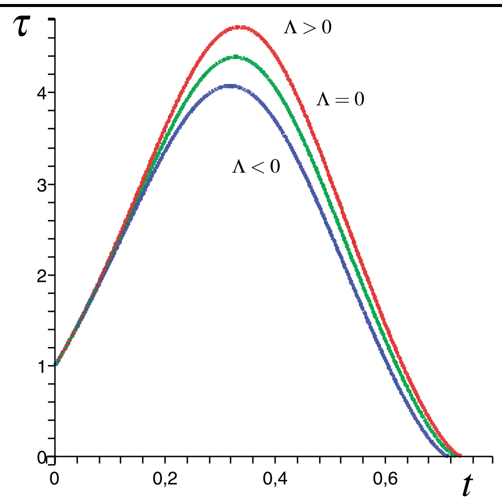

Fig. 3 Evolution of the universe with nontrivial Λ term and $\kappa < 1$

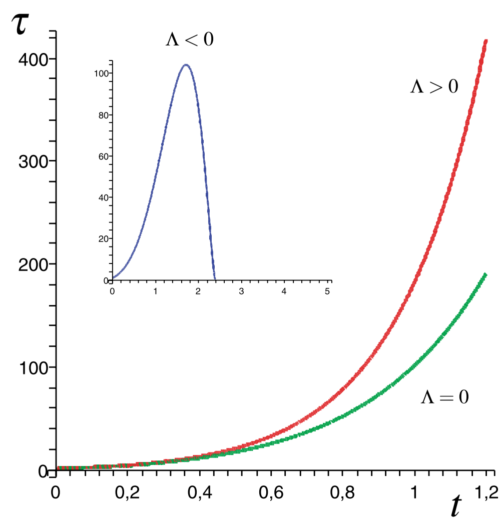

Fig. 4 Evolution of the universe for different values of Λ term with $\kappa > 1$

4 Conclusion

We consider a self consistent system of nonlinear spinor and gravitational fields within the framework of Bianchi type-I cosmological model filled with viscous fluid. The spinor filed nonlinearity is taken to be some power law of the invariants of bilinear spinor forms. Solutions to the corresponding equations are given in terms of the volume scale of the BI space-time, i.e., in terms of $\tau = abc$. The system of equations for determining τ, energy-density of the viscous fluid ε and Hubble parameter H has been worked out. Exact solution to the aforementioned system has been given only for the case of bulk viscosity. As one sees from (2.42) or (2.51), the system in question is a multi-parametric one and may have several solutions depending on the choice of the problem parameters. As one sees, solutions can be non-periodic independent to the sign of Λ term. Given the richness of (3.6) we plan to give qualitative analysis of this system in near future.

References

Armendáriz-Picón, C., Greene, P.B.: Gen. Relativ. Gravit. **35**, 1637 (2003)
Banerjee, A., Duttachoudhury, S.B., Sanyal, A.K.: J. Math. Phys. **26**, 3010 (1985)
Barrow, J.D.: Nucl. Phys. B **310**, 743 (1988)
Belinski, V.A., Khalatnikov, I.M.: J. Experimental Theor. Phys. **69**, 401 (1975)
Berestetski, V.B., Lifshitz, E.M., Pitaevski, L.P.: Quantum Electrodynamics. Nauka, Moscow (1989)
Brill, D., Wheeler, J.: Rev. Mod. Phys. **29**, 465 (1957)
Chimento, L.P.: Phys. Rev. D **68**, 023504 (2003)
Chimento, L.P., Jacubi, A.S., Mèndez, V., Maartens, R.: Class. Quantum Gravity **14**, 3363 (1997)
Desikan, K.: Gen. Relativ. Gravit. **29**, 435 (1997)
Finkelstein, R., LeLevier, R., Ruderman, M.: Phys. Rev. **83**, 326 (1951)
Gavrilov, V.R., Melnikov, V.N., Triay, R.: Class. Quantum Gravity **14**, 2203 (1997)
Guth, A.: Phys. Rev. D **23**, 347 (1981)
Heisenberg, W.: Introduction to the Unified Field Theory of Elementary Particles. Interscience Publ., London (1966)
Huang, W.: J. Math. Phys. **31**, 1456 (1990)
Jacobs, K.C.: Astrophys. J. **153**, 661 (1968)
Kamke, E.: Differentialgleichungen losungsmethoden und losungen. Leipzig, Akademische Verlagsgesellschaft (1957)
Kibble, T.W.B.: J. Math. Phys. **2**, 212 (1961)
Krori, K.D., Mukherjee, A.: Gen. Relativ. Gravit. **32**, 1429 (2000)
Langacker, P.: Phys. Rep. **72**, 185 (1981)
Misner, W.: Nature **214**, 40 (1967)
Misner, W.: Astrophys. J. **151**, 431 (1968)
Murphy, G.L.: Phys. Rev. D **8**, 4231 (1973)
Pacher, T., Stein-Schabas, J.A., Turner, M.S.: Phys. Rev. D **36**, 1603 (1987)
Saha, B.: Phys. Rev. D **64**, 123501 (2001a)
Saha, B.: Mod. Phys. Lett. A **16**, 1287 (2001b)
Saha, B.: Mod. Phys. Lett. A **20**(28), 2127 (2005a), arXiv: gr-qc/0409104
Saha, B.: Romanian Rep. Phys. **57**(1), 7 (2005b)
Saha, B., Rikhvitsky, V.: Physica D **219**, 168 (2006), arXiv: gr-qc/0410056
Saha, B., Shikin, G.N.: J. Math. Phys. **38**, 5305 (1997a)
Saha, B., Shikin, G.N.: Gen. Relativ. Gravit. **29**, 1099 (1997b)
Santos, N.O., Dias, R.S., Banerjee, A.: J. Math. Phys. **26**, 876 (1985)
van Elst, H., Dunsby, P., Tavakol, R.: Gen. Relativ. Gravit. **27**, 171 (1995)
Waga, L., Falcan, R.C., Chanda, R.: Phys. Rev. D **33**, 1839 (1986)
Weinberg, S.: Astrophys. J. **168**, 175 (1972a)
Weinberg, S.: Gravitation and Cosmology. New York, Wiley (1972b)
Zhelnorovich, V.A.: Spinor Theory and Its Application in Physics and Mechanics. Nauka, Moscow (1982)