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Lorentz Transformation of Toroid Polarization

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In a recent paper [1] a modified system of equations of electrodynamics of moving continuous media has been obtained in account of toroid polarizations. In this letter it has been shown that these equations are invariant under Lorentz transformations. The transformation law for the toroid polarizations has also been worked out.

Keywords toroidal moments, Lorentz transformations

In the early fifties, while solving the problem of the multipole radiation of a spatially bounded source, Franz and Wallace [2,3] found a contribution to the electric part of radiation at the expense of magnetization. Further Ya. Zel'dovich [4] pointed out the non-correspondence between the existence of two known multipole sets, Coulomb and magnetic, and the number of form-factors for a spin $-\frac{1}{2}$ charged particles. Following the parity non-conservation law in weak interactions Zel'dovich suggested a third form-factor in the parametrization of the Dirac spinor particle current. As a classical counterpart of this form-factor he introduced *anapole* in connection with the global electromagnetic properties of a toroid coil that are impossible to describe within the charge or magnetic dipole moments in spite of explicit axial symmetry of the toroid coil. In 1963 Shirokov and Cheshkov [5] constructed the parametrization for relativistic matrix elements of currents of charged and spinning particle, which contain the third set of form-factors. Finally, in 1974 Dubovik and Cheskov [6] determined the toroid moment in the framework of classical electrodynamics. Note that *anapole* and *toroid dipole* are not the different names of one and the same thing. They are indeed quite different in nature. For example, the anapole cannot radiate at all while the toroid coil and its point-like model, toroid dipole, can. The matter is that the anapole is some composition of electric dipole and actual toroid dipole giving destructive interference of their radiation.

Recently a principally new type of magnetism known as *aromagnetism* was observed in a class of organic substances, suspended either in water or in other liquids [7]. Later, it was shown that this phenomena of aromagnetism cannot be explained in a standard way, e.g., by ferromagnetism, since the organic molecules do not possess magnetic moments of either orbital or spin origin. It was also shown that the origin of aromagnetism is the interaction of vortex electric field induced by alternative magnetic one with the axial toroid moments in aromatic substances [8].

In a recent work Dubovik and Kuznetsov [9] calculated the toroid moment of Majorana neutrino. It was also pointed out that the magnitude of the toroid dipole moment of a Dirac neutrino (ν_d) is just the half of that of a Majorana one (ν_m) and both of them possesses non-trivial toroid moments even if $m_\nu = 0$ [10]. The study of toroid moments in high energy physics indicates its importance in modern physics. Beside the works mentioned above we would like to refer the paper by Rubin [11], about applications of Toroidal moments in relativistic anyons theory and the need in considering generalized Toroidal 4-moments as effective 4-vector potentials.

The latest theoretical and experimental development force the introduction of toroid moments in the framework of conventional classical electrodynamics that in its part inevitably leads to the modification of the equations of electromagnetism and the equations of motion of particles in external electromagnetic field. Two alternative schemes of introduction of toroid polarizations in the electromagnetic equations and two potential formalism of electrodynamics were given in [1,12].

In this letter we will consider the toroid polarizations subject to Lorentz transformation. For histories sake, we note that one to the earliest attempts to give an alternative description of electrodynamics of moving bodies was undertaken by E Cohn as early as 1902 [13]. Here we will work within the framework of Lorentz transformation.

To begin with we write the Maxwell equations for electromagnetic fields in vacuum, in the presence of extraneous electric charge ρ and electric current, i.e., charge - in - motion, of density $\mathbf{j}$.

$$\mathrm{curl}\mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c}\mathbf{j} \tag{1a}$$

$$\mathrm{div}\mathbf{E} = 4\pi\rho \tag{1b}$$

$$\mathrm{curl}\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0 \tag{1c}$$

$$\mathrm{div}\mathbf{B} = 0 \tag{1d}$$

where $\mathbf{E}$ and $\mathbf{B}$ are the flux densities of electric field and magnetic induction, respectively As we mentioned earlier, the system (1) should be rewritten taking the toroid polarizations into account. The details of the calculations one may find in [1,12]. Here we write the final system as follows:

$$\mathrm{curl}\,\boldsymbol{\mathcal{H}} = \frac{1}{c}\frac{\partial \boldsymbol{\mathcal{D}}}{\partial t} + \frac{4\pi}{c}\mathbf{j} \tag{2a}$$

$$\mathrm{div}\boldsymbol{\mathcal{D}} = 4\pi\rho \tag{2b}$$

$$\mathrm{curl}\boldsymbol{\mathcal{E}} = -\frac{1}{c}\frac{\partial \boldsymbol{\mathcal{B}}}{\partial t} \tag{2c}$$

$$\mathrm{div}\boldsymbol{\mathcal{B}} = 0 \tag{2d}$$

where we define

$$
\begin{aligned}
\mathcal{E} &= \mathbf{E} + 4\pi\mathrm{curl}\mathbf{T}^e, & \mathcal{H} &= \mathbf{H} - 4\pi\mathrm{curl}\mathbf{T}^m, \\
\mathcal{D} &= \mathbf{D} + 4\pi\mathrm{curl}\mathbf{T}^e, & \mathcal{B} &= \mathbf{B} - 4\pi\mathrm{curl}\mathbf{T}^m, \\
\mathcal{D} &= \mathcal{E} + 4\pi\,\mathbf{P}, & \mathcal{H} &= \mathcal{B} - 4\pi\,\mathbf{M}. \\
\mathbf{D} &= \mathbf{E} + 4\pi\,\mathbf{P}, & \mathbf{H} &= \mathbf{B} - 4\pi\,\mathbf{M}.
\end{aligned}
\tag{3}
$$

Indeed, due to the introduction of toroid polarizations, having independent origin in terms of atomic and molecular current and charge distributions, the quantities $\mathbf{B}$ and $\mathbf{D}$ as well as $\mathbf{E}$ and $\mathbf{H}$ lost their initial meaning. The existence of the vorticities $\mathbf{T}^e$ and $\mathbf{T}^m$, generally speaking, can be imputed to the one and the same physical volume.

Finally we introdocuce the two-potential formalism that was developed by us earlier [1,12] defining $\mathcal{B}$ and $\mathcal{E}$ in the following way:

$$\mathcal{B} = \mathrm{curl}\boldsymbol{\alpha}^m + \frac{1}{c}\frac{\partial\boldsymbol{\alpha}^e}{\partial t} + \nabla\varphi^m, \tag{4a}$$

$$\mathcal{E} = \mathrm{curl}\boldsymbol{\alpha}^e - \frac{1}{c}\frac{\partial\boldsymbol{\alpha}^m}{\partial t} - \nabla\varphi^e \tag{4b}$$

Inserting (4) into (2) we find the potential form of the electro-magneto-toroic equations:

$$\Box\,\boldsymbol{\alpha}^m = -\frac{4\pi}{c}\big(\mathbf{j}_{\text{free}} + \frac{\partial\mathbf{P}}{\partial t} + c\,\mathrm{curl}\,\mathbf{M}\big) \tag{5a}$$

$$\Box\,\varphi^m = 0 \tag{5b}$$

$$\Box\,\boldsymbol{\alpha}^e = 0 \tag{5c}$$

$$\Box\,\varphi^e = -4\pi\,(\rho - \mathrm{div}\mathbf{P}) \tag{5d}$$

We now write the transformation law that leaves the systems (5) and (2) Lorentz covariant under

$$\mathrm{div}\,\boldsymbol{\alpha}^{m,e} + (1/c)(\partial\varphi^{e,m}/\partial t) = 0. \tag{6}$$

Connecting the fields in stationary frame (unprimed) with those in moving one (primed) in the following way

$$\mathbf{P} = \gamma\big(\mathbf{P}' + \boldsymbol{\beta}\times\mathbf{M}'\big) - \frac{\gamma-1}{\beta^2}(\mathbf{P}'\cdot\boldsymbol{\beta})\boldsymbol{\beta} \tag{7a}$$

$$\mathbf{M} = \gamma\big(\mathbf{M}' - \boldsymbol{\beta}\times\mathbf{P}'\big) - \frac{\gamma-1}{\beta^2}(\mathbf{M}'\cdot\boldsymbol{\beta})\boldsymbol{\beta} \tag{7b}$$

$$\boldsymbol{\alpha}^m = \boldsymbol{\alpha}^{m\prime} + \gamma\boldsymbol{\beta}\varphi^{e\prime} + \frac{\gamma-1}{\beta^2}(\boldsymbol{\alpha}^{m\prime}\cdot\boldsymbol{\beta})\boldsymbol{\beta} \tag{7c}$$

$$\varphi^e = \gamma\big(\varphi^{e\prime} + \boldsymbol{\beta}\cdot\boldsymbol{\alpha}^{m\prime}\big) \tag{7d}$$

$$\boldsymbol{\alpha}^e = \boldsymbol{\alpha}^{e\prime} + \gamma\boldsymbol{\beta}\varphi^{m\prime} + \frac{\gamma-1}{\beta^2}(\boldsymbol{\alpha}^{e\prime}\cdot\boldsymbol{\beta})\boldsymbol{\beta} \tag{7e}$$

$$\varphi^m = \gamma\big(\varphi^{m\prime} + \boldsymbol{\beta}\cdot\boldsymbol{\alpha}^{e\prime}\big) \tag{7f}$$

$$\rho = \gamma\big(\rho' + \frac{1}{c}(\boldsymbol{\beta}\cdot\mathbf{J}')\big) \tag{7g}$$

$$\mathbf{J} = \mathbf{J}' + \gamma\boldsymbol{\beta}\rho' + \frac{\gamma-1}{\beta^2}(\mathbf{J}'\cdot\boldsymbol{\beta})\boldsymbol{\beta} \tag{7h}$$

one can easily show that the system (5) is Lorentz covariant. Inserting (7c – 7f) into (4) one finds that the vectors $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{B}}$ transform in the following way

$$\begin{aligned} \boldsymbol{\mathcal{E}} &= \gamma(\boldsymbol{\mathcal{E}}' + \boldsymbol{\beta} \times \boldsymbol{\mathcal{B}}') - \frac{\gamma - 1}{\beta^2}(\boldsymbol{\mathcal{E}}' \cdot \boldsymbol{\beta})\boldsymbol{\beta} \\ \boldsymbol{\mathcal{B}} &= \gamma(\boldsymbol{\mathcal{B}}' - \boldsymbol{\beta} \times \boldsymbol{\mathcal{E}}') - \frac{\gamma - 1}{\beta^2}(\boldsymbol{\mathcal{B}}' \cdot \boldsymbol{\beta})\boldsymbol{\beta} \end{aligned} \tag{8}$$

Finally, we write the transformation law for toroid dipole polarizations. In doing so, we underline a very original relation between the vector potential and toroid moment [14]

$$\operatorname{curl}\operatorname{curl}\mathbf{A} = 4\pi c \operatorname{curl}\operatorname{curl}\mathbf{T}\delta(\mathbf{r}) \tag{9}$$

On the other hand we determine

$$\boldsymbol{\mathcal{B}} = \mathbf{B} + \operatorname{curl}\mathbf{T}^m = \operatorname{curl}\mathbf{A} + \operatorname{curl}\mathbf{T}^m.$$

All these facts suggest us to handle $\mathbf{A}$ and $\mathbf{T}$ in the same manner. In other words, $\mathbf{T}^{e,m}$ should form the space part of some four-vectors. To this end, we introduce two four-vectors

$$\Pi^e = (\phi^e, \mathbf{T}^e), \quad \Pi^m = (\phi^m, \mathbf{T}^m) \tag{10}$$

with $\phi^{e,m}$ being the scalar parts of the four-vectors $\Pi^{e,m}$. Now we can write down the transformation law for toroid polarization, which is exactly the same for vector potentials, precisely,

$$\mathbf{T}^m = \mathbf{T}^{m\prime} + \gamma\boldsymbol{\beta}\phi^{m\prime} + \frac{\gamma - 1}{\beta^2}(\mathbf{T}^{m\prime} \cdot \boldsymbol{\beta})\boldsymbol{\beta} \tag{11a}$$

$$\phi^m = \gamma(\phi^{m\prime} + \boldsymbol{\beta} \cdot \mathbf{T}^{m\prime}) \tag{11b}$$

$$\mathbf{T}^e = \mathbf{T}^{e\prime} + \gamma\boldsymbol{\beta}\phi^{e\prime} + \frac{\gamma - 1}{\beta^2}(\mathbf{T}^{e\prime} \cdot \boldsymbol{\beta})\boldsymbol{\beta} \tag{11c}$$

$$\phi^e = \gamma(\phi^{e\prime} + \boldsymbol{\beta} \cdot \mathbf{T}^{e\prime}) \tag{11d}$$

with the additional condition

$$\operatorname{curl}\mathbf{T}^{e,m} = \pm\Big(\frac{1}{c}\frac{\partial \mathbf{T}^{m,e}}{\partial t} + \boldsymbol{\nabla}\phi^{m,e}\Big) \tag{12}$$

to be fulfilled. Naturally arises the question "what do the $\phi^{e,m}$ stand for"? To answer this question let us write the multipole expansions of scalar Φ and vector $\mathbf{A}$ potentials:

$$\begin{aligned} \Phi(\mathbf{r}, t) \approx{}& \frac{1}{R}D^{(0)} + \frac{R_\beta}{R^2}D^{(1)}_\beta + \\ & \frac{3R_\beta R_\gamma - R^2\delta_{\beta\gamma}}{2R^4}\Big(2D^{(2)}_{\beta\gamma} + \frac{1}{3}D^{(0)}_2\delta_{\beta\gamma}\Big) + \cdots \end{aligned} \tag{13a}$$

$$\mathbf{A}_\alpha(\mathbf{r}, t) \approx \frac{1}{cR}\dot{D}^{(1)}_\alpha + \frac{R_\beta}{cR^2}\Big[\dot{D}^{(2)}_{\alpha\beta} - c\epsilon_{\gamma\alpha\beta}M^{(1)}_\gamma + \frac{1}{6}\dot{D}^{(0)}_2\delta_{\alpha\beta}\Big] +$$

$$\frac{3R_\beta R_\gamma - R^2\delta_{\beta\gamma}}{2cR^4}\Bigg[\dot{D}^{(3)}_{\alpha\beta\gamma} - c\Big(\epsilon_{\alpha\beta\mu}M^{(2)}_{\mu\gamma} + \epsilon_{\alpha\gamma\mu}M^{(2)}_{\mu\beta}\Big) +$$
$$\Big(D^{(1)}_{2\beta}\delta_{\alpha\gamma} + \dot{D}^{(1)}_{2\gamma}\delta_{\alpha\beta}\Big) -$$
$$c\Big\{2T^{(1)}_\alpha\delta_{\beta\gamma} - T^{(1)}_\beta\delta_{\alpha\gamma} - T^{(1)}_\gamma\delta_{\alpha\beta}\Big\}\Bigg] \cdot\cdot \tag{13b}$$

where we denote

- $D^{(0)} = \int \rho(\mathbf{r},t)d\mathbf{r}$ - total charge
- $D^{(0)}_2 = \int r^2\,\rho(\mathbf{r},t)d\mathbf{r}$ - scalar mean-square radii
- $D^{(1)}_{2\alpha} = \int r^2 r_\alpha\,\rho(\mathbf{r},t)d\mathbf{r}$ - vector mean-square radii
- $D^{(1)}_\alpha = \int r^{(1)}_\alpha\,\rho(\mathbf{r},t)d\mathbf{r}$ - electric dipole moment
- $D^{(2)}_{\alpha\beta} = \int r^{(2)}_{\alpha\beta}\,\rho(\mathbf{r},t)d\mathbf{r}$ - electric quadrupole moment
- $D^{(3)}_{\alpha\beta\gamma} = \int r^{(3)}_{\alpha\beta\gamma}\,\rho(\mathbf{r},t)d\mathbf{r}$ - electric octupole moment
- $M^{(1)}_\alpha = \frac{1}{2c}\int [\mathbf{r}\times\mathbf{j}]_\alpha d\mathbf{r}$ - magnetic dipole moment
- $M^{(2)}_{\alpha\beta} = \frac{1}{6c}\int \{[\mathbf{r}\times\mathbf{j}]_\alpha r_\beta + [\mathbf{r}\times\mathbf{j}]_\beta r_\alpha\}d\mathbf{r}$ - magnetic quadrupole moment
- $T^{(1)}_\alpha = \frac{1}{10c}\int (r_\alpha r_\beta j_\beta - r^2 j_\alpha)d\mathbf{r}$ - toroid dipole moment

where $r^1_\alpha = r_\alpha$, $r^{(2)}_{\alpha\beta} = \frac{1}{2}(r_\alpha r_\beta - \frac{1}{3}r^2\delta_{\alpha\beta})$ and $r^{(3)}_{\alpha\beta\gamma} = \frac{1}{6}\big(r_\alpha r_\beta r_\gamma - \frac{r^2}{5}(r_\alpha\delta_{\beta\gamma} + r_\beta\delta_{\gamma\alpha} + r_\gamma\delta_{\alpha\beta})\big)$. From (13) follow that the toroid moment contains in the third order of Taylor expansion of $\mathbf{A}$ whereas the third order multipole expansion of Φ consists of scalar mean-square-radii and electric quadrupole moment Thus it makes sense to connect the scalar part of four-vectors Π, i.e., ϕ with the scalar mean-square radii and quadrupole moment depending on the orientation of motion of the torus itself. Here we would like to note that a detailed analysis of the relations between $\mathbf{A}$ and $\mathbf{T}$ indicates that beside the scalar mean-square radii and quadrupole moment the quantity ϕ can also be connected with the dipole moment. Thus we can write down the transformation law for the toroid polarizations as

$$(\mathbf{T}^m)_i = (\mathbf{T}^{m'})_i + \gamma\beta_i D^{(0)}_2 + \frac{\gamma-1}{\beta^2}(\mathbf{T}^{m'}\ \boldsymbol{\beta})\beta_i$$
$$(\mathbf{T}^m)_i = (\mathbf{T}^{m'})_i + \gamma\beta_k D^{(2)}_{ki} + \frac{\gamma-1}{\beta^2}(\mathbf{T}^{m'}\ \boldsymbol{\beta})\beta_i$$

References

[1] V.M. Dubovik and B. Saha, cond-mat/9806043 cond-mat/9806057

[2] W. Franz *Zeitschrift fur Physik* **127**, 363 (1950)

[3] P.R. Wallace *Physical Review* **81**, 493 (1951), **82**, 297 (1951)

[4] Ya.B Zel'dovich. *Zhurnal Eksperimentalnoi i Teoreticheskoi Fiziki*, **33**, 1531 (1957) *Soviet Physics JETP*, **6**, 1184 (1958)
[5] A.A. Cheshkov and Yu.M. Shirokov, *Zhurnal Eksperimentalnoi i Teoreticheskoi Fiziki*, **44**, 1982 (1963); *Soviet Physics JETP*, **17**, 1333 (1963)
[6] V.M Dubovik and A Cheshkov, *Fizika Elementarnikh Chastits i Atomnogo Yadra* **5**, 791 (1974); *Soviet Journal of Particles & Nuclei*, **5**, 318 (1975)
[7] A.A. Spartakov, and N.A. Tolstoi. *Pis'ma v Zhurnal Eksperimentalnoi i Teoreticheskoi Fiziki*, **52**(3), 796 (1990); *JETP Letters*, **52**(3), 161 (1990).
[8] M.A Martsenyuk, and N.M. Martsenyuk. *Pis'ma v Zhurnal Eksperimentalnoi i Teoreticheskoi Fiziki*, **53**(5), 229 (1991); *JETP Letters*, **53**(5), 243 (1991)
[9] V.M. Dubovik, and V.E. Kuznetsov, *JINR Preprint*, E2–96–53 (1996).
[10] E.N. Bukina, V.M Dubovik, and V.E. Kuznetsov, *Yadernaya Fizika*, **61**(6), 1129 (1998); *Physics of Atomic Nuclei*, **61**(6), 1035 (1998).
[11] J.L. Rubin, *Ferroelectrics*, **161**(1–4), 335 (1994).
[12] V.M. Dubovik, M.A. Martsenyuk and B. Saha. *Journal of Theoretical, Mathematical and Computational Physics*, **2**(1), 27 (1999); *ICTP Internal Report*, IC/IR/96/9; cond-mat/9603140.
[13] O. Darrigol, *American Journal of Physics*, **63**(10), 908 (1998).
[14] V.M. Dubovik, and V.V. Tugushev, *Physics Reports*, **187**(4), 145 (1990).